Tutorial 10

Boundary layer theory

1. If the velocity distribution law in a laminar boundary layer over a flat plate is assumes to be of the form $u = ay + by^3$, determine the velocity distribution law.

Solution:

At y = 0, u= 0

At $y = \delta$, u = U

$U = a\delta + b\delta^3$ (a)

At $y = \delta, \frac{du}{dy} = 0$

$\frac{du}{dy} = a + 3by^2$

$a + 3b\delta^2 = 0$

$a = -3b\delta^2$ (b)

From a and b

$U = -3b\delta^3 + b\delta^3$

$b = -\frac{U}{2\delta^3}$

$a = \frac{3U}{2\delta}$

Hence the velocity distribution equation is

$u = \frac{3U}{2\delta}y - \frac{U}{2\delta^3}y^3$

$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$

2. A flat plate of 2m width and 4m length is kept parallel to air flowing at 5m/s. Determine the length of plate over which the boundary layer is laminar and shear stress at the location where boundary layer ceases to be laminar. Take ρ of air = 1.208 kg/m^3 and υ of air = 1.47x10^{-5} m^2/s.

Solution:

Width (b) = 2m

Length (L) = 4m

Velocity (U) = 5m/s

Reynold no. $(Re) = \frac{UL}{\upsilon} = \frac{5x4}{1.47x10^{-5}} = 1.361x10^6$

Re>5x10^5

On the front portion, the boundary layer is laminar and on the rear, it is turbulent.

$Re_x = \frac{Ux}{\upsilon} = 5x10^5$

$\frac{5x}{1.47x10^{-5}} = 5x10^5$
x = 1.47m
Up to 1.47m from the leading edge, the boundary layer is laminar.

$\delta = \frac{5x}{\sqrt{Re_x}} = \frac{5x1.47}{\sqrt{5x10^5}}$ = 0.01039
$C_f = \frac{0.664}{\sqrt{Re_x}} = \frac{0.664}{\sqrt{5x10^5}}$ = 0.000939
Shear stress $(\tau) = \frac{1}{2}C_f\rho U^2 = \frac{1}{2}0.000939x1.208x5^2$ = 0.01418 N/m²

3. For the velocity profile given below, compute the displacement thickness and momentum thickness:

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^2$$

Where U = free stream velocity, u = velocity in boundary layer at a distance y and $\delta =$ boundary layer thickness.

Solution:
$\frac{u}{U} = \frac{3y}{2\delta} - \frac{y^2}{2\delta^2}$
Displacement thickness (δ^*) = ?
Momentum thickness (θ) = ?

$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{3y}{2\delta} + \frac{y^2}{2\delta^2}\right) dy$
$= \left| y - \frac{3y^2}{4\delta} + \frac{y^3}{6\delta^2} \right|_0^\delta$
$= \delta - \frac{3\delta}{4} + \frac{\delta}{6}$
$= \frac{5\delta}{12}$

$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(\frac{3y}{2\delta} - \frac{y^2}{2\delta^2}\right)\left(1 - \frac{3y}{2\delta} + \frac{y^2}{2\delta^2}\right) dy$
$= \int_0^\delta \left(\frac{3y}{2\delta} - \frac{9y^2}{4\delta^2} + \frac{3y^3}{4\delta^3} - \frac{y^2}{2\delta^2} + \frac{3y^3}{4\delta^3} - \frac{y^4}{4\delta^4}\right) dy$
$= \int_0^\delta \left(\frac{3y}{2\delta} - \frac{11y^2}{4\delta^2} + \frac{3y^3}{2\delta^3} - \frac{y^4}{4\delta^4}\right) dy$
$= \left| \frac{3y^2}{4\delta^2} - \frac{11y^3}{12\delta^3} + \frac{3y^4}{8\delta^3} - \frac{y^5}{20\delta^4} \right|_0^\delta$
$= \frac{3\delta}{4} - \frac{11\delta}{12} + \frac{3\delta}{8} - \frac{\delta}{20}$
$= \frac{19\delta}{120}$

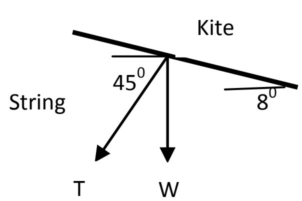
Drag and lift

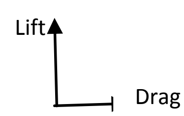
4. A flat plate 2mx2m moves at 40km/hr in stationary air of density 1.25kg/m^3. If the coefficient of drag and lift are 0.2 and 0.8 respectively, find the lift force, the drag force, the resultant force and the power required to keep the plate in motion.

Solution:
Area of plate (A) = 4m^2
Velocity (V) = 40 km/hr = $\frac{40x1000}{60x60}$ =11.11m/s
Density of air (ρ) = 1.25kg/m^3
Coefficient of drag (C_D) = 0.2
Coefficient of lift (C_L) = 0.8
Lift force (F_L) =?
Drag force (F_D) = ?
Resultant force (R) = ?
Power (P) =?

$F_L = \frac{1}{2}C_L\rho AV^2 = \frac{1}{2}x0.8x1.25x4x11.11^2$ = 246.86 N
$F_D = \frac{1}{2}C_D\rho AV^2 = \frac{1}{2}x0.2x1.25x4x11.11^2$ = 61.71N
$R = \sqrt{F_L^2 + F_D^2} = \sqrt{246.86^2 + 61.71^2}$ = 254.45N
P = F_D V = 61.71x11.11 = 685W

5. A kite weighs 0.9N and has an area of 7400 cm^2. The tension in the kite string is 3.3 N when the string makes an angle of 45^0 with the horizontal. For a wind of 30km/hr, what are the coefficients of lift and drag if the kite assumes an angle of 8^0 with the horizontal? Consider the kite essentially a flat plate and density of air = 1.2kg/m^3.





Solution:
Weight of kite (W) = 0.9N
Area of kite (A) = 7400 cm^2 = 0.74 m^2
Velocity (V) = 30 km/hr = $\frac{30x1000}{60x60}$ =8.33m/s
Density of air (ρ) = 1.2kg/m^3
Tension (T) = 3.3N
Coefficient of lift (C_L) = ?

Coefficient of drag (C_D) = ?

Forces in X-dir
F_D = 3.3cos45 = 2.33N
Force in Y-dir
F_L = 3.3sin45+0.9 = 3.23N

$F_L = \frac{1}{2} C_L \rho A V^2$
$3.23 = \frac{1}{2} x C_L x 1.2 x 0.74 x 8.33^2$
C_L = 0.104

$F_D = \frac{1}{2} C_D \rho A V^2$
$2.33 = \frac{1}{2} x C_D x 1.2 x 0.74 x 8.33^2$
C_D = 0.075

6. Calculate the weight of a ball of diameter 50mm which is just supported in a vertical air stream which is flowing at a velocity of 10 m/s. Take density of air = 1.25kg/m^3 and kinematic viscosity = 15 stokes.

Solution:
Diameter of ball (D) = 50mm = 0.05m
Area of ball (A) = $\frac{\pi}{4} x 0.05^2$ = 0.001963 m^2
Density of air (ρ) = 1.25kg/m^3
kinematic viscosity (ν) = 15 stokes = $15x10^{-4}$ m^2/s
Velocity (V) = 10m/s
Weight of ball = ?
Reynold no. $(Re) = \frac{VD}{\upsilon} = \frac{10x0.05}{15x10^{-4}}$ = 333
For Re between 5 to 1000, coefficient of drag (C_D) = 0.4
Weight of ball = Drag force
$= \frac{1}{2} C_D \rho A V^2 = \frac{1}{2} x 0.4 x 1.25 x 0.001963 x 10^2$ = 0.049N

7. A metallic sphere of sp.gr. 8.0 falls in an oil of density 800 kg/m^3. The diameter of the sphere is 10mm and it attains a terminal velocity of 50mm/s. Find the viscosity of the oil in Poise.

Solution:
Density of sphere (ρ_s) = 8x1000 = 8000 kg/m^3
Density of oil (ρ_o) = 800 kg/m^3
Diameter of sphere (D) = 0.01m
Terminal velocity (V) = 0.05m/s
Viscosity of oil (μ) = ?

Weight of sphere = Buoyant force on sphere + Drag force

$$\rho_s g x \frac{1}{6}\pi D^3 = \rho_0 g x \frac{1}{6}\pi D^3 + 3\pi\mu D V$$

$$8000x9.81x\frac{1}{6}\pi x0.01^3 = 800x9.81x\frac{1}{6}\pi x0.01^3 + 3\pi\mu x0.01x0.05$$

μ = 7.848 PaS = 78.48 Poise

The expression for drag force is valid for Re<0.2.

$Re = \frac{\rho V D}{\mu} = \frac{800x0.05x0.01}{7.848}$ = 0.05

8. A metallic ball of diameter 5mm drops in a fluid of sp.gr. 0.8 and viscosity 30 poise. The sp.gr. of metallic ball is 9.0. Find (a) the force exerted by the fluid on the ball, (b) the pressure drag and skin friction drag, and (c) the terminal velocity of the ball in the fluid.

Solution:

Diameter of ball (D) = 0.005m

Density of fluid (ρ) = 0.8x1000 = 800 kg/m^3

Viscosity of fluid (μ) = 30 poise = 30/10 = 3 PaS

Density of ball (ρ_b) = 9x1000 = 9000 kg/m^3

Drag force (F_D) = ?

Pressure drag and friction drag = ?

Terminal velocity (V) = ?

Weight of ball (W) = Drag force (F_D) + Buoyant force on ball (F_B)

$F_D = W - F_B$

$= \rho_b g x \frac{1}{6}\pi D^3 - \rho g x \frac{1}{6}\pi D^3 = 9000x9.81x\frac{1}{6}\pi x0.005^3 - 800x9.81x\frac{1}{6}\pi x0.005^3$ = 0.005265N

Pressure drag = $\frac{1}{3}F_D = \frac{1}{3}x0.005265$= 0.001755 N

Friction drag = $\frac{2}{3}F_D = \frac{2}{3}x0.005265$= 0.00351N

$$F_D = 3\pi\mu D V$$

$$0.005265 = 3\pi x3x0.005V$$

V = 0.0372 m/s

Checking the Reynold's no.

$Re = \frac{\rho V D}{\mu} = \frac{800x0.0372x0.005}{3}$ = 0.05

As Re<0.2, above expression for F_D is valid.

9. A jet plane which weighs 19620N has a wing area of 25m^2. It is flying at a speed of 200km/hr. When the engine develops 588.6KW, 70% of this power is used to overcome the drag resistance of the wing. Calculate the coefficient of lift and coefficient of drag for the wing. Take density of air = 1.25 kg/m^3.

Solution:

Weight of plane (W) = 19620N
Wing area (A) = 25m^2
Speed (V) = 200 km/hr = $\frac{200x1000}{3600}$ = 55.55m/s
Power = 588.6KW = 588600W
Power used to overcome drag resistance (P) = 0.7x588600 =412020W
Density of air (ρ) = 1.25 kg/m^3
Coefficient of drag (C_D) = ?
Coefficient of lift (C_L) = ?

P = F_D x V
412020 = F_D x 55.55
F_D = 7417.1 N

$$F_D = \frac{1}{2} C_D \rho A V^2$$

$$7417.1 = \frac{1}{2} x C_D x 1.25 x 25 x 55.55^2$$

C_D = 0.154

Lift force (F_L) = Weight of plane = 19620N

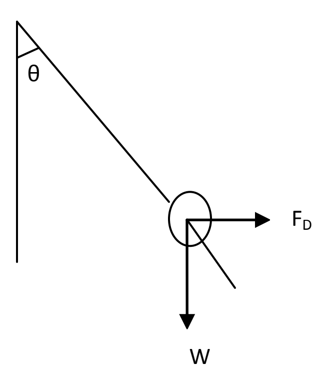
$$F_L = \frac{1}{2} C_L \rho A V^2$$

$$19620 = \frac{1}{2} x C_L x 1.25 x 25 x 55.55^2$$

C_L = 0.407

10. A 10mm ball of relative density 1.2 is suspended from a string, in air flowing at a velocity of 10m/s. Determine the angle which the string will make with the vertical. Take ρ of air = 1.208 kg/m^3 and viscosity of air = 1.85x10^{-5} Pa-S. Also compute the tension in the string.

Solution:
Diameter of ball (d) = 10mm
Velocity (V) = 10m/s
Angle(θ) = ?



Area of sphere (A) = $\frac{\pi}{4} x 0.01^2$ = 7.85x10^{-5} m^2
Weight of sphere (W) = $\gamma Vol = 1.2 x 9810 x \frac{1}{6} x \pi x 0.01^3$ = 0.006163 N

Reynold no. $(Re) = \frac{\rho V d}{\mu} = \frac{1.208x10x0.01}{1.85x10^{-5}} = 6530$

For Re between 1000-100000, C_D = 0.5

$F_D = \frac{1}{2} C_D \rho A V^2 = \frac{1}{2} 0.5x1.208x7.85x10^{-5}x10^2$ = 0.002371

$tan\theta = \frac{F_D}{W} = \frac{0.002371}{0.006163}$

θ = 21^0

Tension in the string = $\sqrt{F_D^2 + W^2} = \sqrt{0.002371^2 + 0.006163^2}$ = 0.0066N

11. Determine the rate of deceleration that will be experienced by a blunt nosed projectile of drag coefficient 1.22 when it is moving horizontally at 1600 km/hr. The projectile has a diameter of 0.5m and weighs 3000N. Take ρ of air = 1.208 kg/m^3.

Solution:

Velocity (V) = 1600 km/hr = $\frac{1600x1000}{3600}$ = 444.44 m/s

Drag coeff. (C_D) = 1.22

Diameter of projectile (d) = 0.5m

Area of sphere (A) = $\frac{\pi}{4} x0.5^2$ = 0.1963 m^2

Weight (W) = 3000N

Deceleration (a) = ?

$F_D = \frac{1}{2} C_D \rho A V^2$

$F_D = -ma = -\frac{W}{g} a$

$-\frac{W}{g} a = \frac{1}{2} C_D \rho A V^2$

$-\frac{3000}{9.81} a = \frac{1}{2} x1.22x1.208x0.1963x444.44^2$

a = -93.4 m/s^2

12. An aeroplane weighing 22500N has a wing area of 22.5m^2 and span of 12m. What is the lift coefficient if it travels at 320 km/hr in the horizontal direction? Also compute the value of circulation and angle of attack measured from zero lift axis.

Solution:

Velocity (V) = 320 km/hr = $\frac{320x1000}{3600}$ = 88.89 m/s

Wing area (A) = 22.5m^2

Weight = 22500 N = Lift force (F_L)

Lift coefficient (C_L) =?

Angle of attack (θ) =?

Circulation(Γ) = ?

$F_L = \frac{1}{2} C_L \rho A V^2$

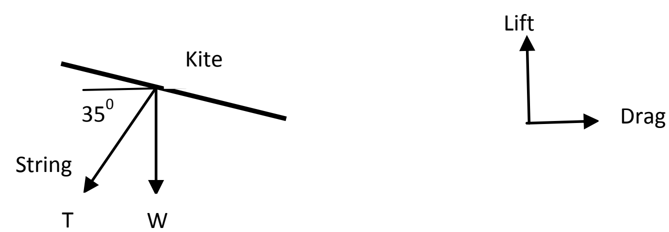
$22500 = \frac{1}{2} C_L x1.208x22.5x88.89^2$

$C_L = 0.2095$

$C_L = 2\pi sin\theta$
$0.2095 = 2\pi sin\theta$
$\theta = 1.911^0$
$\Gamma = \pi CV sin\theta = \pi x \frac{22.5}{12} x 88.89 sin 1.911$ = 17.46 m^2/s

13. A kite, which may be assumed to be a flat plate and mass 1kg, soars at an angle to the horizontal. The tension in the string holding the kite is 60N when the wind velocity is 50 km/h horizontally and the angle of string to the horizontal direction is 35^0. The density of air is 1.2 kg/m^3. Calculate the drag coefficient for the kite in the given position if the lift coefficient in the same position is 0.45. Both coefficients have been based on the full area of the kite.

Solution:



Solution:
Mass of kite = 1kg
Weight of kite (W) = 1x9.81 N = 9.81N
Area of kite = A
Velocity (V) = 50 km/hr = $\frac{50x1000}{60x60}$ =13.88/s
Density of air (ρ) = 1.2kg/m^3
Tension (T) = 60N
Coefficient of lift (C_L) = 0.45
Coefficient of drag (C_D) = ?

Forces in X-dir (Drag)
F_D = Tcos35 = 60cos35 =49.14N
Force in Y-dir (Lift)
F_L = Tsin35+9.81 = 60sin35+9.81=44.22N

$F_L = \frac{1}{2} C_L \rho A V^2$
$44.22 = \frac{1}{2} x 0.45 x 1.2 x A x 13.88^2$
A = 0.85 m^2

$F_D = \frac{1}{2} C_D \rho A V^2$

$49.14 = \frac{1}{2} x C_D x 1.2 x 0.85 x 13.88^2$

C_D = 0.5

14. A steel sphere of 5mm diameter falls in a glycerin at a terminal velocity of 0.05m/s. Assume Stoke's law is applicable, determine (a) dynamic viscosity of glycerin, (b) drag force and (c) coefficient of drag. Take sp wt of steel and glycerin as 75 KN/m^3 and 12.5 KN/m^3 respectively.

Solution:

Solution:

Diameter of ball (D) = 5mm = 0.005m

Sp wt of steel (γ_{steel}) = 75 KN/m^3

Sp wt of glycerin (γ_{gly}) = 12.5 KN/m^3

Terminal velocity (V)= 0.05m/s

Viscosity of fluid (μ) = ?

Drag force (F_D) = ?

Coefficient of drag (C_D) = ?

Weight of sphere (W) = Drag force (F_D) + Buoyant force on the sphere(F_B)

$F_D = W - F_B$

$= \gamma_{steel} x \frac{1}{6} \pi D^3 - \gamma_{gly} x \frac{1}{6} \pi D^3$

$= 75000 x \frac{1}{6} \pi x 0.005^3 - 12500 x \frac{1}{6} \pi x 0.005^3$ = 0.00409N

From Stoke's law

$$F_D = 3\pi\mu DV$$

$$0.00409 = 3\pi x \mu x 0.005 x 0.05$$

μ= 1.7NS/m^2

Reynold's no.

$Re = \frac{\rho VD}{\mu} = \frac{\left(\frac{12500}{9.81}\right) x 0.05 x 0.005}{1.7}$ = 0.18

As Re<0.2, Stoke's is valid.

$C_D = \frac{24}{Re} = \frac{24}{0.18}$ = 133

Dimensional and model analysis

15. Using Rayleigh's method, derive an expression for flow through orifice (Q) in terms of density of liquid (ρ), diameter of the orifice (D) and the pressure difference (P).

Solution:

$Q = f(\rho, D, P)$

$Q = K\rho^a D^b P^c$

Writing dimensions

$L^3T^{-1} = K(ML^{-3})^a(L)^b(ML^{-1}T^{-2})^c$

$L^3T^{-1} = KM^{a+c}L^{-3a+b-c}T^{-2c}$

Equating the powers of M, L and T

a+c = 0

-3a+b-c = 3

-2c = -1

Solving above equations

c = 1/2

a = -c= -1/2

b = 3a+c+3 = $-\frac{3}{2} + \frac{1}{2} + 3$ =2

Substituting values of a, b and c

$Q = K\rho^{-1/2}D^2P^{1/2} = \ KD^2\sqrt{P/\rho}$

16. Assuming the drag force exerted by a flowing fluid (F) is a function of the density (ρ), viscosity (μ), velocity of fluid (V) and a characteristics length of body (L), show by using Rayleigh's method that $F = C\rho A\frac{V^2}{2}$ where A is area and C is constant.

Solution:

$F = f(\rho, \mu, V, L)$

$F = K\ \rho^a\mu^bV^cL^d$

Writing dimensions

$MLT^{-2} = K(ML^{-3})^a(ML^{-1}T^{-1})^b(LT^{-1})^c(L)^d$

$MLT^{-2} = KM^{a+b}L^{-3a-b+c+d}T^{-b-c}$

Equating the powers of M, L and T

a+b = 1

-3a-b+c+d = 1

-b-c = -2

There are 3 equations and 4 unknowns. As we have to get expression in terms of ρ, V and L (Area), we can express the powers of these three variables i.e. a, c and d in terms of b.

a = 1-b
c = 2-b
Substituting a and c in second eqaution
-3+3b-b+2-b+d = 1
or d = 2-b

Substituting the values of a, c and d

$$F = K\,\rho^{1-b}\mu^{b}V^{2-b}L^{2-b}$$
$$= K\rho V^2L^2\rho^{-b}\mu^{b}V^{-b}L^{-b}$$
$$= K\rho V^2L^2\left(\frac{\rho VL}{\mu}\right)^{-b}$$
$$= [K(Re)^{-b}]\rho AV^2$$
$$= [2K(Re)^{-b}]\rho A\frac{V^2}{2}$$
$$= C\rho A\frac{V^2}{2}$$

17. Power input to a propeller (P) is expressed in terms of density of air (ρ), diameter (D), velocity of the air stream (V), rotational speed (ω), viscosity (μ) and speed of sound (C). Show that
$P = c\rho\omega^3D^5$ where c = constant. Use Rayleigh's method.

Solution:
$$P = f(\rho, D, V, \omega, \mu, C)$$
$$P = K\rho^aD^bV^c\omega^d\mu^eC^f$$

Writing dimensions
$$ML^2T^{-3} = K(ML^{-3})^a(L)^b(LT^{-1})^c(T^{-1})^d(ML^{-1}T^{-1})^e(LT^{-1})^f$$
$$ML^2T^{-3} = KM^{a+e}L^{-3a+b+c-e+f}T^{-c-d-e-f}$$

Equating the powers of M, L and T
a+e = 1
-3a+b+c-e+f = 2
-c-d-e-f = -3

There are 3 equations and 6 unknowns. As we have to get expression in terms of ρ, ω and D, we can express the powers of these three variables i.e. a, b and d in terms of remaining variables.

Solving above equations,

a = 1-e
d = 3-c-e-f
substituting a and d in second eqaution
-3+3e+b+c-e+f = 2
Or, b = 5-c-2e-f

Substituting the values of a, b and c
$P = K\rho^a D^b V^c \omega^d \mu^e C^f = K\rho^{1-e} D^{5-c-2e-f} V^c \omega^{3-c-e-f} \mu^e C^f$
$= K\rho D^5 \omega^3 \, (\rho^{-e} D^{-2e} \omega^{-e} \mu^e)(D^{-c} V^c \omega^{-c})\left(D^{-f} \omega^{-f} C^f\right)$
$= K\rho D^5 \omega^3 \left(\frac{\rho D^2 \omega}{\mu}\right)^{-e} \left(\frac{V}{D\omega}\right)^c \left(\frac{C}{D\omega}\right)^f$
Representing $= K \left(\frac{\rho D^2 \omega}{\mu}\right)^{-e} \left(\frac{V}{D\omega}\right)^c \left(\frac{C}{D\omega}\right)^f$ by c
$P = c\rho\omega^3 D^5$

18. If the resistance to motion of a sphere through a fluid (R) is a function of the density (ρ), viscosity (μ) of the fluid, and the radius (r) and velocity (u) of the sphere, develop a relationship of R using Buckingham's π theorem.
(Take u, r and ρ as repeating variables and take the dimension of shear stress for R)

Solution:
$f1(R, \rho, \mu, r, u) = 0$
Total number of variables = 5
No. of fundamental dimensions = 3
No. of π terms = 5-3= 2
$f(\pi_1, \pi_2) = 0$ (I)

Choose u, r and ρ as repeating variables.

First π term
$\pi_1 = u^{a1} r^{b1} \rho^{c1} R$ (II)

Writing dimensions
$M^0 L^0 T^0 = (LT^{-1})^{a1} (L)^{b1} (ML^{-3})^{c1} ML^{-1}T^{-2}$
Equating the powers of M, L and T
c1+1 =0
a1+b1-3c1-1 = 0
-a1-2 = 0

c1 = -1, a1 = -2
Substituting a1 and c1 in second equation
-2+b1+3-1=0

b1 = 0
Substituting the values of a1, b1 and c1 in II
$\pi_1 = u^{-2} r^0 \rho^{-1} R$
$\pi_1 = \frac{R}{\rho u^2}$

Second π term
$\pi_2 = u^{a2} r^{b2} \rho^{c2} \mu$ (III)

Writing dimensions
$M^0 L^0 T^0 = (LT^{-1})^{a2} (L)^{b2} (ML^{-3})^{c2} ML^{-1}T^{-1}$
Equating the powers of M, L and T
c2+1 =0
a2+b2-3c2-1 = 0
-a2-1 = 0

a2 = -1, c2 = -1
Substituting a2 and c2 in second equation
-1+b2+3-1 =0
b2 =-1

Substituting the values of a2, b2 and c2 in III
$\pi_2 = u^{-1} r^{-1} \rho^{-1} \mu$
$\pi_2 = \frac{\mu}{ur\rho}$

Substituting values of π_1 and π_2 in I
$f\left(\frac{R}{\rho u^2}, \frac{\mu}{ur\rho}\right) = 0$
$\frac{R}{\rho u^2} = f\left(\frac{\mu}{ur\rho}\right)$
$R = \rho u^2 f\left(\frac{\mu}{ur\rho}\right)$

19. The pressure difference (ΔP) in a pipe of diameter (D) and length (L) due to viscous flow depends on the velocity of fluid (V), viscosity (μ) and density (ρ). Using Buckingham's π theorem, show that $\Delta P = \frac{\mu VL}{D^2} f(Re)$ where $Re = \frac{\rho DV}{\mu}$ is Reynold's number.
(Take D, V and ρ as repeating variables)

Solution:
$f1(\Delta P, D, L, V, \mu, \rho) = 0$
Total number of variables = 6
No. of fundamental dimensions = 3

No. of π terms = 6-3= 3
$f(\pi_1, \pi_2, \pi_3\,) = 0$ (I)

Choose D, V and μ as repeating variables.

First π term
$\pi_1 = D^{a1} V^{b1} \rho^{c1} \Delta P$ (II)

Writing dimensions
$M^0 L^0 T^0 = (L)^{a1} (LT^{-1})^{b1} (ML^{-3})^{c1} ML^{-1}T^{-2}$
Equating the powers of M, L and T
c1+1 =0
a1+b1-3c1-1 = 0
-b1- 2 = 0

c1 = -1, b1 = -2

a1-2+3-1=0
a1 = 0

Substituting the values of a1, b1 and c1 in II

$\pi_1 = D^0 V^{-2} \rho^{-1} \Delta P$
$\pi_1 = \frac{\Delta P}{\rho V^2}$

Second π term
$\pi_2 = D^{a2} V^{b2} \rho^{c2} L$ (III)

Writing dimensions
$M^0 L^0 T^0 = (L)^{a2} (LT^{-1})^{b2} (ML^{-3})^{c2} L$
Equating the powers of M, L and T
c2 = 0
a2+b2-3c2+1 = 0
b2 = 0
a2 = -1
Substituting the values of a2, b2 and c2 in III

$\pi_2 = D^{-1} V^0 \rho^0 L$
$\pi_2 = \frac{L}{D}$

Third π term

$\pi_3 = D^{a3}V^{b3}\rho^{c3}\mu$ (IV)

Writing dimensions

$M^0L^0T^0 = (L)^{a3}(LT^{-1})^{b3}(ML^{-3})^{c3}ML^{-1}T^{-1}$

Equating the powers of M, L and T

c3+1 =0

a3+b3-3c3-1 =0

-b3-1 = 0

c3 = -1

b3 = -1

a3-1+3-1 = 0

a3 =-1

Substituting the values of a3, b3 and c3 in (IV)

$\pi_3 = D^{-1}V^{-1}\rho^{-1}\mu$

$\pi_3 = \frac{\mu}{\rho DV}$

Substituting the values of π_1, π_2 and π_3 in (I)

$f\left(\frac{\Delta P}{\rho V^2}, \frac{L}{D}, \frac{\mu}{\rho DV}\right) = 0$

$\frac{\Delta P}{\rho V^2} = f\left(\frac{L}{D}, \frac{\mu}{\rho DV}\right)$

Multiplying first π term by 1/ π_2 and 1/ π_3 and expressing the product as a function of 1/π_3

$\frac{\Delta P}{\rho V^2}\frac{D}{L}\frac{\rho DV}{\mu} = f\left(\frac{\rho DV}{\mu}\right)$

$\Delta P = \frac{\mu VL}{D^2} f(Re)$

20. Show by dimensional analysis that the power P required to operate a test tunnel is given by

$$P = \rho L^2V^3\phi\left(\frac{\mu}{\rho LV}\right)$$

where ρ is density of fluid, μ is viscosity, V is fluid mean velocity, P is the power required and L is the characteristics tunnel length.

Solution:

$f1(P, \rho, L, V, \mu) = 0$

Total number of variables = 5

No. of fundamental dimensions = 3

No. of π terms = 5-3= 2

$f(\pi_1, \pi_2) = 0$ (I)

Choose ρ, L and V as repeating variables.

First π term

$\pi_1 = \rho^{a1} L^{b1} V^{c1} P$ (II)

Writing dimensions

$M^0 L^0 T^0 = (ML^{-3})^{a1} (L)^{b1} (LT^{-1})^{c1} ML^2 T^{-3}$

Equating the powers of M, L and T

a1+1 =0

-3a1+b1+c1+2 = 0

-c1-3 = 0

Solving

a1 = -1, c1 = -3

-3x-1+b1-3+2 = 0

b1 = -2

Substituting the values of a1, b1 and c1 in II

$$\pi_1 = \rho^{-1} L^{-2} V^{-3} P$$

$$\pi_1 = \frac{P}{\rho L^2 V^3}$$

Second π term

$\pi_2 = \rho^{a2} L^{b2} V^{c2} \mu$ (III)

Writing dimensions

$M^0 L^0 T^0 = (ML^{-3})^{a2} (L)^{b2} (LT^{-1})^{c2} ML^{-1} T^{-1}$

Equating the powers of M, L and T

a2+1 = 0

-3a2+b2+c2-1=0

-c2-1 = 0

Solving

a2 = -1, c2 = -1

-3x-1+b2-1-1 = 0

b2 = -1

Substituting the values of a2, b2 and c2 in III

$$\pi_2 = \rho^{-1} L^{-1} V^{-1} \mu$$

$$\pi_2 = \frac{\mu}{\rho L V}$$

Substituting the values of π_1 and π_2 in I

$$f\left(\frac{P}{\rho L^2 V^3}, \frac{\mu}{\rho L V}\right) = 0$$

$$\frac{P}{\rho L^2 V^3} = \phi\left(\frac{\mu}{\rho L V}\right)$$

$$P = \rho L^2 V^3 \phi\left(\frac{\mu}{\rho L V}\right)$$

21. A pipe of diameter 1.8m is required to transport oil of sp.gr. 0.8 and viscosity 0.04 poise at the rate of 4 m^3/s. Tests were conducted on a 20cm diameter pipe using water at 20^0C. Find the velocity and rate of flow in the model. Viscosity of water at 20^0C = 0.01 poise.

Solution:
Diameter of prototype (D_p) = 1.8m
Density of oil (ρ_p) = 0.8x1000 = 800kg/m^3
Viscosity of prototype (μ_p) = 0.04 poise = 0.004 PaS
Discharge for prototype (Q_p) = 4 m^3/s
Velocity of prototype (V_p) = $\frac{Q_p}{A_p} = \frac{4}{\frac{\pi}{4}x1.8^2}$ = 1.572 m/s
Diameter of model (D_m) = 0.2m
Density of water (ρ_m) = 1000kg/m^3
Viscosity of water (μ_m) = 0.01 poise = 0.001 PaS
Velocity of model (V_m) = ?
Rate of flow in the model (Q_m) = ?

From Reynolds' model law
Re)model = Re) Prototype

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p}$$
$$\frac{1000xV_m x0.2}{0.001} = \frac{800x1.572x1.8}{0.004}$$

V_m = 2.83 m/s
Q_m = V_m A_m = $2.83x\frac{\pi}{4}x0.2^2$= 0.0889 m^3/s

22. A ship 250m long moves in seawater, whose density is 1030 kg/m^3. A 1:125 model of this ship is to be tested in wind tunnel. The velocity of air in the wind tunnel around the model is 20m/s and the resistance of the ship is 50N. Determine the velocity and resistance of the ship in seawater. The density of air is 1.24 kg/m^3. Take the kinematic viscosity of seawater and air as 0.012 stokes and 0.018 stokes respectively.

Solution:
Length of prototype (L_p) = 250m
Density of seawater (ρ_p) = 1030 kg/m^3
Kinematic viscosity of seawater (υ_p) = 0.012 stokes = 0.012x10^{-4} m^2/s
Length of model (L_m) = $\frac{1}{125}x250$ =2m
Density of air (ρ_m) = 1.24 kg/m^3
Kinematic viscosity of air (υ_m) = 0.018 stokes = 0.018x10^{-4} m^2/s
Velocity of model (V_m) = 20m/s
Resistance of model (F_m) = 50N
Velocity of prototype (V_p) = ?

Resistance of prototype (F_p) = ?

From Reynolds' model law
Re)model = Re) Prototype

$$\frac{V_m L_m}{\upsilon_m} = \frac{V_p L_p}{\upsilon_p}$$
$$\frac{20x2}{0.018x10^{-4}} = \frac{V_p x250}{0.012x10^{-4}}$$

V_p = 0.1066 m/s

Resistance = mass x acceleration = $\rho L^3 \frac{V}{t} = \rho L^2 \frac{L}{t} V = \rho L^2 V^2$
$$\frac{F_p}{F_m} = \frac{\rho_p L_p^2 V_p^2}{\rho_m L_m^2 V_m^2}$$
$$\frac{F_p}{50} = \frac{1030x250^2x0.1066^2}{1.24x2^2x20^2}$$
F_p = 18436 N

23. In 1:30 model of a spillway, the velocity and discharge are 1.5m/s and 2 m^3/s. Find the corresponding velocity and discharge in the prototype.

Solution:
Linear scale ratio (L_r) = 1/30
Velocity of model (V_m) = 1.5m/s
Discharge of model (Q_m) = 2 m^3/s
Velocity of prototype (V_p) = ?
Discharge of prototype (Q_p) = ?

From Froude model law,
Fr)model = Fr)prototype

$$\frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_p}}$$
$$\frac{1.5}{\sqrt{9.81x\frac{1}{30}L_p}} = \frac{V_p}{\sqrt{9.81L_p}}$$
V_p = 8.216 m/s

$$\frac{Q_p}{Q_m} = \frac{A_p V_p}{A_m V_m} = \frac{L_p^2 V_p}{L_m^2 V_m}$$
$$\frac{Q_p}{2} = \frac{(30L_m)^2 8.216}{L_m^2 x1.5}$$
Q_p = 9859 m^3/s

24. A spillway model is to be built geometrically similar scale of 1/40 across a flume of 50cm width. The prototype is 20m high and the maximum head on it is expected to be 2m. (a) What height of the model and what head on the model should be used? (b) If the flow over the model at a particular head is 10 lps, what flow per m length of the prototype is expected? (c) If the negative pressure in the model is 150mm, what is the negative pressure in the prototype?

Solution:
Scale ratio for length (L_r) = 1/40
Width of model (B_m) = 0.5m
Height of prototype (H_p) = 20m
Head on prototype (Hd_p) = 2m
a) Height of model (H_m) = ?
Head on model (Hd_m) = ?

$$L_r = \frac{H_m}{H_p}$$

$$\frac{1}{40} = \frac{H_m}{20}$$

H_m = 0.5m

$$L_r = \frac{Hd_m}{Hd_p}$$

$$\frac{1}{40} = \frac{Hd_m}{2}$$

Hd_m = 0.05m

b) Flow through model (Q_m) = 10 lps = 0.01 m^3/s
Flow through prototype (Q_p) = ?

$$\frac{Q_m}{Q_p} = L_r^{2.5}$$

$$\frac{0.01}{Q_p} = \left(\frac{1}{40}\right)^{2.5}$$

Q_p = 101.2 m^3/s
Width of prototype (B_p)

$$L_r = \frac{B_m}{B_p}$$

$$\frac{1}{40} = \frac{0.5}{B_p}$$

B_p= 20m

Discharge per unit width = 101.2/20 = 5.06 m^3/s

c) Negative pressure head in model (P_m) = -0.15m

Negative pressure head in prototype (P_p) = ?

$$L_r = \frac{P_m}{P_p}$$
$$\frac{1}{40} = \frac{-0.15}{P_p}$$

P_p = -6m

25. The pressure drop in an aeroplane model of size 1/50 of its prototype is 4 N/cm^2. The model is tested in water. Find the corresponding pressure drop in prototype. Take density of air = 1.24 kg/m^3. The viscosity of water is 0.01 poise while the viscosity of air is 0.00018 poise.

Solution:
Linear scale ratio (L_r) = 1/50
Pressure drop in model (P_m) = 4 N/cm^2 = 4x10^4 N/m^2
Density of air (ρ_p) = 1.24 kg/m^3
Density of water (ρ_m) = 1000 kg/m^3
Viscosity of water (μ_m) = 0.01 poise = 0.001 PaS
Viscosity of air (μ_p) = 0.00018 poise = 0.000018 PaS
Pressure drop in prototype (P_p) = ?

As the problem involves both viscous and pressure force, we have to use both Reynolds and Euler model law.
From Reynolds' model law
Re)model = Re) Prototype

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$
$$\frac{1000 x V_m x \frac{1}{50} L_p}{0.001} = \frac{1.24 x V_p L_p}{0.000018}$$

V_m = 3.44 V_p

From Euler model law
Eu) model = Eu) prototype

$$\frac{V_m}{\sqrt{P_m/\rho_m}} = \frac{V_p}{\sqrt{P_p/\rho_p}}$$
$$\frac{3.44 V_p}{\sqrt{\frac{4x10^4}{1000}}} = \frac{V_p}{\sqrt{\frac{P_p}{1.24}}}$$

P_p = 4.2 N/m^2