CHAPTER

4

BUOYANCY AND FLOATATION

4.1. Buoyancy. 4.2. Centre of buoyancy. 4.3. Types of equilibrium of floating bodies. 4.4. Metacentre and metacentric height. 4.5. Determination of metacentric height—Analytical method—experimental method. 4.6. Oscillation (rolling) of a floating body. Highlights—Objective Type questions—Theoretical Questions—Unsolved Examples.

4.1 Buoyancy

Whenever a body is immersed wholly or partially in a fluid it is subjected to an upward force which tends to lift (or buoy) it up. This *tendency for an immersed body to be lifted up in the fluid, due to an upward force opposite to action of gravity is known as* **buoyancy**. The force tending to lift up the body under such conditions is known as *buoyant force* or *force of buoyancy* or *upthrust*. The magnitude of the buoyant force can be determined by ***Archimedes' principle*** which states as follows:

"*When a body is immersed in a fluid either wholly or partially, it is buoyed or lifted up by a force, which is equal to the weight of fluid displaced by the body.*"

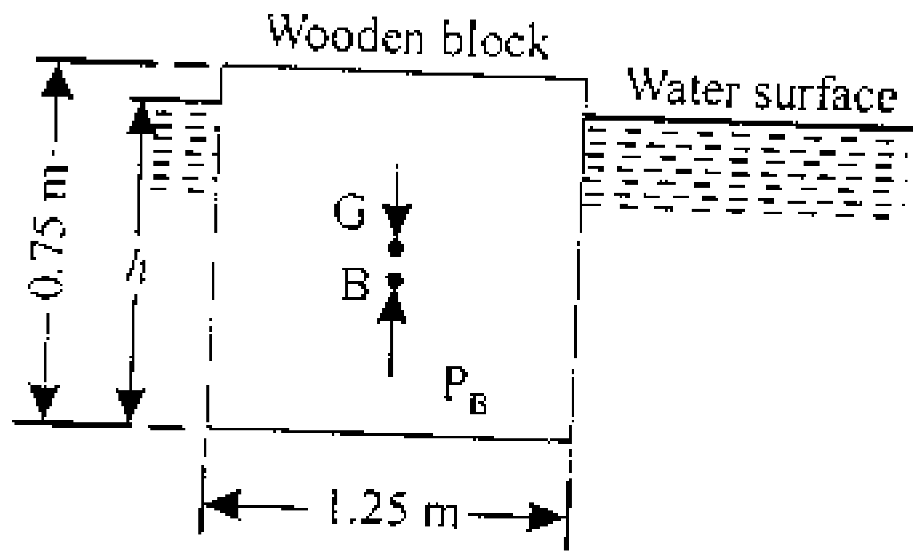
4.2 Centre of Buoyancy

The point of application of the force of buoyancy on the body is known as the **centre of buoyancy.** It is *always the centre of gravity of the volume of fluid displaced.*

Example 4.1. *A wooden block of width 1.25 m, depth 0.75 m and length 3.0 m is floating in water. Specific weight of the wood is 6.4 kN/m³. Find*

(*i*) *Volume of water displaced, and*

(*ii*) *Position of centre of buoyancy.*



P_B = Buoyant force

Fig. 4.1

Solution. Width of the wooden block = 1.25 m

Depth of the wooden block = 0.75 m

Length of the wooden block = 3.0 m

Volume of the block = $1.25 \times 0.75 \times 3 = 2.812$ m³

Specific weight of wood, $w = 6.4$ kN/m³

Weight of the block = $6.4 \times 2.812 = 18$ kN

(*i*) **Volume of water displaced:**

For equilibrium the weight of water displaced

= Weight of wooden block = 18 N

Volume of water displaced

$$= \frac{\text{Weight of water displaced}}{\text{Weight density of water}}$$

$$= \frac{18}{9.81} = \mathbf{1.835\ m^3\ (Ans)}$$

($\because$ Weight density of water = 9.81 kN/m³) or $\quad 1.25 \times h \times 3.0 = 1.835$

(where h = depth of wooden block in water)

$$\therefore \quad h = \frac{1.835}{1.25 \times 3.0} = 0.489 \text{ m}$$

Hence centre of buoyancy $= \frac{0.489}{2} =$ **0.244 from the base (Ans.)**

Example 4.2. *A wooden block of specific gravity 0.7 and having a size of 2 m × 0.5m × 0.25 m floating in water. Determine the volume of concrete of specific weight 25 kN/m³, that may be placed which will immerse the (i) block completely in water and (ii) block and concrete completely water.*

Solution. Size of the block = 2 m × 0.5 m × 0.25 m

∴ Volume of the block = 0.25 m³

Specific gravity of the block = 0.7

Specific gravity of the block = 0.7 × 9.81 = 6.867 kN/m³

Weight of the block = 6.867 × 0.25 = 1.716 kN

(∵ Specific weight of water = 9.81 kN/m³)

Let W_c = Weight of concrete required to be placed on the block, and

V_c = Volume of concrete required to be placed on the block.

Total weight of the block = $W_c + 1.716$ kN ...(*i*)

(*i*) Immersion of the block only:

When the block is completely immersed, the volume of water displaced = 0.25 m³

∴ Upward thrust at the time of complete immersion

= 0.25 × 9.81 = 2.45 kN ...(*ii*)

Now equating (*i*) and (*ii*), we get

$$W_c + 1.716 = 2.45$$

or $$W_c = 0.734 \text{ kN}$$

Volume of concrete, $V_c = \frac{\text{weight}}{\text{sp. weight}} = \frac{0.734}{25} =$ **0.0294 m⁴ (Ans.)**

(*ii*) Immersion of block and concrete:

Total weight of the block $= 25\,V_c + 1.716$ ...(*i*)

and upward thrust $= (V_c + 0.25) \times 9.81$ ...(*ii*)

Equating (*i*) and (*ii*), we get

$$25V_c + 1.716 = (V_c + 0.25) \times 9.81$$

or $25\,V_c + 1.716 = 9.81V_c + 2.45$ or $15.19\,V_c = 0.734$

or $V_c =$ **0.0483 m³ (Ans.)**

Example 4.3. *Find the density of a metallic body which floats at the interface of mercury of specific gravity 13.6 and water such that 35 percent of its volume is submerged in mercury and 65 percent in water.*

Solution. Let, V = Volume of the body, m³.

Then volume of body submerged in mercury

$$= \frac{35}{100} \times V = 0.35\ V\ \text{m}^3$$

Volume of body submerged in water

$$= \frac{65}{100} \times V = 0.65\ V\ \text{m}^3$$

The body will be in equilibrium when,

Total buoyant (upward) force = weight of the body

But, total buoyant force = force of buoyancy due to water + force of buoyancy due to mercury

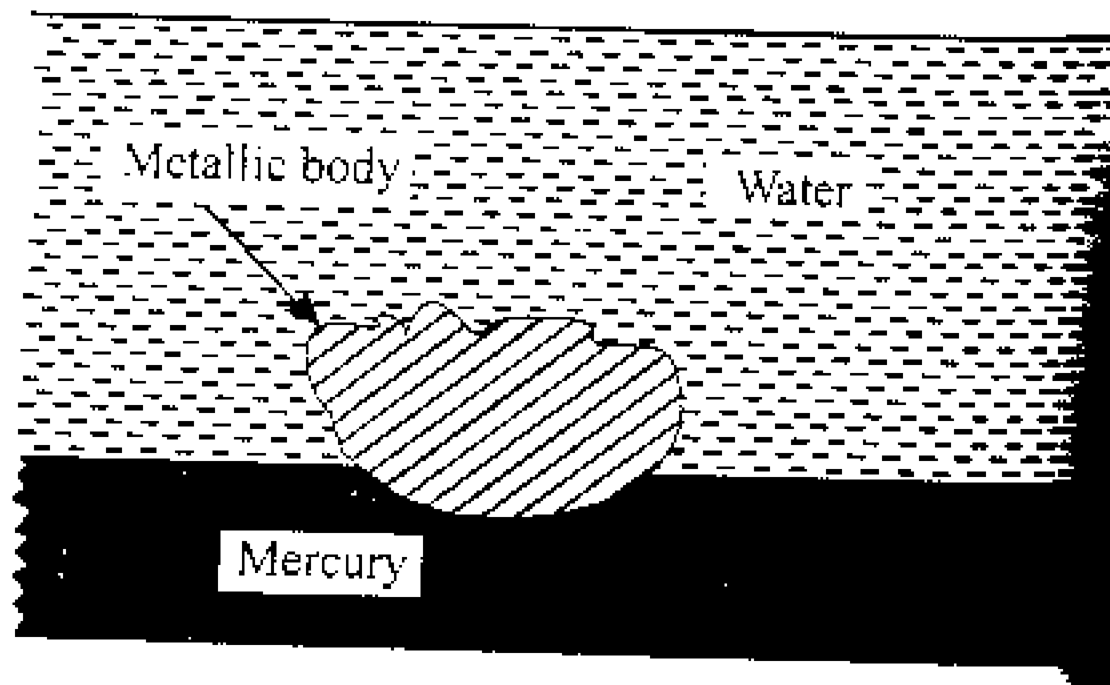


Fig. 4.2

= weight of water displaced by the body + weight of mercury displaced by the bo

= (weight density of water × volume of water displaced) + (weight density of merc × volume of mercury displaced)

$= 9.81 \times 0.65\ V (\text{kN}) + 13.6 \times 9.81 \times 0.35\ V (\text{kN})$

and, weight of the body = weight density × volume of the body

$= w_{body} \times V$

(where w_{body} = weight density of the metallic body)

For equilibrium, we have

$$9.81 \times 0.65\ V + 13.6 \times 9.81 \times 0.35\ V = w_{body} \times V$$

$\therefore\ w_{body}$ = **53.07 kN/m³ (Ans.)**

Example 4.4. *A metallic cube 30 cm side and weighing 450 N is lowered into a tank conta a two-fluid layer of water and mercury. Determine the position of block at mercury-water interface when it has reached equilibrium.*

(Madras University)

Solution. Refer Fig. 4.3. The metallic cube sinks beneath the water surface and comes to rest at the water-mercury interface.

As per principle of floatation, we have weight of cubical block = buoyant force

= weight of water and mercury displaced by the block.

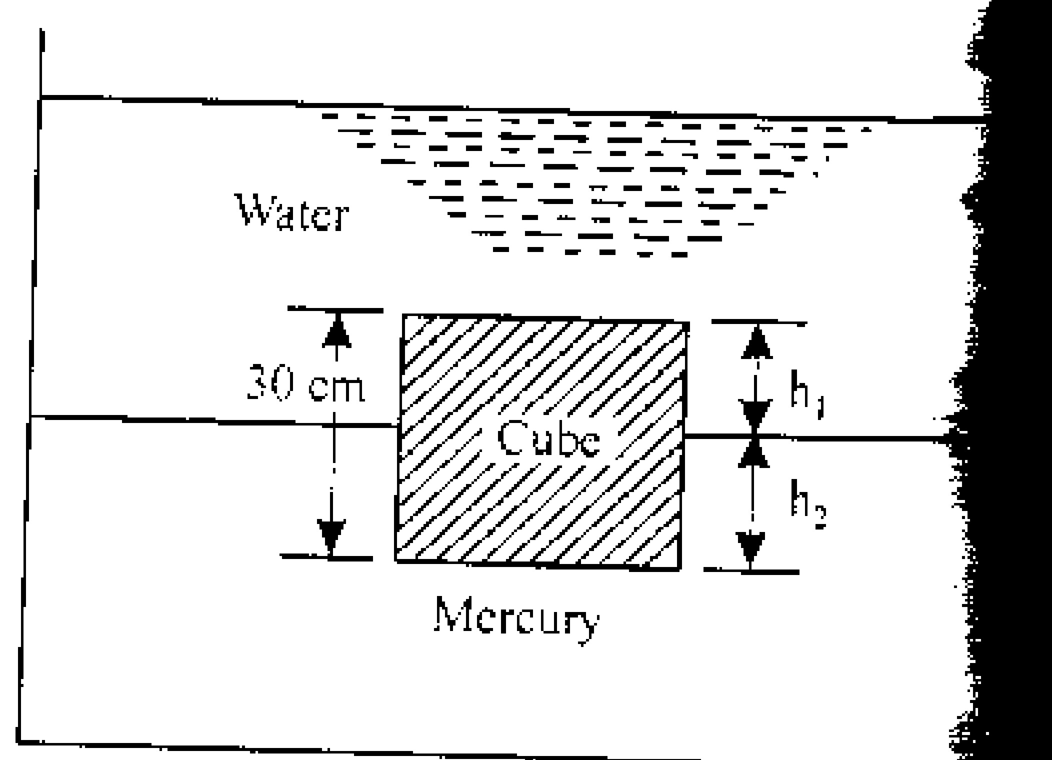


Fig. 4.3

Thus, $450 = 9810\ (h_1 \times 0.3 \times 0.3) + 9810 \times 13.6\ (h_2 \times 0.3 \times 0.3)$

$= (h_1 + 13.6\ h_2) \times (9810 \times 0.3 \times 0.3)$

or $$(h_1 + 13.6\ h_2) = \frac{450}{9810 \times 0.3 \times 0.3} = 0.509\ \text{m}$$

Also, $h_1 + h_2 = 0.3$ m

From (*i*) and (*ii*), we have the depth of cube below the water-mercury interface,

$$h_2 = \frac{(0.509 - 0.3)}{12.6} = \mathbf{0.01658\ m\ or\ 16.58\ mm\ (Ans.)}$$

Example 4.5. *A 8 cm side cube weighing 4 N is immersed in a liquid of relative contained in a rectangular tank of cross-sectional area 12 cm × 12 cm. If the tank con* *to a height of 6.4 cm before the immersion determine the levels of the bottom of the cube and the liquid surface.*

Solution. Refer Fig. 4.4

Let, h_1 = Depth to which the bottom of the cube falls below original liquid surface (cm), and

h_2 = Height of rise of liquid above the original liquid surface (cm), and

$(h_1 + h_2)$ = Depth and submergence of the cube (cm).

Volume L = Volume M

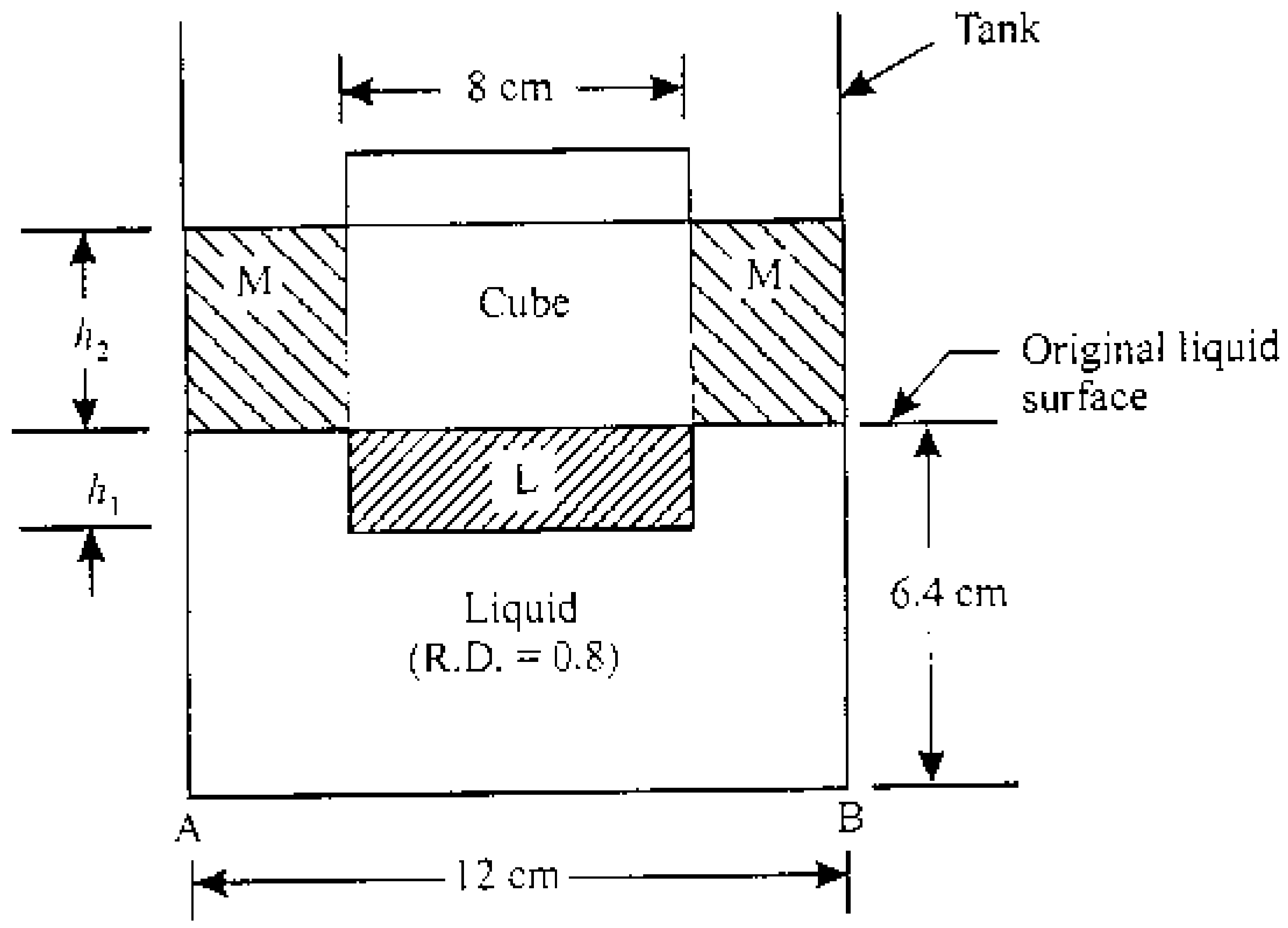


Fig. 4.4

$$8 \times 8 \times h_1 = (12^2 - 8^2) \times h_2$$

or $$h_1 = 1.25\, h_2$$

Weight of the cube, $W = 4\text{N}$ ...(Given)

$$W = \text{Buoyant force} = \frac{(8 \times 8) \times (h_1 + h_2) \times 0.8 \times 9810}{10^6}$$

or $$4 = 0.5023\,(h_1 + h_2)$$

or $$4 = 0.5023\,(1.25\,h_2 - h_2) = 1.13\,h_2$$

∴ $$h_2 = 3.54 \text{ cm}$$

$$h_1 = 4.425 \text{ cm}$$

Level of bottom of cube above plane AB = $6.4 - h_1 = 6.4 - 4.425$ = **1.975 cm (Ans.)**

Level of the liquid surface above plane AB = $6.4 + h_2 = 6.4 + 3.54$ = **9.94 cm (Ans.)**

Example 4.6. *A cube 50 cm side is inserted in a two-layer fluid with specific gravity 1.2 and 0.9. The upper and lower halves of the cube are composed of materials with specific gravity 0.6 and 1.4 respectively. What is the distance of the top of cube above interface?* **(U.P.S.C)**

Solution. Refer Fig. 4.5

Weight of cube = $[S_1 (= 0.6) \times 9.81 \times 0.5 \times 0.5 \times 0.25)] + [S_2 (= 1.4) \times 9.81 \times 0.5 \times 0.5 \times 0.25]$

$= 1.226$ kN

Let h = Height of top of the cube above the interface.

Then. Buoyant force = Weight of lighter and heavier liquids displaced by the block

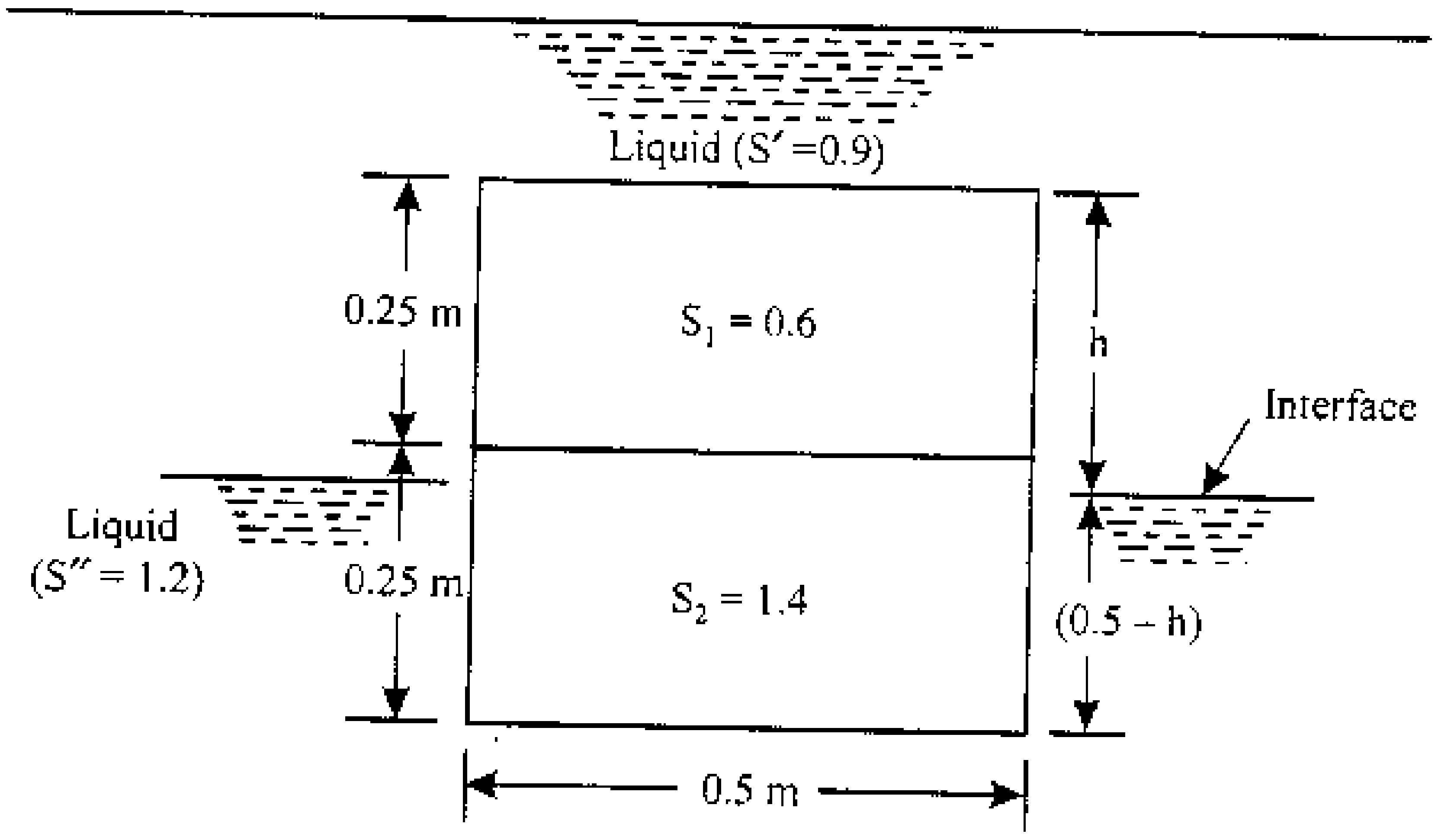


Fig. 4.5

$$= [S'(= 0.91) \times 9.81 \times 0.5 \times 0.5 \times h] + [S''(= 1.2) \times 9.81 \times 0.5 \times 0.5\,(0.5 - h)]$$
$$= 2.207\,h + 1.471 - 2.943\,h = -0.736\,h = 1.471$$

As per principle of floatation, we have, weight of block = buoyant force

i.e. $1.226 = -0.736\,h + 1.471$

$$\therefore \quad h = \frac{(1.471 - 1.226)}{0.736} = 0.333 \text{ m or } 33.3 \text{ cm}$$ **(Ans.)**

Example 4.7. *A spherical object of 1.45 m diameter is completely immersed in a water reservoir and chained to the bottom. If the chain has a tension of 5.20 kN, find the weight of the object when it is taken out of the reservoir into the air.*

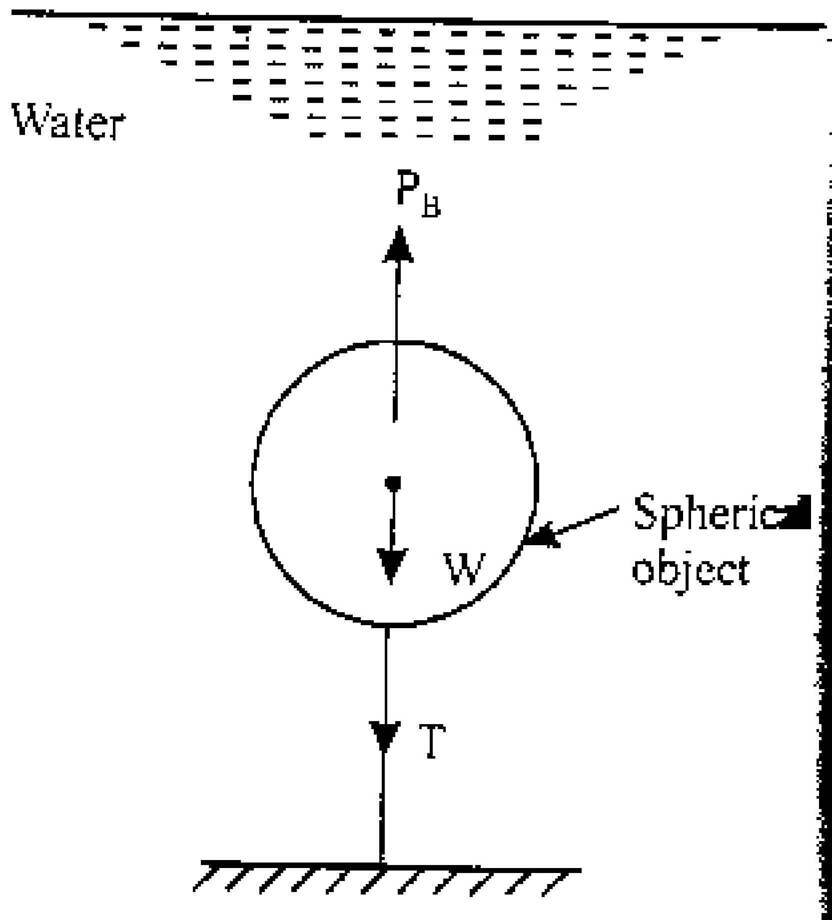


Fig. 4.6

Solution. *Given:* d = 1.45 m; T = 5.20 kN.

Weight of the object, W:

Buoyant force, $P_B = W$ (weight of the object) + T (tension in the chain)

$$\therefore \quad W = P_B - T$$
$$= \frac{4}{3}\pi \times \left(\frac{1.45}{2}\right)^3 \times 9.81 - 5.20$$
$$= \mathbf{10.46\ kN}$$ **(Ans.)**

Example 4.8. *A cylinder of mass 10 kg and area of cross-section 0.1 m² is tied down with string in a vessel containing two liquids as shown in figure 4.7. Calculate gauge pressure on the the cylinder bottom and the tension in the string. Density of water = 1000 kg/m³. Specific gravity of A = 0.8. Specific gravity of B (water) = 1.0* (***Gate, 1998***)

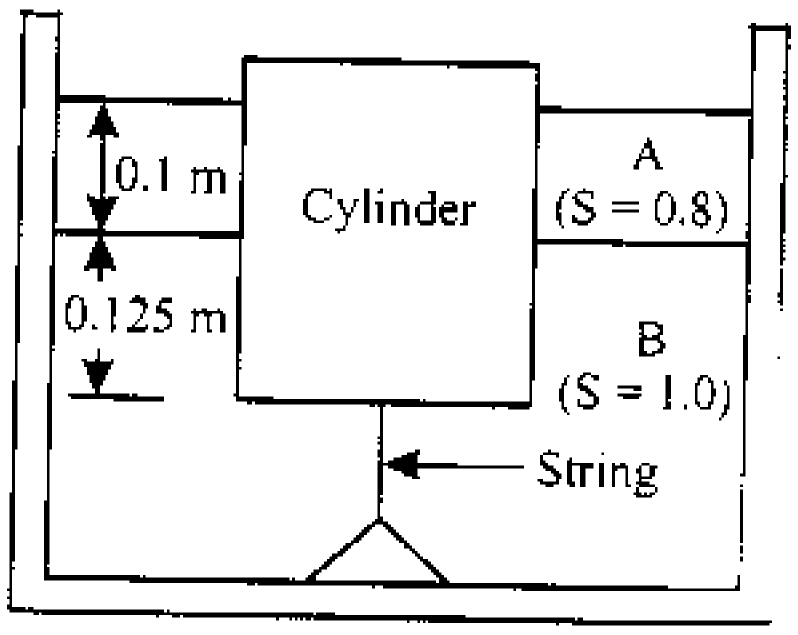


Fig. 4.7

Solution. *Given*: Mass of cylinder, m = 10 kg

Area of cross-section = 0.1m²

Density of water (liquid B)

= 1000 kg/m³

Density of liquid A = 0.8 × 1000 = 800 kg/m³

Tension in string, T:

Volume of liquid A displaced $= 0.1 \times 0.1 = 0.01\ m^3$

$\therefore$ Mass of liquid A displaced, $m_A = 0.01 \times 800 = 8$ kg

$\therefore$ Volume of liquid B displaced $= 0.1 \times 0.125 = 0.0125\ m^3$

$\therefore$ Mass of liquid B displaced $m_B = 0.0125 \times 1000 = 12.5$ kg

Total mass of liquid displaced $= m_A + m_B = 8 + 12.5 = 20.5$ kg

Upward thrust $= 20.5 \times 9.81 = 201.1$ N

Weight of cylinder $= mg = 10 \times 9.81 = 98.1$ N

Net upward thrust $= 201.1 - 98.1 = 103$ N

$\therefore$ Tension in the string, **T = 103 N (Ans.)**

Pressure (gauge) on the cylinder bottom, p:

$$p = \frac{\text{Net upward thrust}}{\text{Area of cross-section}} = \frac{103}{0.1} = \mathbf{1030\ N/m^2\ (Ans.)}$$

4.3. Types of Equilibrium of Floating Bodies

The equilibrium of floating bodies is of the following types:

1. Stable equilibrium
2. Unstable equilibrium, and
3. Neutral equilibrium.

4.3.1. Stable Equilibrium

When a body is given a small angular displacement (*i.e.* tilted slightly), by some external force, then it returns back to its original position due to the internal forces (the weight and the upthrust), such an equilibrium is called *stable equilibrium*.

4.3.2. Unstable Equilibrium

If the body does not return to its original position from the slightly displaced angular position and heels farther away, when given a small angular displacement, such an equilibrium is called an *unstable equilibrium*.

4.3.3. Neutral Equilibrium

If a body, when given a small angular displacement, occupies a new position and remains at rest in this new position, it is said to possess a *neutral equilibrium*.

4.4. Metacentre and Metacentric Height Metacentre:

Fig. 4.8 (a) shows body floating in a liquid in a state of equilibrium. When it is given a small angular displacement (see Fig. 4.8 (b)) it starts oscillating about some point (M). This point, *about which the body starts oscillating*, is called ***metacentre.***

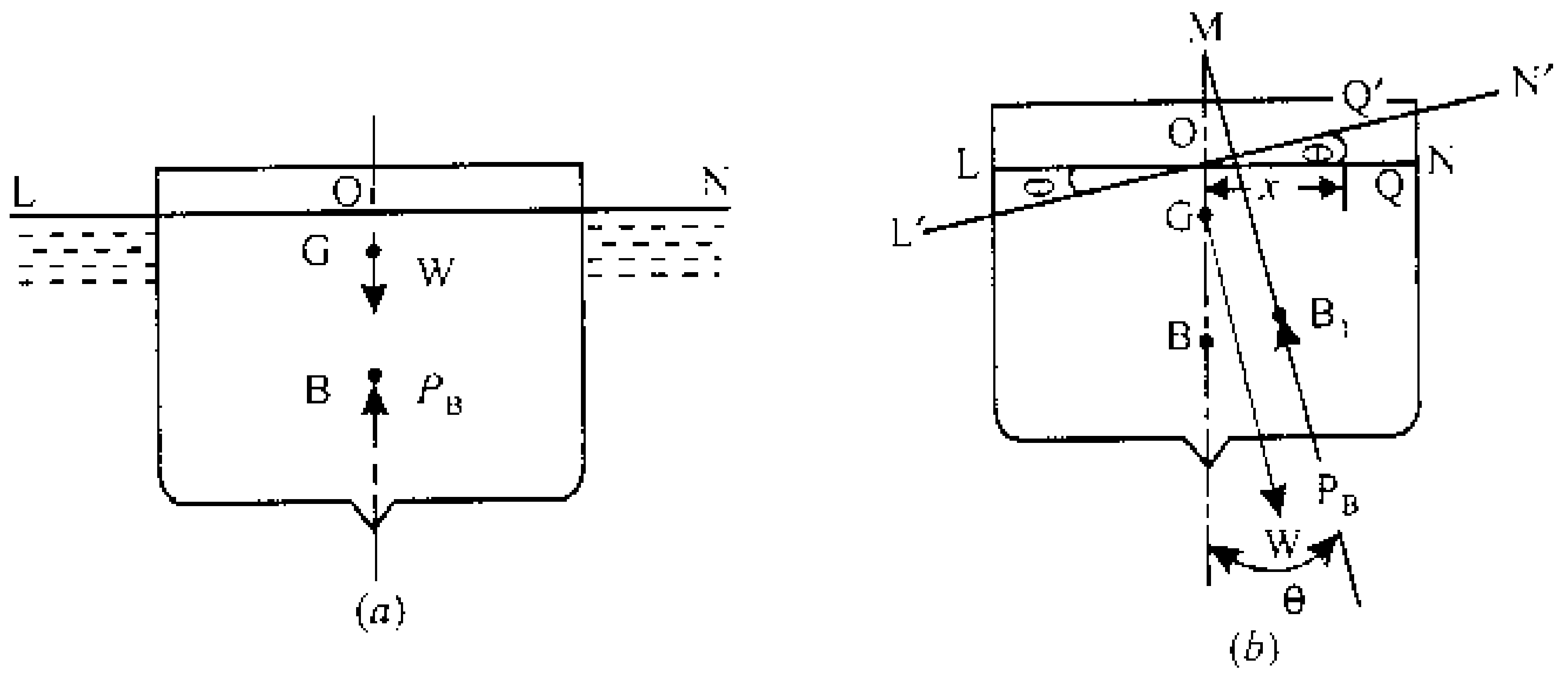


Fig. 4.8

The ***metacentre*** may also be defined as *a point of intersection of the axis of body passing through c.g.(G) and, original centre of buoyancy (B) and a vertical line passing through the centre of buoyancy (B_1) of the tilted position of the body.*

The position of metacentre, M remains practically constant for the small angle of tilt θ.

Metacentric height:

The distance between the centre of gravity of a floating body and the metacentre (*i.e.* distance GM as shown in Fig.4.8 (*b*) is called *metacentric height.*

- *For stable equilibrium,* the position of metacentre M remains higher than c.g. of the body, G.
- *For unstable equilibrium,* the position of metacentre M remains *lower* than G.
- *For neutral equilibrium,* the position of metacentre M *coincides* with G.

4.5 Determination of Metacentric Height

The metacentric height may be determined by the following two methods:

1. Analytical method, and
2. Experimental method.

4.5.1 Analytical Method

Refer Fig. 4.8 (*b*). It shows the tilted position of the floating body, the line L'ON' represents the water surface. The portion N'ON of the body is submerged and the portion L'OL is lifted because of tilting. As a result of this, the centre of buoyancy changes it position from B to B_1. The intersection of axis of the body and the vertical line through B_1, locates the metacentre, M of the body.

To find the metacentric height GM consider an elementary cylindrical prism QQ' of portion N'ON at a distance 'x' from O. Let the area of this elementary prism be δA. The height of this elementary prism is given by $x.\theta$. The volume of this elementary prism is given by

$$\delta V = x.\theta.\delta A. \quad ...(i)$$

The upward force or buoyancy force acting at this prism (δP_B) is given by

$$\delta P_B = w.\delta V = w.\, x.\theta.\delta A \quad ...(ii)$$

(where w = unit weight of liquid)

The moment of this buoyancy force about O

$$x.\delta P = w.\theta.x^2.\delta A \quad ...(iii)$$

For the total portion N'ON, this moment is given by

$$\int x.\, dP_B = \int w.\theta.\, x^2.dA = w\theta \int x^2\, dA \quad ...(iv)$$

or

$$\int x.dP_B = w\theta.I$$

(where I = moment of inertia of the sectional area at the water line about the axis through O)

$\int x.dP_B$ gives the change in moment due to buoyancy.

Now $$\int x.dP_B \; (= w.\theta.I) = P_B \times BB_1$$

(where P_B = the total force of buoyancy)

But $$BB_1 = BM \times \theta \text{ and } P_B = W = w \times V$$

$\therefore$ $$w.\theta.I = w.V.BM.\theta \quad \text{or} \quad BM = \frac{1}{V} \quad ...(4.1)$$

Now metacentric height, $GM = BM \pm BG$

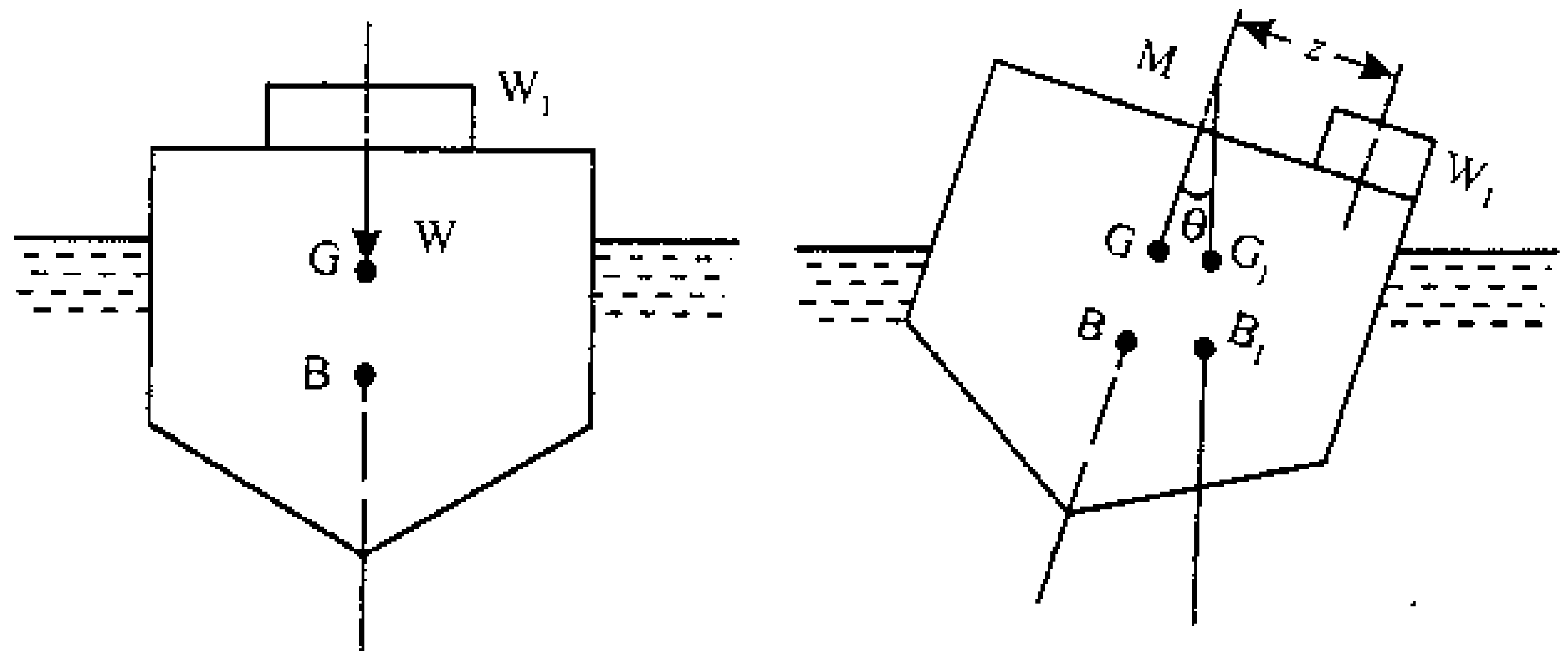
+ ve sign : when G is *lower* than B

− ve sign : when G is *higher* than B

4.5.2. Experimental Method

Refer Fig. 4.9.

In this method, a known weight W_1 is shifted by a distance, z across the axis of tilt. The change in moment due to this shift is $W_1 z$. Let the angle of tilt be θ. This angle of tilt may be measured experimentally by using a plumb bob. The change in moment due to this tilt is equal to $W.GG_1$ or $W.GM \tan\theta$.



(a) (b)

Fig. 4.9. Experimental method for determination of metacentric height.

$$\therefore \quad W_1.z = W.Gm.\tan\theta \quad \text{or} \quad GM = \frac{W_1.z}{W.\tan\theta} \quad \ldots(4.2)$$

If, l = Length of plumb bob, and

d = Displacement of the plumb bob,

then, $$\tan\theta = \frac{d}{l}$$

and, metacentric height is given by

$$GM = \frac{W_1.z.l}{W.d} \quad \ldots(4.3)$$

Example 4.9. *A wooden block of specific gravity 0.75 floats in water. If the size of the block is 1 m × 0.5 m × 0.4 m, find its metacentric height.*

Solution. Size (or dimensions) of the block = 1 m × 0.5 m × 0.4 m

Specifice gravity of wood = 0.75

Specific weight $w = 0.75 \times 9.81 = 7.36\ kN/m^3$

Weight of wooden block = specific weight × volume

$= 7.36 \times 1 \times 0.5 \times 0.4 = 1.472$ kN

Let depth of immersion = h metres.

Weight of water displaced

= specific weight of water × volume of the wood submerged in water

$= 9.81 \times 1 \times 0.5 \times h = 4.9\ h$ kN

Now for equilibrium,

Weight of wooden block = weight of water displaced; $1.472 = 4.9\ h$

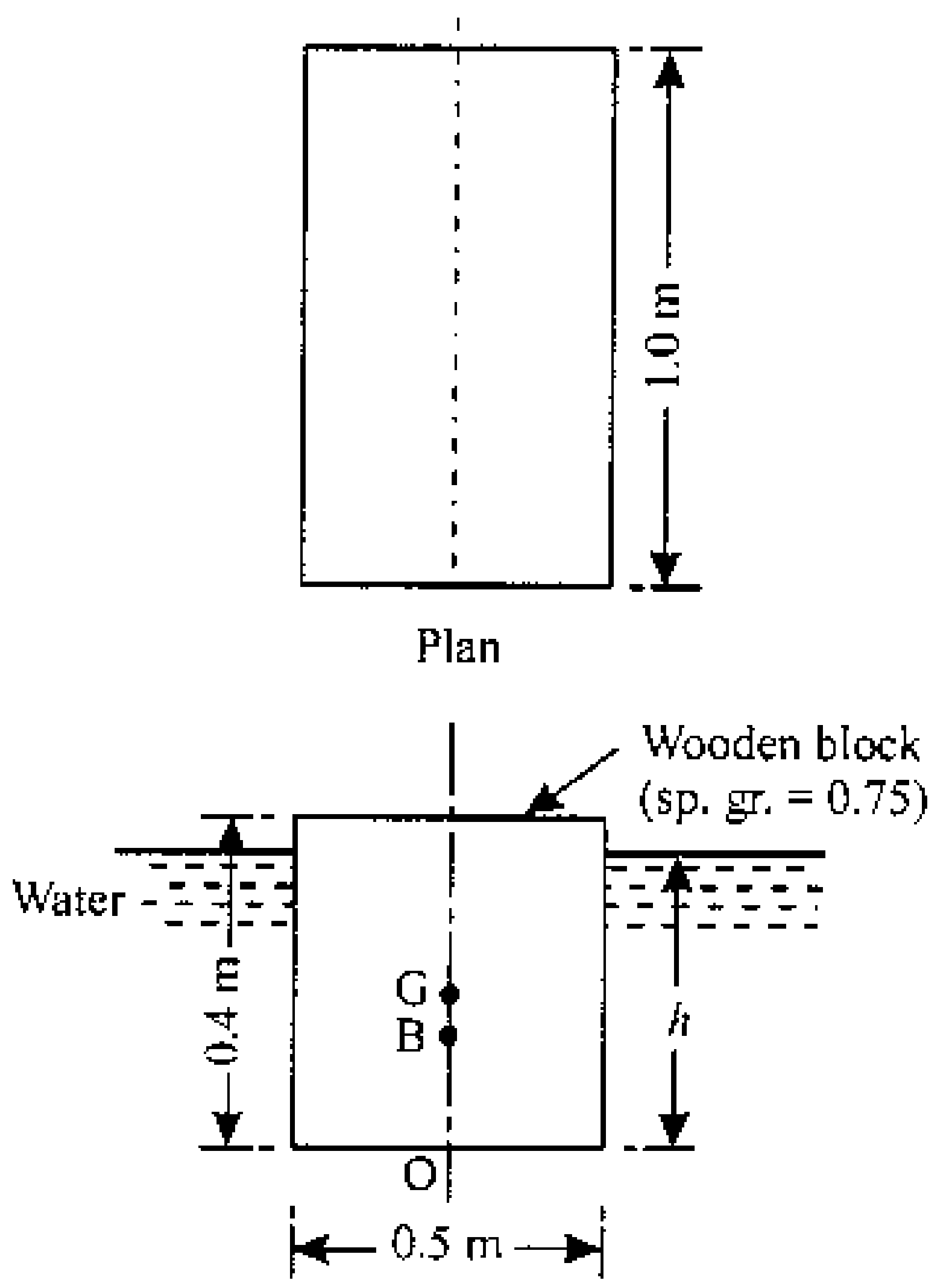


Fig. 4.10

or $$h = \frac{1.472}{4.9} = 0.3 \text{ m}$$

$\therefore$ Distance of centre of buoyancy from bottom *i.e.*,

$$OB = \frac{h}{2} = \frac{0.3}{2} = 0.15 \text{ m}$$

and $$OG = \frac{0.4}{2} = 0.2 \text{ m}$$

$\therefore$ $$BG = OG - OB = 0.2 - 0.15 = 0.05 \text{ m}$$

Also $$BM = \frac{I}{V}$$

Where I = Moment of inertia of a rectangular section and

$$= \frac{1 \times 0.5^3}{12} = 0.014 \text{ m}^4$$

and, V = Volume of water displaced (or volume of wood in water)

$$= 1 \times 0.5 \times h = 1 \times 0.5 \times 0.3 = 0.15 \text{ m}^3$$

$$BM = \frac{I}{V} = \frac{0.0104}{0.15} = 0.069 \text{ m}$$

We know that the metacentric height,

$$GM = BM - BG \ (\because \ G \text{ is above } B)$$
$$= 0.069 - 0.05 = \mathbf{0.019 \text{ m (Ans.)}}$$

Example 4.10. *A solid cylinder 2 m in diameter and 2m high is floating in water with its axis vertical. If the specific gravity of the material of cylinder is 0.65 find its metacentric height. State also whether the equilibrium is stable or unstable.*

Solution. *Given:* Diameter of cylinder, $d = 2$ *m*; height of cylinder, $h = 2m$; specific gravity = 0.65

Depth of cylinder in water = sp. gravity × h

$$= 0.65 \times 2.0 = 1.3 \ m$$

Distance of centre of buoyancy (B) from O, *i.e.*,

$$OB = \frac{1.3}{2} = 0.65 \text{ m}$$

Distance of centre of gravity (G) from O, *i.e.*,

$$OG = \frac{2.0}{2} = 1.0 \text{ m}$$

$$BG = OG - OB = 1.0 - 0.65 = 0.35 \text{ m}$$

Also $$BM = \frac{1}{V}$$

Where, I = Moment of inertia of the plan of the body about $Y - Y$

$$= \frac{\pi}{64} d^4 = \frac{\pi}{64} \times 2^4 = 0.785 \text{ m}^4$$

and, V = Volume of cylinder of water

$$= \frac{\pi}{4} d^2 \times \text{depth of cylinder in water}$$

$$= \frac{\pi}{4} \times 2^2 \times 1.3 = 4.084 \text{ m}^3$$

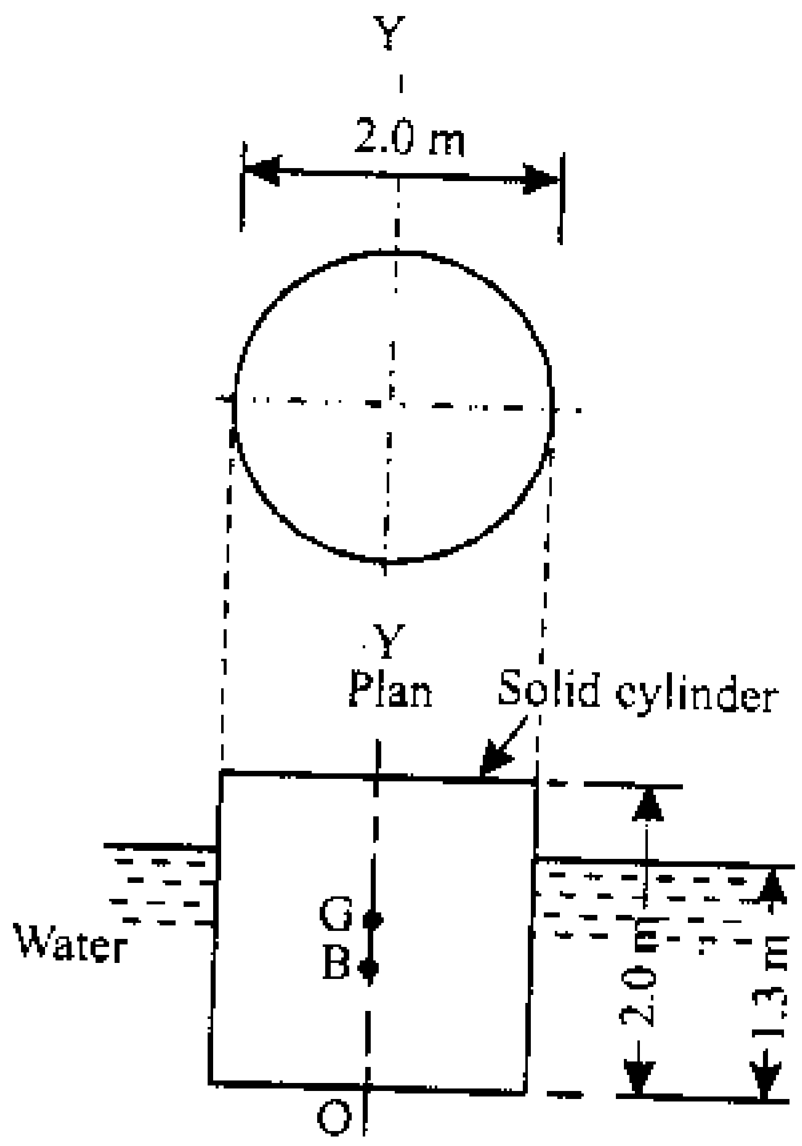


Fig. 4.11

$$\therefore \quad BM = \frac{I}{V} = \frac{0.785}{4.084} = 0.192 \text{ m}$$

We know that the metacentric height,

$$GM = BM - BG = 0.192 - 0.35$$

$$= \mathbf{-0.158 \text{ m (Ans.)}}$$

–ve sign means that the metacentric (M) is below the centre of gravity (G). Thus **the cylinder** is in **unstable equilibrium. (Ans.)**

Example 4.11. *A weight of 100 kN is moved through a distance of 8 metres across the **deck** of a pontoon of 7500 kN displacement floating in water. This makes a pendulum 2.5 metres long to **move** through 120 mm horizontally. Calculate the metacentric height of the pontoon.*

Solution. Weight of the movable load, $W_1 = 100$ kN

Distance through which load is moved, $z = 8$ m

Weight of pontoon, $W = 7500$ kN

Length of the plumb bob, $l = 2.5$ m

Displacement of the plumb bob, $d = 120$ mm $= 0.12$ m

Let GM = metacentric height of the pontoon.

Using the relation,

$$GM = \frac{W_1 . z . l}{W . d} = \frac{100 \times 8 \times 2.5}{7500 \times 0.12} = \mathbf{2.22 \text{ m (Ans.)}}$$

Example 4.12. *A body has the cylindrical upper portion of* 4m *diameter and 2.4 m deep. The lower portion, which is curved, displaces a volume of 800 litres of water and its centre of buoyancy is situated 2.6 m below the top of the cylinder. The centre of gravity of the whole body is 1.6* m *below the top of the cylinder and the total displacement of water is 52 kN. Find the metacentric height of the body.*

Solution. *Given:* Diameter of body, $d = 4$ m

Depth of cylindrical portion = 2.4 m

Volume of curved portion = 800 litres = 0.8 m^3

Distance between centre of buoyancy of curved portion and top of body,

$$OB_1 = 2.6 \text{ m}$$

Distance between centre of gravity of the whole body and top of the cylinder, $OG = 1.6$ m

Total volume of water displaced, $V = \frac{52}{9.81} = 5.3 \text{ m}^3$

Volume of water displaced by the cylindrical portion = 5.3 – 0.8 = 4.5 m^3

If y is the distance between the water surface and the top of the body then

$$4.5 = \frac{\pi}{4} \times 4^2 \times (2.4 - y)$$

$$y = 2.4 - \frac{4.5 \times 4}{\pi \times 4^2} = 2.04 \text{ m}$$

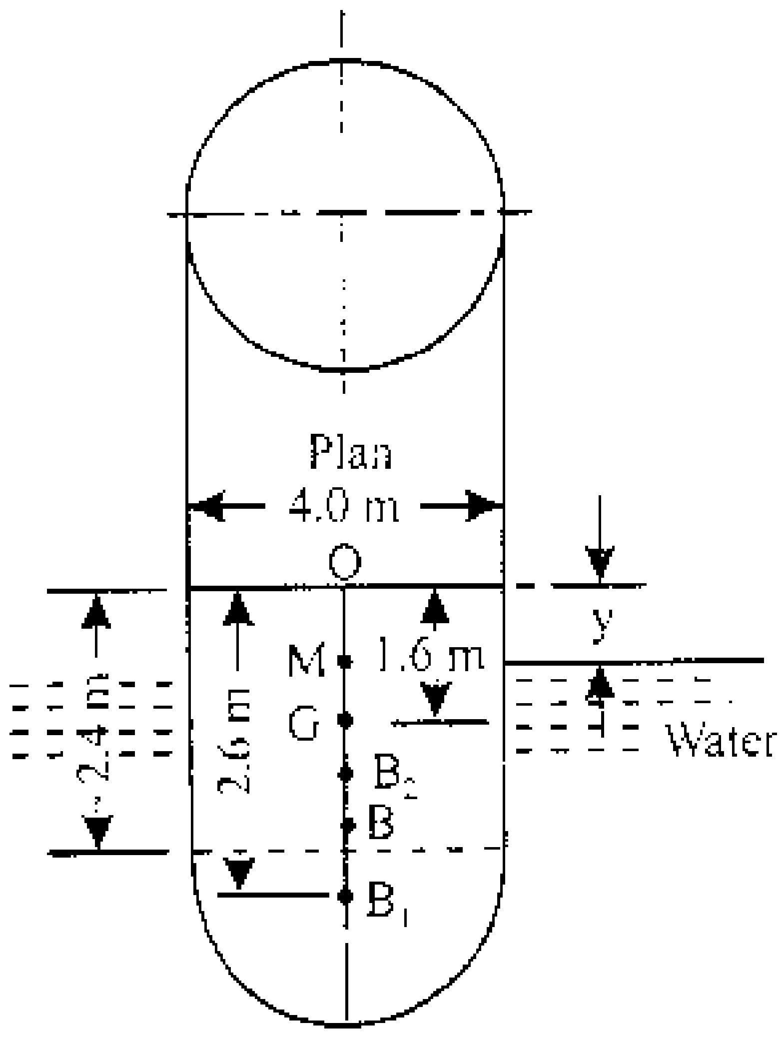


Fig. 4.12

Distance of the centre of buoyancy of the cylindrical portion from the top of the body,

$$OB_2 = y + \left(\frac{2.4 - y}{2}\right) = 2.04 + \frac{2.4 - 2.04}{2} = 2.22 \text{ m}$$

If B be the centre of buoyancy of the whole body then,

$$OB = \frac{(0.8 \times 2.6) + (4.5 \times 2.22)}{0.8 + 4.5} = 2.277 \text{ m}$$

Now $BG = OB - OG = 2.277 - 1.6 = 0.677$ m

Now
$$BM = \frac{I}{V}$$

where, I = moment of inertia of the cylindrical portion (top portion) about its *c.g.*

$$= \frac{\pi}{64} \times 4^4 \text{ m}^4 = 12.566 \text{ m}^4$$

and
$$V = 5.3 \text{ m}^3 \text{ (already calculated above)}$$

$$BM = \frac{12.566}{5.3} = 2.37 \text{ m}$$

Metacentric height, $GM = BM - BG = 2.37 - 0.677 =$ **1.693 m (Ans.)**

Example 4.13. *A solid cube of sides 0.5 m each is made of a material of relative density 0.5. The cube floats in a liquid of relative density 0.95 with two of its faces horizontal. Examine its stability.*

(AMIE Summer, 1997)

Solution. *Given*: Side of the cube = 0.5 m; specific gravity of cube material = 0.5, relative density of liquid = 0.95.

Depth of cube in liquid,

$$h = \frac{0.5 \times 0.5}{0.95} = 0.263 \text{ m}$$

Distance of centre of buoyancy (B) from O,

i.e.
$$OB = \frac{0.263}{2} = 0.1315 \text{ m}$$

Distance of centre of gravity (G) from O, *i.e.*

$$OG = \frac{0.5}{2} = 0.25 \text{ m}$$

$$BG = OG - OB = 0.25 - 0.1315 = 0.1185 \text{ m}$$

B lies below G.

$$BM = \frac{I}{V}$$

where, I = moment of inertia of the plane of the body about YY

$$= \frac{1}{12}(0.5)(0.5)^3 = 0.005208 \text{ m}^3$$

V = Volume of liquid displaced

$$= 0.5 \times 0.5 \times 0.263 = 0.06575 \text{ m}^3$$

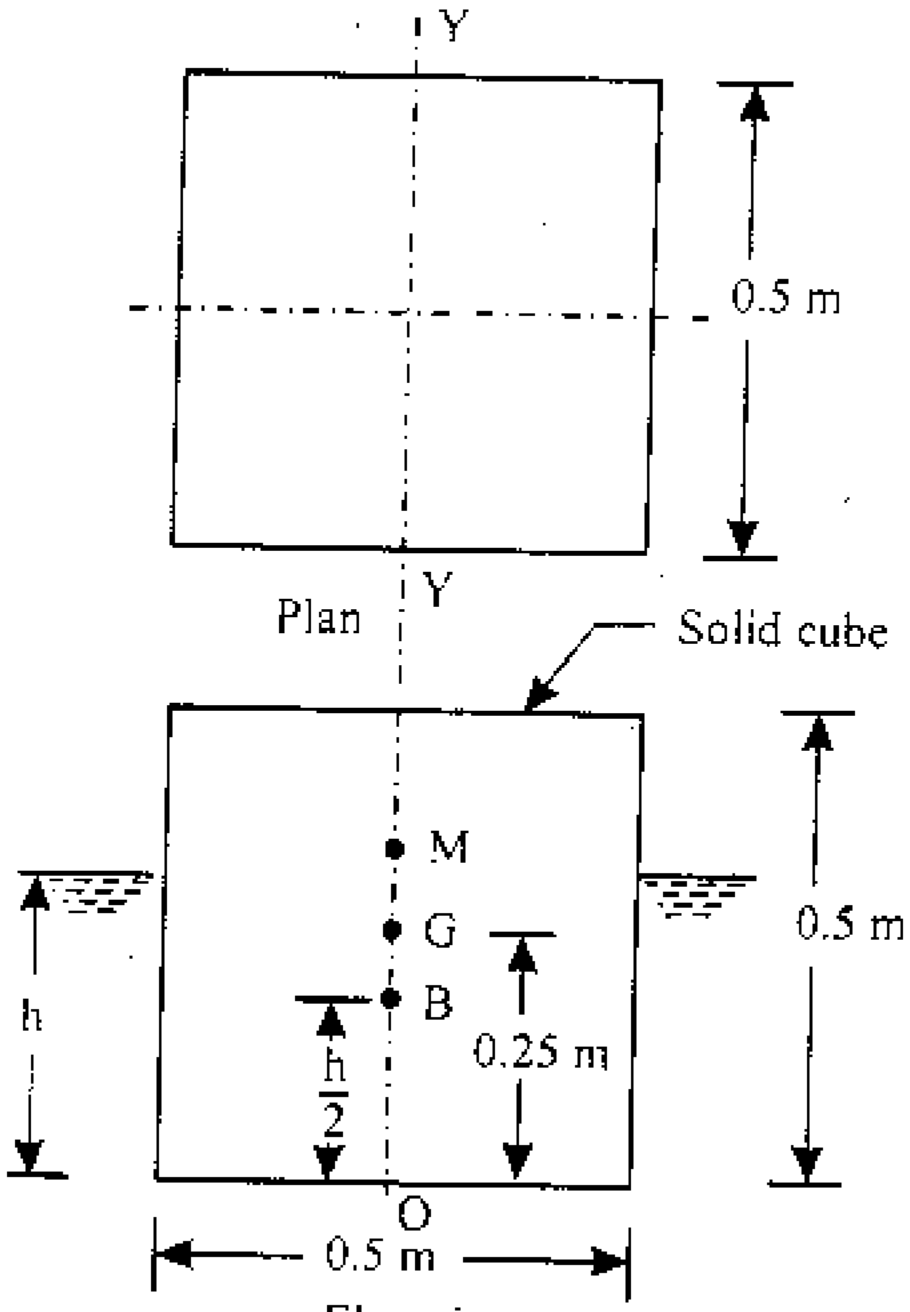


Fig. 4.13

$$\therefore \quad BM = \frac{I}{V} = \frac{0.005208}{0.06575} = 0.0792 \text{ m}$$

Metacentric height $GM = BM - BG = 0.0792 - 0.1185 = -0.0393$ m

–ve sign means that the metacentre (M) is below the centre of gravity (G). Thus the cube will be **unstable.**

Example 4.14. *A hollow cylinder closed at both ends has an outside diameter of 1.25 m, length 3.5 m and specific weight 75 kN/m³. If cylinder is to float just in stable equilibrium in sea water (specific weight 10 kN/m³) find its minimum permissible thickness.*

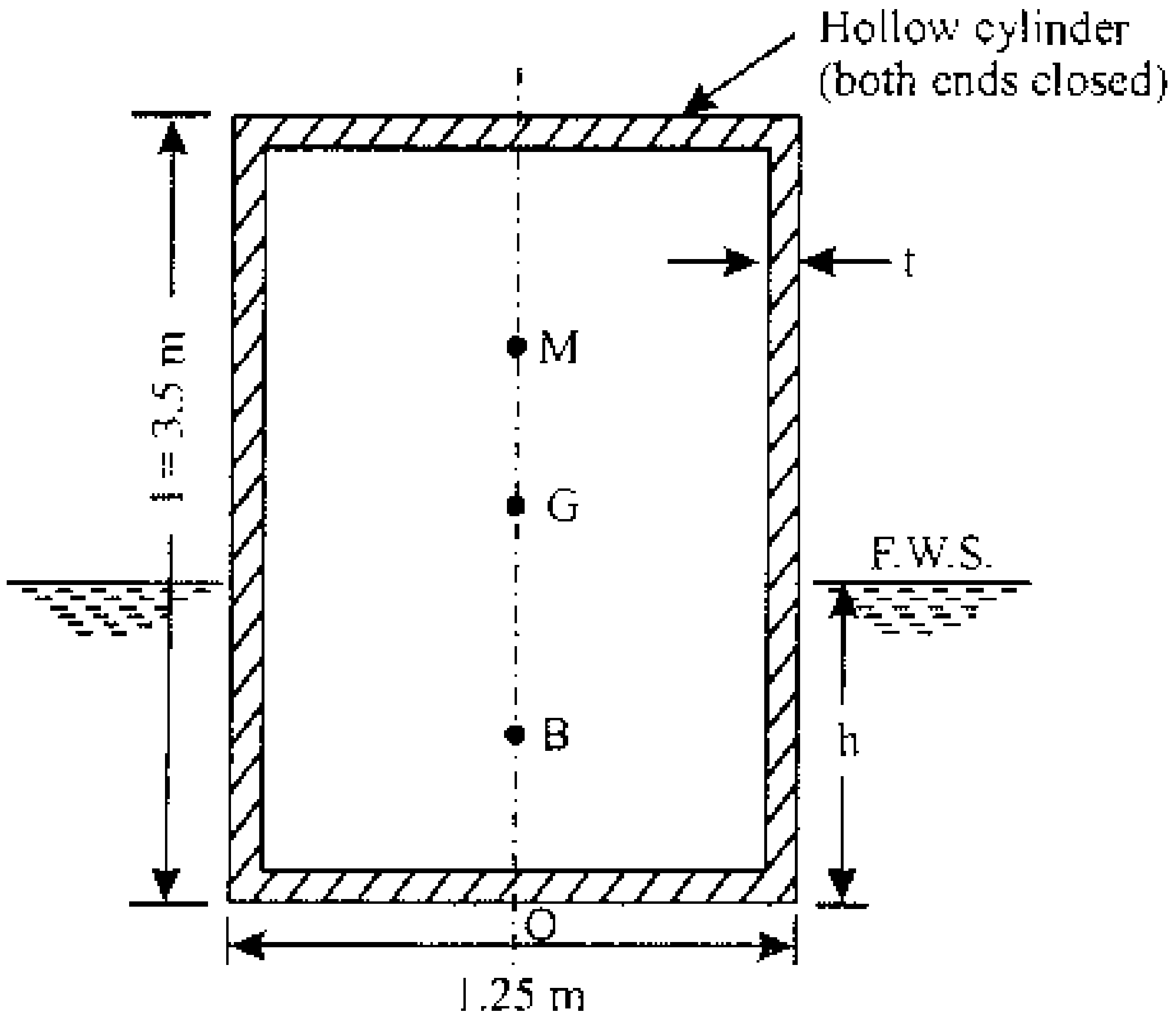


Fig. 4.14

Solution. *Given*: $d = 1.25$ m, $l = 3.5$ m, $w_c = 75$ kN/m³; $w_w = 10$ kN/m³

Minimum permissible thickness, *t*:

Let h = depth of immersion, m.

Weight of sea water displaced

$$= \frac{\pi}{4} d^2\, h \times w_w$$

$$= \frac{\pi}{4} \times 1.25^2 \times h \times 10 = 12.27\, h \text{ kN}$$

Weight of cylinder = volume of cylinder × specific weight.

= (volume of two end sections + volume of circular portion) × specific weight

$$= \left[2 \times \frac{\pi}{4} d^2 \times t + \frac{\pi}{4} \left\{ d^2 - (d - 2t)^2 \right\} l \right] \times w_c$$

$$= \left[2 \times \frac{\pi}{4} d^2 t + \pi\, dtl \right] \times w_c \qquad \text{(assuming } t << l\text{)}$$

(Ignoring term involving t^2)

$$\left[2 \times \frac{\pi}{4} \times 1.25^2 \times t + \pi \times 1.25 \times t \times 3.5 \right] \times 75$$

$$= 1215\, t \text{ kN}$$

Under equilibrium conditions, weight of cylinder = weight of sea water displaced.

$$12.27\, h = 1215\, t \quad \text{or} \quad h = 99\, t$$

Volume of cylinder under water or volume of sea water displaced,

$$V = \frac{1215t}{10} = 121.5t$$

If M is the metacentre, then

$$BM = \frac{I}{V} = \frac{\frac{\pi}{64} \times 1.25^4}{121.5t} = \frac{0.00099}{t}$$

$$OB = \frac{h}{2} = \frac{99t}{2} = 49.5t$$

$$OG = \frac{3.5}{2} = 1.75 \text{ m}$$

$$BG = OG - OB = 1.75 - 49.5\, t$$

For the cylinder to float just in stable equilibrium,

$$BG = BM; \quad \text{or} \quad 1.75 - 49.5\, t = \frac{0.00099}{t}$$

or $$49.5\, t^2 - 1.75\, t + 0.00099 = 0$$

or $$t = \frac{1.75 \pm \sqrt{(1.75)^2 - 4 \times 49.5 \times 0.00099}}{2 \times 49.5} = \frac{1.75 \pm 1.69}{99}$$

$= 0.0347$ m or 6.06×10^{-4} m

Hence, minimum permissible thickness = 6.06×10^{-4} m or **0.606 mm (Ans.)**

Example 4.15. *A solid of 200 mm diameter and 800 mm length has its base 20 mm thick and of specific gravity 6. The remaining part of the cylinder is of specific gravity 0.6. State, if it can float vertically in water.*

Solution. *Given:* Dia. of cylinder, = 200 mm = 0.2 m

Area of cylinder, $A = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2$

Length of cylinder = 800 mm = 0.8 m

Thickness of base = 20 mm = 0.02 m

Sp. gr. of base = 6, sp. gr. of remaining portion = 0.6

Distance between combined centre of gravity (G) and the bottom of the cylinder (O),

$$OG = \frac{\left[A \times 0.78 \times 0.6\left(\frac{0.78}{2} + 0.02\right)\right] + \left[A \times 0.02 \times 6 \times \frac{0.02}{2}\right]}{(A \times 0.78 \times 0.6) + (A \times 0.02 \times 6)}$$

(where A = area of cylinder)

$$= \frac{0.1919 + 0.0012}{0.468 + 0.12} = 0.3284 \text{ m (or 328.4 mm)}$$

Combined sp. gr. of the cylinder

$$= \frac{(0.78 \times 0.6) + (0.02 \times 6)}{0.78 + 0.02} = 0.735$$

Depth of immersion of the cylinder

$= 0.8 \times 0.735 = 0.588$ m

Distance of centre of buoyancy from the bottom of the cylinder,

$$OB = \frac{0.588}{2} = 0.294 \text{ (or 29.4 mm)}$$

$$BG = OG - OB = 0.3284 - 0.294$$

$= 0.0344$ m (or 34.4 mm)

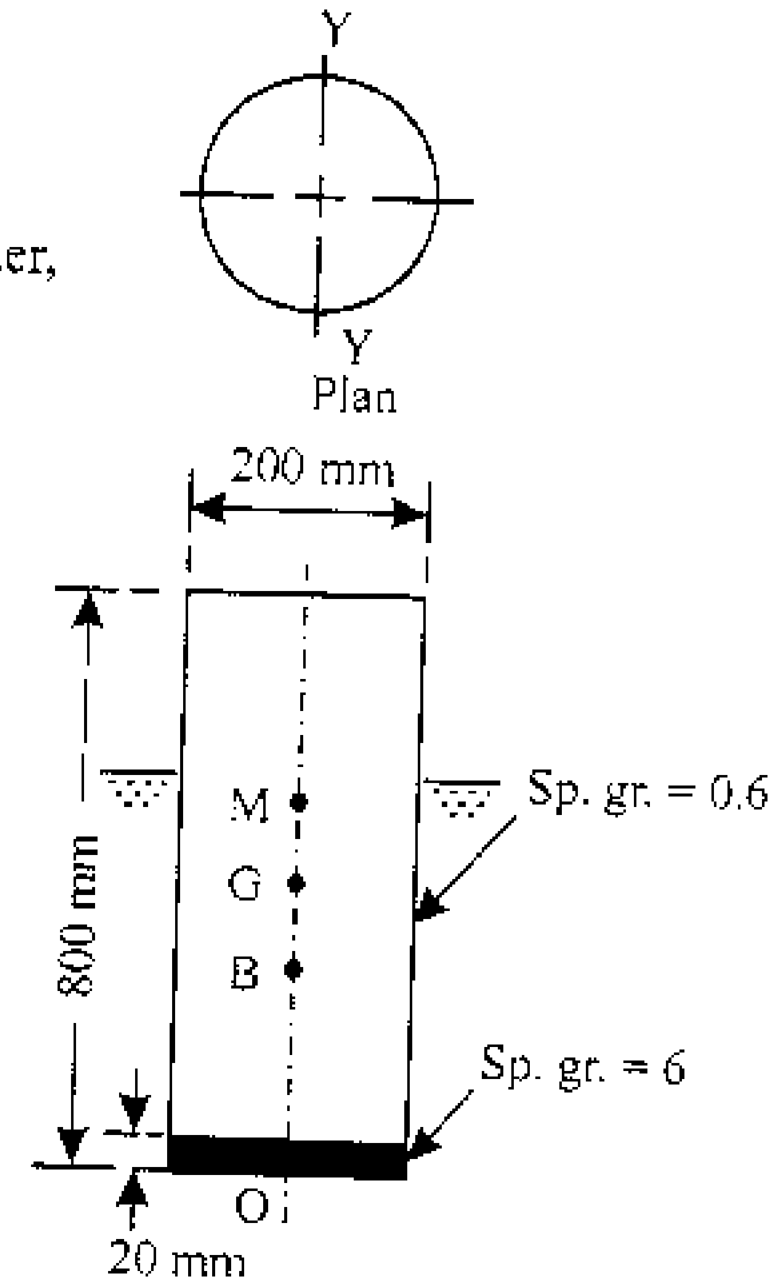


Fig. 4.15

Now, $$BM = \frac{I}{V}$$

where, I = moment of inertia of circular section

$$= \frac{\pi}{64} \times 0.2^4 = 2.5 \times 10^{-5} \pi \text{ m}^4$$

and, V = volume of water displaced

$$= \frac{\pi}{4} \times 0.2^2 \times 0.588 = 0.00588\,\pi$$

$\therefore$ $$BM = \frac{2.5 \times 10^{-5} \pi}{0.00588 \pi}$$

$= 0.00425$ m or 4.25 mm

Now metacentric height,

$$GM = BM - BG = 4.25 - 34.4 \text{ mm} = -30.15 \text{ mm}$$

Negative sign means that the metacentre (M) is below the centre of gravity (G). Thus the cylinder is in *unstable equilibrium and it* ***cannot float*** *vertically in water.* **(Ans.)**

Example 4.16. *An 80 mm diameter composite solid cylinder consists of an 80 mm diameter, 20 mm thick metallic plate having specific gravity 4.0 attached at the lower end of an 80 mm diameter wooden cylinder of specific gravity 0.8. Find the limits of the length of the wooden portion so that the composite cylinder can float in stable equilibrium in water (specific gravity 1.0) with its axis vertical.* **(AMIE Winter, 1998)**

Solution. Refer Fig. 4.16. *Given*: $d = 80$ mm; $a = 20$ mm; $S_1 = 4$; $S_2 = 0.8$

Limits of the length of the wooden portion:

The cylinder will float vertically in water if its metacentric height GM is +ve. To find the metacentric height, the locations of centre of gravity G and centre of buoyancy B of the combined cylinder is to be found.

The distance of the centre of gravity of the solid cylinder from O is given by:

$$OG = \frac{\left[\frac{\pi}{4}d^2 \times a \times S_1\right] \times \frac{a}{2} + \left[\frac{\pi}{4}d^2 \times b \times S_2\right] \times \left[a + \frac{b}{2}\right]}{\left[\frac{\pi}{4}d^2 \times a \times S_1 + \frac{\pi}{4}d^2 \times b \times S_2\right]}$$

$$= \frac{\left(a \times S_1 \times \frac{a}{2}\right) + b \times S_2\left(a + \frac{b}{2}\right)}{a \times S_1 + b \times S_2} = \frac{\left(20 \times 4 \times \frac{20}{2}\right) + b \times 0.8\left(20 + \frac{b}{2}\right)}{20 \times 4 + b \times 0.8}$$

$$= \left[\frac{800 + 0.8b\left(20 + \frac{b}{2}\right)}{80 + 0.8b}\right] = \left[\frac{1000 + 20b + \frac{b^2}{2}}{100 + b}\right] \quad ...(i)$$

(Dividing numerator and denominator by 0.8 and simplifying)

Let, height of immersion of cylinder = $(h + a)$

Also, Weight of cylinder = weight of water displaced

or $\quad \frac{\pi}{4}d^2 \times a \times S_1 + \frac{\pi}{4}d^2 \times b \times S_2$

$$= \frac{\pi}{4}d^2 (h + a) \times S_{water}$$

$$a \times S_1 + b \times S_2 = (h + a) \qquad (\because S_{water} = 1)$$

or $\quad 20 \times 4 + b \times 0.8 = (h + a)$

i.e. $\quad h + a = 80 + 0.8\,b \quad ...(ii)$

Now, $OB = \frac{h + a}{2} = \frac{80 + 0.8b}{2} = 40 + 0.4b \quad ...(iii)$

$$BG = OG - OB = \left[\frac{1000 + 20b + \frac{b^2}{2}}{100 + b}\right] - (40 + 0.4b) \quad \text{[From (i) and (iii)]}$$

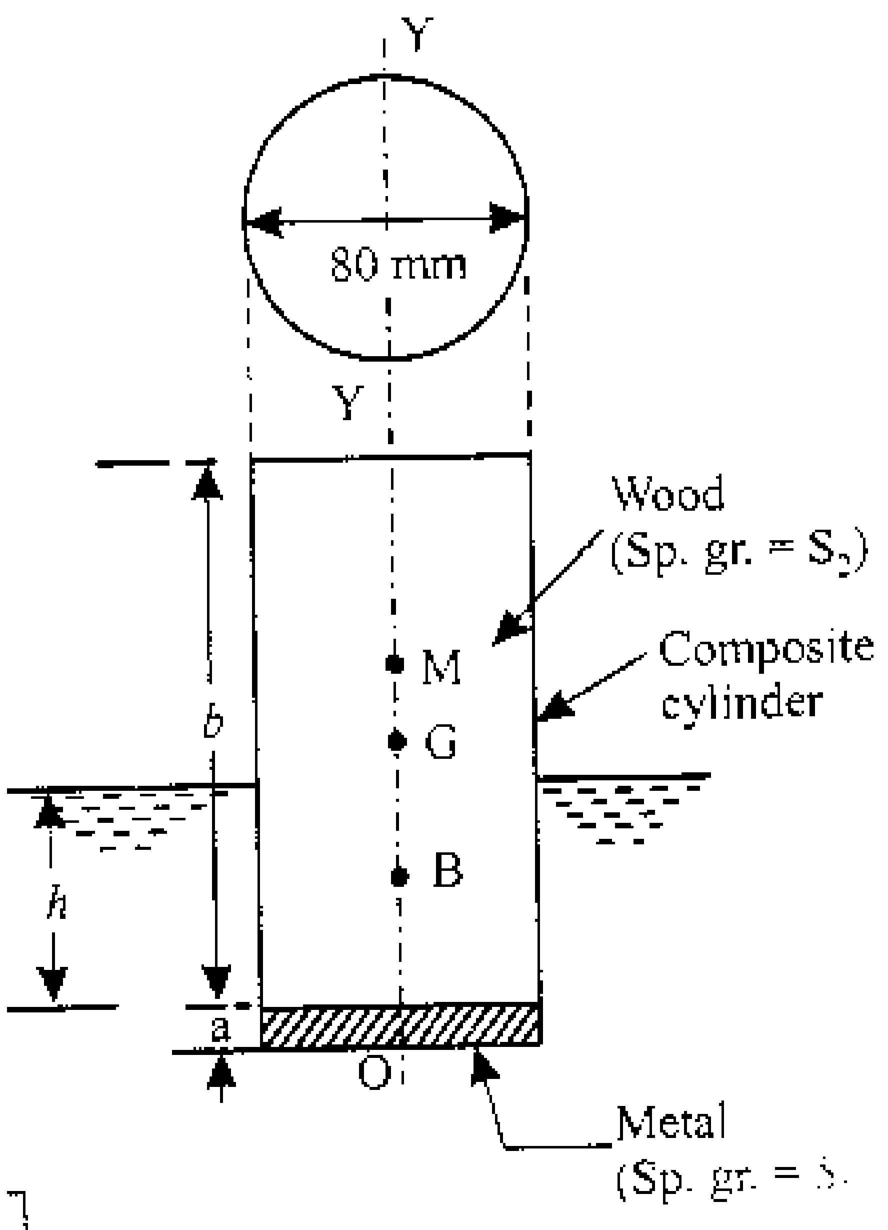


Fig. 4.16

$$= \frac{1}{100 + b)}\left[\left(1000 + 20b + \frac{b^2}{2}\right) - (4000 + 80b + 0.4b^2)\right]$$

$$= \frac{1}{(100+b)}[0.1b^2 - 60b - 3000] \quad ...(iv)$$

I = second moment of area of the section about $YY = \frac{\pi d^4}{64}$

V = volume of water displaced = $\frac{\pi}{4} d^2 (h + a)$

$$BM = \frac{I}{V} = \frac{\pi d^4/64}{\frac{\pi}{4} d^2 (h+a)} = \frac{\pi d^4}{64} \times \frac{4}{\pi d^2 (h+a)} = \frac{d^2}{16(h+a)}$$

$$= \frac{(80)^2}{16(80+0.8b)}, \text{ using } (ii) \text{ for } (h+a)$$

$$= \frac{400}{80+0.8b} = \frac{500}{100+b} \quad ...(v)$$

$$GM = BM - BG = \frac{500}{(100+b)} - \frac{1}{(100+b)}[0.1b^2 - 60b - 3000] \quad [\text{using } (iv)] \text{ and } (v)]$$

or $$GM = \frac{3500 + 60b - 0.1b^2}{(100+b)} \quad ...(vi)$$

It should be +ve and in the limit = 0

i.e. $0.1b^2 - 60b - 3500 = 0$

or $b^2 - 600b - 35000 = 0$

or $$b = \frac{600 + \sqrt{(600)^2 + 4 \times 35000}}{2}, \text{ taking +ve value of } b$$

or $b = 653.55$ mm. *This is the* **upper limit** *for b.* **(Ans.)**

The **lower limit** *for b* will be $b = h$, and from eqn. (*ii*), we have

$h + a = 80 + 0.8\,b$

$b + 20 = 80 + 0.8\,b$

$\boldsymbol{b = 300}$ **mm (Ans.)**

It may be checked from eqn. (*ii*) that it gives a + ve value of GM.

Example 4.17. *A hollow wooden cylinder of specific gravity 0.6 has an outer diameter of 600 mm and an inner diameter of 300 mm. It is required to float in oil of specific gravity 0.9. Calculate:*

(*i*) *The maximum length (height) of the cylinder so that it shall be stable when floating with it axis vertical;*

(*ii*) *The depth to which it will sink.*

Solution. Outer diameter of cylinder, D = 600 mm = 0.6 m

Inner diameter of cylinder, d = 300 mm = 0.3 m

Specific weight = 0.6 × 9.81 = 5.886 kN/m^3

∴ Weight of cylinder = volume of cylinder × specific weight

$$= \pi/4\,(D^2 - d^2) \times l \times 5.886 = \frac{\pi}{4}(0.6^2 - 0.3^2) \times l \times 5.886 = 1.248\,l \text{ kN}$$

(where, l = length/height of the cylinder)

This also represents the weight of oil displaced.

∴ Volume of oil displaced, $V = \frac{1.248 l}{0.9 \times 9.81} = 0.1413\,l$

i.e. Volume of cylinder immersed in oil, $V = 0.1413l$

Depth of immersion, $h = \dfrac{\text{volume of cylinder under oil}}{\text{cross-section area of cylinder}}$

$$= \frac{0.1413l}{\frac{\pi}{4}(0.6^2 - 0.3^2)} = 0.666l$$

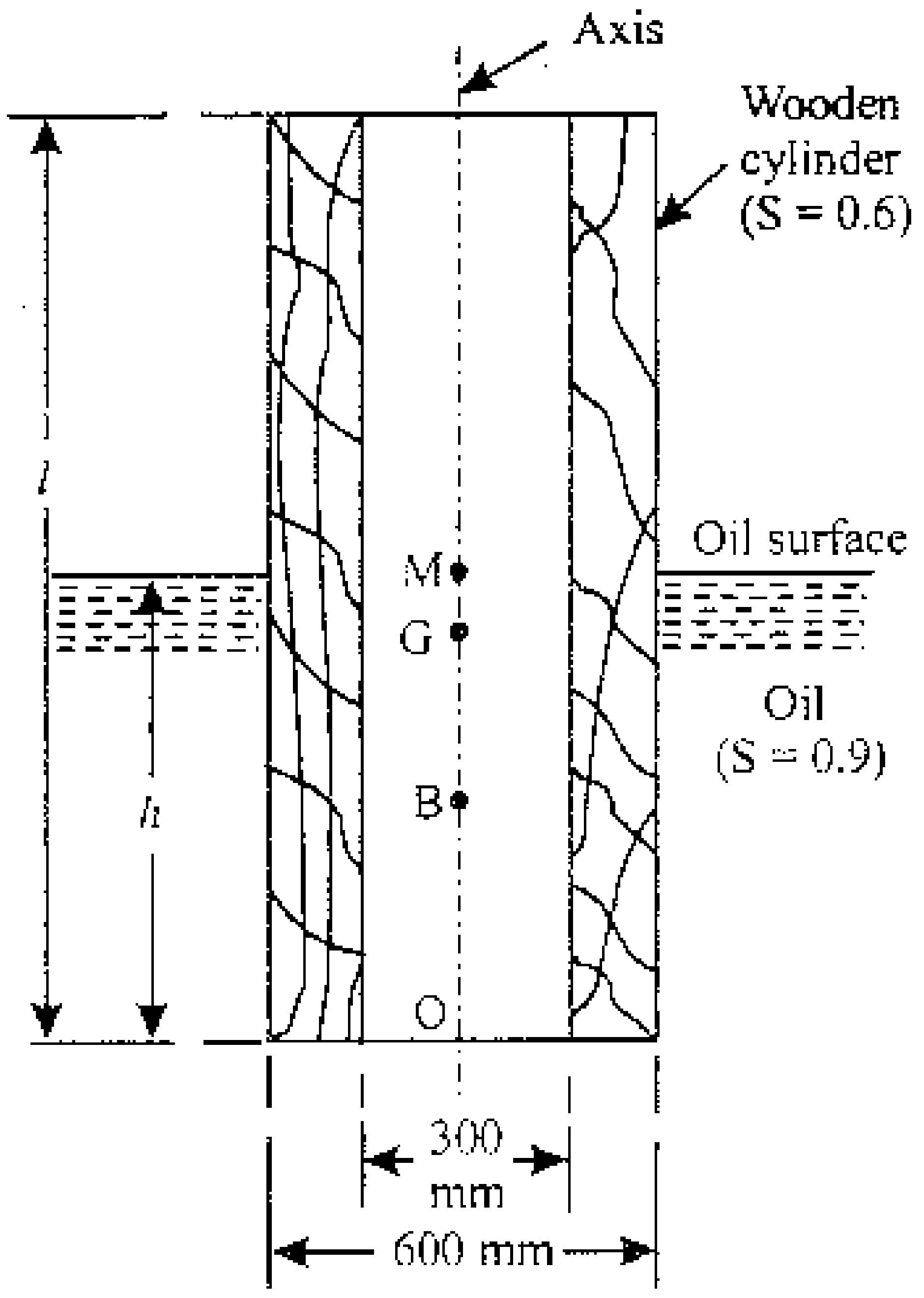


Fig. 4.17

Height of centre of buoyancy (B) from O,

i.e. $OB = \dfrac{h}{2} = \dfrac{0.666l}{2} = 0.333\,l$

If M is the metacentre, then

$$BM = \frac{I}{V} = \frac{\frac{\pi}{64}(0.6^4 - 0.3^4)}{0.1413\,l} = \frac{0.0422}{l}$$

$$OM = OB + BM = 0.333\,l + \frac{0.0422}{l}$$

Distance of centre of gravity (G) from the point O,

$$OG = \frac{l}{2} = 0.5\,l$$

For stable equilibrium, M should be at a level than G, *i.e.* $OM > OG$

$$0.333\,l + \frac{0.0422}{l} > 0.5\,l$$

$$\frac{0.0422}{l} > 167\,l \quad \text{or}$$

$> 0.167\,l^2$

$$0.167\,l^2 < 0.0422 \quad \text{or} \quad l < \left(\frac{0.0422}{0.167}\right)^{1/2} < 0.503 \text{ m}$$

l_{max} = **0.503 m (Ans.)**

$h = 0.666\,l = 0.666 \times 0.503 =$ **0.335 m (Ans.)**

mple 4.18. *A rectangular pontoon 12 m long, 9 m wide and 3 m deep weighs 1380 kN and a water. The pontoon carries on its upper deck a boiler 6 m diameter and weighing 864 kN. of gravity of each unit coincides with geometrical centre of the arrangement and lies in the cal line.*

hat is the metacentric height?

the arrangement stable?

pecific weight of sea water = 10 kN/m³

ion. Total weight of the arrangement, $W = 1380 + 864 = 2244$ kN

lso represent the weight of water displaced.

e of sea water displaced,

$$V = \frac{\text{weight of water displaced}}{\text{specific weight of water}} = \frac{2244}{10} = 224.4 \text{ m}^3$$

ume of the arrangement under water, $V = 224.4$ m³

of immersion,

$$h = \frac{\text{volume of the arrangement under water}}{\text{cross-sectional area of the pontoon}}$$

$$= \frac{224.4}{9 \times 12} = 2.077 \text{ m}$$

Distance of centre of buoyancy (B) from the base point O, $OB = \frac{2.077}{2} = 1.0385$ m

Let M be the metacentre.

Then,
$$BM = \frac{I}{V} = \frac{\frac{1}{12} \times 12 \times 9^3}{224.4} = 3.248 \text{ m}$$

$$OM = OB + BM = 1.0385 + 3.248 = 4.286 \text{ m}$$

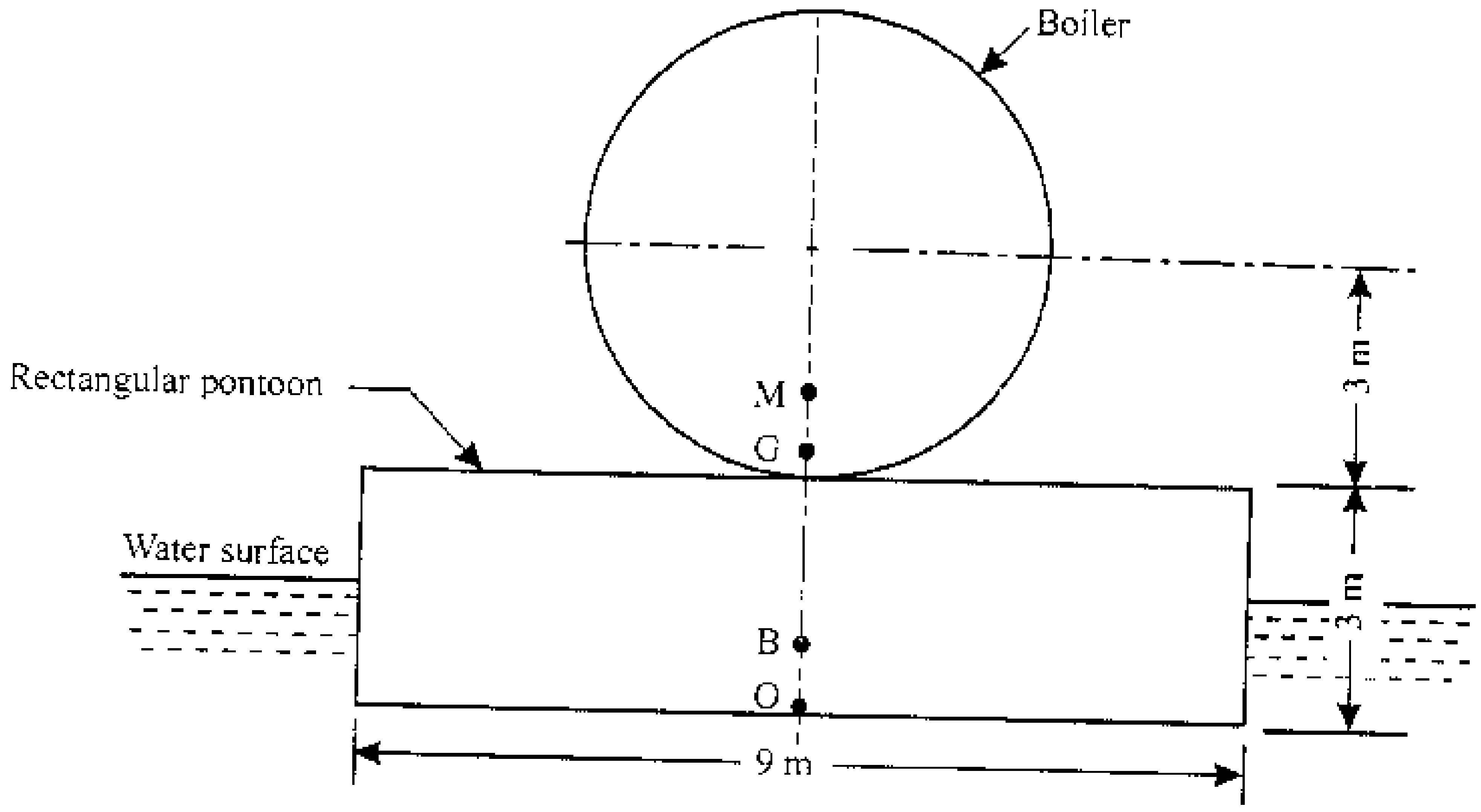


Fig. 4.18

To find the position of combined centre of gravity above the base point O, taking moments about O, we get

$$1380 \times 1.5 + 864 \times 6 = 2244 \times OG$$

$$\therefore \quad OG = \frac{1380 \times 1.5 + 864 \times 6}{2244} = 3.232 \text{ m}$$

(*i*) **Metacentric height, GM:** $GM = OM - OG = 4.286 - 3.232 =$ **1.054 m (Ans.)**

(*ii*) **Stability of the arrangement:**

Since $OM > OG$, M is at a higher level than G.

Hence the arrangement is **stable (Ans.)**

Example 4.19. *A buoy having a diameter of 2.4 m and length 1.95 m is floating with its axis vertical in sea water (specific weight = 10 kN/m³). Its weight is 16.5 kN and a load of 1.65 kN is placed centrally at its top. If the buoy is to remain in stable equilibrium, find the maximum permissible height of the centre of gravity of the load above the top of the buoy.*

Solution. *Given:* Diameter of the buoy, $d = 2.4$ m; length of the buoy, $l = 1.95$ m

Weight of the buoy, $W_{buoy} = 16.5$ kN

Weight placed at the top of the buoy, $W = 1.65$ kN

Specific weight of sea water = 10 kN/m^3

Total weight of the arrangement $W_t = W_{buoy} + W$

$$= 16.5 + 1.65 = 18.15 \text{ kN}$$

This is also the weight of water displaced by the arrangement.

Volume of water displaced,

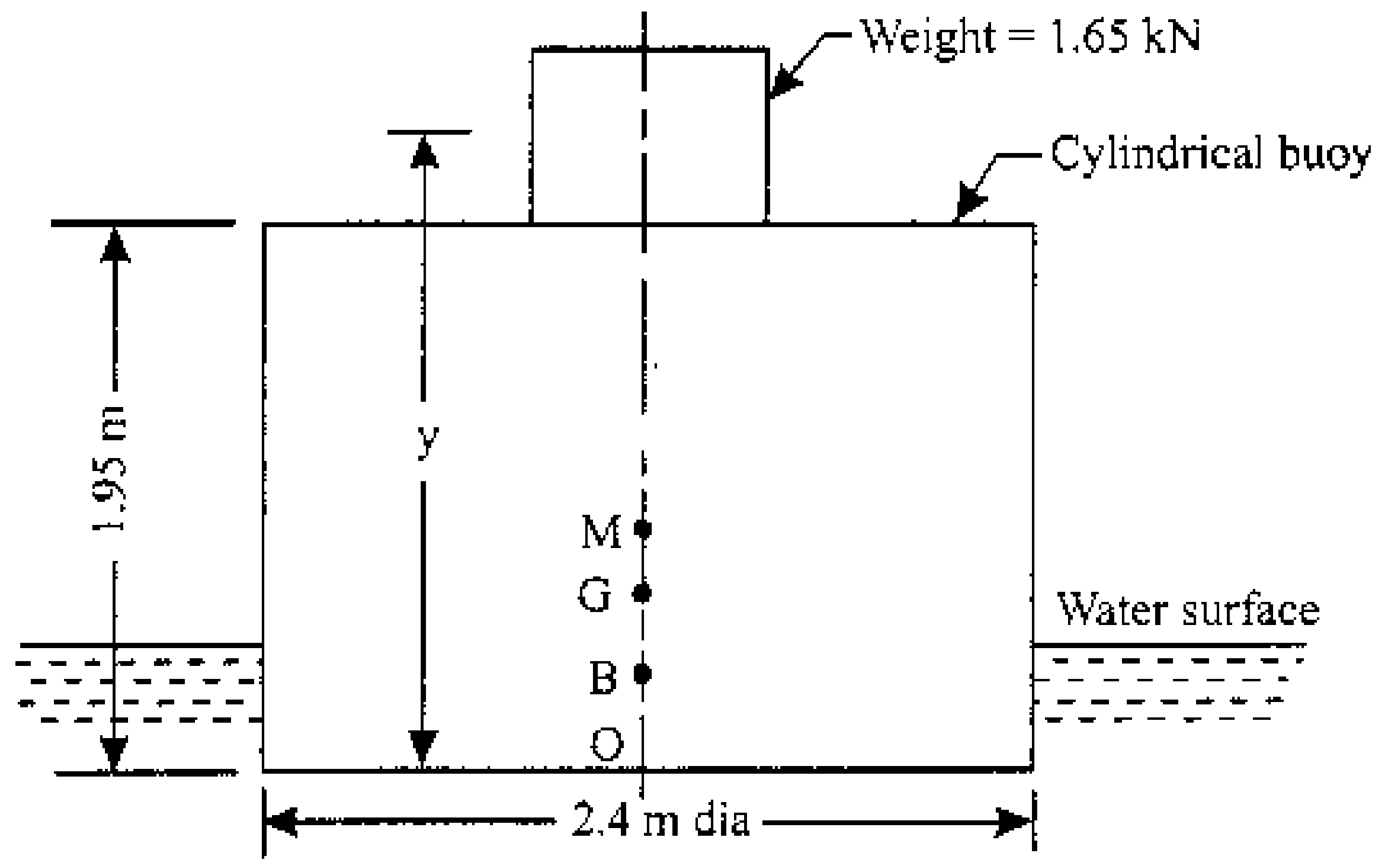


Fig. 4.19

$$V = \frac{W_t}{\text{sp. weight of water}}$$

$$= \frac{18.15}{10} = 1.815 \text{ m}^3$$

Depth of immersion,

$$h = \frac{\text{Volume of water displaced}}{\text{Cross-sectional area of the buoy}}$$

$$= \frac{1.815}{\pi/4 \times 2.4^2} = 0.4 \text{ m}$$

Height of centre of buoyancy (B) above base point O,

$$OB = \frac{h}{2} = \frac{0.4}{2} = 0.2 \text{ m}$$

If M is the metacentre, then

$$BM = \frac{I}{V} = \frac{\pi/64 \times 2.4^4}{1.815} = 0.897 \text{ m}$$

$$OM = OB + BM = 0.2 + 0.897 = 1.097 \text{ m}$$

Let, y = Height of centre of gravity of the load above the base O.

To find the position of combined centre of gravity above the base point O, taking moments about O, we get

$$16.5 \times \frac{1.95}{2} + 1.65 \times y = 18.15 \times OG$$

or $$16.087 + 1.65\,y = 18.15 \times OG$$

or $$OG = \frac{16.087 + 1.65\,y}{18.15} = 0.886 + 0.091y$$

The equilibrium will be stable when $OM > OG$ *i.e.* $1.097 > (0.886 + 0.091y)$

or $$0.211 > 0.091y \text{ or } 0.091y < 0.211 \text{ or } y < 2.318 \text{ m}$$

But the height of buoy = 1.95 m

The height of centre of gravity of the load above the buoy should *not be greater than* (2.31
= **0.368 m (Ans.)**

Example 4.20. *A wooden cylinder (sp. gravity = 0.54) of diameter d and length l is required to float in oil (sp. gravity = 0.81). Find the l/d ratio for the cylinder to float with its longitudinal axis vertical in oil.*

Solution. *Given:* Diameter of the cylinder = d; length of the cylinder = l;
Sp. gravity, S_1 = 0.54; sp. gravity of oil, S_2 = 0.81.

$\frac{\mathbf{l}}{\mathbf{d}}$ **ratio:**

Let h = depth of cylinder immersed in oil.

Now, weight of cylinder = weight of oil displaced...(principle of buoyancy)

or $$\frac{\pi}{4}d^2\, l \times S_1 = \frac{\pi}{4}d^2\, h \times S_2$$

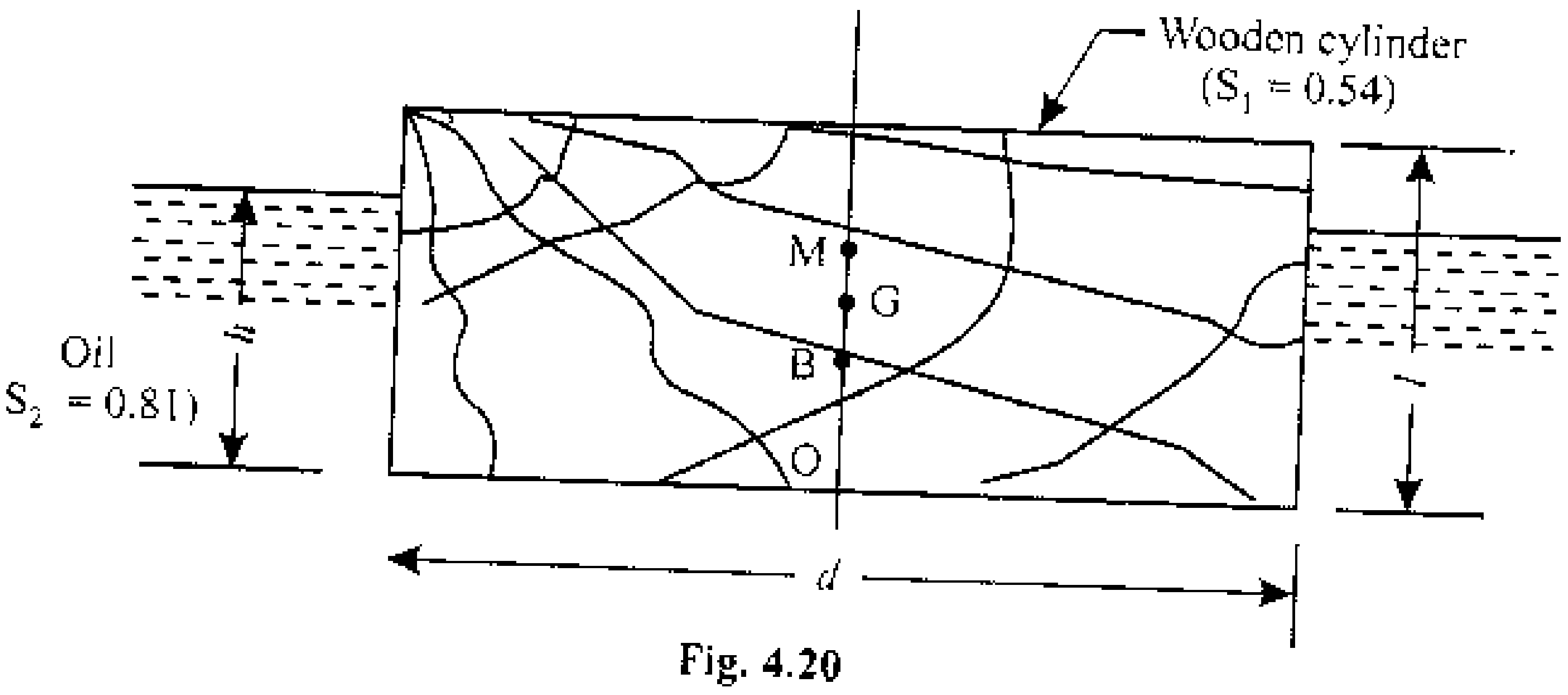


Fig. 4.20

$$l \times 0.54 = h \times 0.81 \quad \therefore \quad h = \frac{0.54\, l}{0.81} = \frac{2}{3}l$$

$\therefore$ The distance of centre of buoyancy B from O, $OB = \frac{h}{2} = \frac{1}{3}l$

The distance of centre of gravity G from base point O, $OG = l/2$

$\therefore$ If M is the metacentre, then

$$BM = \frac{I}{V} = \frac{\frac{\pi}{64}d^4}{\frac{\pi}{4}d^2 h} = \frac{d^2}{16h} = \frac{d^2}{16 \times \frac{2}{3}l} = \frac{3d^2}{32l}$$

$$\therefore \quad OM = OB + BM = \frac{1}{3}l + \frac{3d^2}{32l}$$

For stable equilibrium, $OM > OG$ or $\left(\frac{1}{3}l + \frac{3d^2}{32l}\right) > \frac{l}{2}$

or $$\frac{3d^2}{32l} > \left(\frac{l}{2} - \frac{l}{3}\right) \text{ or } \frac{3d^2}{32l} > \frac{l}{6}$$

or $$\frac{3d^2}{32l^2} > \frac{1}{6} \text{ or } \frac{18}{32} > \frac{l^2}{d^2}$$

or $$\frac{l^2}{d^2} < \frac{18}{32} \text{ or } \frac{l}{d} < \left(\frac{18}{32}\right)^{1/2} \quad \text{or} \quad (9/16)^{1/2} \text{ or } 3/4$$

$$\therefore \quad \frac{l}{d} < 3/4 \textbf{ (Ans.)}$$

Example 4.21. *A log of wood 0.9 m in diameter and 7.5 long is floating in river water. If the specific gravity of log is 0.7, what is the depth of the wooden log in water?*

Solution. *Given:* Diameter of the wooden log, $d = 0.9$ m;

Length of the log, $l = 7.5$ m

Specific gravity, $S = 0.7$

Weight of the log $= (0.7 \times 9.81) \times \frac{\pi}{4} d^2 l = 0.7 \times 9.81 \times \frac{\pi}{4} \times 0.9^2 \times 7.5 = 32.76$ kN

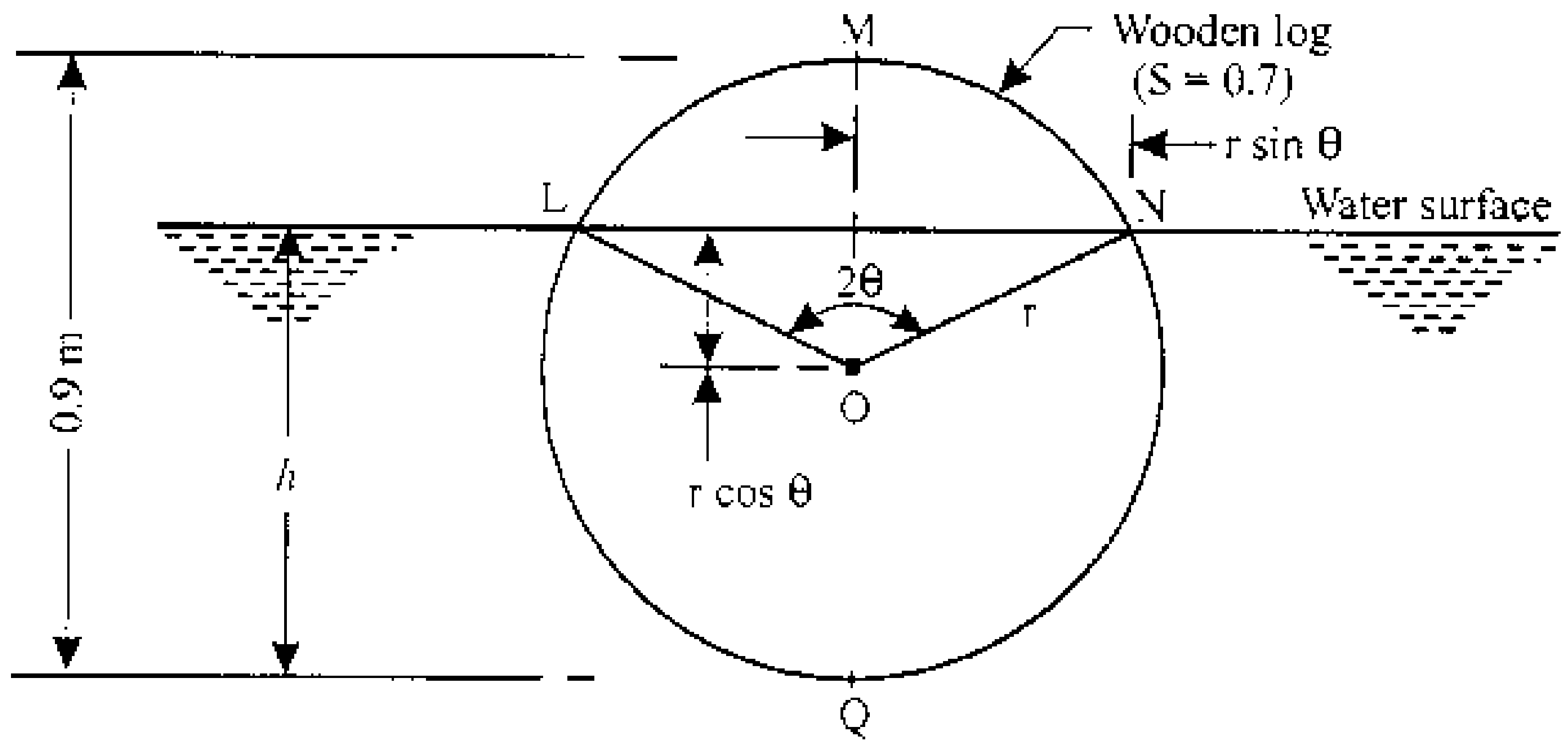


Fig. 4.21

This also represents the weight of water displaced.

Volume of water displaced $= \frac{32.76}{9.81} = 3.34 \text{ m}^3$

Let h = depth of immersion.

$\therefore$ Volume of log inside water = volume of water displaced = 3.34 m³

$$3.34 = \text{area } LQNL \times 7.5$$

Area $\quad LQNL = \frac{3.34}{7.5} = 0.4453 \text{ m}^2$

But area $\quad LQNL$ = area $LQNOL$ + area LON

$$= \pi r^2 \left[\frac{360 - 2\theta}{360}\right] + \frac{1}{2} \times 2r\sin\theta \times r\cos\theta$$

$$= \pi r^2 \left(1 - \frac{\theta}{180}\right) - r^2 \sin\theta \cos\theta$$

$\therefore$ $$0.4453 = \pi \times 0.45^2 \left(1 - \frac{\theta}{180}\right) + 0.45^2 \sin\theta \cos\theta$$

or $$0.4453 = 0.6362 - 0.003534\,\theta + 0.2025 \sin\theta \cos\theta$$

or $$0.003534\,\theta - 0.2025 \sin\theta \cos\theta = 0.6362 - 0.4453 = 0.1909$$

or $$\theta - \frac{0.2025}{0.003534} \sin\theta \cos\theta = \frac{0.1909}{0.003534}$$

or $$\theta - 57.3 \sin\theta \cos\theta = 54.02$$

or $$\theta - 57.3 \sin\theta \cos\theta - 54.02 = 0$$

By hit and trial, we get $\quad \theta \simeq 71.5°$

$\therefore$ Depth of wooden log in water, $h = r + r\cos\theta$

$= 0.45 + 0.45 \cos 71.5°$ or $h =$ **0.593 m (Ans.)**

Example 4.22. *A float valve regulates the flow of oil of specific gravity 0.8 in a cistern. The spherical float is 150 mm in diameter. AOB is a weightless link carrying the float at one end, and a valve at the other end which closes the pipe through which oil flows into the cistern. The link is mounted in a frictionless hinge at 0 and angle AOB is 135°. The length of OA is 200 mm, and the distance between the centre of the float and hinge is 500 mm. When the flow is stopped AO will be vertical. The valve is to be pressed on to the seat with a force of 10 N to completely stop the flow of oil in to the cistern. It was observed that the flow of oil is stopped when the free surface of oil in the cistern is 350* mm *bleow the hinge. Determine the weight of the float.* **[UPSC Engg. Services]**

Solution. Refer to Fig. 4.22.

Specific gravity of oil = 0.8

Diameter of the float, d = 150 mm = 0.15 m

$\angle AOB = 135°$

Weight of the float, W:

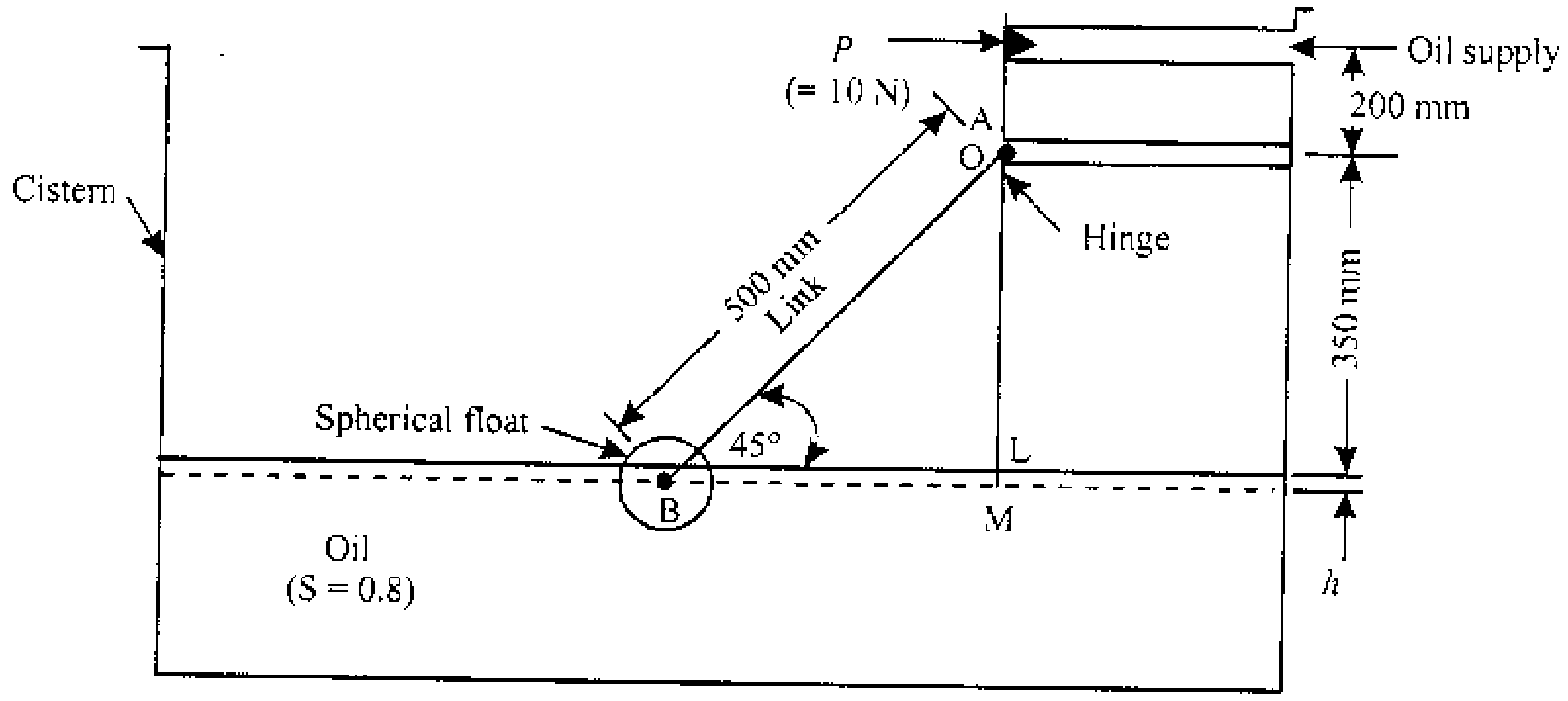


Fig. 4.22

When the oil flow is stopped, the level of oil is as shown in Fig. 4.22; the centre of float is below the level of oil by a depth h.

In ΔOBM : $\sin 45° = \dfrac{OM}{OB} = \dfrac{OL + LM}{OB} = \dfrac{0.35 + h}{0.5}$

$0.5 \sin 45° = 0.35 + h$ or $h = 0.00355$ m

Volume of oil displaced $= \dfrac{2}{3}\pi r^3 + \pi r^2 \times h$

$= \dfrac{2}{3} \times \pi \left(\dfrac{0.15}{2}\right)^3 + \pi + \left(\dfrac{0.15}{2}\right)^2 \times 0.00355 = 0.000946 \text{ m}^3$

Buoyant force = weight of oil displaced

= volume of oil displaced × sp. gravity of oil

= 0.000946 × (0.8 × 9.81) = 0.007424 kN or 7.42 N

Since the buoyant force and the weight of the float passes through the same vertical line, therefore,

Net force on float = buoyant force – weight of float = 7.42 – W

Taking moments about the hinge O, we get

$$P \times 0.2 = (7.42 - W) \times BM$$

or $\quad 10 \times 0.2 = (7.42 - W) \times 0.5 \cos 45°$

$$\therefore \quad W = 7.42 - \frac{10 \times 0.2}{0.5 \cos 45°} = 1.76 \text{ N } i.e. \ W = \mathbf{1.76 \text{ N (Ans.)}}$$

Example 4.23. *A cylindrical buoy is 2 m in diameter and 2.5m long and weighs 22 kN. The specific weight of sea water is 10.25 kN/m³. Show that the buoy does not float with its axis vertical? What minimum pull should be applied to a chain attached to the centre of the base to keep the buoy vertical?* **[UPSC; AMIE Summer, 1999]**

Solution. *Given:* Diameter of the buoy, $d = 2$ m;

Length of the buoy, $l = 2.5$ m;

Weight of the buoy, $W = 22$ kN;

Specific weight of sea water = 10.25 kN/m³.

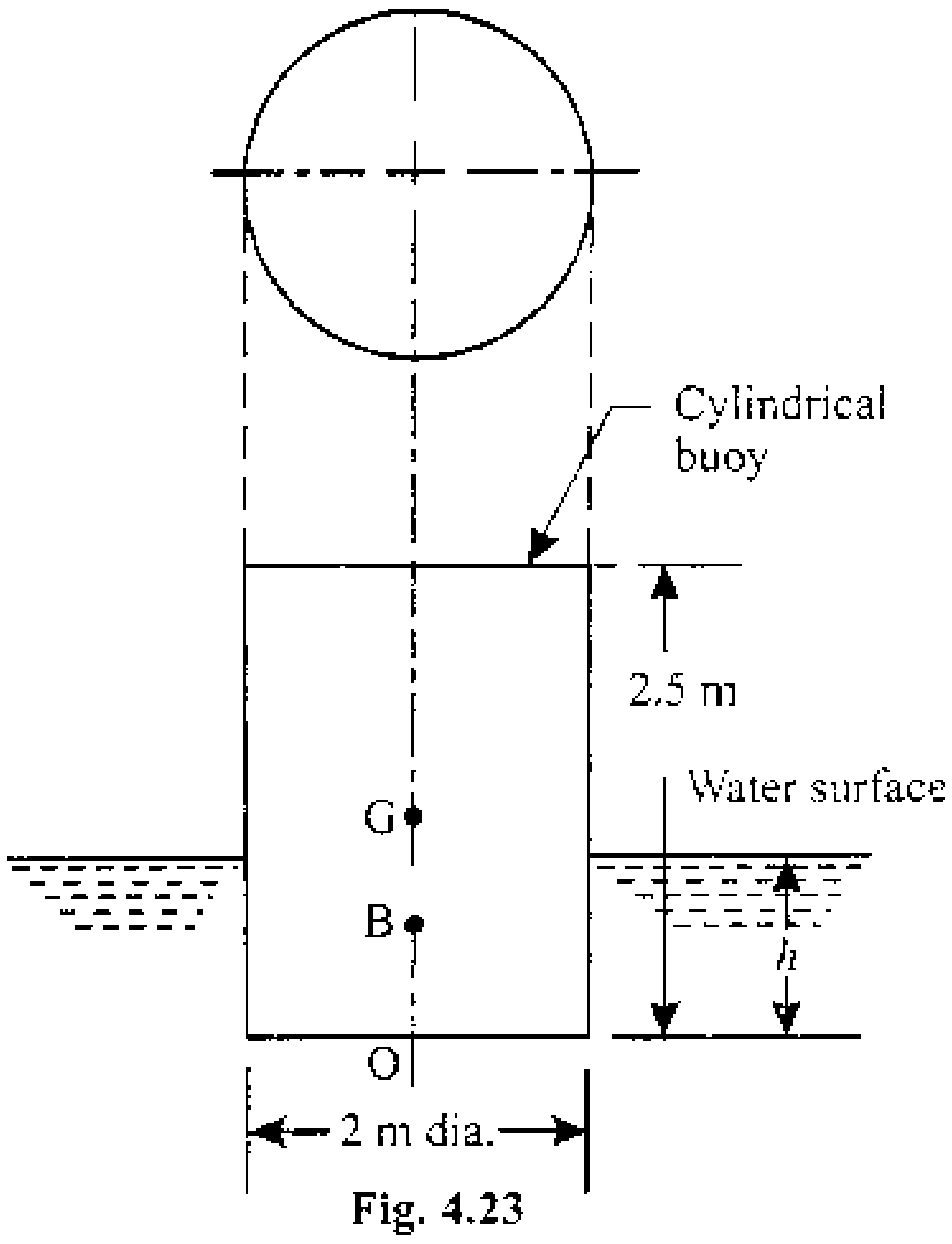


Fig. 4.23

Part I: *To show that the buoy does not float with its axis vertical:*

$$V = \frac{\text{Weight of water displaced}}{\text{Specific weight of water}}$$

(Weight of the buoy = Weight of water displaced)

$$= \frac{22}{10.25} = 2.146 \text{ m}^3$$

i.e., Volume of buoy immersed in water

$$= 2.146 \text{ m}^3$$

Let h = depth of immersion.

Then, $$h = \frac{\text{volume of buoy immersed in water}}{\text{cross-sectional area of the buoy}}$$

$$= \frac{2.146}{(\pi/4) \times 2^2} = 0.683 \text{ m}$$

Distance of centre of buoyance (B) from the base O,

$$OB = \frac{h}{2} = \frac{0.683}{2} = 0.342 \text{ m}$$

Let M be the metacentre,

Then $$BM = \frac{I}{V} = \frac{(\pi/64) \times 2^4}{2.146} = 0.366 \text{ m}$$

$\therefore OM = OB + BM = 0.342 + 0.366 = 0.708$ m

Distance of centre of gravity (G) from the base point O, $OG = \frac{2.5}{2} = 1.25$ m

Since $OM < OG$, therefore, the buoy is **unstable** when floating with axis vertical. **(Ans.)**

Part II: *Minimum pull required to keep the buoy vertical:*

Let T = minimum pull (kN) which should be applied to chain attached to the centre of the base to keep the buoy vertical.

Total downward force = $W + T = (22 + T)$

Displaced volume of water $= \left(\frac{22 + T}{10.25}\right)$ m³

Depth of immersion, $$h' = \frac{22 + T}{10.25 \times (\pi/4) \times 2^2} = \frac{22 + T}{32.2} \text{ m}$$

$$\therefore \quad OB' = \frac{h'}{2} = \frac{22 + T}{2 \times 32.2} = \frac{22 + T}{64.4} \text{ m}$$

$$B'M' = \frac{I}{V} = \frac{(\pi/64) \times 2^4}{\frac{22 + T}{10.25}} = \frac{8.05}{22 + T}$$

To find new centre of gravity G' due to self weight acting at G and tension T in the chain, taking moments about point O, we get

$$22 \times 1.25 = (22 + T) \times OG'$$

$$\therefore \quad OG' = \frac{22 \times 1.25}{22 + T}$$

$$B'G' = OG' - OB' = \frac{22 \times 1.25}{22 + T} - \frac{22 + T}{64.4}$$

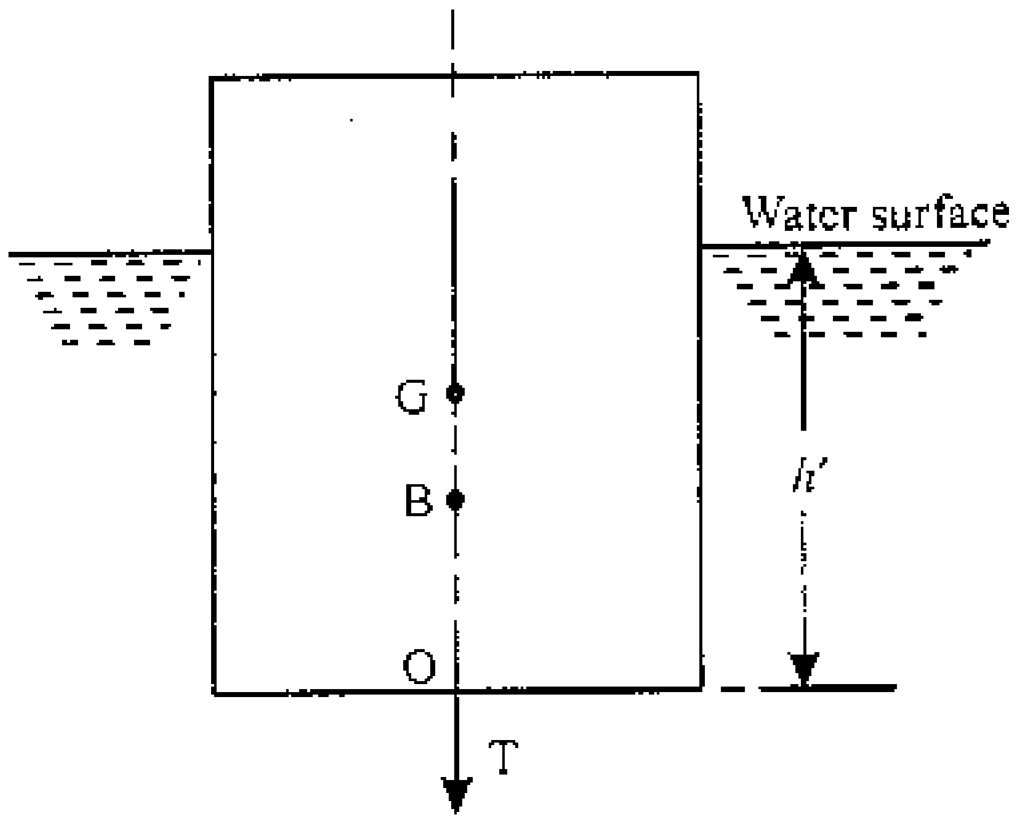


Fig. 4.24

For stable equilibrium M' must lie above G', *i.e.*

$$B'M' > B'G'$$

$$\therefore \quad \frac{8.05}{22 + T} > \left[\frac{22 \times 1.25}{22 + T} - \frac{22 + T}{64.4}\right]$$

or $$\left[\frac{8.05}{22 + T} + \frac{22 + T}{64.4}\right] > \frac{22 \times 1.25}{22 + T}$$

or $$\frac{8.05 \times 64.4 + (22 + T)^2}{(22 + T) \times 64.4} > \frac{22 \times 1.25}{22 + T}$$

or $$518.42 + (22 + T)^2 > 22 \times 1.25 \times 64.4$$

or $$(22 + T)^2 > [22 \times 1.25 \times 64.4 - 518.42] \quad \text{or} \quad > 1252.58$$

or $$22 + T > 35.39 \text{ or } T > 13.39 \text{ kN}$$

Hence, *minimum pull in the chain required to keep the buoy vertical* = **13.39 kN (Ans.)**

Example 4.24. *A solid cone floats in a liquid with its apex downwards. The specific gravity of the material of the cone and the liquid are* 0.7 *and* 0.95 *respectively. Determine the least apex angle of cone for stable equilibrium.*

Solution. Specific weight of cone = 0.7 × 9.81 = 6.87 kN/m^2

Specific weight of liquid = 0.95 × 9.81 = 9.32 kN/m^2

Let, H = Height of the cone,

h = Height of cone immersed in liquid, and

2α = Apex angle of the cone.

Weight of the cone = Volume × sp. weight

$$= \frac{1}{3}\pi R^2 H \times 6.87 = \frac{1}{3}\pi H^3 \tan^2\alpha \times 6.87 \text{ kN}$$

$$\left[\because \tan\alpha = \frac{R}{H} \; i.e. \; R = H \tan\alpha \text{ and } r = h \tan\alpha\right]$$

Weight of liquid displaced $= \frac{1}{3}\pi r^2 h \times 9.32 = \frac{1}{3}\pi h^3 \tan^2\alpha \times 9.32$ kN

Now, weight of cone = weight of liquid displaced (since the cone is floating)

i.e., $\frac{1}{3}\pi H^3 \tan^2\alpha \times 6.87 = \frac{1}{3}\pi h^3 \tan^2\alpha \times 9.32$

$$\therefore \quad l = H\left[\frac{6.87}{9.32}\right]^{1/3} = 0.9\ H$$

...ce of centre of buoyancy from the apex, $OB = \frac{3}{4}h = \frac{3}{4} \times 0.9H = \mathbf{0.675H}$

...ce of centre of gravity G from the apex, $OG = \frac{3}{4}H = 0.75H$

...ble equilibrium, the metacentre (M) should ... G or may *coincide* with G.

$BG \le BM$ or $OG - OB \le BM$ ...(*i*)

$$BM = \frac{I}{V}$$

..., I = moment of inertia of the circular sec- ... the liquid level

$$= \frac{\pi r^4}{4} = \frac{\pi \times h^4 \tan^4 \alpha}{4}$$

V = volume of liquid displaced

$$= \frac{1}{3}\pi h^3 \tan^2 \alpha$$

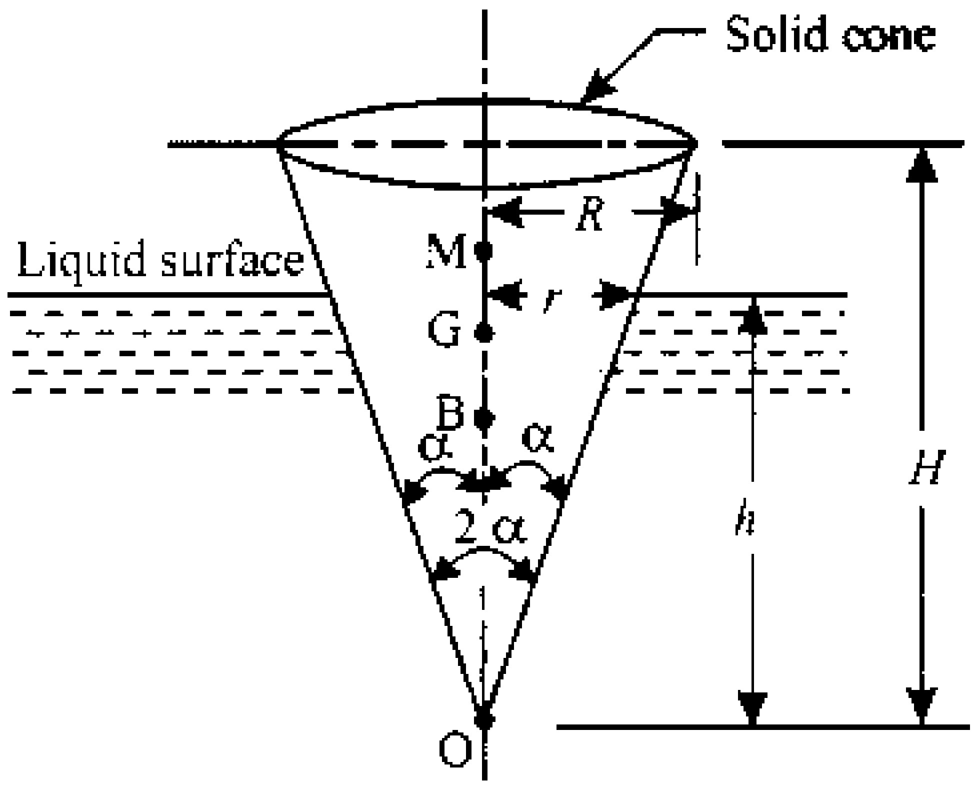


Fig. 4.25

...ting various values in (*i*), we get

$$0.75\ H - 0.675\ H \le \frac{\left(\frac{\pi h^4 \tan^4 \alpha}{4}\right)}{\left(\frac{1}{3}\pi h^3 \tan^2 \alpha\right)} \quad \text{or} \quad 0.75\ H \le \frac{h^4 \tan^4 \alpha}{4} \times \frac{3}{\pi h^3 \tan^2 \alpha}$$

$\therefore$ $0.075\ H \le 0.75\ h \tan^2\alpha$

$0.075\ H \le 0.75 \times (0.9\ H) \tan^2\alpha \quad (\because\ h = 0.9\ H)$

$$\tan^2 \alpha \ge \frac{0.75}{0.75 \times 0.9} \ge 0.111$$

$\tan \alpha \ge 0.333$ or $\alpha \ge 18° 24'$

...st apex angle, 2α = **36° 48′ (Ans.)**

...mple 4.25. *A cone of specific gravity S, is floating in water with its apex downwards. It has ... and vertical height H. Show that for stable equilibrium of cone,*

$$(i)\ \sec^2 \alpha > \frac{H}{h} \qquad (ii)\quad H < \left[\frac{R^2 S^{1/3}}{1 - S^{1/3}}\right]^{1/2}$$

... h is the depth of immersion and α is the half apex angle.

...on. Radius of the cone = R; height of the cone = H;

...gravity of the cone = S

h = Depth of immersion, 2α = apex angle,

r = Radius of cone at the water surface, O = apex of the cone,

G = $C.g.$ of the cone,

B = Centre of buoyancy, and

M = Position of metacentre.

$OG = 3/4\ H$; $OB = 3/4\ h$

$\sec^2\alpha > \frac{\mathbf{H}}{\mathbf{h}}$:

where, I = Moment of inertia = $\dfrac{\pi r^4}{4}$,

and

V = Volume of water displaced

$= \dfrac{1}{3}\pi r^2 h$

$$BM = \frac{\frac{\pi r^4}{4}}{\frac{1}{3}\pi r^2 h} = \frac{3}{4} \times \frac{r^2}{h}$$

Substituting $r = h \tan \alpha$, we get

$$BM = \frac{3}{4} \times \frac{h^2 \tan^2 \alpha}{h}$$

$$= \frac{3}{4}\, h \tan^2 \alpha \left(\because \frac{r}{h} = \tan \alpha \right)$$

$$OM = OB + BM = \frac{3}{4}h + \frac{3}{4}h \tan^2 \alpha$$

$$= \frac{3}{4}h(1 + \tan^2 \alpha) = \frac{3}{4}h \sec^2 \alpha$$

For stable equilibrium,

$BM > BG$ or $OM > OG$

$$= \frac{3}{4}h \sec^2 \alpha > \frac{3}{4}H \text{ or } Sec^2\alpha > \frac{H}{h}$$

...**Proved (Ans.)**

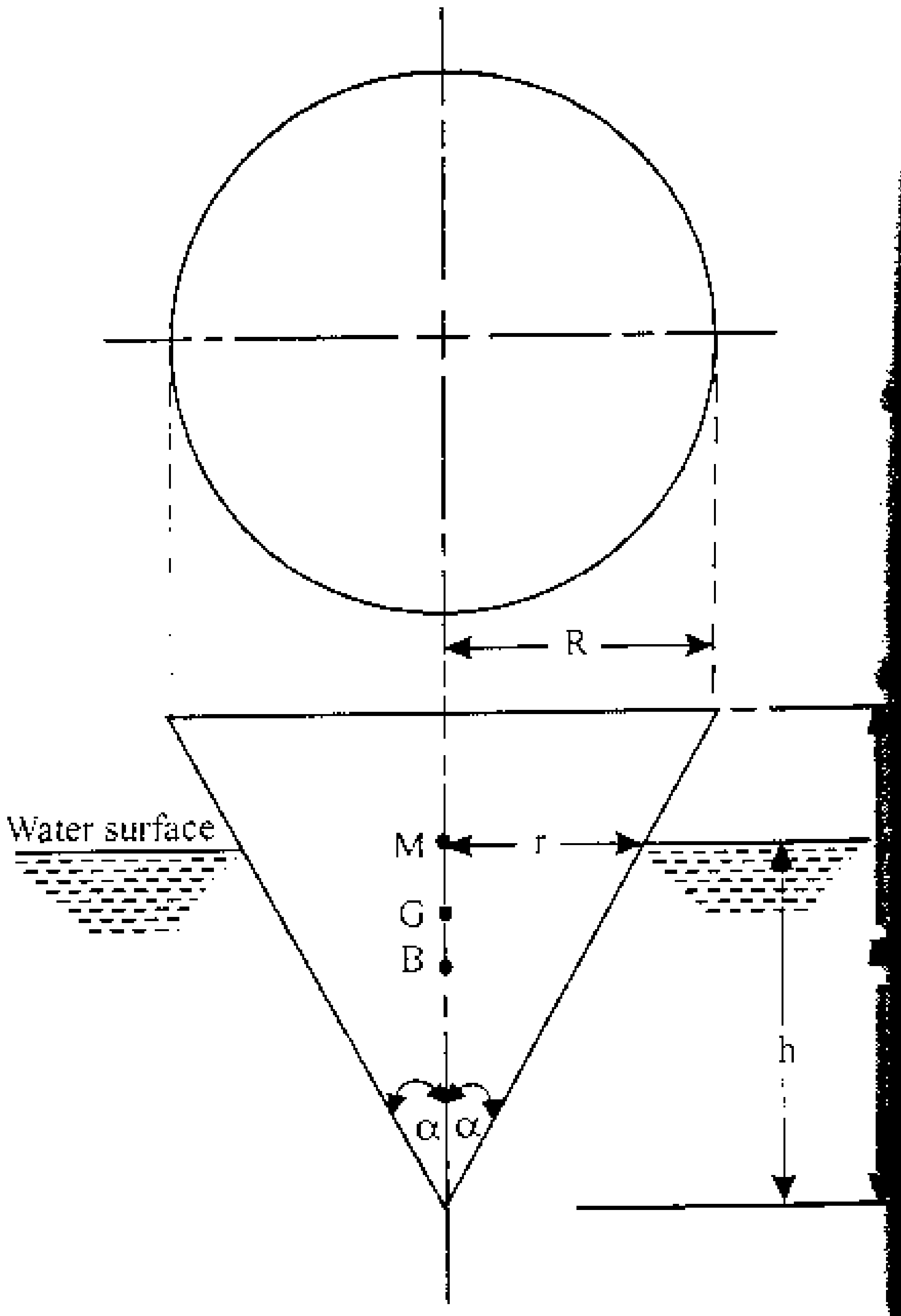


Fig. 4.26

(*ii*) $\mathbf{H} < \left(\dfrac{\mathbf{R^2 S^{1/3}}}{\mathbf{1 - S^{1/3}}} \right)^{1/2}$:

As per principle of floatation,

Weight of cone = weight of water displaced or $\dfrac{1}{3}\pi R^2 H \times w_c = \dfrac{1}{3}\pi r^2 h \times w$

(where w_c and w are the specific weights of cone material and water respectively).

Substituting $R = H \tan \alpha$ and $r = h \tan \alpha$, we get

$$\frac{1}{3}\pi H^2 \tan^2 \alpha \times H \times w_c = \frac{1}{3}\pi h^2 \tan^2 \alpha \times h \times w$$

$$H^3 \tan^2\alpha \times S = h^3 \tan^2\alpha \qquad \left(\because \frac{w_c}{w} = S \right.$$

or $h^3 = H^3 S$ or $h = H.S^{1/3}$

(where S is the sp. gravity of cone material).

From the relations $\text{Sec}^2\alpha > \dfrac{H}{h}$ and $h = H.S^{1/3}$ (derived above), we have $\text{Sec}^2\alpha > \dfrac{1}{S^{1/3}}$

or $(1 + \tan^2 \alpha) > \dfrac{1}{S^{1/3}}$ or $\tan^2\alpha > \dfrac{1}{S^{1/3}} - 1$ or $\tan^2 \alpha > \dfrac{1 - S^{1/3}}{S^{1/3}}$

Substituting $\tan \alpha = \dfrac{R}{H}$, we get $\dfrac{R^2}{H^2} > \dfrac{1 - S^{1/3}}{S^{1/3}}$

$$\text{or} \quad \frac{H^2}{R^2} < \frac{S^{1/3}}{1 - S^{1/3}} \quad \text{or} \quad H^2 < \frac{R^2 S^{1/3}}{1 - S^{1/3}}$$

$$\text{or} \quad H < \left(\frac{R^2 S^{1/3}}{1 - S^{1/3}}\right)^{1/2} \quad \text{...Proved (Ans.)}$$

Example 4.26. *A solid cone (S = 0.8) diameter 36 cm and height 30 cm floats with its vertex downwards in water as shown in fig. 4.27. Is this cone in stable equilibrium?*

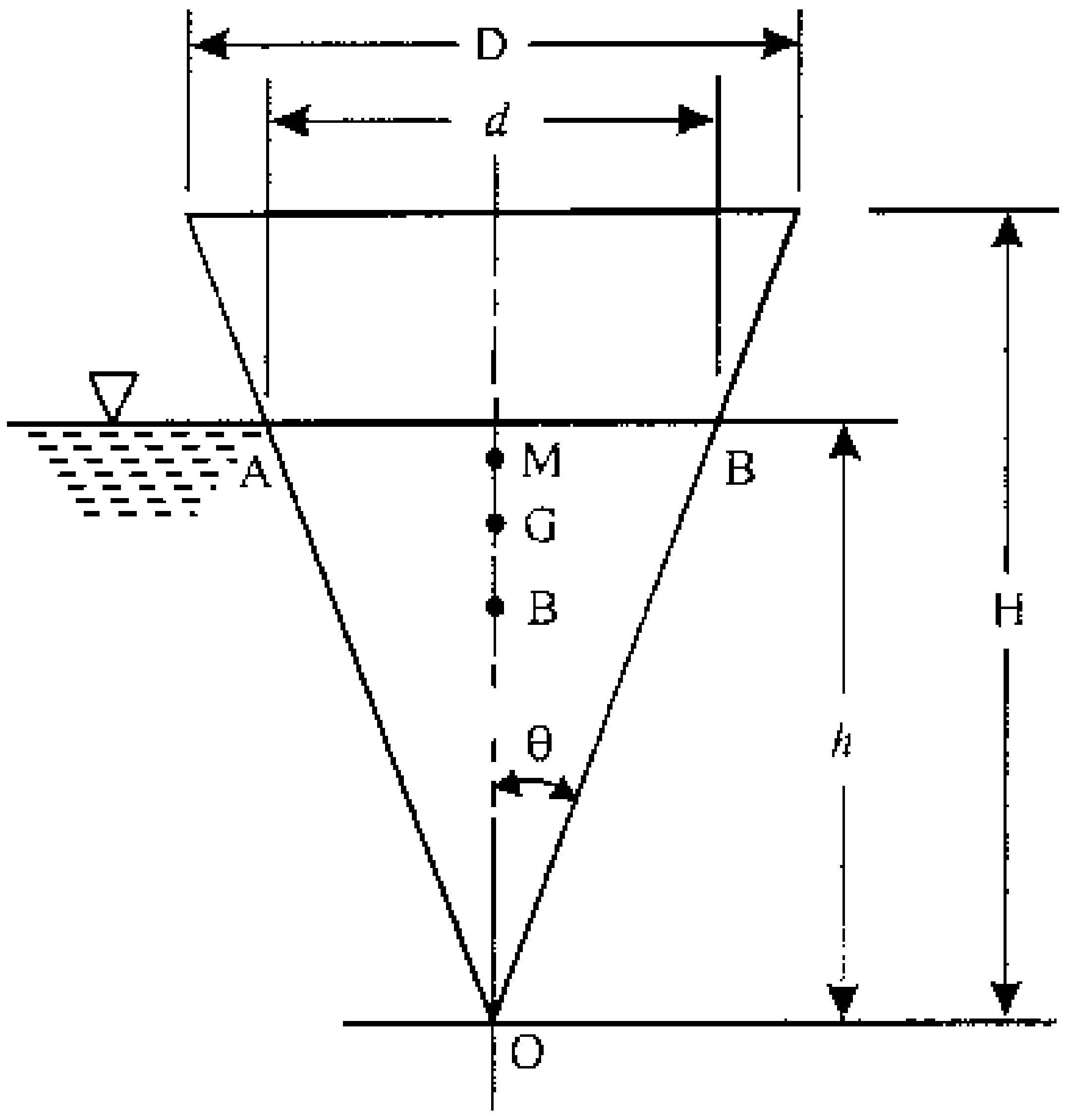


Fig. 4.27

Solution. *Given:* D = 36 *cm*, *H* = 30 *cm*; S = 0.8

Let, θ = Semivertex angle,

Then, $\tan\theta = \frac{18}{30} = 0.6$

∴ $\theta = \tan^{-1}(0.6) = 30.96°$

Diameter of the cone at water surface,

$$d = 2y \tan\theta \quad \left(\because \frac{d/2}{h} = \tan\theta\right)$$

Weight of cone = Weight of water displaced

$$\frac{1}{3} \times \pi \left(\frac{D}{2}\right)^2 \times H \times (w \times S) = \frac{1}{3}\pi\left(\frac{d}{2}\right)^2 \times h \times w$$

$$\therefore \quad D^2HS = d^2y$$

$$= (2h\tan\theta)^2 \times h = 4h^3\tan^2\theta$$

$$= 4h^3\left(\frac{D}{2H}\right)^2 \qquad \left(\because \frac{D/2}{H} = \tan\theta\right)$$

or, $h^3 = H^3S$

or, $h = HS^{1/3}$

$$= 30 \times (0.8)^{1/3} = 27.85 \text{ cm}$$

If *B* is the centre of buoyancy,

$$OB = \frac{3}{4}h = \frac{3}{4} \times 27.85 = 20.89 \text{ cm}$$

$$OG = \frac{3}{4}H = \frac{3}{4} \times 30 = 22.5 \text{ cm}$$

Now, $d = 2h\tan\theta = 2 \times 27.85 \times 0.6 = 33.42$ cm

$$BM = \frac{I}{V} = \frac{(\pi d^4)/64}{\frac{1}{3}\pi\left(\frac{d}{2}\right)^2 \times h} = \frac{3}{16}\left(\frac{d^2}{h}\right)$$

$$= \frac{3}{16} \times \frac{(33.42)^2}{27.85} = 7.52 \text{ cm}$$

$$OM = OB + BM$$

$$= 20.89 + 7.52 = 28.41 \text{ cm}$$

$$OG = 22.5 \text{ cm}$$

$MG = OM - OG = 28.41 - 22.5 = 5.91$ cm

i.e., M is above G by 5.91 cm

Hence the cone is under **stable equilibrium. (Ans.)**

Example 4.27. *A ship 63 m long and 9 m broad has a displacement of 16000 kN. When a weight of 200 kN is moved across the deck through a distance of 5.4 m, the ship is tilted through 5°. The second moment of area of the water line section about its fore–and–oft axis is 75 per cent of that of circumscribing rectangle, and centre of buoyancy is 2.1m below the water line. Determine.*

(*i*) *The metacentric height, and* (*ii*) *The position of centre of gravity of ship.*

Take specific weight of sea water = 10.25 kN/m³

Solution. Length of the ship, $l = 63$ m

Breadth of the ship, $b = 9$ m

Displacement, $W = 16000$ kN

Angle of tilt, $\theta = 5°$

Movable weight, $W_1 = 200$ kN

Distance moved by W_1, $z = 5.4$ m

(*i*) **Metacentric height, GM:**

We know, $GM = \dfrac{W_1 . z}{W \tan\theta}$ (eqn. 4.2)

$$= \frac{200 \times 5.4}{16000 \times \tan 5°} = 0.77 \text{m}$$

i.e. GM = **0.77 m (Ans.)**

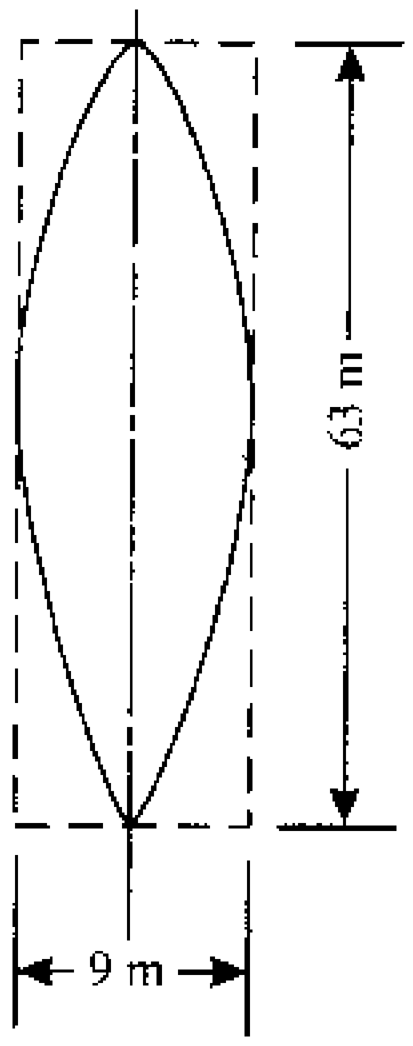


Fig. 4.28

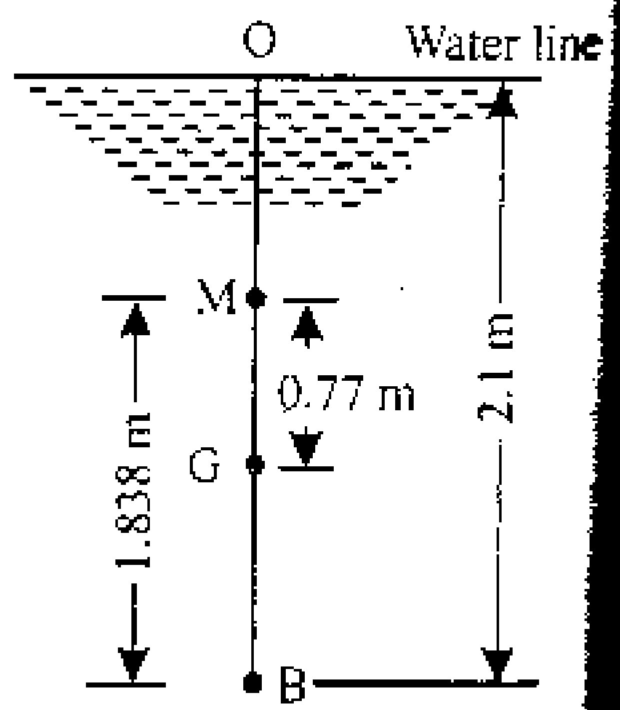


Fig. 4.29

(*ii*) **The position of centre of gravity of the ship:**

Distance between the metacentre M and the centre of buoyancy is given by, $BM = \dfrac{I}{V}$

where, I = Second moment of area of the water line section

$$= 0.75 \times \left(\frac{63 \times 9^3}{12}\right) = 2870 \ m^4$$

and, V = Volume of water displaced by the vessel

$$= \frac{\text{Weight of the vessel}}{\text{Specific weight of vessel}} = \frac{16000}{10.25} = 1561 \text{ m}^3$$

$\therefore$ $BM = \dfrac{2870}{1561} = 1.838 \text{m}$

Now, $OG = OM + MG = (OB - MB) - MG$

$= (2.1 - 1.838) + 0.77 = 1.032$ m

i.e. OG = **1.032 m** (below the water line) **(Ans.)**

4.6. Oscillation (Rolling) of a Floating Body

It has been observed that whenever a body, floating in a liquid is given a small angular displacement, it starts oscillating about its metacentre M (see Fig. 4.30) in the same manner as a pendulum oscillates about its point of suspension.

Let, W = Weight of floating body,

θ = Angle (in radians) through which the body is depressed,

α = Angular acceleration of the body in rad./s^2,

T = Time of rolling (*i.e.* one complete oscillation) in seconds,

k = Radius of gyration about G, and

I = Moment of inertia of the body about its centre of gravity G

$= \frac{W}{g} k^2$

GM = Metacentric height of the body.

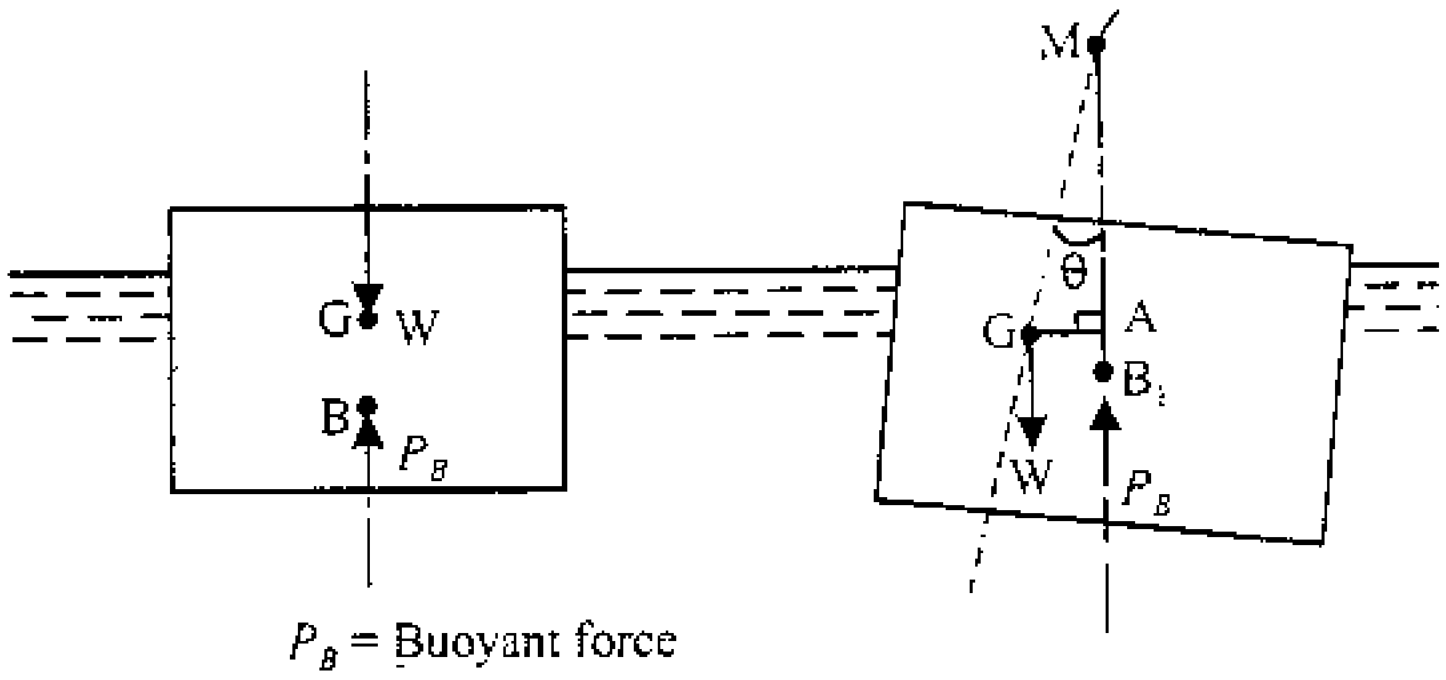


Fig. 4.30

When the force which has caused angular displacement is removed the only force acting on the is due to the restoring couple due to the weight W of the body and the force of buoyancy P_B.

$\therefore$ *Restoring couple* $= W \times GA$

$$= W \times GM \tan\theta \qquad \ldots(i)$$

$$= W \cdot GM \cdot \theta$$

[assuming θ to be small ($\tan\theta = \theta$)]

Angular acceleration of the body, $\alpha = -\frac{d^2\theta}{dt^2}$

–ve sign indicates that the force is acting in such a way that it tends to *decrease* the angle θ.

Also, *inertia torque* = moment of inertia × angular acceleration

$$= I.\alpha = -\frac{W}{g} k^2 \times \frac{d^2\theta}{dt^2} \qquad \ldots(ii)$$

Equating (*i*) and (*ii*), we get $W.\ GM.\theta = -\frac{W}{g} k^2 \times \frac{d^2\theta}{dt^2}$ or $\frac{W}{g} k^2 \frac{d^2\theta}{dt^2} + W.GM.\theta = 0$

Dividing both sides by W, we get $\frac{k^2}{g} \times \frac{d^2\theta}{dt^2} + GM.\theta = 0$

Again, dividing both sides by $\frac{k^2}{g}$, we get $\frac{d^2\theta}{dt^2} + \frac{GM.g\theta}{k^2} = 0$

The above equation is a differential equation of second degree, whose solution is:

$$Q = C_1 \sin\left[\sqrt{\frac{GM.g}{k^2}} \times t\right] + C_2 \cos\left[\sqrt{\frac{GM.g}{k^2}} \times t\right] \qquad \ldots(iii)$$

where C_1 and C_2 are constants of integration.

The values of C_1 and C_2 are obtained from the following boundary conditions:

1. At $t = 0, \theta = 0$

$C_2 = 0$ [By substitution of $t = 0, \theta = 0$ in (*iii*)]

2. At $t = \frac{T}{2}, \theta = 0$

$$\therefore \qquad 0 = C_1 \sin\left[\sqrt{\frac{GM.g}{k^2}} \times \frac{T}{2}\right]$$

Since C_1 cannot be equal to zero, therefore

$$\sin\left[\sqrt{\frac{GM.g}{k^2}} \times \frac{T}{2}\right] = 0 \quad \text{or} \quad \sqrt{\frac{GM.g}{k^2}} \times \frac{T}{2} = \pi \qquad (\because \sin \pi = 0)$$

or
$$T = 2\pi\sqrt{\frac{k^2}{GM.g}} \qquad \text{...(4.4)}$$

Example 4.28. *A ship of weight 32000 kN is floating in sea water. The centre of buoyancy is 1.6 meters below its centre of gravity. The moment of inertia of the ship area at the water level is 8320 m^4. If the radius of gyration of the ship is 3.2 m find its period of rolling.*

Take sp. weight of sea water = 10.1 kN/m³

Solution. *Given:* Weight of the ship, $W = 32000$ kN

Distance between centre of buoyancy and centre of gravity, $BG = 1.6\ m$

Moment of inertia, $I = 8320\ m^4$

Radius of gyration, $k = 3.2\ m$

Period of rolling of the ship, *T*:

Volume of sea water displaced,

$$V = \frac{\text{weight}}{\text{specific weight of sea water}} = \frac{32000}{10.1} = 3168.3\,m^3$$

Using the relation, $BM = \frac{I}{V} = \frac{8320}{3168.3} = 2.626\,m$

Also, the metacentric height, $GM = BM - BG = 2.626 - 1.6 = 1.026\ m$

Now using the relation, $T = 2\pi\sqrt{\frac{k^2}{GM.g}} = 2\pi\sqrt{\frac{3.2^2}{1.028 \times 9.81}} = 6.33\,\text{s}$ **(Ans.)**

HIGHLIGHTS

1. The tendency for an immersed body to be lifted up in the fluid, due to an upward fo[rce] opposite to action of gravity is known as *buoyancy*.
2. The floating bodies may have the following types of equilibrium:
   (*i*) Stable equilibrium,
   (*ii*) Unstable equilibrium, and
   (*iii*) Neutral equilibrium.
3. The metacentre is defined as a point of intersection of the axis of body passing through c[entre of gravity] (G) and original centre of buoyancy (B), and a vertical line passing through the centre [of] buoyancy (B_1) of the titled position of the body.
4. The distance between the centre of gravity (G) of a floating body and the metacentre (M) is called *metacentric height*.
5. The metacentric height (GM) by experimental method is given by:

$$GM = \frac{W_1.z.l}{W.d}\left(= \frac{W_1.z}{W\tan\theta}\right)$$

where, W_1 = Known weight,

z = Distance through which W_1 is shifted across the axis of the tilt,

l = Displacement of the plumb bob, and

θ = Angle of tilt $\left(\tan\theta = \frac{d}{l}\right)$.

6. Time of rolling, $T = 2\pi\sqrt{\frac{k^2}{GM.g}}$

where, k = Radius of gyration about *c.g.* (G), and

GM = Metacentric height of the body.

OBJECTIVE TYPE QUESTIONS

Choose the Correct Answer.

1. The tendency for an immersed body to be lifted up in the fluid, due to an upward force opposite to the action of gravity is known as
   (*a*) buoyancy
   (*b*) centre of buoyancy
   (*c*) buoyant force
   (*d*) none of the above.
2. The magnitude of the buoyant force can be determined by
   (*a*) Newton's second law of motion
   (*b*) Archimedes' principles
   (*c*) Principle of moments
   (*d*) none of the above.
3. When a body is immersed in a fluid, partially or completely, the force of buoyancy is equal to
   (*a*) the weight of the body
   (*b*) the weight of the fluid displaced by the body
   (*c*) the weight of the volume of the fluid equal to the volume of body
   (*d*) none of the above.
4. The point of application of the force of buoyancy on the body is known as
   (*a*) centre of gravity
   (*b*) centre of buoyancy
   (*c*) metacentre
   (*d*) none of the above.
5. "When a body is immersed in a fluid either wholly or partially, it is buoyed or lifted up by a force which is equal to the weight of fluid displaced by the body".
   This principle was enunciated by
   (*a*) Archimedes (*b*) Newton
   (*c*) Pascal (*d*) Kirchhoff.
6. A floating body is in stable equilibrium when
   (*a*) the metacentre is below its centre of gravity
   (*b*) the metacentre is above its centre of gravity
   (*c*) the metacentric height is zero.
   (*d*) its centre of gravity is below the centre of buoyancy.
7. An ice-cube is floating in glass of water. As the cube melts the water level
   (*a*) remain constant (*b*) falls
   (*c*) rises (*d*) none of the above.
8. If the position of metacentre M remains lower than *c.g.* of the body, G, the body will remain in a state of
   (*a*) stable equilibrium
   (*b*) unstable equilibrium
   (*c*) neutral equilibrium
   (*d*) any of the above.
9. Metacentric height can be determined by
   (*a*) only analytical method
   (*b*) only experimental method
   (*c*) both (*a*) and (*b*)
   (*d*) none of the above.
10. If a body does not return to its original position from the slightly displaced angular position and heels farther away, when given a small angular displacement; such an equilibrium is called
    (*a*) stable equilibrium
    (*b*) unstable equilibrium
    (*c*) neutral equilibrium
    (*d*) any of the above.