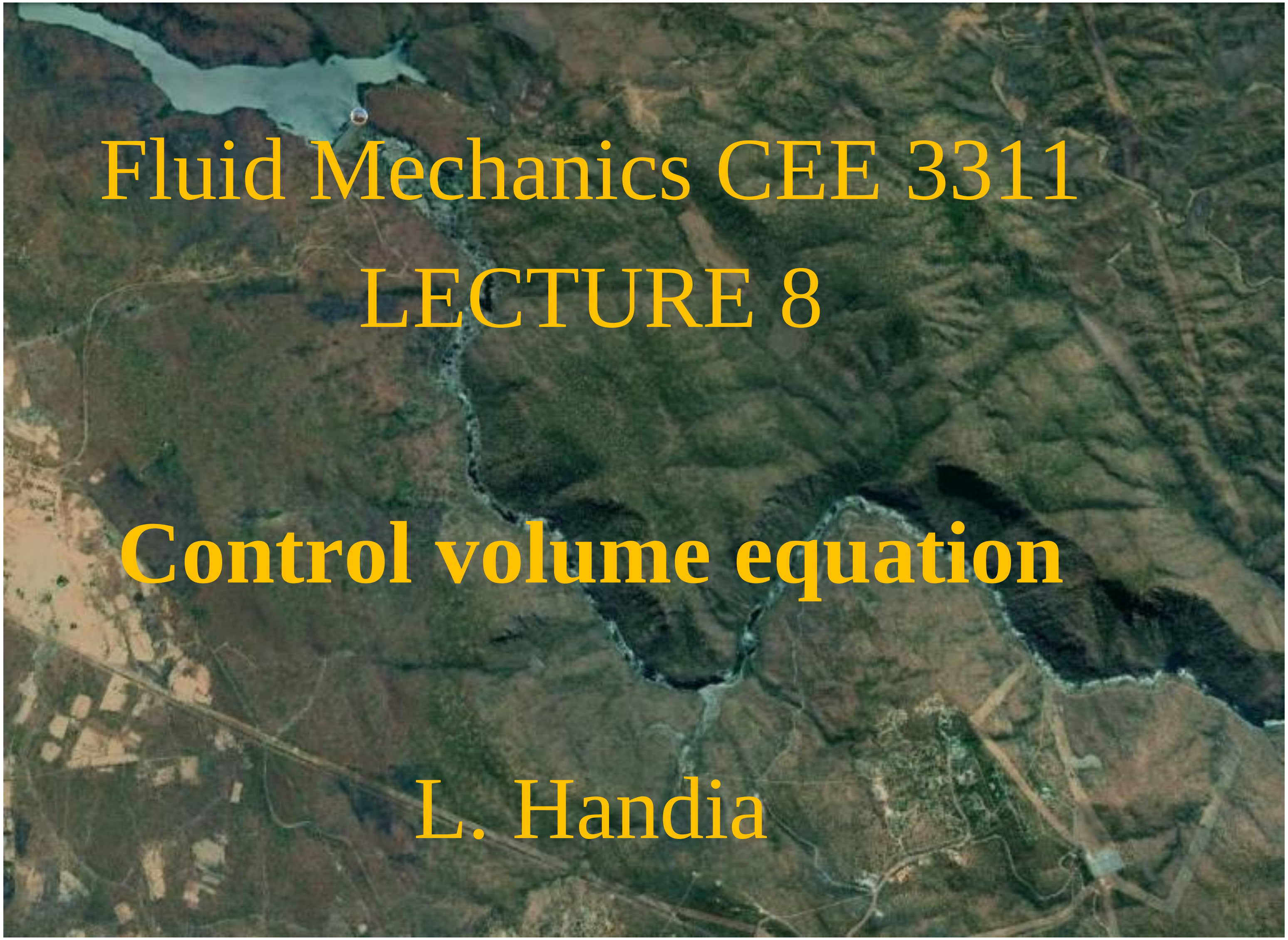

Fluid Mechanics CEE 3311

LECTURE 8

Control volume equation

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Introduction

- In Lecture 1 it has been stated that for a continuous medium 3 conservation laws are valid:
  - Conservation of mass
  - Conservation of energy
  - Conservation of momentum

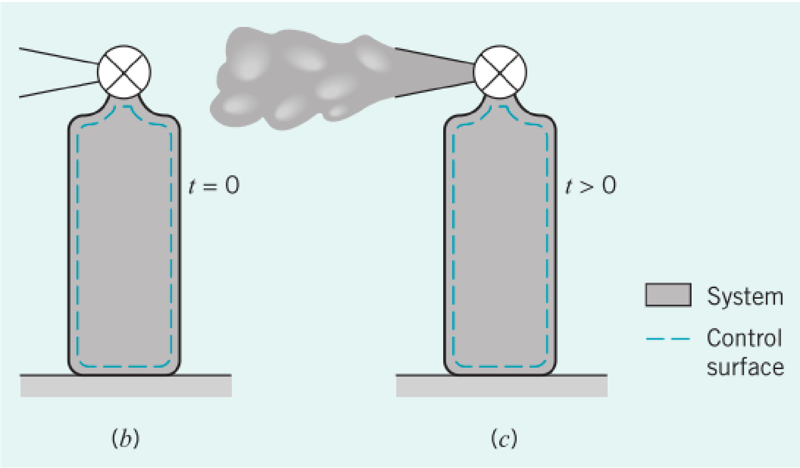
System and control volume

In employing the basic laws, two modes of application may be adopted

1. System approach
2. Control volume approach

System approach

- A fluid system refers to a specific mass of fluid within the boundaries defined by a closed surface.
- The shape of the system, and so the boundaries, may change with time, as when liquid flows through a constriction or when gas is compressed; as a fluid moves and deforms so the system containing it moves and deforms.
- The size and shape of a system is entirely optional.



Control volume approach

- A control volume refers to a *fixed* region in space, which does not move or change shape.
- It is usually chosen as a region that fluid flows into and out of.
- Its *closed boundaries* are called the *control surface*. Again, the size and shape of a control volume is entirely optional, although the boundaries are often chosen to coincide with some solid or other natural flow boundaries.
- Actually, the control surface may be in motion through space relative to an absolute frame of reference; this is acceptable provided the motion is limited to constant-velocity translation.

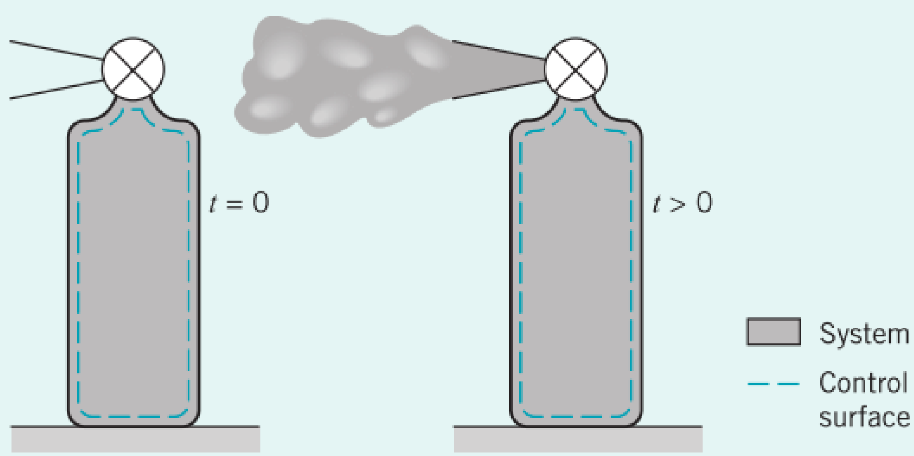
Another way of defining system and control volume

As is discussed in Chapter 1, a fluid is a type of matter that is relatively free to move and interact with its surroundings. As with any matter, a fluid's behavior is governed by fundamental physical laws which are approximated by an appropriate set of equations. The application of laws such as the conservation of mass, Newton's laws of motion, and the laws of thermodynamics form the foundation of fluid mechanics analyses. There are various ways that these governing laws can be applied to a fluid, including the system approach and the control volume approach. By definition, a ***system*** is a collection of matter of fixed identity (always the same atoms or fluid particles), which may move, flow, and interact with its surroundings. A ***control volume,*** on the other hand, is a volume in space (a geometric entity, independent of mass) through which fluid may flow.

Example: fire extinguisher

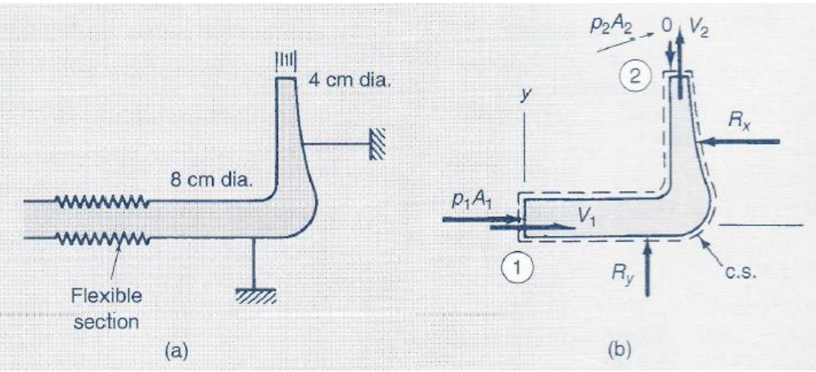
(a)

Physically these represent the time rate of change of mass within the system and the time rate of change of mass within the control volume, respectively. We choose our system to be the fluid within the tank at the time the valve was opened ($t = 0$) and the control volume to be the tank itself as shown in Fig. E4.7*b*. A short time after the valve is opened, part of the system has moved outside of the control volume as is shown in Fig. E4.7*c*. The control volume remains fixed. The limits of integration are fixed for the control volume; they are a function of time for the system.

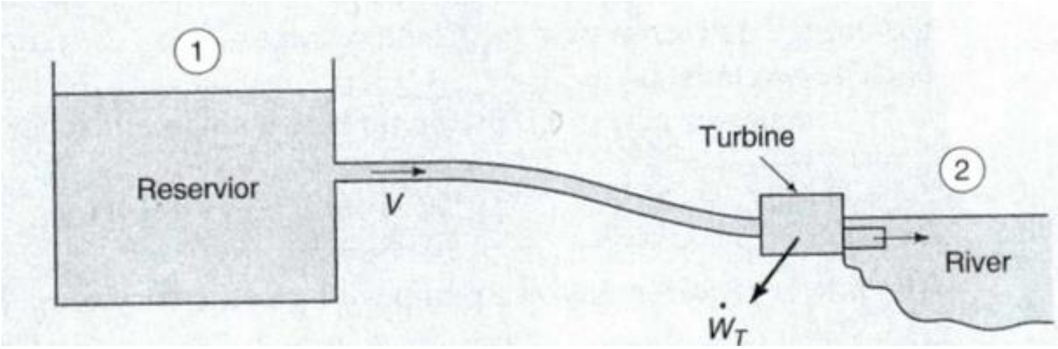


(b) (c)

Other Examples



We have selected a control volume that surrounds the bend as shown in Fig b.



The control volume to be used extends from section 1 to section 2

Introduction

- We shall now derive a general *relationship* between a *system and a control volume* that provides an important basis for the equations of continuity, energy, and momentum for moving fluids.
- This relationship is derived from what is commonly referred to as the ***control volume approach***, more formally known as the ***Reynolds transport theorem***.
- Addressing the motion of fluid as it moves through a given region, the control volume approach is also called the ***Eulerian approach***, in contrast to the Lagrangian approach; control volume does not move

Derivation of the control volume equation

- Extensive properties (N) are mass M, momentum M V, and energy E.
- Intensive properties (η) are extensive properties per unit mass. η is eta.
- Thus
  - ❑Mass/unit mass $[unity]$
  - ❑Momentum/unit mass $[V]$
  - ❑Energy/unit mass $[e]$

Derivation of the control volume equation

- The relationship between intensive and extensive properties is denoted by

$$N = \int \eta \, dm = \int \eta \, \rho \, dV$$

where dm = differential mass

dV = differential volume

- The integral is over the *volume occupied by the system* at a given instant.
- The cv equation is derived by considering the *rate of change* of an extensive property of the *system* of fluid that is *flowing through* the *cv*.

Derivation of the control volume equation

- Let N represent the total amount of some fluid property, such as mass, energy, or momentum, *contained within specified boundaries* at a *specified time*.
- The specified boundaries will be either those of a system, indicated by a subscript SYS, or those of a control volume, indicated by a subscript CV.

Derivation of the control volume equation

- Consider the general flow situation in Fig. below.
- At time t, the boundaries of the system and the control volume were *chosen to coincide*, so $(N_{SYS})_t = (N_{CV})_t$.

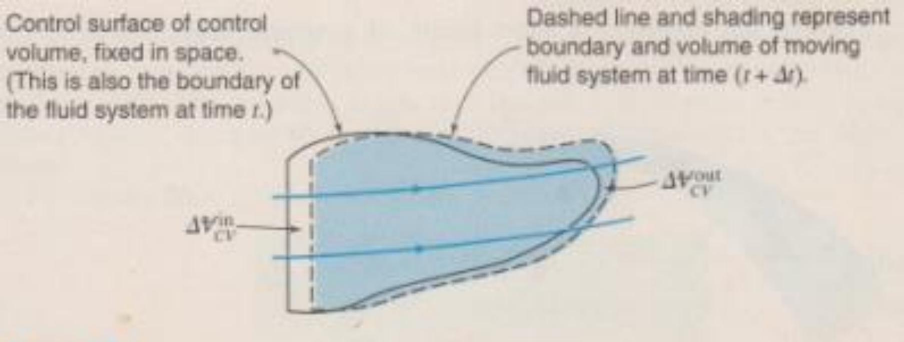


Figure 4.7
Fluid system, control volume, and differences

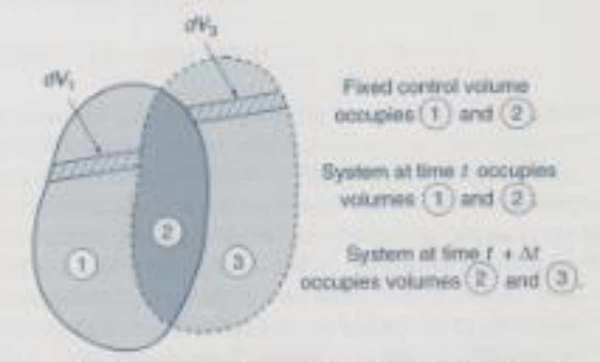


Figure 4.4 System and fixed control volume.

Derivation of the control volume equation

- At instant Δt later, the *system has moved* a little *through the control volume* and possibly slightly *changed its shape*; a small amount of new fluid ΔV_{CV}^{in} has entered the control volume, and another small amount of system fluid ΔV_{CV}^{out} has left the control volume, where V represents volume.

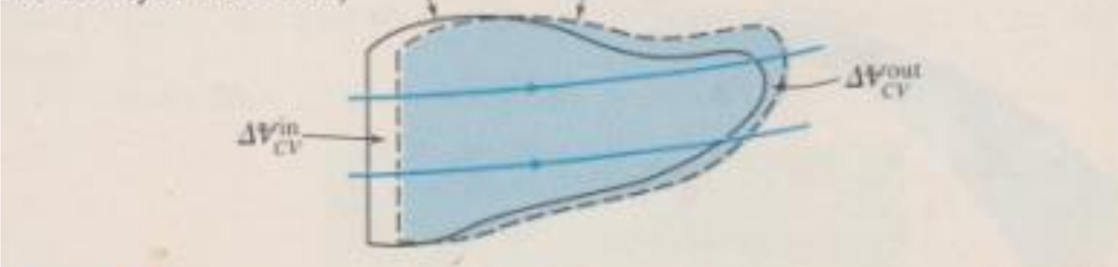


Figure 4.7
Fluid system, control volume, and differences

Derivation of the control volume equation

- These small volumes carry small amounts of property N with them, so that ΔV_{CV}^{in} enters and ΔV_{CV}^{out} leaves the control volume.

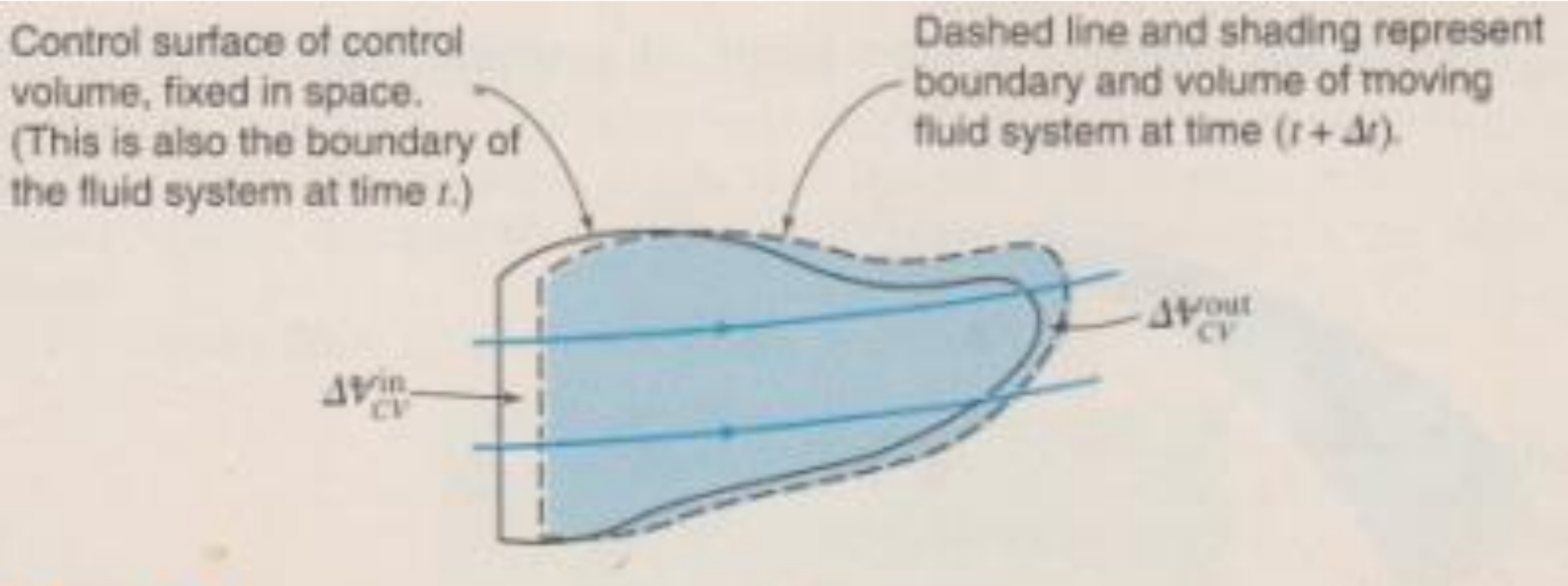


Figure 4.7
Fluid system, control volume, and differences

Derivation of the control volume equation

- In Fig 8.1 the velocity vectors and area vectors are also indicated.
- *Area vectors* are always pointing in *outward direction* perpendicular to that area

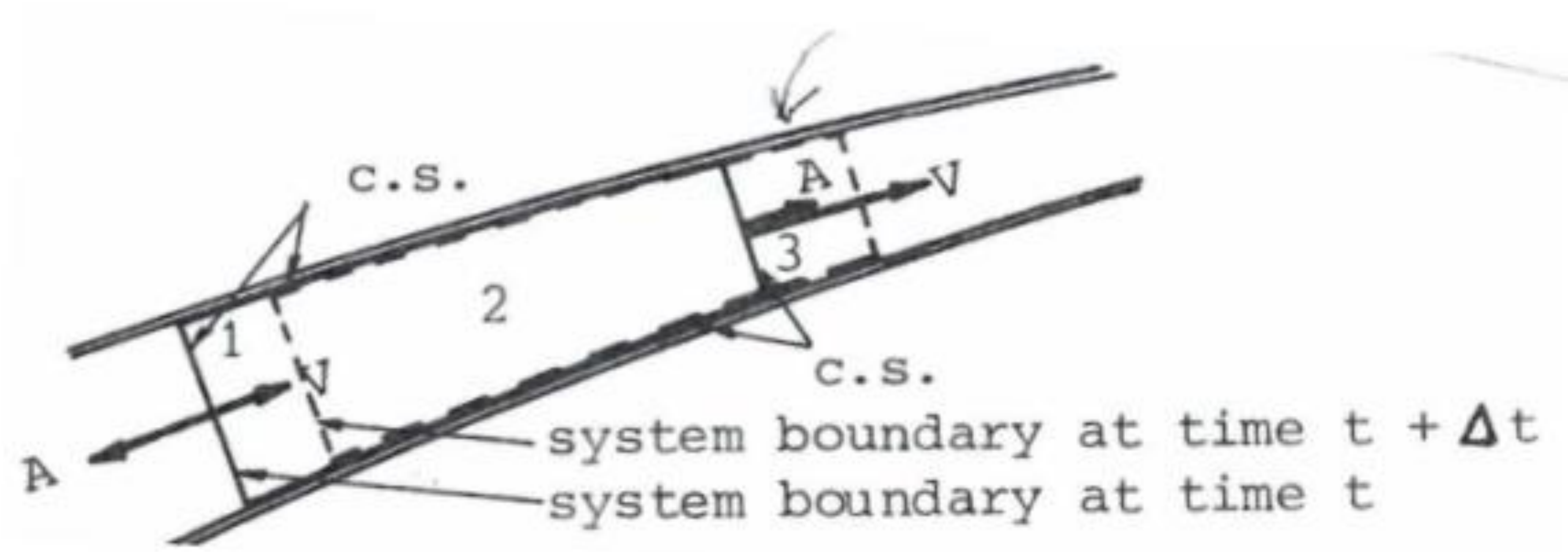


Fig 8.1 notation for derivation of cv equation

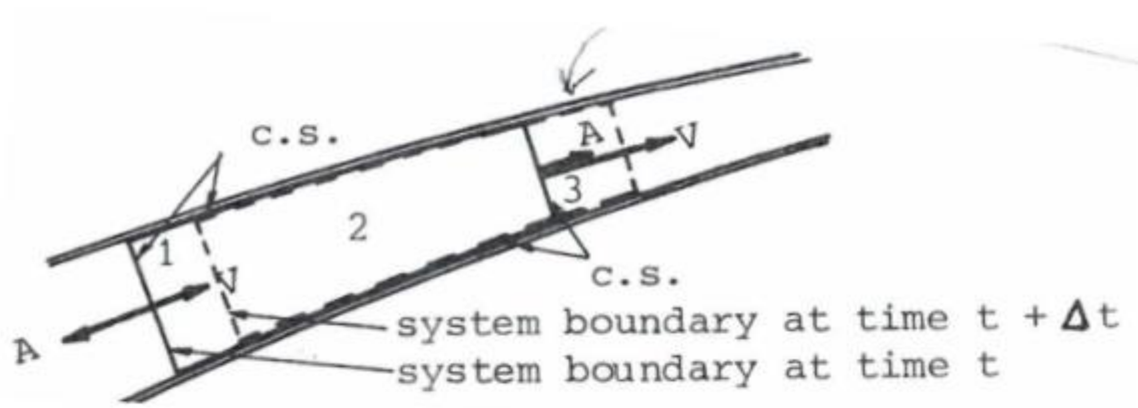
Derivation of the control volume equation

- The rate of change with respect to time of an arbitrary extensive property N of the system will be given by

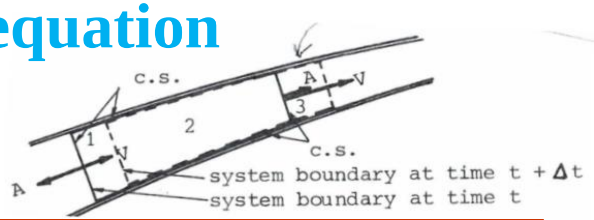
- $$\frac{dN_{SYS}}{dt} = \lim_{\Delta t \to 0} \frac{N_{SYS_{t+\Delta t}} - N_{SYS_t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{(N_2 + N_3)_{t+\Delta t} - (N_1 + N_2)_t}{\Delta t}$$

- Rearranging terms yields

- $$\frac{dN_{SYS}}{dt} = \lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$$



Derivation of the control volume equation



- $$\frac{dN_{SYS}}{dt} = \lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$$

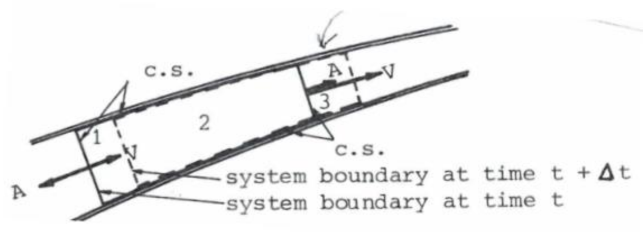
- $\frac{dN_{SYS}}{dt}$ is the rate of change of the total amount of any extensive property N within the moving system.
- $\lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t}$ is the rate of change of the same property, but contained within the fixed control volume.
- $\lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$ is the net rate of outflow of N passing through the control surface.
- So Equation states that the difference between the rate of change of N within the system and that within the control volume is equal to the net rate of outflow from the control volume.

$$\frac{dN_{SYS}}{dt} - \lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$$

See alternative derivation

Derivation of the control volume equation

- $\frac{dN_{SYS}}{dt} = \lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$

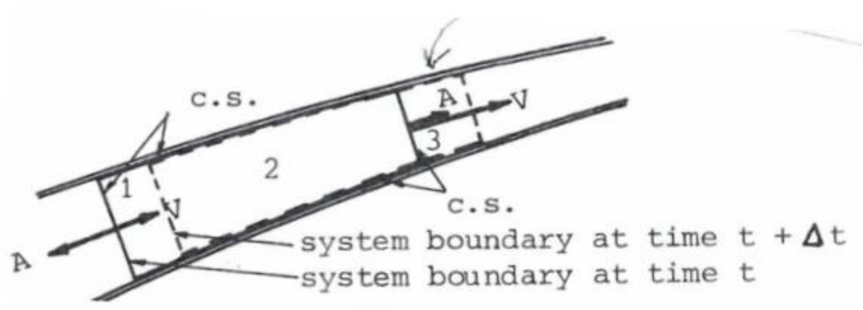


- $\lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t}$ represents the *rate of change* of the property N in *region 2*, but as $\Delta t \to 0$, region 2 approaches that of the cv i.e., negligible movement or change from *original cv* as $\Delta t \to 0$.
- In other words, this term is the rate of change with respect to time of the extensive property N of the fluid *inside the cv* at time t.
- Therefore

$$\lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t} = \frac{dN_{cv}}{dt} = \frac{d}{dt} \int_{cv} \eta \, \rho \, dV$$

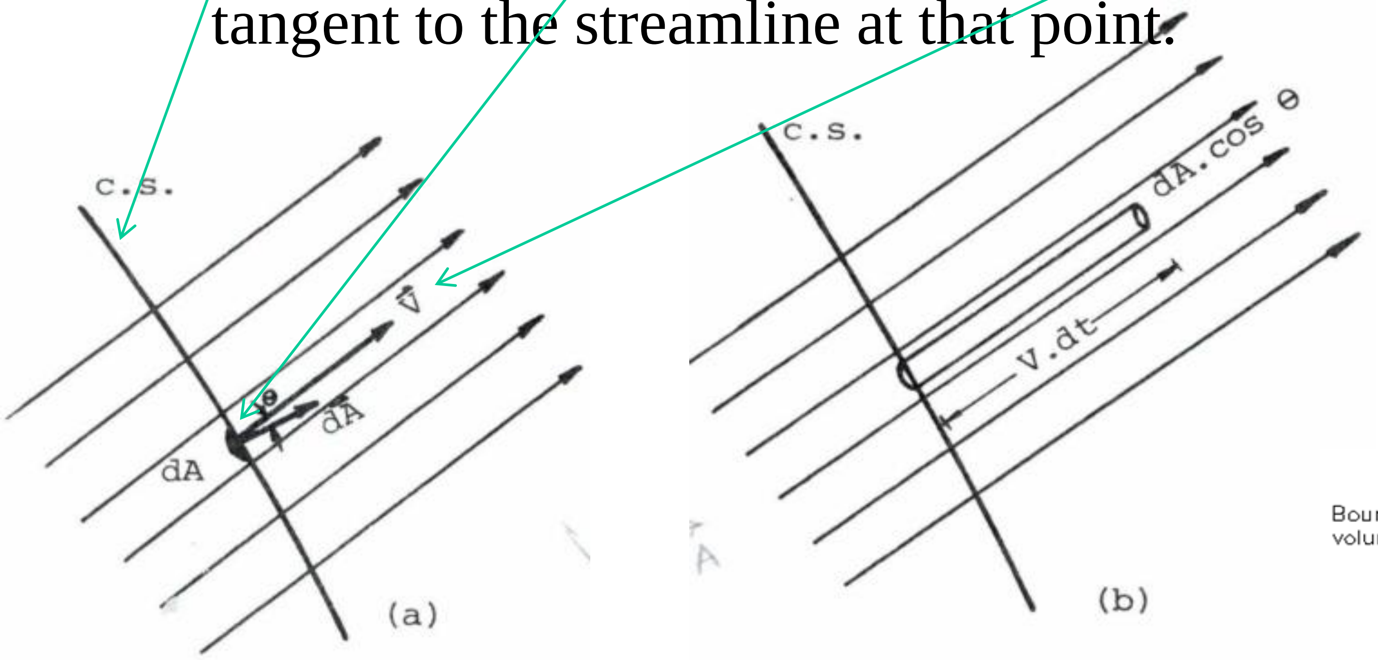
Derivation of the control volume equation

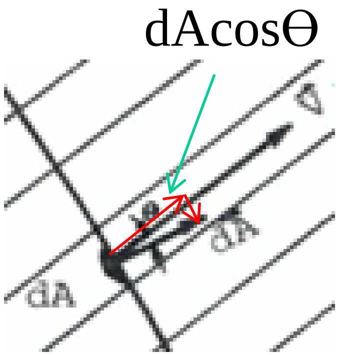
- $\lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$ can be analysed in the following manner
  - ❑ $N_{3,t+\Delta t}$ = amount of property N that has *passed out of the cv* in time Δt
  - ❑ $N_{1,t}$ = amount of property N that has *passed into the cv* in time Δt
- Thus $\lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$ is the *net rate of outflow* of N from the cv at time t.
- This term can be written in a more compact way.



Derivation of the control volume equation

- Consider a steady flow velocity field and a portion of a cs.
- In Fig (a) dA is the interface of fluid that is just touching the cs at time t.
- In Fig (b) the interface has moved v.dt along a direction tangent to the streamline at that point.





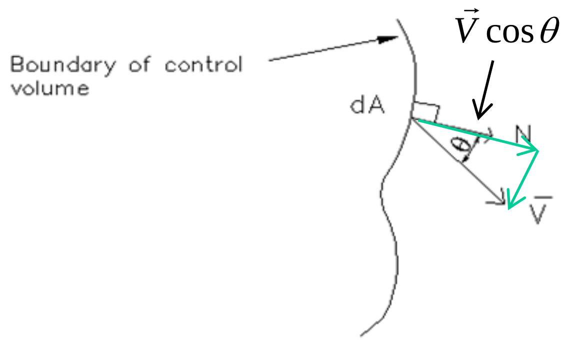
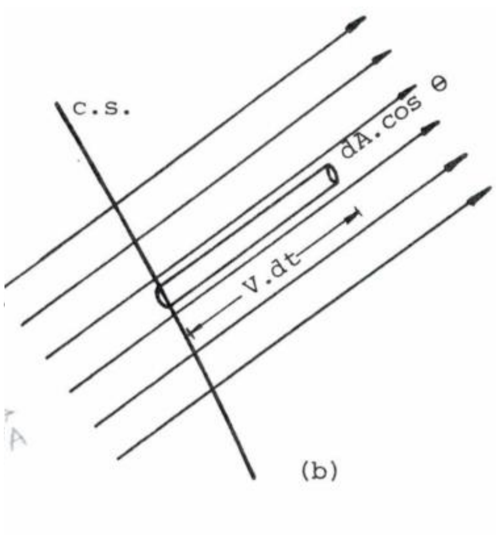
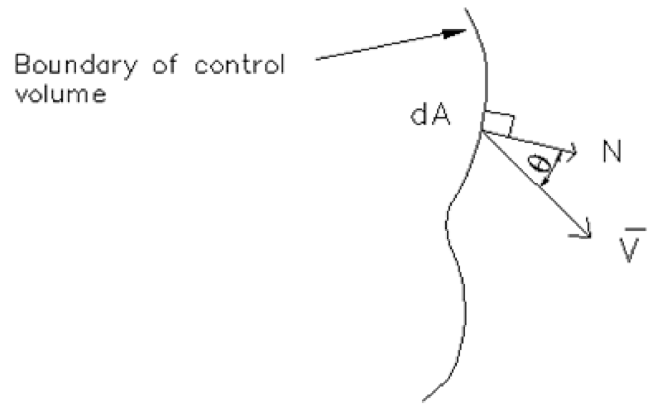


Fig. Flow across the cs a) at time t, b) at time t + dt

Derivation of the control volume equation

- Volume of the stream tube $dV = lA$
- $dV = (v \cdot dt)dA\cos\theta = (\vec{v}\,d\vec{A})\,dt$
- θ is the angle the velocity vector makes with a unit vector normal to the area





- $\vec{v}\,d\vec{A}$ is called the vector dot product of the velocity vector $\vec{v}$ and area vector $d\vec{A}$.
- dV is the volume that has crossed dA of the cs in time dt.

Derivation of the control volume equation

- $dV = (v \cdot dt)dA\cos\theta = (\vec{v}\, d\vec{A})\, dt$

- $\frac{dV}{dt} = (\vec{v}\, d\vec{A})$

- Putting this into $\frac{dN}{dt} = \frac{\int \eta\, \rho\, dV}{dt}$

- $\frac{dN}{dt} = \frac{\int \eta\, \rho\, dV}{dt} = \int_{cs} \eta\, \rho\, (\frac{dV}{dt}) = \int_{cs} \eta\, \rho\, (\vec{v}\, d\vec{A})$= net rate of flow of an extensive property N out of the control volume through the control surface, which is precisely equal to $\lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$

Derivation of the control volume equation

Combining the equations of cv and cs gives (into $\frac{dN_{SYS}}{dt} = \lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$)

$$\frac{dN_{sys}}{dt} = \frac{d}{dt} \int_{cv} \eta \, \rho \, dV + \int_{cs} \eta \, \rho \, (\vec{v} \, d\vec{A})$$

This is the control volume equation

Note: dV for control volume & dA for control surface

Alternative derivation of $\frac{dN_{SYS}}{dt} = \lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$

Let X represent the total amount of some fluid property, such as mass, energy, or momentum, contained within specified boundaries at a specified time. The specified boundaries will be either those of a system, indicated by a subscript S, or those of a control volume, indicated by a subscript CV. Consider the general flow situation of Fig. 4.7. At time t, the boundaries of the system and the control volume were chosen to coincide, so $(X_S)_t = (X_{CV})_t$. At instant Δt later, the system has moved a little through the control volume and possibly slightly changed its shape; a small amount of new fluid ΔV_{CV}^{in} has entered the control volume, and another small amount of system fluid ΔV_{CV}^{out} has left the control volume, where V represents volume. These

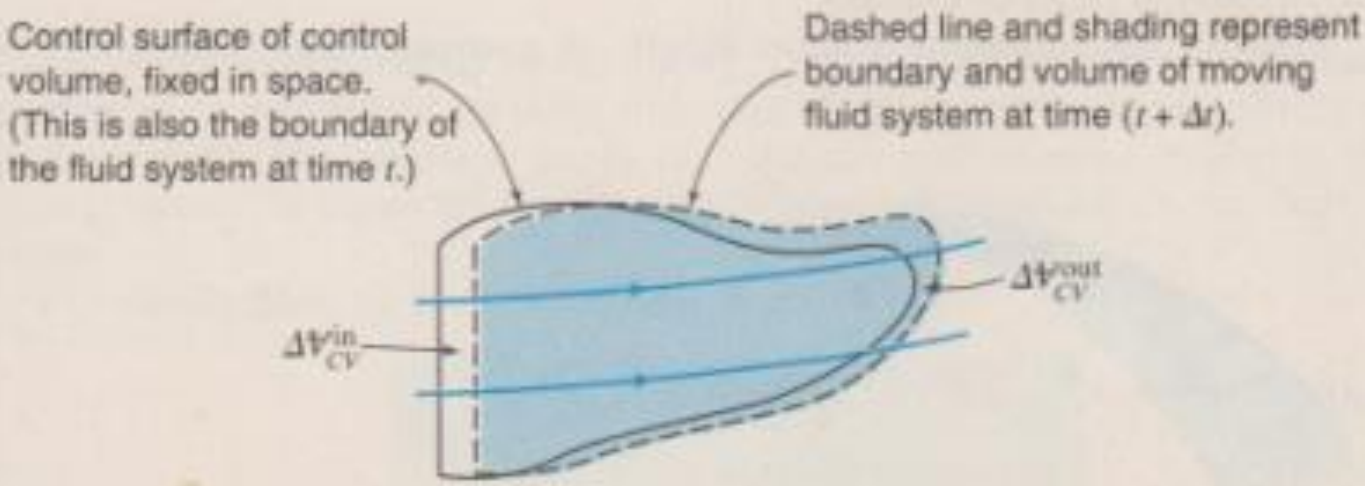
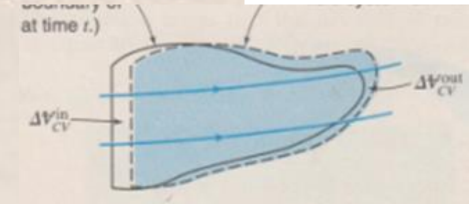


Figure 4.7
Fluid system, control volume, and differences

Alternative derivation of $\frac{dN_{SYS}}{dt} = \lim_{\Delta t \to 0} \frac{N_{2,t+\Delta t} - N_{2,t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{N_{3,t+\Delta t} - N_{1,t}}{\Delta t}$

small volumes carry small amounts of property X with them, so that ΔX_{CV}^{in} enters and ΔX_{CV}^{out} leaves the control volume. Comparing X in the various volumes, we see that



$$(X_S)_{t+\Delta t} = (X_{CV})_{t+\Delta t} + \Delta X_{CV}^{out} - \Delta X_{CV}^{in}$$

Subtracting the equation for t from that for $t + \Delta t$, we obtain

$$(X_S)_{t+\Delta t} - (X_S)_t = (X_{CV})_{t+\Delta t} - (X_{CV})_t + \Delta X_{CV}^{out} - \Delta X_{CV}^{in}$$

Since $X_{s,t} = X_{CV,t}$

or

$$\Delta X_S = \Delta X_{CV} + \Delta X_{CV}^{out} - \Delta X_{CV}^{in} \qquad (4.8)$$

and dividing by Δt and letting $\Delta t \to 0$, we get

$$\frac{dX_S}{dt} = \frac{dX_{CV}}{dt} + \frac{dX_{CV}^{out}}{dt} - \frac{dX_{CV}^{in}}{dt} \qquad (4.9)$$

These equations will be used in subsequent studies of continuity, energy, and momentum. The left-hand side of Eq. (4.9) is the rate of change of the total amount of any extensive property X within the moving system. The next term, dX_{CV}/dt, is the rate of change of the same property, but contained within the fixed control volume. The last two terms are the net rate of outflow of X passing through the control surface. So Eq. (4.9) *states that the difference between the rate of change of X within the system and that within the control volume is equal to the net rate of outflow from the control volume.*