Structural Analysis III

Compatibility of Displacements

&

Principle of Superposition

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1. Introduction

1.1 Background

In the case of 2-dimensional structures there are three equations of statics:

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum M = 0$$

Thus only three unknowns (reactions etc.) can be solved for using these equations alone. Structures that cannot be solved through the equations of static equilibrium alone are known as statically indeterminate structures. These, then, are structures that have more than 3 unknowns to be solved for.

In order to solve statically indeterminate structures we must identify other knowns about the structure. These other knowns are usually displacements, such as those at the supports. When these are combined with the principle of superposition, indeterminate structures can be solved.

2. Compatibility of Displacements

2.1 Description

When a structure is loaded it deforms under that load. Points that were connected to each other remain connected to each other, though the distance between them may have altered due to the deformation. All the points in a structure do this is such a way that the structure remains fitted together in its original configuration.

Compatibility of displacement is thus:

> *Displacements are said to be compatible when the deformed members of a loaded structure continue to fit together.*

Thus, compatibility means that:

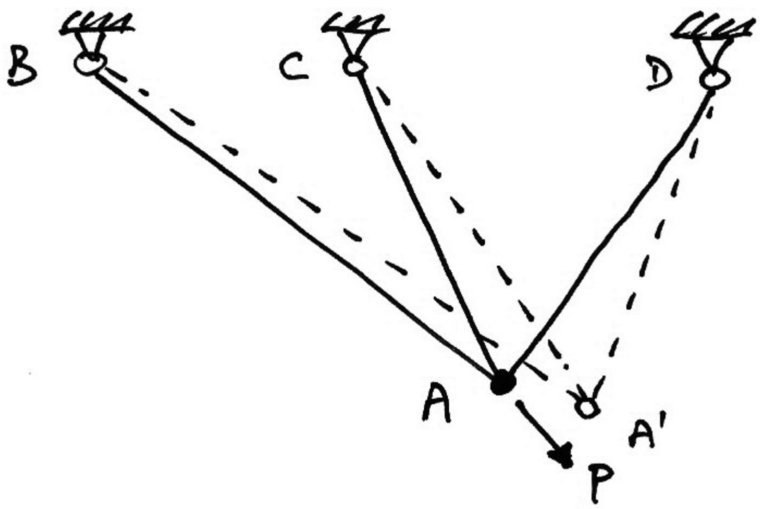
- Two initially separate points do not move to another common point;
- Holes do not appear as a structure deforms;
- Members initially connected together remain connected together.

This deceptively simple idea is very powerful when applied to indeterminate structures.

2.2 Examples

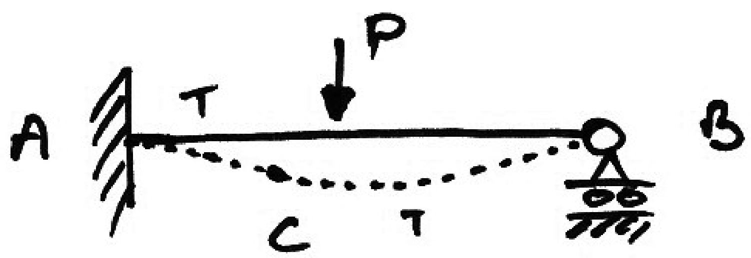
Truss

The following truss is indeterminate. Each of the members has a force in it and consequently undergoes elongation. However, by compatibility of displacements, the elongations must be such that the three members remain connected after loading, even though the truss deforms and Point *A* moves to Point *A'*. This is an extra piece of information (or 'known') and this helps us solve the structure.



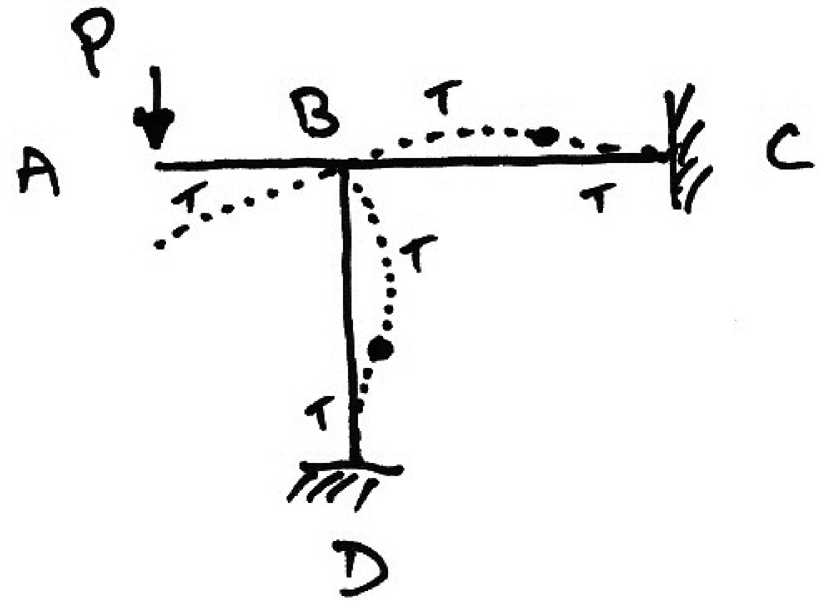
Beam

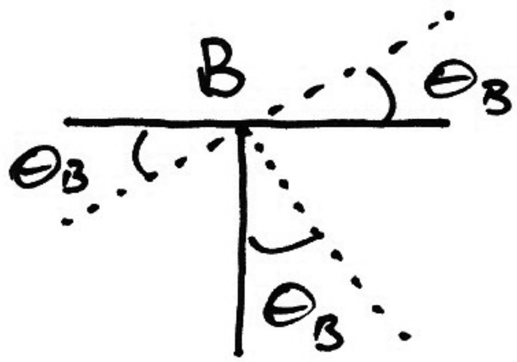
The following propped cantilever is an indeterminate structure. However, we know by compatibility of displacements that the deflection at point B is zero before and after loading, since it is a support.



Frame

The following frame has three members connected at joint *B*. The load at *A* causes joint *B* to rotate anti-clockwise. The ends of the other two members connected at *B* must also undergo an anti-clockwise rotation at *B* to maintain compatibility of displacement. Thus all members at *B* rotate the same amount, θ_B, as shown below.



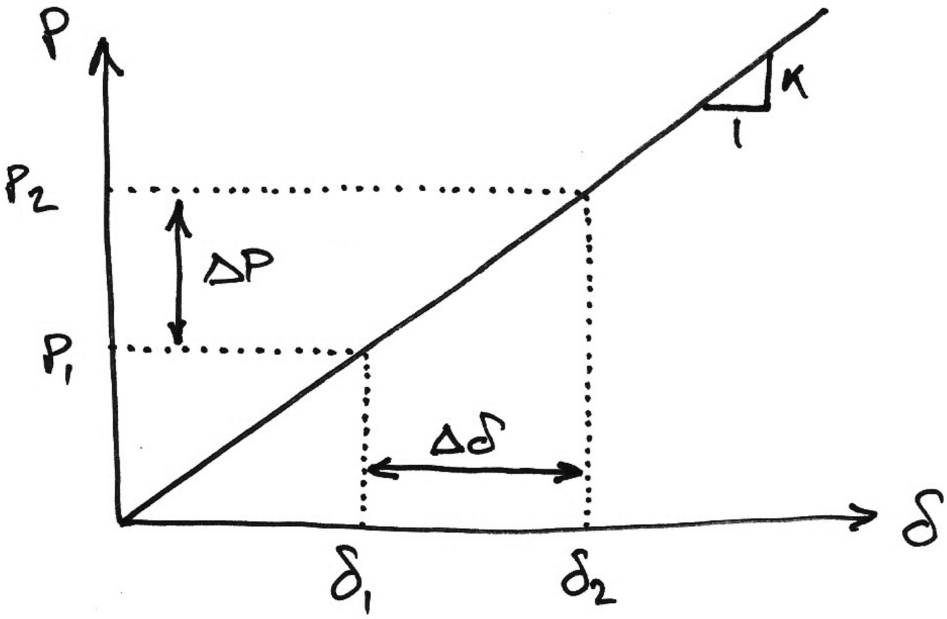


Joint *B*

3. Principle of Superposition

3.1 Development

For a linearly elastic structure, load, *P*, and deformation, δ, are related through stiffness, *K*, as shown:



For an initial load on the structure we have:

$$P_1 = K \cdot \delta_1$$

If we instead we had applied ΔP we would have gotten:

$$\Delta P = K \cdot \Delta\delta$$

Now instead of applying ΔP separately to P_1 we apply it after P_1 is already applied. The final forces and deflections are got by adding the equations:

$$P_1 + \Delta P = K \cdot \delta_1 + K \cdot \Delta\delta$$
$$= K\left(\delta_1 + \Delta\delta\right)$$

But, since from the diagram, $P_2 = P_1 + \Delta P$ and $\delta_2 = \delta_1 + \Delta\delta$, we have:

$$P_2 = K \cdot \delta_2$$

which is a result we expected.

This result, though again deceptively 'obvious', tells us that:

- Deflection caused by a force can be added to the deflection caused by another force to get the deflection resulting from both forces being applied;
- The order of loading is not important (ΔP or P_1 could be first);
- Loads and their resulting load effects can be added or subtracted for a structure.

This is the **Principle of Superposition**:

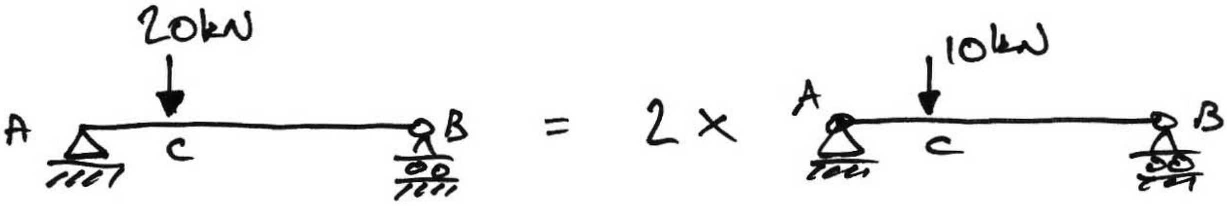
For a linearly elastic structure, the load effects caused by two or more loadings are the sum of the load effects caused by each loading separately.

Note that the principle is limited to:

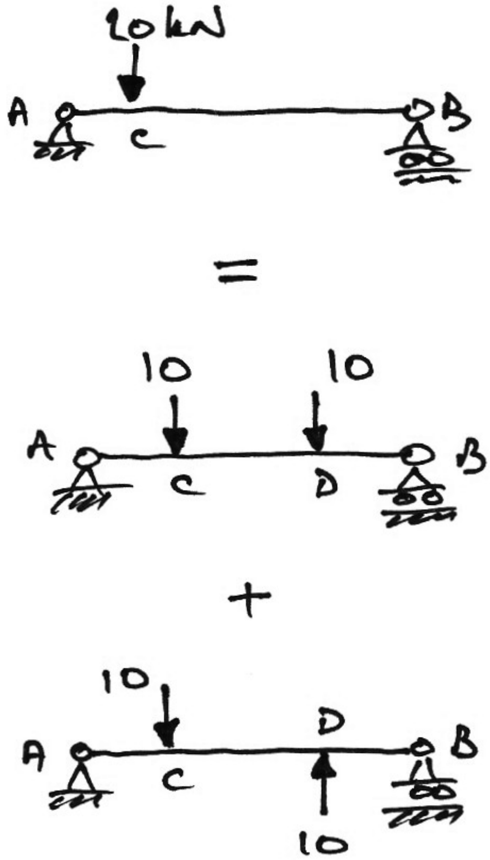
- Linear material behaviour only;
- Structures undergoing small deformations only (linear geometry).

3.2 Example

If we take a simply-supported beam, we can see that its solutions can be arrived at by multiplying the solution of another beam:



The above is quite obvious, but not so obvious is that we can also break the beam up as follows:



Thus the principle is very flexible and useful in solving structures.

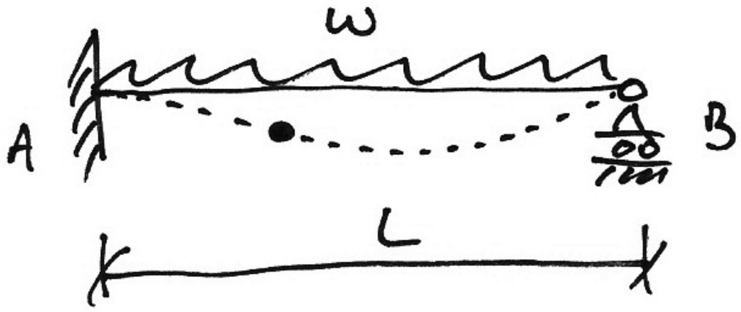
4. Solving Indeterminate Structures

4.1 Introduction

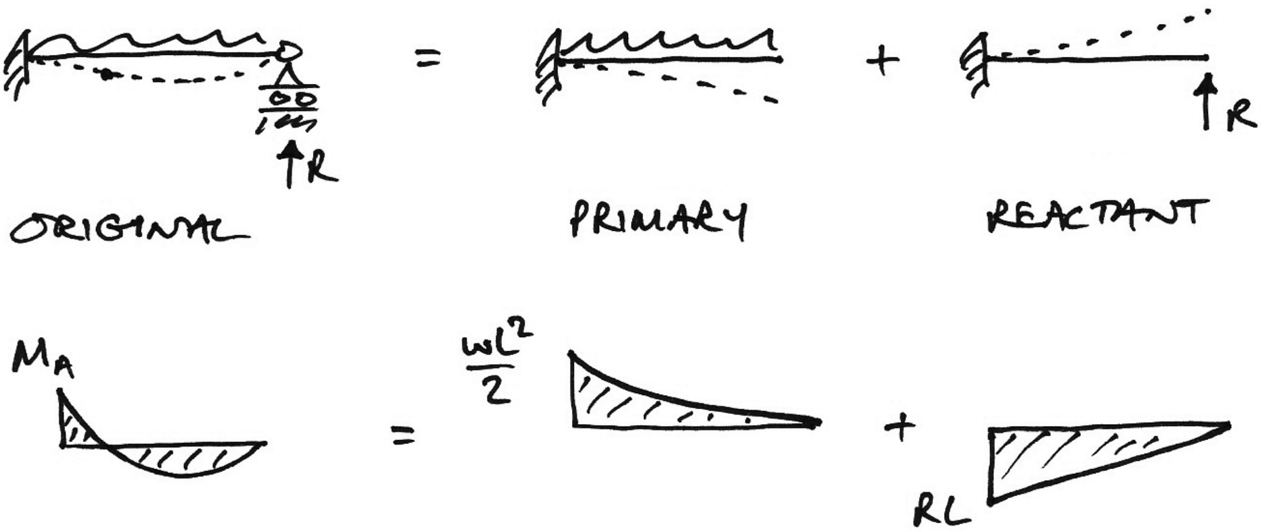
Compatibility of displacement along with superposition enables us to solve indeterminate structures. Though we'll use more specialized techniques they will be fundamentally based upon the preceding ideas. Some simple example applications follow.

4.2 Example: Propped Cantilever

Consider the following propped cantilever subject to UDL:



Using superposition we can break it up as follows (i.e. we choose a redundant):



Next, we consider the deflections of the primary and reactant structures:

Now by compatibility of displacements for the original structure, we know that we need to have a final deflection of zero after adding the primary and reactant deflections at *B*:

$$\delta_B = \delta_B^P + \delta_B^R = 0$$

From tables of standard deflections, we have:

$$\delta_B^P = +\frac{wL^4}{8EI} \text{ and } \delta_B^R = -\frac{RL^3}{3EI}$$

In which downwards deflections are taken as positive. Thus we have:

$$\begin{aligned}\delta_B &= +\frac{wL^4}{8EI} - \frac{RL^3}{3EI} = 0 \\ \therefore R &= \frac{3wL}{8}\end{aligned}$$

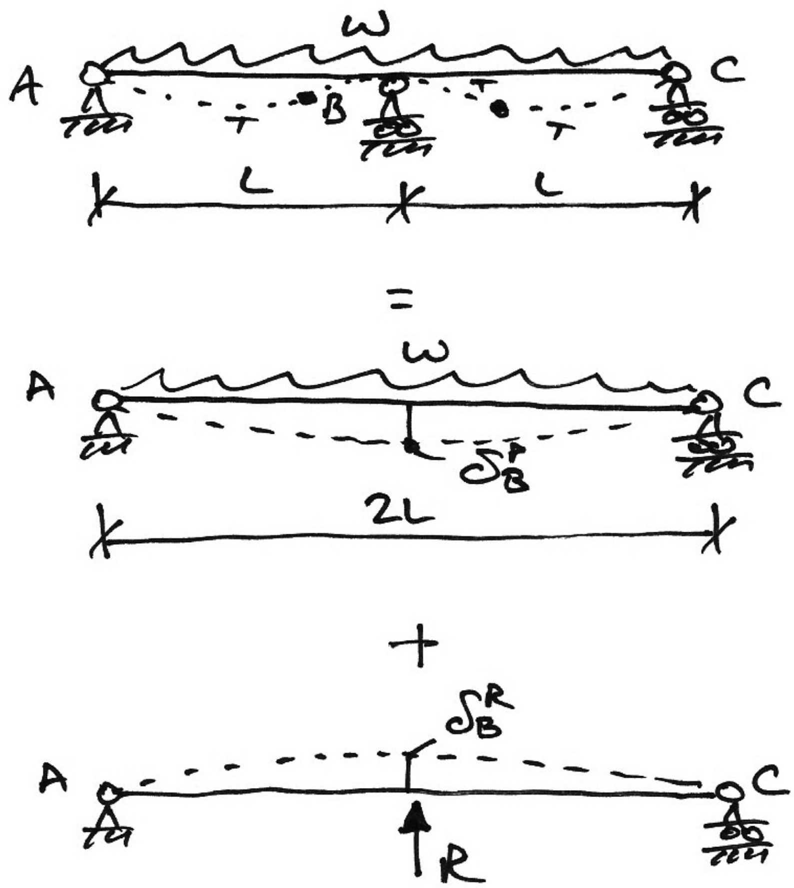
Knowing this, we can now solve for any other load effect. For example:

$$\begin{aligned}M_A &= \frac{wL^2}{2} - RL \\ &= \frac{wL^2}{2} - \frac{3wL}{8}L \\ &= \frac{4wL^2 - 3wL^2}{8} \\ &= \frac{wL^2}{8}\end{aligned}$$

Note that the $wL^2/8$ term arises without a simply-supported beam in sight!

4.3 Example: 2-Span Beam

Considering a 2-span beam, subject to UDL, which has equal spans, we break it up using the principle of superposition:



Once again we use compatibility of displacements for the original structure to write:

$$\delta_B = \delta_B^P + \delta_B^R = 0$$

Again, from tables of standard deflections, we have:

$$\delta_B^P = +\frac{5w(2L)^4}{384EI} = +\frac{80wL^4}{384EI}$$

And:

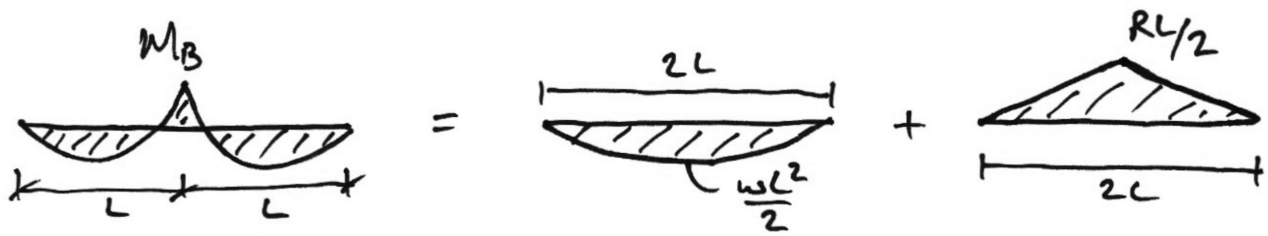
$$\delta_B^R = -\frac{R(2L)^3}{48EI} = -\frac{8RL^3}{48EI}$$

In which downwards deflections are taken as positive. Thus we have:

$$\begin{aligned}\delta_B &= +\frac{80wL^4}{384EI} - \frac{8RL^3}{48EI} = 0 \\ \frac{8R}{48} &= \frac{80wL}{384} \\ R &= \frac{10wL}{8}\end{aligned}$$

Note that this is conventionally not reduced to $5wL/4$ since the other reactions are both $3wL/8$. *Show this as an exercise.*

Further, the moment at *B* is by superposition:



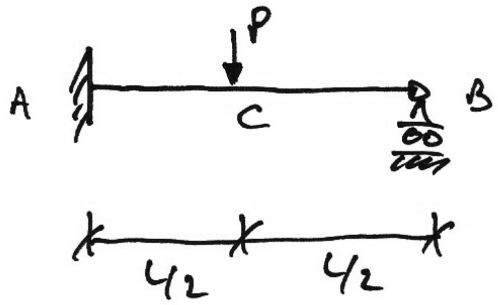
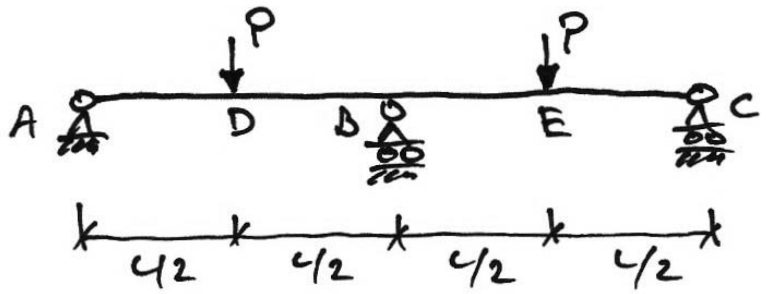
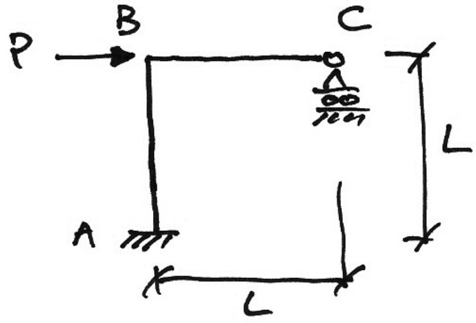
Hence:

$$\begin{aligned}M_B &= \frac{RL}{2} - \frac{wL^2}{2} = \frac{10wL}{8}\cdot\frac{L}{2} - \frac{wL^2}{2} = \frac{10wL^2 - 8wL^2}{16} \\ &= \frac{wL^2}{8}\end{aligned}$$

And again $wL^2/8$ arises!

5. Problems

Use compatibility of displacement and the principle of superposition to solve the following structures. In each case draw the bending moment diagram and determine the reactions.

1.	
2.	
3.	 This one is tricky: choosing the reaction at C gives $R = 3P/8$.