Fluid Mechanics CEE 3311
LECTURE 9

Conservation of mass

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The continuity principle is based on the conservation of mass.
In this case extensive property N=mass
intensive property $\eta=1$
Then control volume equation becomes

$$\frac{dN_{sys}}{dt} = \frac{d}{dt}\int_{cv} \eta\, \rho\, dV + \int_{cs} \eta\, \rho\, (\vec{v}\, d\vec{A})$$

$$\frac{d(\mathrm{Mass})}{dt} = \frac{d}{dt}\int_{cv} \rho dV + \int_{cs} \rho \vec{v}\overrightarrow{dA}$$

$$\frac{d(\text{Mass})}{dt} = \frac{d}{dt}\int_{cv} \rho dV + \int_{cs} \rho \vec{v}\,\overrightarrow{dA}$$

If the flow is steady, there results

$$0 = 0 + \int_{cs} \rho \vec{v}\,\overrightarrow{dA}$$

$$\int_{cs} \rho \vec{v}\,\overrightarrow{dA} = 0$$

For one dimensional flow $\int_{cs} \rho \vec{v}\,\overrightarrow{dA}$ can be written as $\sum \rho \vec{v}\vec{A}$

For steady one dimensional flow the formula diminishes to

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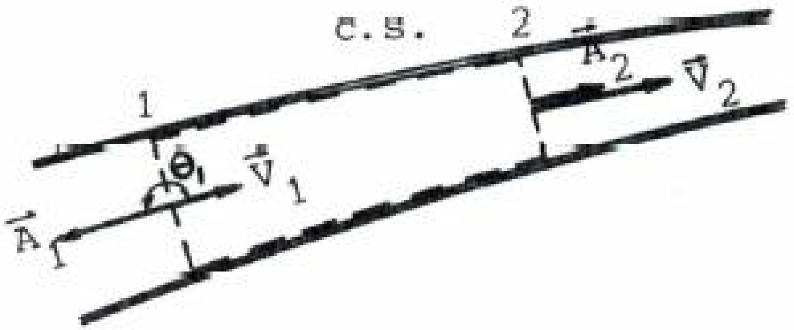
This formula may be used, e.g., in a steady flow case in a conduit



$$\sum \rho \vec{v}\vec{A} = -\rho_1 A_1 v_1 + \rho_2 A_2 v_2 = 0$$

for $\overrightarrow{v_1}\overrightarrow{A_1} = A_1 v_1 \cos 180° = -A_1 v_1$

and $\overrightarrow{v_2}\overrightarrow{A_2} = A_2 v_2 \cos 0° = +A_2 v_2$

The continuity equation takes the form

$$\rho_2 A_2 v_2 = \rho_1 A_1 v_1$$

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$$\rho_2 A_2 v_2 = \rho_1 A_1 v_1$$

If the density is constant *in the control volume*, the continuity equation then reduces to

$$A_1 v_1 = A_2 v_2$$

This form of the equation is used quite often, particularly with liquids and low speed gas

EXAMPLE 4.1

Water flows at a uniform velocity of 3 m/s into a nozzle that reduces the diameter from 10 cm to 2 cm (Fig. E4.1). Calculate the water's velocity leaving the nozzle and the flow rate.

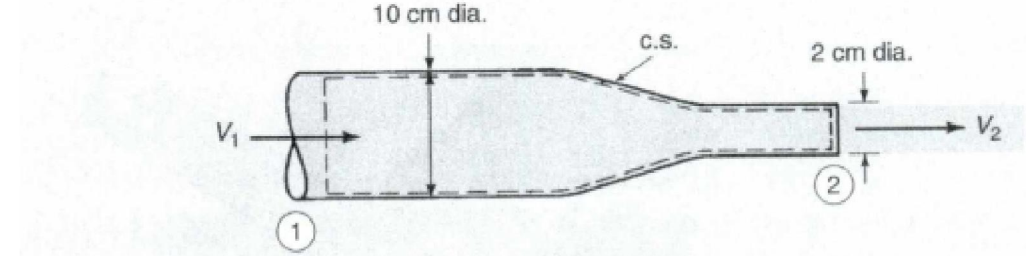


Figure E4.1

Solution: The control volume is selected to be the inside of the nozzle as shown. Flow enters the control volume at section 1 and leaves at section 2. The simplified continuity equation (4.3.6) is used:

$$A_1V_1 = A_2V_2$$

$$\therefore\ V_2 = V_1\frac{A_1}{A_2}$$

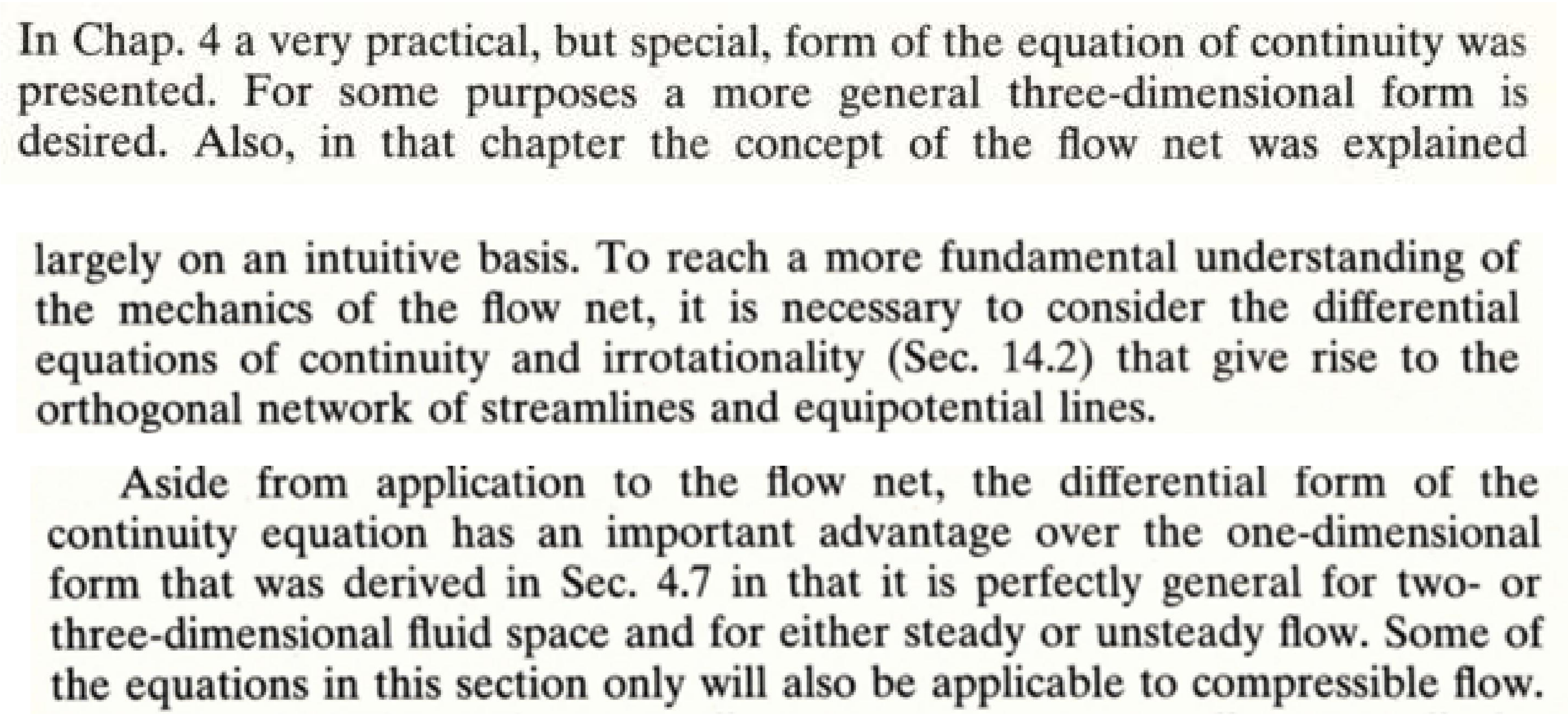
$$= 3\,\frac{\pi \times 0.1^2/4}{\pi \times 0.02^2/4} = 75\ \text{m/s}$$

The flow rate, or discharge, is found to be

$$Q = V_1A_1$$

$$= 3 \times \pi \times 0.1^2/4 = 0.0236\ \text{m}^3/\text{s}$$

DIFFERENTIAL EQUATION OF CONTINUITY



In Chap. 4 a very practical, but special, form of the equation of continuity was presented. For some purposes a more general three-dimensional form is desired. Also, in that chapter the concept of the flow net was explained largely on an intuitive basis. To reach a more fundamental understanding of the mechanics of the flow net, it is necessary to consider the differential equations of continuity and irrotationality (Sec. 14.2) that give rise to the orthogonal network of streamlines and equipotential lines.

Aside from application to the flow net, the differential form of the continuity equation has an important advantage over the one-dimensional form that was derived in Sec. 4.7 in that it is perfectly general for two- or three-dimensional fluid space and for either steady or unsteady flow. Some of the equations in this section only will also be applicable to compressible flow.

DIFFERENTIAL EQUATION OF CONTINUITY

Figure 14.1 shows three coordinate axes x, y, z mutually perpendicular and fixed in space. Let the velocity components in these three directions be u, v, w, respectively. Consider now a small parallelepiped, having sides Δx, Δy, Δz. In the x direction the rate of mass flow into this box through the left-hand face is approximately $\rho u \, \Delta y \, \Delta z$, this expression becoming exact in the limit as the box is shrunk to a point. The corresponding rate of mass flow out of the box through the right-hand face is $\{\rho u + [\partial(\rho u)/\partial x] \, \Delta x\} \, \Delta y \, \Delta z$. Thus the net rate of mass flow into the box in the x direction is $-[\partial(\rho u)/\partial x] \, \Delta x \, \Delta y \, \Delta z$.

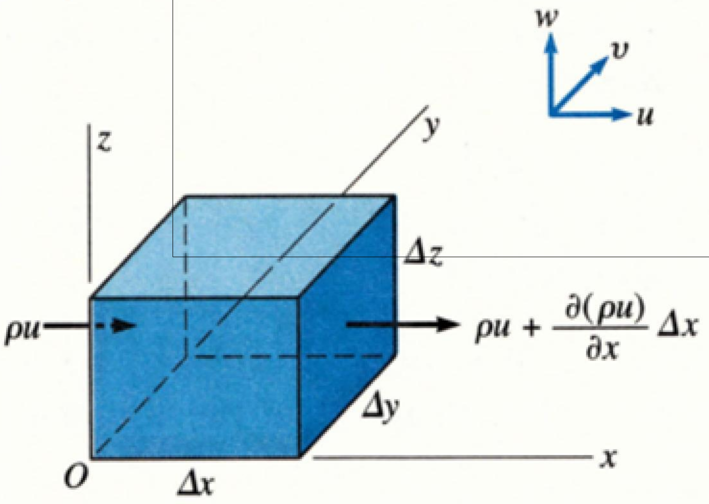


Figure 14.1

$$\frac{d(\text{Mass})}{dt} = \frac{d}{dt}\int_{cv} \rho dV + \int_{cs} \rho \vec{v} d\overrightarrow{A}$$

DIFFERENTIAL EQUATION OF CONTINUITY

$$\frac{d(\mathrm{Mass})}{dt} = \frac{d}{dt}\int_{cv} \rho \, dV + \int_{cs} \rho \vec{v} \, d\vec{A}$$

Similar expressions may be obtained for the y and z directions. The sum of the rates of mass inflow in the three directions must equal the time rate of change of the mass in the box, or $(\partial\rho/\partial t)\,\Delta x\,\Delta y\,\Delta z$. Summing up, applying the limiting process, and dividing both sides of the equation by the volume of the parallelepiped, which is common to all terms, we get

$$-\frac{\partial(\rho u)}{\partial x}\Delta x\Delta y\Delta z - \frac{\partial(\rho v)}{\partial y}\Delta x\Delta y\Delta z - \frac{\partial(\rho w)}{\partial z}\Delta x\Delta y\Delta z = \frac{\partial\rho}{\partial t}\Delta x\Delta y\Delta z$$

Unsteady compressible flow:

$$-\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z} = \frac{\partial\rho}{\partial t} \qquad (14.1)$$

which is the equation of continuity in its most general form. This equation as well as the other equations in this section are, of course, valid regardless of whether the fluid is a real one or an ideal one. If the flow is steady, ρ does not vary with time, but it may vary in space. Since $\partial(\rho u)/\partial x = \rho(\partial u/\partial x) + u(\partial\rho/\partial x)$, it follows that for steady flow the equation may be written as

Steady compressible flow:

$$u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad (14.2)$$

DIFFERENTIAL EQUATION OF CONTINUITY

Steady compressible flow:
$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad (14.2)$$

In the case of an incompressible fluid (ρ = constant), whether the flow is steady or not, the equation of continuity becomes

Steady incompressible flow:
$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \qquad (14.3)$$

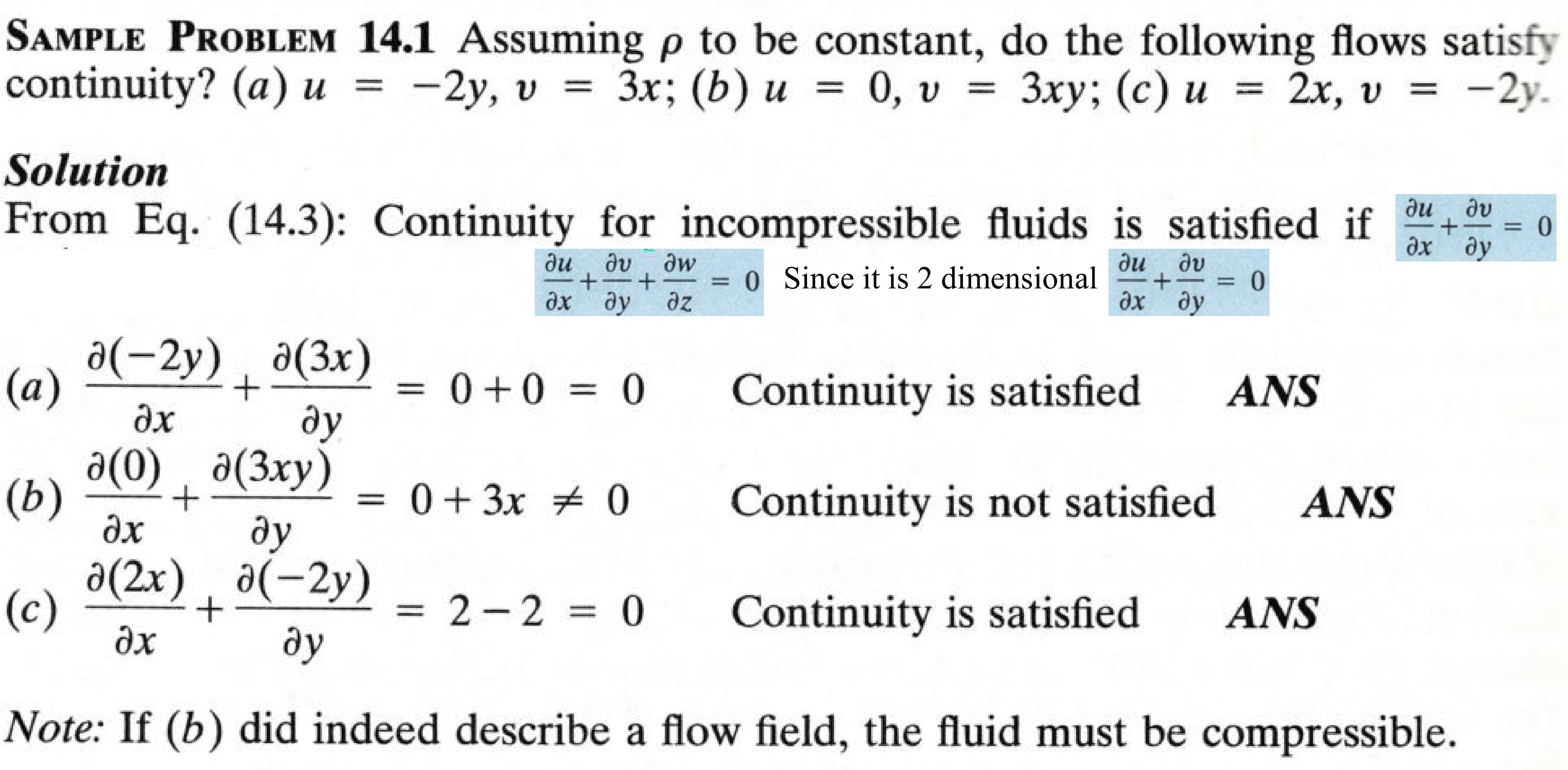
For two-dimensional flow, application of the same procedure to an elemental volume in polar coordinates yields for steady flow the following equations:

Steady compressible flow:
$$\frac{1}{r}(\rho v_r) + \frac{\partial}{\partial r}(\rho v_r) + \frac{\partial}{r\,\partial \theta}(\rho v_t) = 0 \qquad (14.4)$$

Steady, incompressible flow:
$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_t}{r\,\partial \theta} = 0 \qquad (14.5)$$

where v_r and v_t represent the velocities in the radial and tangential[1] directions, respectively.

DIFFERENTIAL EQUATION OF CONTINUITY



Sample Problem 14.1 Assuming ρ to be constant, do the following flows satisfy continuity? (*a*) $u = -2y$, $v = 3x$; (*b*) $u = 0$, $v = 3xy$; (*c*) $u = 2x$, $v = -2y$.

Solution

From Eq. (14.3): Continuity for incompressible fluids is satisfied if $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ Since it is 2 dimensional $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$(a)\quad \frac{\partial(-2y)}{\partial x} + \frac{\partial(3x)}{\partial y} = 0 + 0 = 0 \qquad \text{Continuity is satisfied} \qquad \textbf{\textit{ANS}}$$

$$(b)\quad \frac{\partial(0)}{\partial x} + \frac{\partial(3xy)}{\partial y} = 0 + 3x \neq 0 \qquad \text{Continuity is not satisfied} \qquad \textbf{\textit{ANS}}$$

$$(c)\quad \frac{\partial(2x)}{\partial x} + \frac{\partial(-2y)}{\partial y} = 2 - 2 = 0 \qquad \text{Continuity is satisfied} \qquad \textbf{\textit{ANS}}$$

Note: If (*b*) did indeed describe a flow field, the fluid must be compressible.