Fluid Mechanics CEE 3311

LECTURE 14

Conservation of momentum

Applications of the momentum equation

L. Handia

A: Force exerted on pressure conduits

- The momentum equation can be used to compute the force exerted on pressure conduits such as reducers and bends
- $(F_{BF})_x$ and $(F_{BF})_y$ are the components of the force which the *bend exerts on the water*.
- The usual convention is to consider the direction in which the flow is occurring as the positive direction.
- The force of the *fluid on the bend* is, of course, equal and opposite to that of the bend on the fluid.

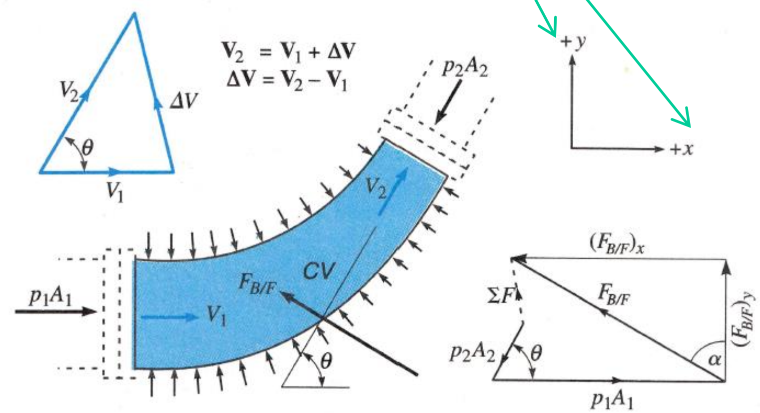


Figure 6.4
Forces on the fluid in a reducing bend.

A1: Force exerted on pressure conduits

Momentum equation can be simplified considerably if a device has entrances and exits across which the flow may be assumed to be uniform and if the flow is steady.

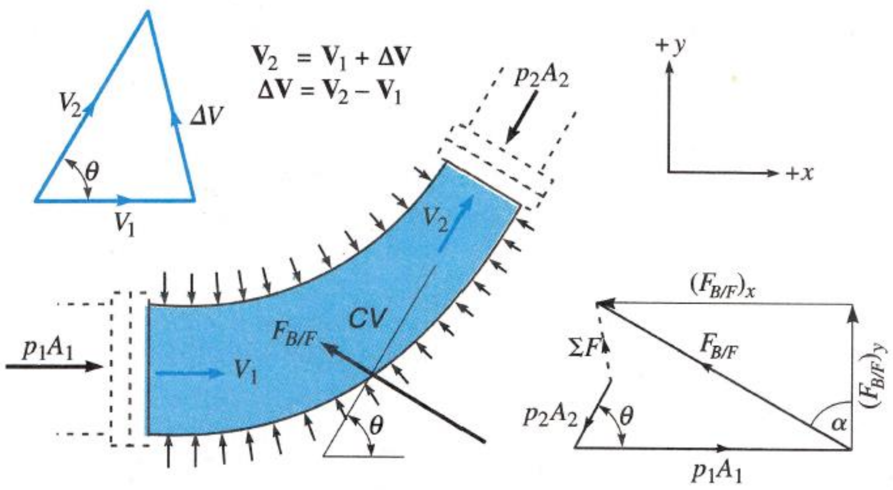


Figure 6.4
Forces on the fluid in a reducing bend.

Assuming the flow in the horizontal plane so that the weight can be neglected, applying the momentum equation by *summing up forces* acting on the fluid in the x direction, and *equating them to the change in fluid momentum* in the x direction gives i.e., $\sum F_x = \frac{d(mv_x)}{dt} = \rho Q(\Delta v)$

$$\sum F_x = \frac{d}{dt}\int \rho v_x \, dV - \beta_1 \rho \bar{v}_{1x}\bar{v}_1 A_1 + \beta_2 \rho \bar{v}_{2x}\bar{v}_2 A_2$$ 13.6a

$$\sum F_x = -(F_{BF})_x + P_1A_1 - P_2A_2\cos\theta$$

$$\frac{d(mv_x)}{dt} = \rho Q(\Delta v) = \rho\, v_1(-A_1v_1) + \rho\, v_2\cos\theta(A_2v_2)$$

A1: Force exerted on pressure conduits

$$-(F_{BF})_x + P_1A_1 - P_2A_2\cos\theta = \rho\, v_1(-A_1v_1) + \rho\, v_2\cos\theta(A_2v_2) \quad 14.16$$

14.17

$$P_1A_1 - P_2A_2\cos\theta - (F_{BF})_x = \rho\, v_1(-A_1v_1) + \rho\, v_2\cos\theta(A_2v_2) = \rho Q(\, v_2\cos\theta - v_1)$$

Which, when rewritten for the force we wish to find, becomes

$$(F_{BF})_x = P_1A_1 - P_2A_2\cos\theta - \rho Q(\, v_2\cos\theta - v_1) \quad 14.18$$

Similarly , in y direction

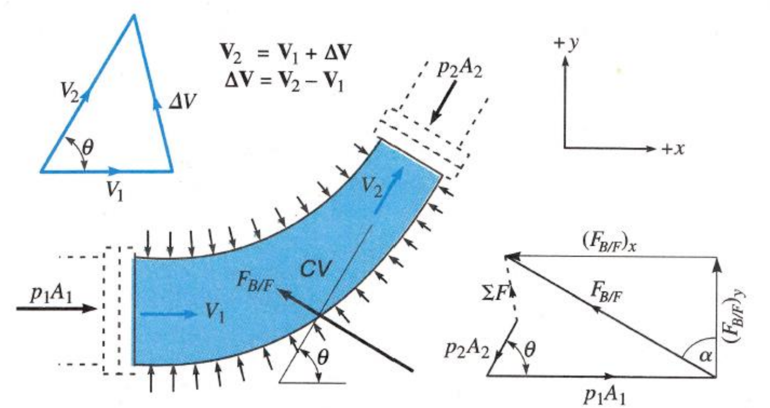


Figure 6.4
Forces on the fluid in a reducing bend.

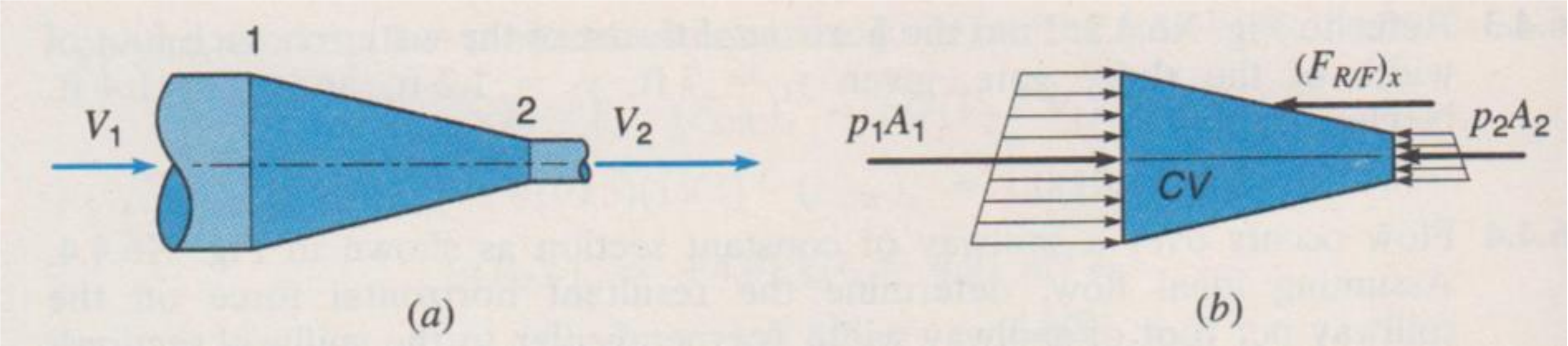
$$\sum F_y = 0 - P_2A_2\sin\theta + (F_{BF})_y = 0 + \rho\, v_2\sin\theta(A_2v_2) = \rho Q(\, v_2\sin\theta - 0) \quad 14.19$$

Which, when rewritten becomes

$$(F_{BF})_y = P_2A_2\sin\theta + \rho Q\, v_2\sin\theta \quad 14.20$$

A2: Force exerted on reducer

- Analysis is similar to that of a bend



A: Force exerted on a bend

Example

Water flows through a horizontal pipe bend and exits into the atmosphere (Fig. E4.11a). The flow rate is 0.01 m³/s. Calculate the force in each of the rods holding the pipe bend in position. Neglect body forces and viscous effects.

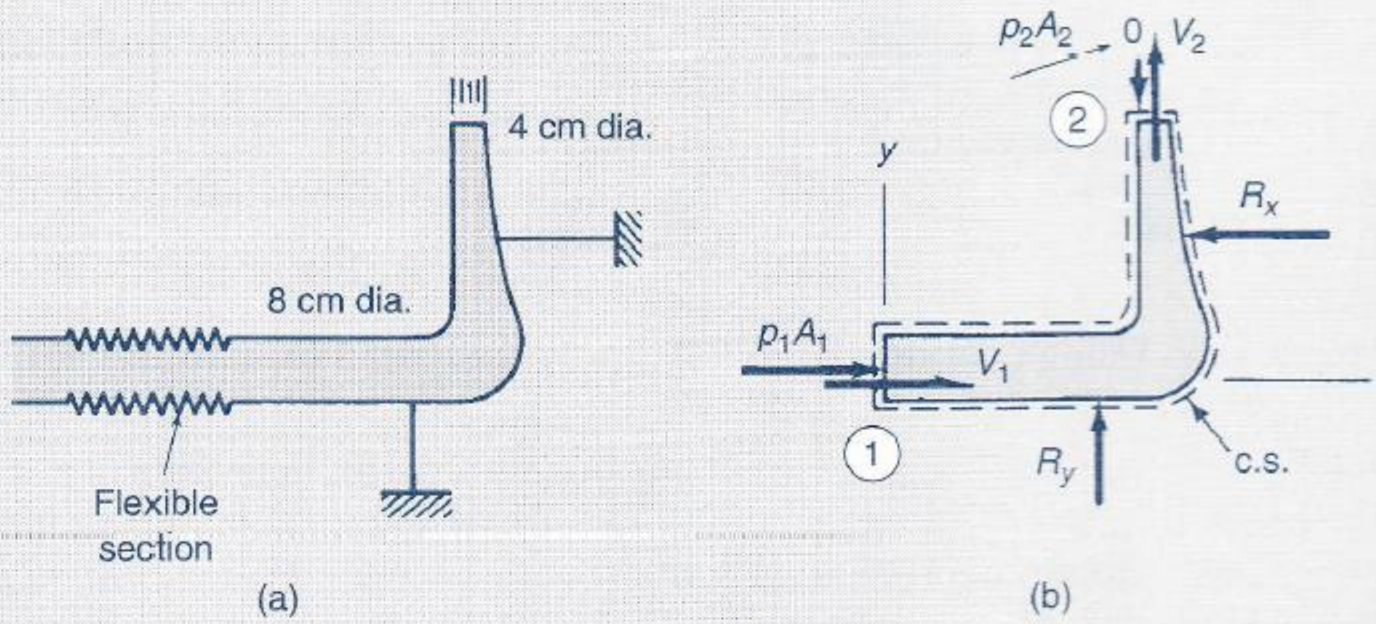
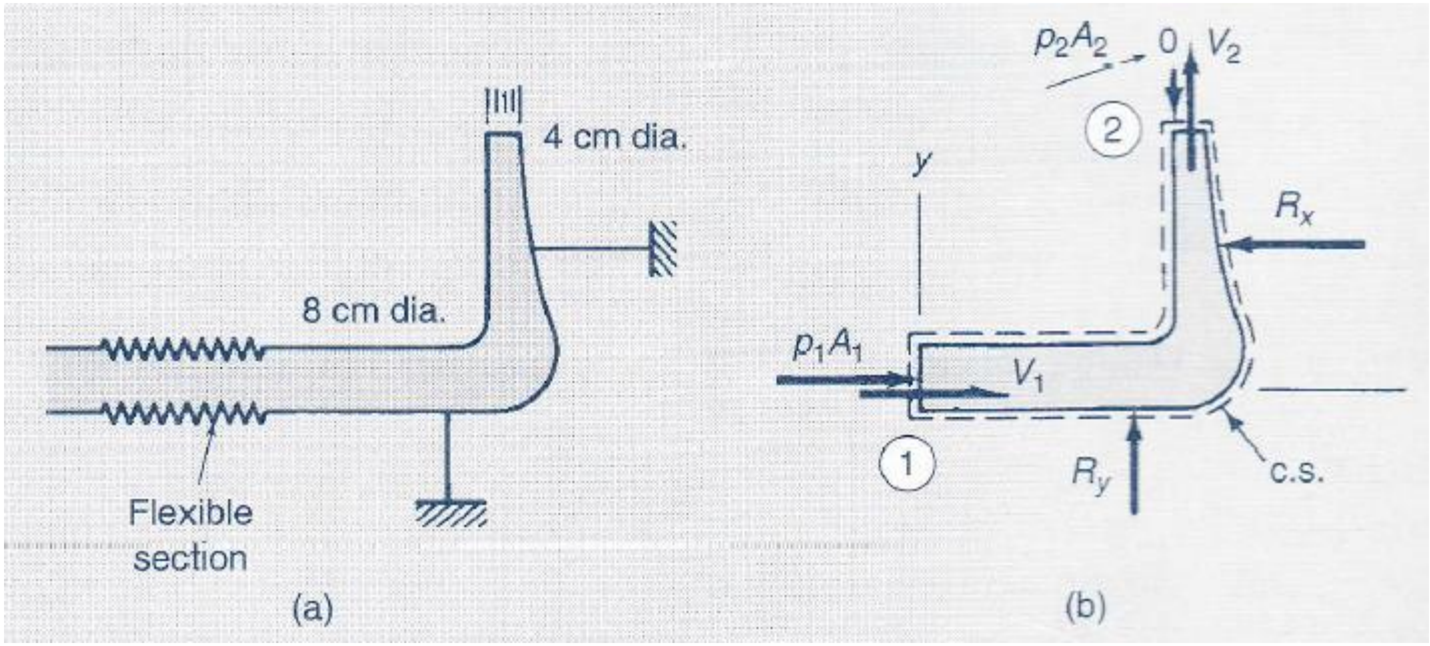


Figure E4.11

A: Force exerted on a bend

Example

Solution: We have selected a control volume that surrounds the bend as shown in Fig. E4.11b. Since the rods have been cut, the forces that the rods exert on the control volume are included. The pressure forces at the entrance and exit of the control volume are also shown. The flexible section is capable of resisting the interior pressure but it transmits no axial force or moment. The body force (weight of the control volume) does not act in the x- or y-direction but normal to it. Therefore, no other forces are shown. The average velocities are found to be



Forces on a bend

$$V_1 = \frac{Q}{A_1} = \frac{0.01}{\pi(0.08)^2/4} = 1.99 \text{ m/s}$$

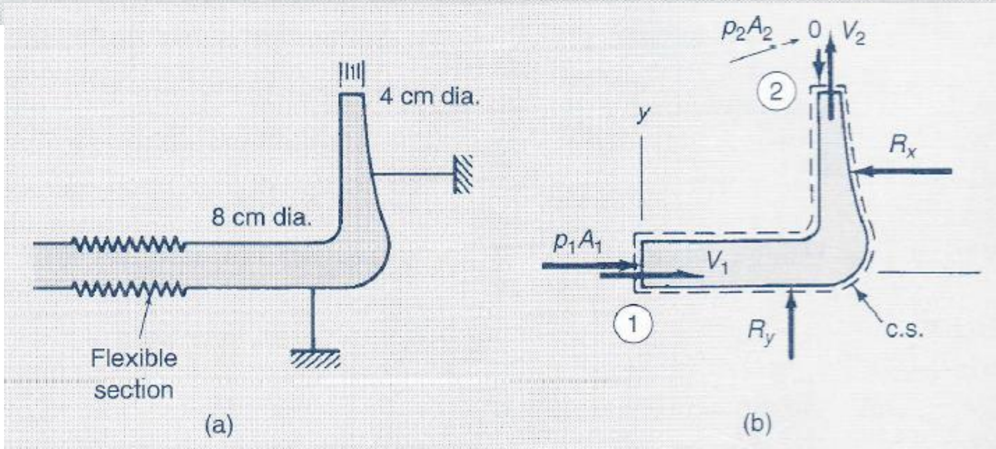
$$V_2 = \frac{Q}{A_2} = \frac{0.01}{\pi(0.04)^2/4} = 7.96 \text{ m/s}$$

Before we can calculate the forces R_x and R_y we need to find the pressures p_1 and p_2. The pressure p_2 is zero because the flow exits into the atmosphere. The pressure at section 1 can be determined using the energy equation or the Bernoulli equation. Neglecting losses between sections 1 and 2, the energy equation gives

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} = \frac{V_2^2}{2g} + \frac{\cancelto{0}{p_2}}{\gamma}$$

$$\frac{p_1}{\rho g} + z_1 + \propto_1 \frac{v_1^2}{2g} + h_p = \frac{p_2}{\rho g} + z_2 + \propto_2 \frac{v_2^2}{2g} + h_t + h_l$$

$$\therefore\ p_1 = \frac{\gamma}{2g}(V_2^2 - V_1^2) = \frac{9810}{2 \times 9.81}(7.96^2 - 1.99^2) = 29\,700 \text{ Pa}$$



A: Force exerted on a bend

Now we can apply the momentum equation (4.5.6) in the x-direction to find R_x and in the y-direction to find R_y:

$$-(F_{BF})_x + P_1A_1 - P_2A_2\cos\theta = \rho\, v_1(-A_1v_1) + \rho\, v_2\cos\theta(A_2v_2) \quad 14.16$$

$$\sum F_x = \frac{d(mv_x)}{dt} = \rho Q(\Delta v)$$

$$(F_{BF})_x = P_1A_1 - P_2A_2\cos\theta - \rho Q(v_2\cos\theta - v_1) \quad 14.18$$

Better to use this equation or 14.16 than 14.18

x-direction:

$$p_1A_1 - R_x = \dot{m}(V_{2x} - V_{1x})$$ ($V_{2x} = 0$)

$$29\,700 \times \frac{\pi}{4} \times (0.08)^2 - R_x = 1000 \times 0.01 \times (-1.99)$$

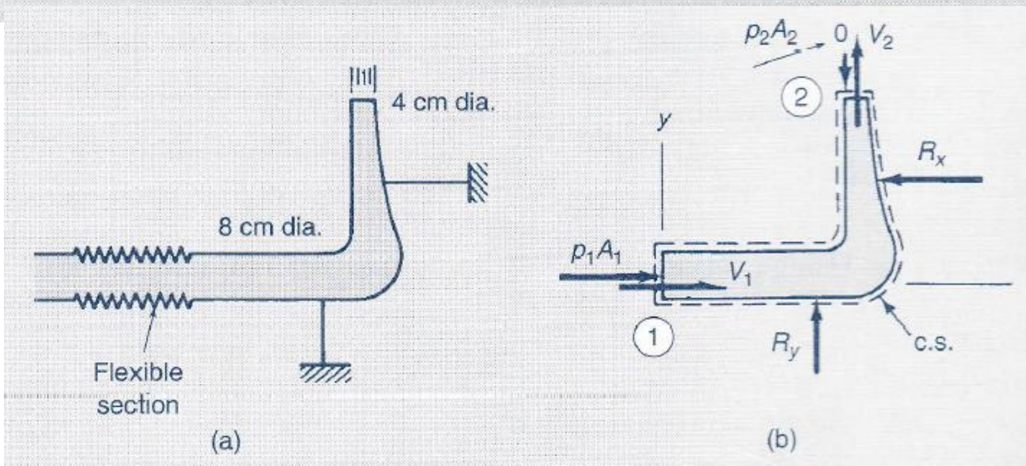
$$\therefore R_x = 169 \text{ N}$$

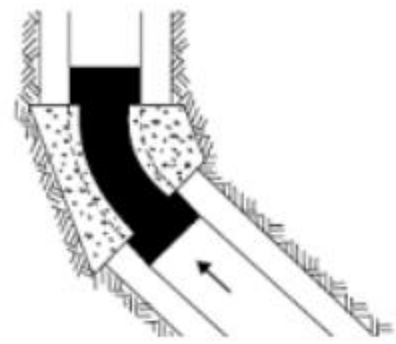
y-direction:

$$R_y - P_2A_2 = \dot{m}(V_{2y} - V_{1y})$$ ($P_2A_2 = 0$, $V_{1y} = 0$)

$$\therefore R_y = 1000 \times 0.01 \times 7.96 = 79.6 \text{ N}$$

Note that we have assumed uniform profiles and steady flow and used $\dot{m} = \rho Q$. These are the usual assumptions if information is not given otherwise.





Bend in horizontal plane anchorage

B: Forces on gates

An example of free surface flow in a rectangular channel is shown below.

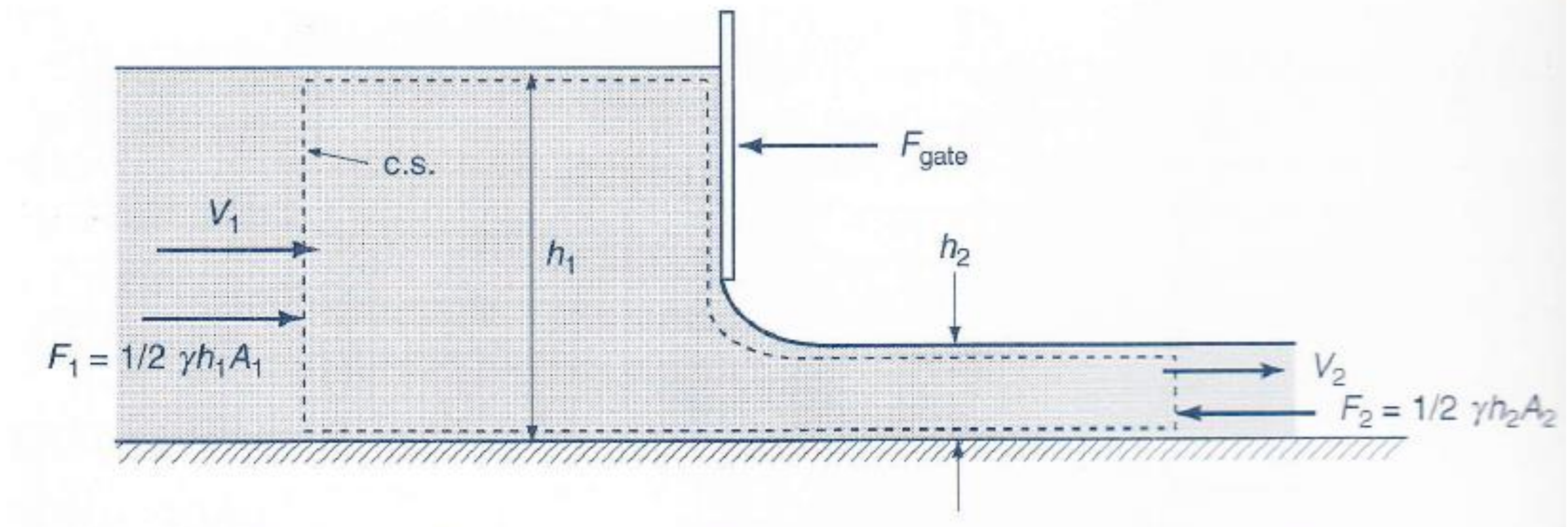


Figure 4.12 Force of the flow on a gate in a free-surface flow.

If we want to determine the force of the gate on the flow, the following expression can be derived from the momentum equation

$$\sum F_x = F_1 - F_2 - F_{gate} = \rho\, v_1(-A_1 v_1) + \rho\, v_2(A_2 v_2)$$

$$= \rho Q(\, v_2 - v_1)$$

B: Forces on gates

$$\sum F_x = F_1 - F_2 - F_{gate} = \rho\, v_1(-A_1 v_1) + \rho\, v_2(A_2 v_2)$$

$$= \rho Q(\, v_2 - v_1)$$

$$F_1 - F_2 - F_{gate} = \rho Q(\, v_2 - v_1)$$

$$F_{gate} = F_1 - F_2 - \rho Q(\, v_2 - v_1)$$

14.21

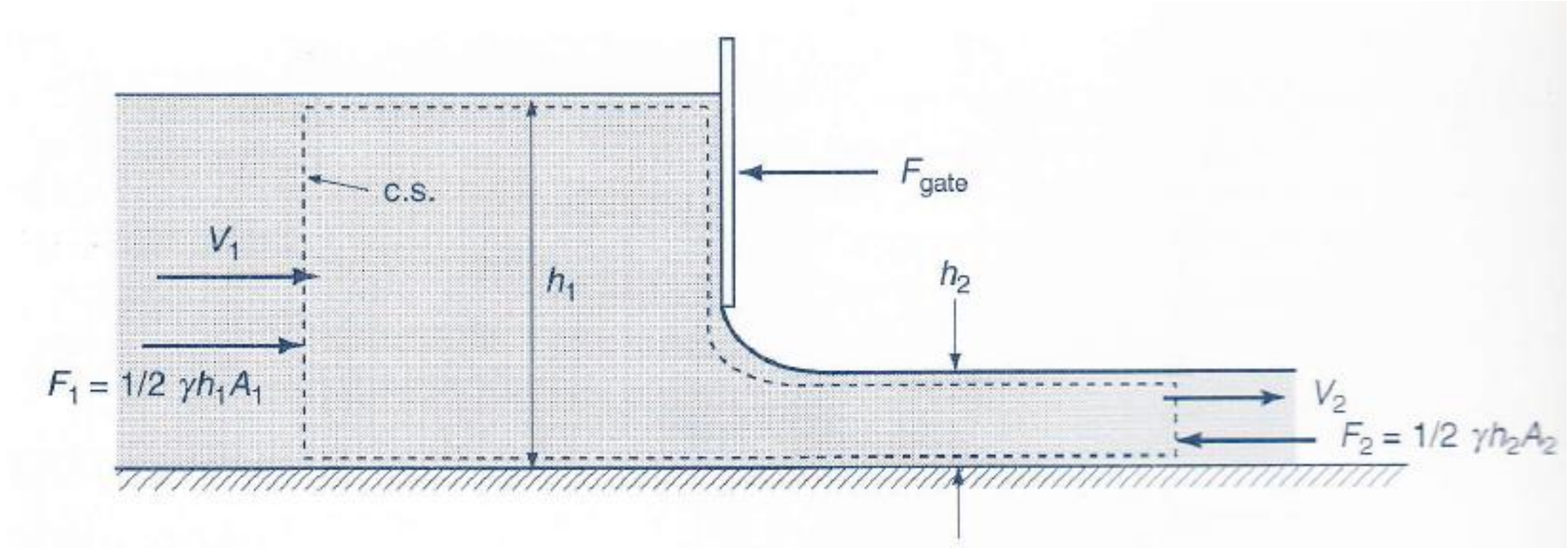


Figure 4.12 Force of the flow on a gate in a free-surface flow.

B: Forces on gates

Example

Sample Problem 6.1 The water passage shown in Fig. S6.1 is 10 ft (3 m) wide normal to the plane of the figure. Determine the horizontal force acting on the shaded structure. Assume ideal flow.

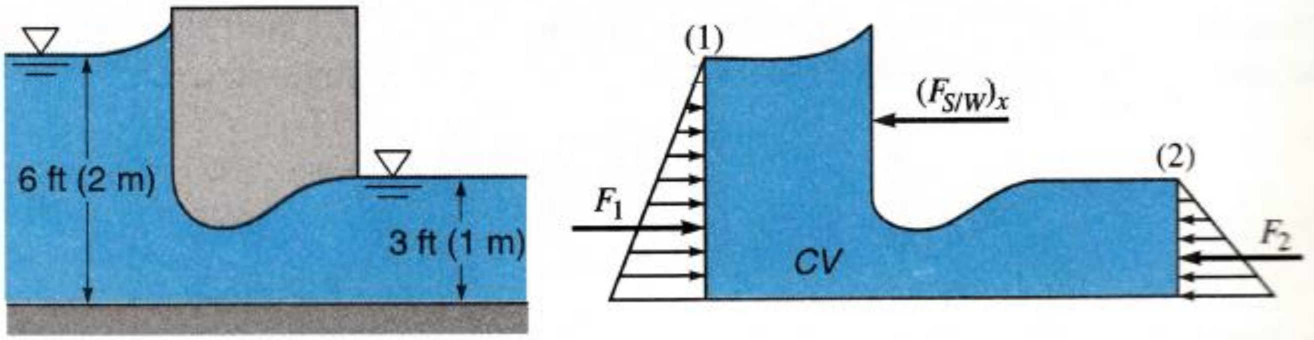
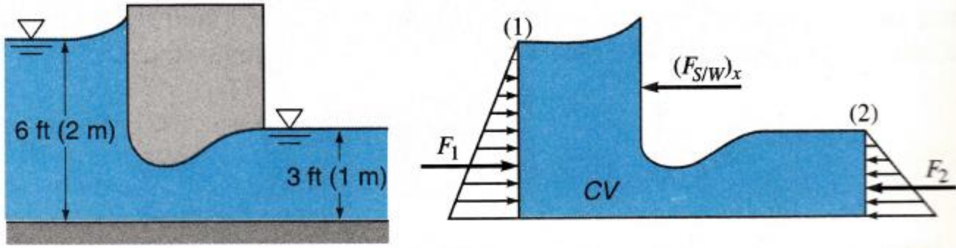


Figure S6.1

B: Forces on gates



Solution

In free-surface flow such as this where the streamlines are parallel, the water surface is coincident with the hydraulic grade line. Writing an energy equation from the upstream section to the down-stream section,

$$\frac{p_1}{\rho g} + z_1 + \propto_1 \frac{v_1^2}{2g} + h_p = \frac{p_2}{\rho g} + z_2 + \propto_2 \frac{v_2^2}{2g} + h_t + h_l$$

Free surface at 1 and 2 means atmospheric pressure i.e., zero gauge pressure. No pump, turbine and negligible head loss

Solution (SI units)

Energy:
$$2 + \frac{V_1^2}{2(9.81)} = 1 + \frac{V_2^2}{2(9.81)} \qquad (3)$$

Continuity: $A_1 v_1 = A_2 v_2$
$$2(3)V_1 = 1(3)V_2 \qquad (4)$$

Substituting Eq. (4) into Eq. (3) yields

$$V_1 = 2.56 \text{ m/s}, \qquad V_2 = 5.11 \text{ m/s}$$

$$Q = A_1V_1 = A_2V_2 = 15.34 \text{ m}^2/\text{s}$$

B: Forces on gates

Solution

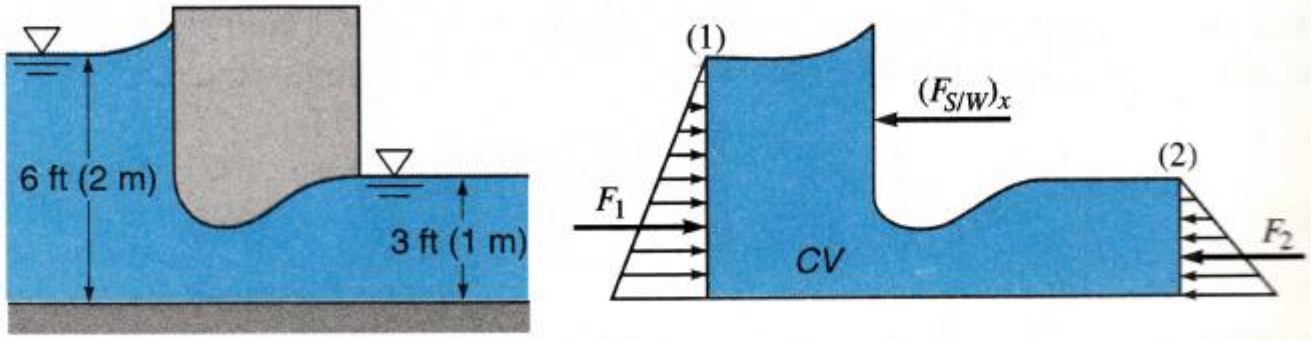


Figure S6.1

$F =$

Applying the impulse–momentum equation (6.7*a*) to the free-body diagram,

$$F_{gate} = F_1 - F_2 - \rho Q(v_2 - v_1)$$ 14.21

$F = \gamma \bar{h} A = p_c A$

$$F_1 - F_2 - (F_{S/W})_x = \rho Q(V_2 - V_1)$$

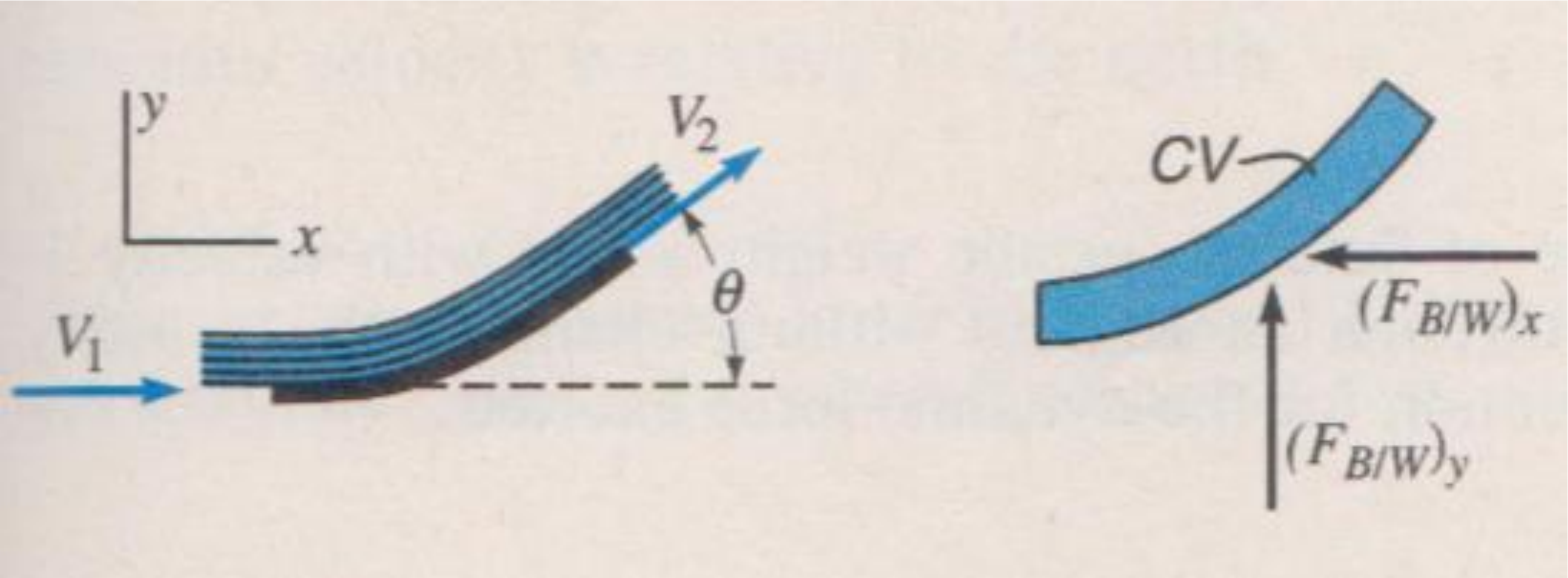
$$9.81(1)(2)(3) - 9.81(0.5)(1)(3) - (F_{S/W})_x = 1.0(15.34)(5.11 - 2.56)$$

$$(F_{S/W})_x = +4.91 \text{ kN} = 4.91 \text{ kN} \leftarrow$$

So $(F_{W/S})_x = 4.91 \text{ kN} \rightarrow$ ***ANS***

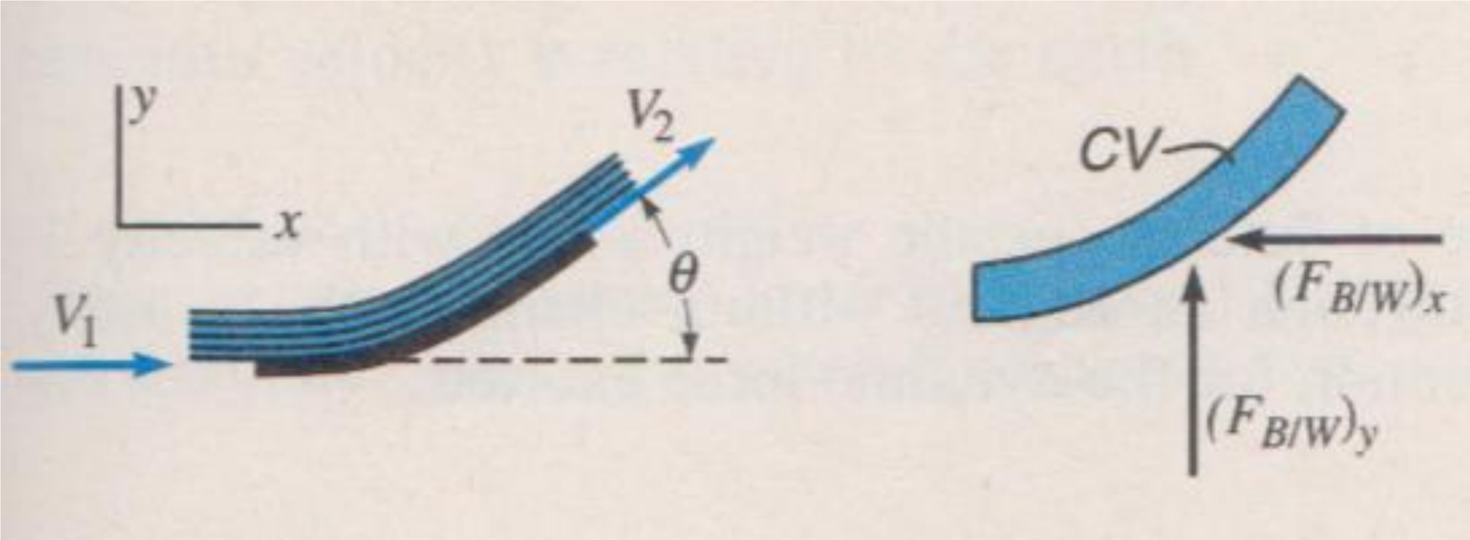
C: Jet deflected by a ***stationary*** vane or blade (deflector)

- The application of the momentum equation to deflectors forms an integral part of the analysis of many turbomachines such as *turbines, pumps and compressors*



C: Jet deflected by a ***stationary*** vane or blade (deflector)

- The main difference with preceeding sections is that with the vane or blade, the fluid is in contact with the atmosphere; hence the gage pressures in the jet are zero and the *PA* forces disappear!!!!
- Another difference is that in many types of fluid machinery where vanes or blades are used, the velocities are often so high that the neglect of friction may introduce a sizeable error. In such cases, for accurate results, friction should be considered.
- This is usually handled by prescribing a reduction in the velocity of the flow between its arrival and departure points on the blade

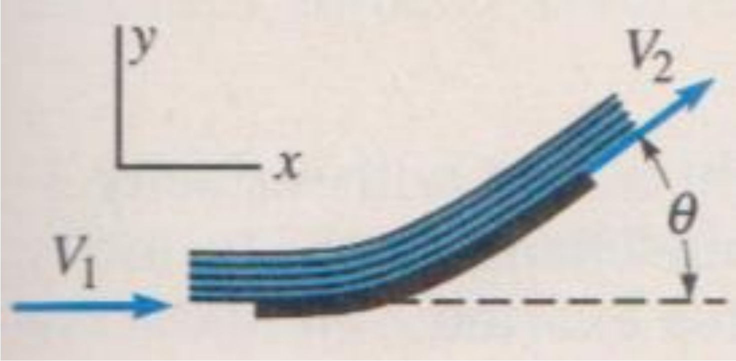


C: Jet deflected by a ***stationary*** vane or blade (deflector)

$$\sum F_x = \frac{d}{dt}\int \rho v_x \, dV - \beta_1 \rho \bar{v}_{1x}\bar{v}_1 A_1 + \beta_2 \rho \bar{v}_{2x}\bar{v}_2 A_2$$

13.6a

$$-(F_{BW})_X = -\rho v_{1x} v_1 A_1 + \rho v_{2x} v_2 A_2$$

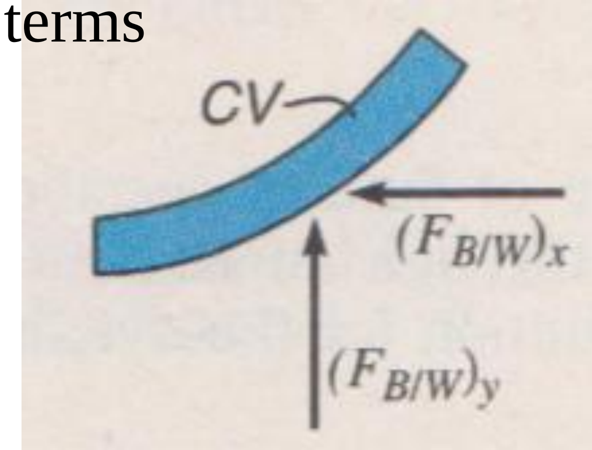


$$continuity \quad v_1 A_1 = v_2 A_2 = Q$$

$$-(F_{BW})_X = \rho Q(v_{2x} - v_{1x}) = \rho Q(v_2 \cos\theta - v_1)$$

Hence $(F_{BW})_X$ assumed direction is correct since terms in brackets is negative

$$if \quad we \ assume \ that \ v_1 = v_2$$



$$-(F_{BW})_X = \rho Q(v\cos\theta - v)$$

C: Jet deflected by a ***stationary*** vane or blade (deflector)

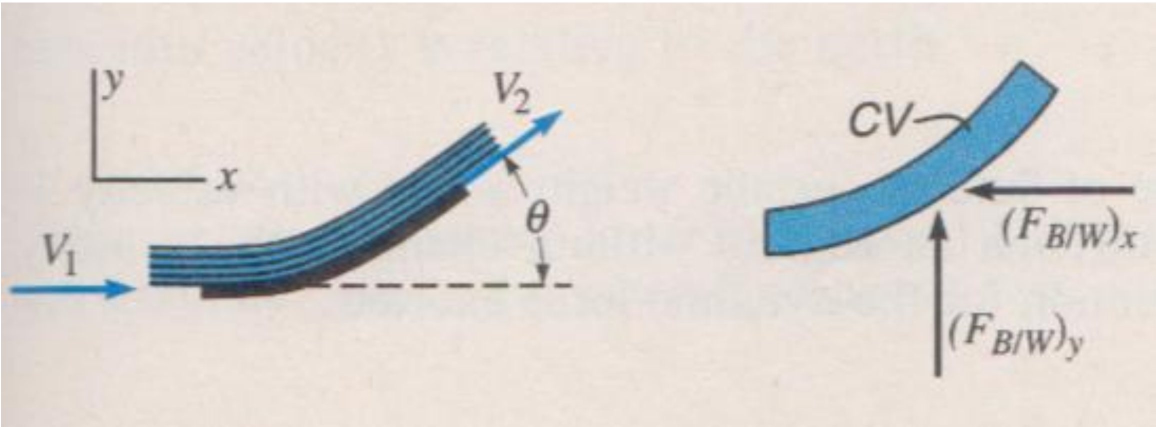
Applying Eq. 13.6b along the y-axis

$$\sum F_y = \frac{d}{dt}\int \rho v_y \, dV - \beta_1 \rho \bar{v}_{1y}\bar{v}_1 A_1 + \beta_2 \rho \bar{v}_{2y}\bar{v}_2 A_2 \qquad 13.6b$$

$$+(F_{BW})_y = -\rho v_{1y} v_1 A_1 + \rho v_{2y} v_2 A_2$$

$$+(F_{BW})_X = \rho Q(v_{2y} - v_{1y}) = \rho Q(v_2 \sin\theta - 0)$$

$$(F_{BW})_X = \rho Q v_2 \sin\theta$$



D: Jet deflected by a ***moving*** vane or blade

- For a *moving vane*, e.g., a turbine runner or wind vane, the same type of analysis as in the previous section can be carried out, except that it would be convenient to let the cv move with the vane. For that case the flow is steady.

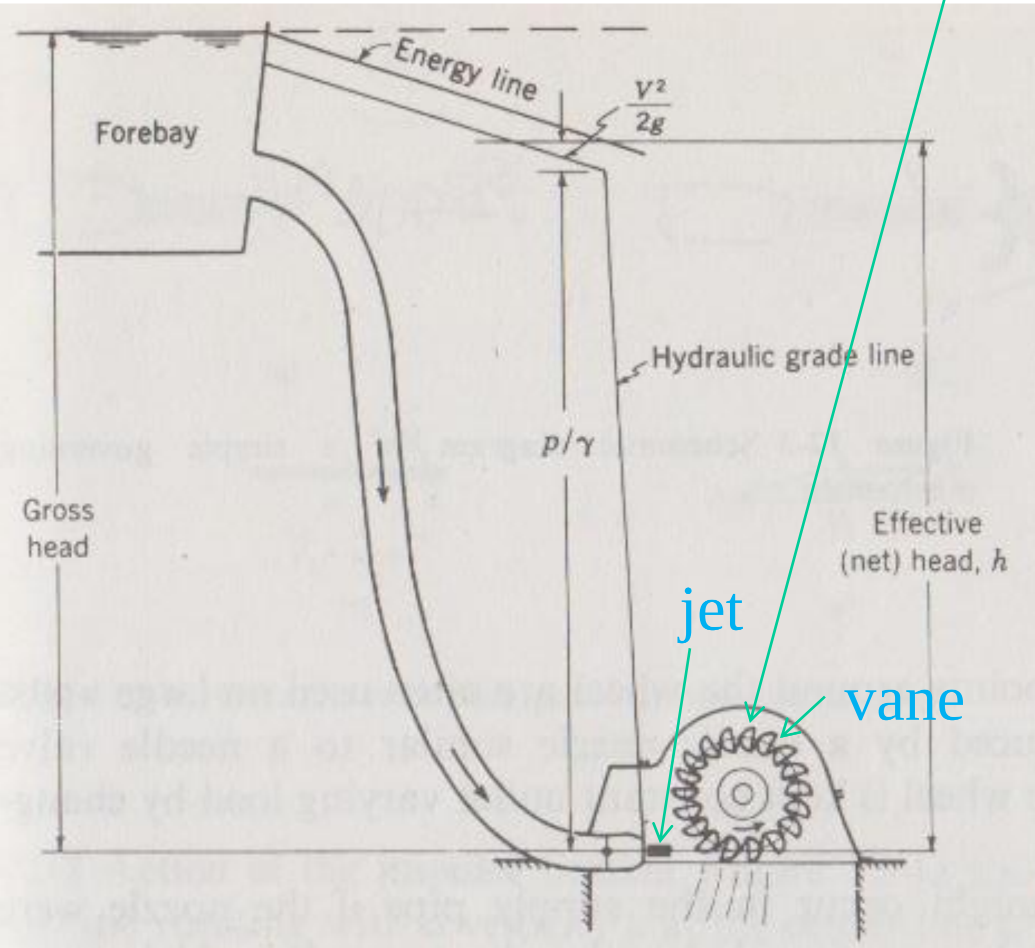


Figure 12-1 Definition sketch for impulse-turbine installation.

D: Jet deflected by a ***moving*** vane or blade

- In Fig 14.2a the *absolute* velocities, i.e. velocities with respect to the earth, are drawn, while Fig 14.2b the *relative* velocity of the jet, i.e. the velocity of the jet with respect to the moving vane, is represented.

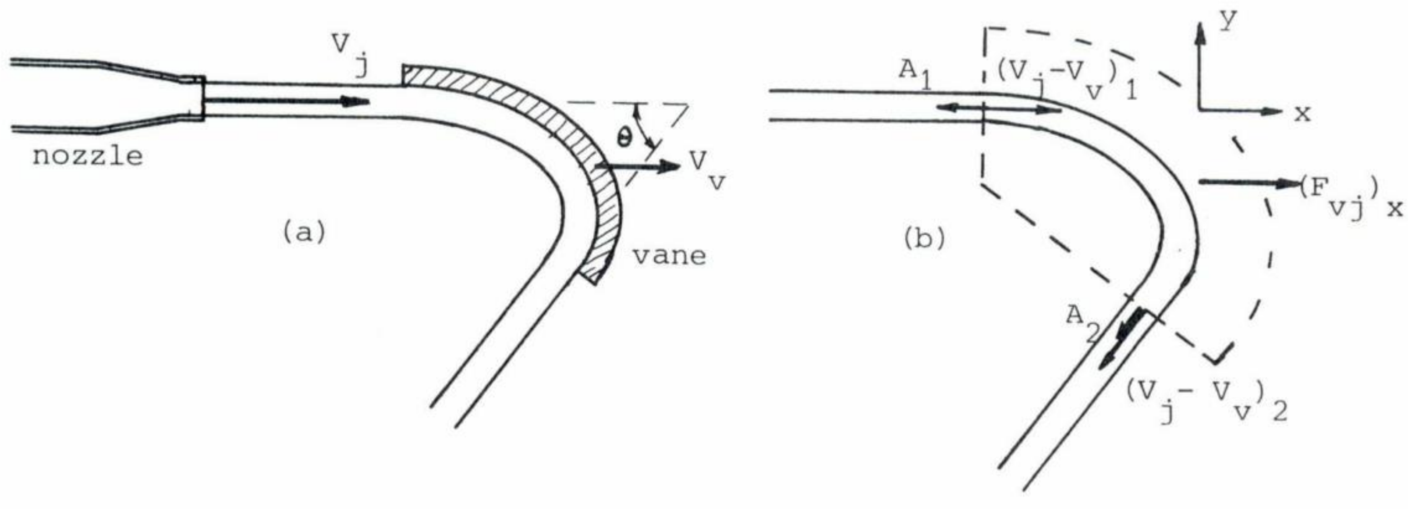
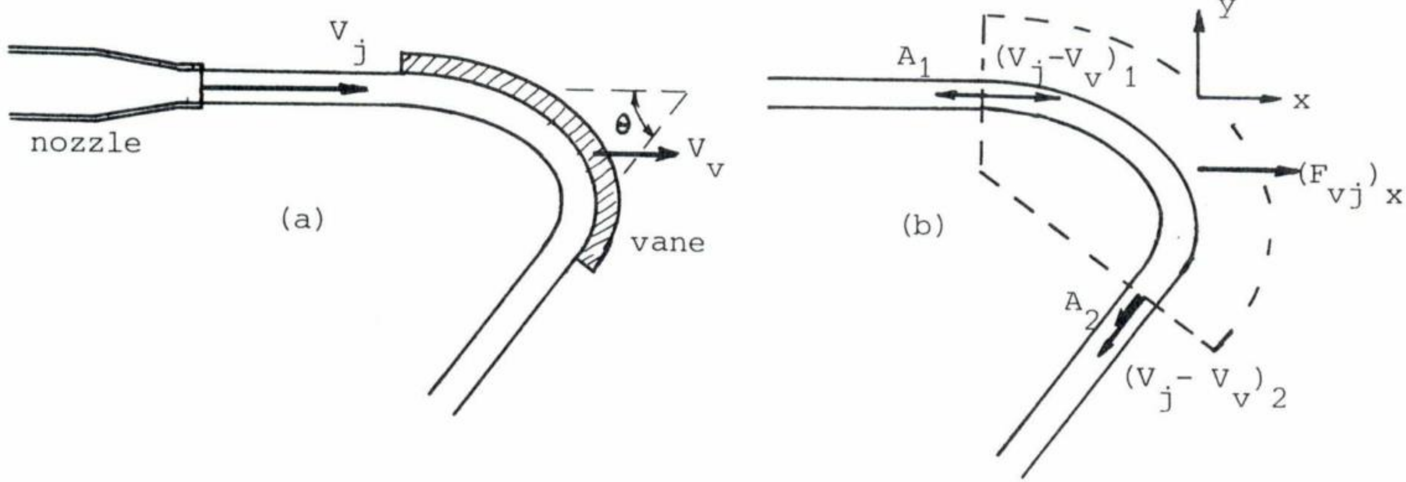


Fig 14.2 Jet deflected by a moving vane



The force applied to the jet by the vane in the x direction is then ($\sum F_s + \sum F_b = \frac{d}{dt}\int_{cv} v\,\rho\,dV + \int_{cs} v\,\rho\,(\vec{v}\,d\vec{A})$ 13.4)

$$(F_{vj})_x = 0 + \int_{cs} \rho\, v_x(\vec{v}\,d\vec{A}) = \rho\,(v_j - v_v)_{1x}(-A\quad)(v_j - v_v)_1) + \rho\,(v_j - v_v)_{2x}(-A\quad)(v_j - v_v)_2)$$

$$= -\rho\,A(v_j - v_v)^2 \quad - \rho\,A(v_j - v_v)^2 \cos\theta$$

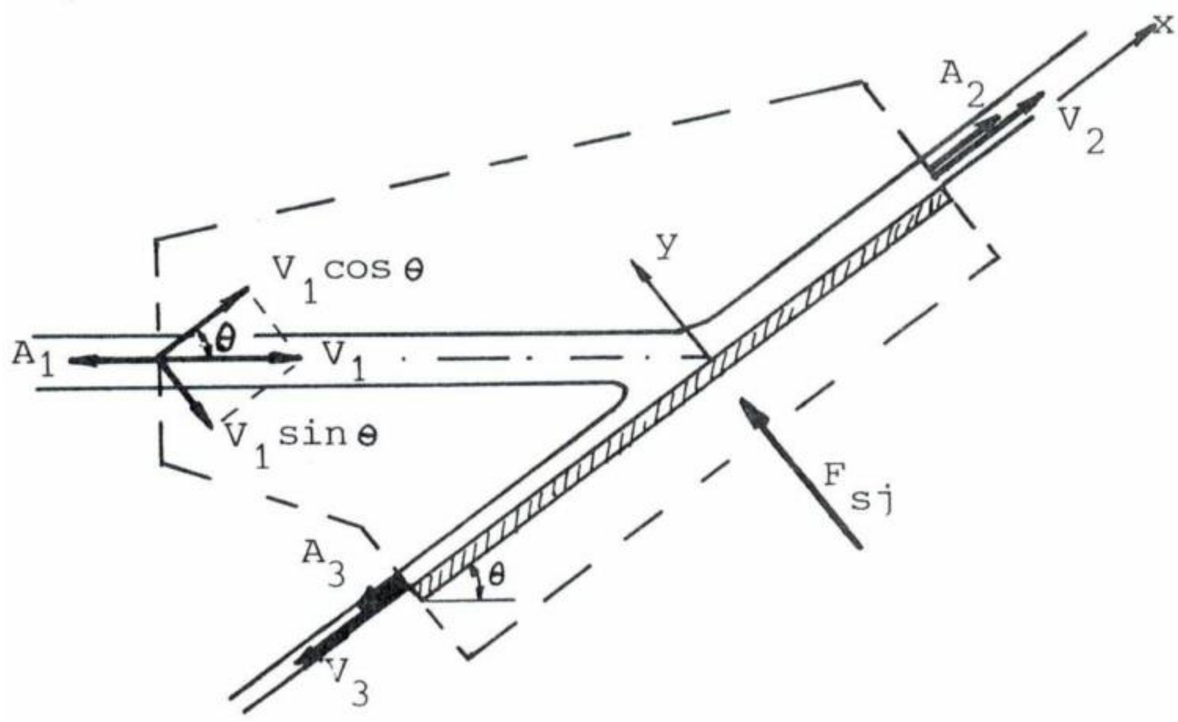
14.5

The terms between brackets are always positive

The terms between brackets are always positive, thus $(F_{vj})_x$ is negative. This means that the force applied to the jet is working from right to left. However, the force which the jet exerts on the vane, $(F_{jv})_x$, is working from left to right.

The power delivered by the vane is equal to the product of the force on the vane and the speed of the vane. The resulting power is then $(F_{vj})_x . V_v$. Obviously, no power results unless the vane speed is greater than zero and less than the velocity of the approaching jet.

E: Jet striking a surface



Assuming friction is negligible, $v_1 = v_2 = v_3$ 14.6

since $A_1 = A_2 + A_3$

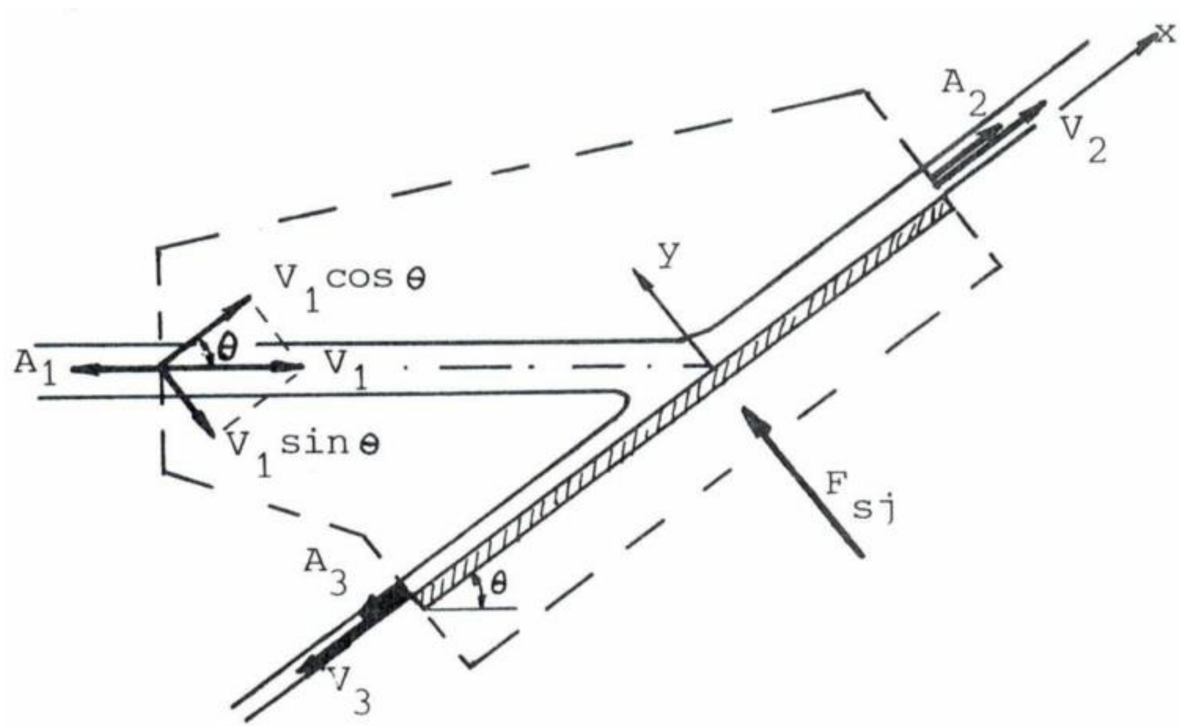
Applying the continuity equation

$$A_1 v_1 = A_2 v_2 + A_3 v_3$$ 14.7

Hence, combining 14.6 & 14.7 gives

$A_1 v_1 = A_2 v_2 + A_3 v_3 \rightarrow A_1 v = A_2 v + A_3 v \rightarrow$

$A_1 = A_2 + A_3$ 14.8



Applying the momentum equation in x-direction

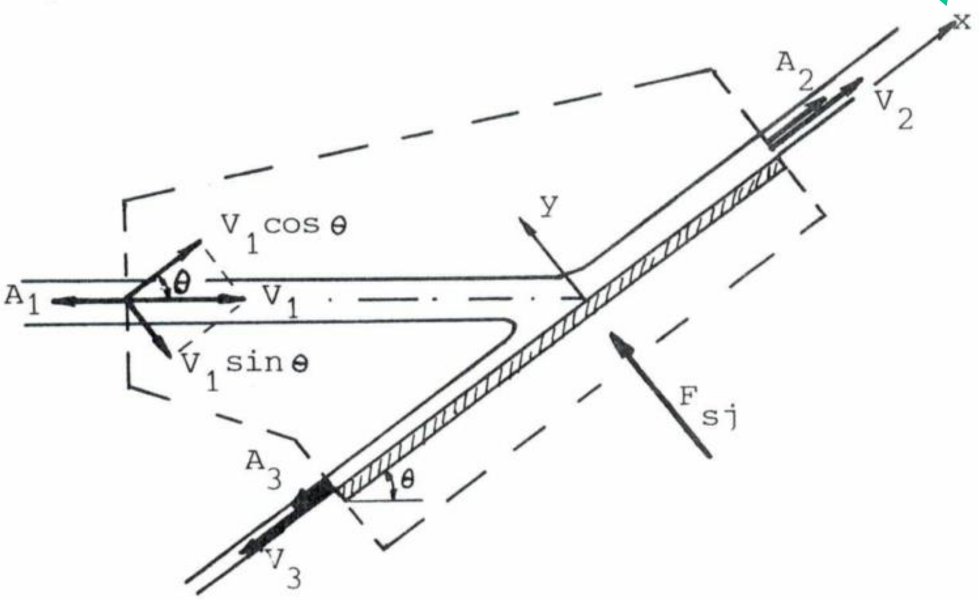
$(\sum F_s + \sum F_b = \frac{d}{dt}\int_{cv} v\,\rho\, dV + \int_{cs} v\,\rho\,(\vec{v}\, d\vec{A})\,)$ 13.4

$$\sum F_x = 0 + \rho(v_1 \cos\theta)(-A_1 v_1) + \rho v_2 (A_2 v_2) + \rho(-v_3)(A_3 v_3) \quad 14.9$$

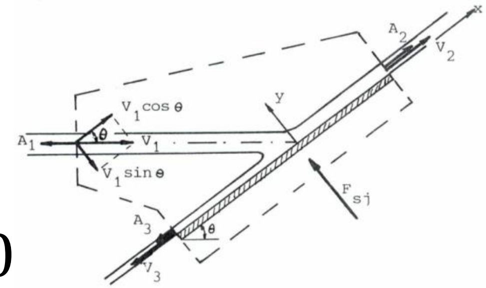
$\sum F_x = 0$, since friction is assumed negligible, thus Eqn 14.9 becomes

$$\sum F_x = \rho(v_1 \cos\theta)(-A_1 v_1) + \rho v_2 (A_2 v_2) - \rho v_3 (A_3 v_3) = 0$$

$$-A_1 \cos\theta + A_2 - A_3 = 0 \quad 14.10$$



Adding 14.10 and 14.8 yields



$$A_1v_1 = A_2v_2 + A_3v_3 \rightarrow A_1v = A_2v + A_3v \rightarrow A_1 = A_2 + A_3$$

$$\sum F_x = \rho(v_1\cos\theta)(-A_1v_1) + \rho v_2(A_2v_2) - \rho v_3(A_3v_3) = 0$$

$$-A_1\cos\theta + A_2 - A_3 = 0$$

$$\begin{array}{r} -A_1\cos\theta + A_2 - A_3 = 0 \\ \underline{-A_1 \qquad + A_2 + A_3 = 0} \\ -A_1(1+\cos\theta) + 2A_2 = 0 \end{array}$$

Hence

$$A_2 = \frac{1}{2}A_1(1+\cos\theta) \qquad 14.11$$

$$A_3 = \frac{1}{2}A_1(1-\cos\theta) \qquad 14.12$$

Eqns 14.11 & 14.12 can also be written in terns of discharges

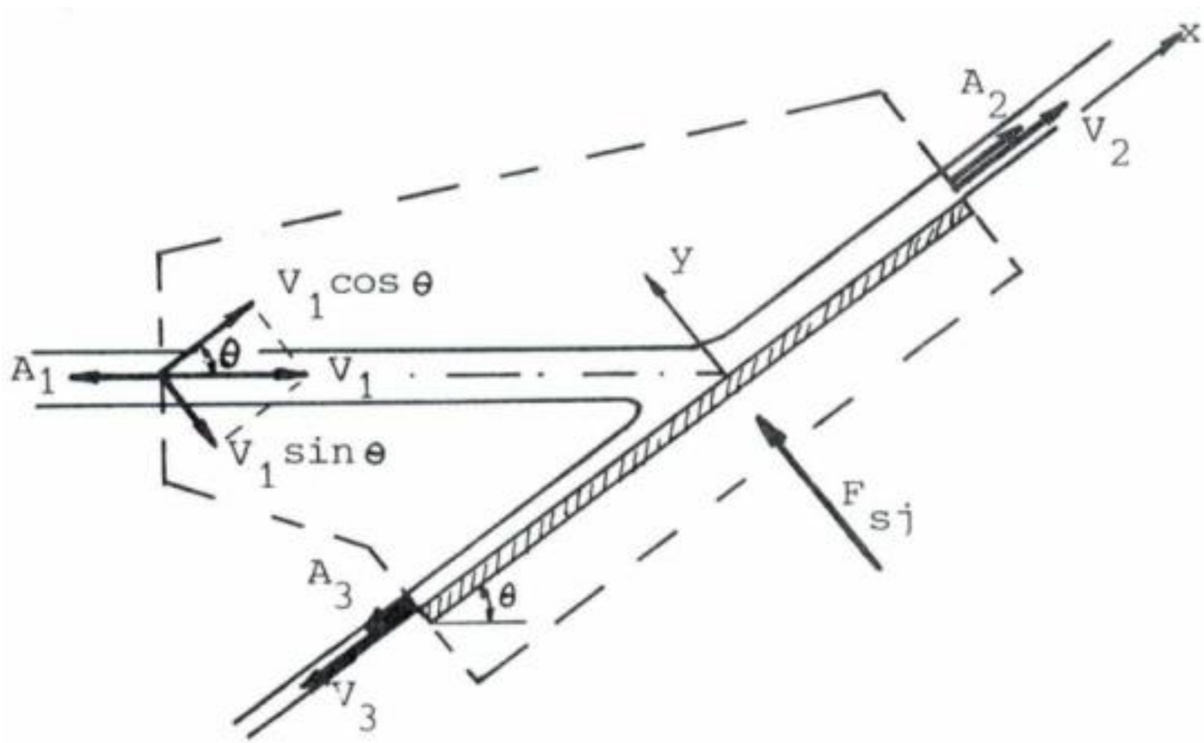
$$Q_2 = \frac{1}{2}Q_1(1+\cos\theta) \qquad 14.13$$

$$Q_3 = \frac{1}{2}Q_1(1-\cos\theta) \qquad 14.14$$

The momentum eqn in the y-direction

$(\sum F_s + \sum F_b = \frac{d}{dt}\int_{cv} v\,\rho\,dV + \int_{cs} v\,\rho\,(\vec{v}\,d\vec{A})$ 13.4 $)$

$$(F_{sj})_y = 0 + \rho(-v_1 \sin\theta)(-A_1 v_1) = \rho A_1 {v_1}^2 \sin\theta \qquad 14.15$$



Forces on gates

Solution

Note that the momentum principle will not permit one to obtain the vertical component of the force of the water on the shaded structure, because the pressure distribution along the bottom of the channel is unknown. The pressure distribution along the boundary of the structure and along the bottom of the channel can be estimated by sketching a flow net and applying Bernoulli's principle. The horizontal and vertical components of the force can be found by computing the integrated effect of the pressure-distribution diagram.

Summary

$$\sum F = \dot{m}(\Delta v) = \rho Q(\Delta v)$$