CHAPTER

5

FLUID KINEMATICS

5.1. Introduction. 5.2. Description of fluid motion.—Langrangian method— Eulerian method. 5.3. Types of fluid flow—steady and unsteady flows—uniform and non-uniform flows–one, two and three dimensional flows—rotational and irrotational flows—laminar and turbulent flows—compressible and incompressible flows. 5.4. Types of flow lines-path line—stream line—stream tube—streak line— 5.5. Rate of flow or discharge. 5.6. Continuity equation. 5.7. Continuity equation in cartesian co-ordinates. 5.8. Equation of continuity in polar co-ordinates. 5.9. Circulation and vorticity. 5.10. Velocity potential and stream function—velocity potential—stream function—relation between stream function and velocity potential. 5.10. Flow nets—methods of drawing the flow nets—uses and limitations of flow nets. Highlights—Theoretical Questions—Objective Type Questions—Unsolved Examples.

5.1. Introduction

Fluid kinematics may be defined as follows:

"***Fluid kinematics** is a branch of 'Fluid mechanics' which deals with the study of velocity and acceleration of the particles of fluids in motion and their distribution in space without considering any force or energy involved.*

The motion of fluid can be described fully by an expression describing the location of a fluid particle in space at different times thus enabling determination of the magnitude and direction of velocity and acceleration in the flow field at any instant of time.

In the chapter we shall deal with the conception of fluid flow in general.

5.2. Description of Fluid Motion

The motion of fluid particles may be described by the following methods:

1. Langrangian method.
2. Eulerian method.

5.2.1. Langrangian Method

In this method, the observer *concentrates on the movement of a single particle*. The path taken by the particle and the changes in its velocity and acceleration are studied.

In the Cartesian system, the position of the fluid particle in space (x, y, z) at any time t from a position (a, b, c) at time $t = 0$ shall be given as:

$$x = f_1\,(a, b, c, t)$$
$$y = f_2\,(a, b, c, t)$$
$$z = f_3\,(a, b, c, t)$$

The velocity and acceleration components (obtained by taking derivatives with respect to time) are given by:

Velocity components:

$$\left.\begin{aligned} u &= \frac{\partial x}{\partial t} \\ v &= \frac{\partial y}{\partial t} \\ w &= \frac{\partial z}{\partial t} \end{aligned}\right\} \quad \text{...(5.2)}$$

Acceleration components :

$$\left.\begin{aligned} a_x &= \frac{\partial^2 x}{\partial t^2} \\ a_y &= \frac{\partial^2 y}{\partial t^2} \\ a_z &= \frac{\partial^2 z}{\partial t^2} \end{aligned}\right\} \quad \text{...(5.3)}$$

At any point, the resultant velocity or acceleration shall be the *resultant* of three components of respective quantity at that point.

$\therefore$ Resultant velocity, $$V = \sqrt{u^2 + v^2 + w^2} \quad \text{...(5.4)}$$

Acceleration, $$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad \text{...(5.5)}$$

Similarly, other quantities like pressure, density, etc. can be found.

This method entails the following *shortcomings:*

1. Cumbersome and complex.
2. The equations of motion are very difficult to solve and the motion is hard to understand.

5.2.2. Eulerian Method

In Eulerian method, the observer *concentrates on a point in the fluid system.* Velocity, acceleration other characteristics of the fluid at that particular point are studied.

This method is almost *exclusively used* in fluid mechanics, especially because of its mathematical *plicity.* In fluid mechanics, we are not concerned with the motion of each particle, but we study general state of motion at various points in the fluid system.

The velocities at any point (x, y, z) can be written as

$$\left.\begin{aligned} u &= f_1(x, y, z, t) \\ v &= f_2(x, y, z, t) \\ w &= f_3(x, y, z, t) \end{aligned}\right\} \quad \text{...(5.6)}$$

The components of acceleration of the fluid particle can be worked out by partial differentiation ollows:

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy + \frac{\partial u}{\partial z}.dz + \frac{\partial u}{\partial t}.dt$$

$$a_x = \frac{\partial u}{dt} = \left(\frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{dz}.\frac{dz}{dt} \right) + \frac{\partial u}{\partial t}.\frac{dt}{dt}$$

$$\left.\begin{aligned} a_x &= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) + \frac{\partial u}{\partial t} \\ \text{Similarly,} \quad a_y &= \frac{dv}{dt} = \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) + \frac{\partial v}{\partial t} \\ \text{and,} \quad a_z &= \frac{dw}{dt} = \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) + \frac{\partial w}{\partial t} \end{aligned}\right\} \quad \text{...(5.7)}$$

Now, resultant velocity: $V = \sqrt{u^2 + v^2 + w^2}$ ...(5.8)

Resultant acceleration, $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$ ...(5.9)

In vector notation,

Velocity vector: $V = ui + vj + wk$ ...(5.10)

Acceleration vector: $a = \dfrac{dV}{dt} = \left(u\dfrac{\partial V}{\partial x} + v\dfrac{\partial V}{\partial y} + w\dfrac{\partial V}{\partial z} \right) + \dfrac{\partial V}{\partial t}$ ...(5.11)

$$a = a_x i + a_y j + a_z k \quad ...(5.12)$$

and $|V| = \sqrt{u^2 + v^2 + w^2}$ ...(5.13)

$$|a| = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad ...(5.14)$$

Vectorially,

$$a = (V.\nabla)\, V + \frac{\partial V}{\partial t} \quad ...(5.15)$$

The velocity, in general, is a function of space(s) and time (t) *i.e.*

$$V = f(x, y, z, t)$$

or

$$V = f(s, t)$$

and the acceleration,

$$a = \frac{dV}{dt} = \frac{\partial V}{\partial s} \cdot \frac{ds}{dt} + \frac{\partial V}{\partial t}$$

$$\therefore \quad a = V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t} \quad ...(5.16)$$

Thus the acceleration consists of the two parts:

(*i*) $V\dfrac{\partial V}{\partial s}$: This part is due to change in position or movement and is called **convective acceleration.**

$$\therefore \text{ Convective acceleration } = V\frac{\partial V}{\partial s} = \frac{1}{2}\frac{\partial (V)^2}{\partial s} \quad ...(5.17)$$

= terms in the parenthesis of Eqn. (5.7)

(*ii*) $\dfrac{\partial V}{\partial t}$: This part is with respect to time at a given location and is called **local** (or *temporal*) **acceleration**.

$$\therefore \text{ Local acceleration } = \frac{\partial V}{\partial t} \quad ...(5.18)$$

$$= \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \text{ in Eqn. (5.7)}$$

Tangential and normal acceleration: Refer to Fig. 5.1

When the motion is curvilinear eqn. 5.16 gives the *tangential acceleration*. A particle moving in a curved path will always have a normal acceleration $a_n = \dfrac{V^2}{r}$ towards the centre of the curved path (r being the radius of the path), though its tangential acceleration (a_s) may be zero as in the case of uniform circular motion.

For motion along a curved path, in general,

$$a = a_s + a_n$$

$$= \left(V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t} + \frac{V^2}{r} \right) \quad ...(5.19)$$

5.3. Types of Fluid Flow

Fluids may be *classified* as follows:

1. Steady and unsteady flows
2. Uniform and non-uniform flows
3. One, two and three dimensional flows
4. Rotational and irrotational flows
5. Laminar and turbulent flows
6. Compressible and incompressible flows.

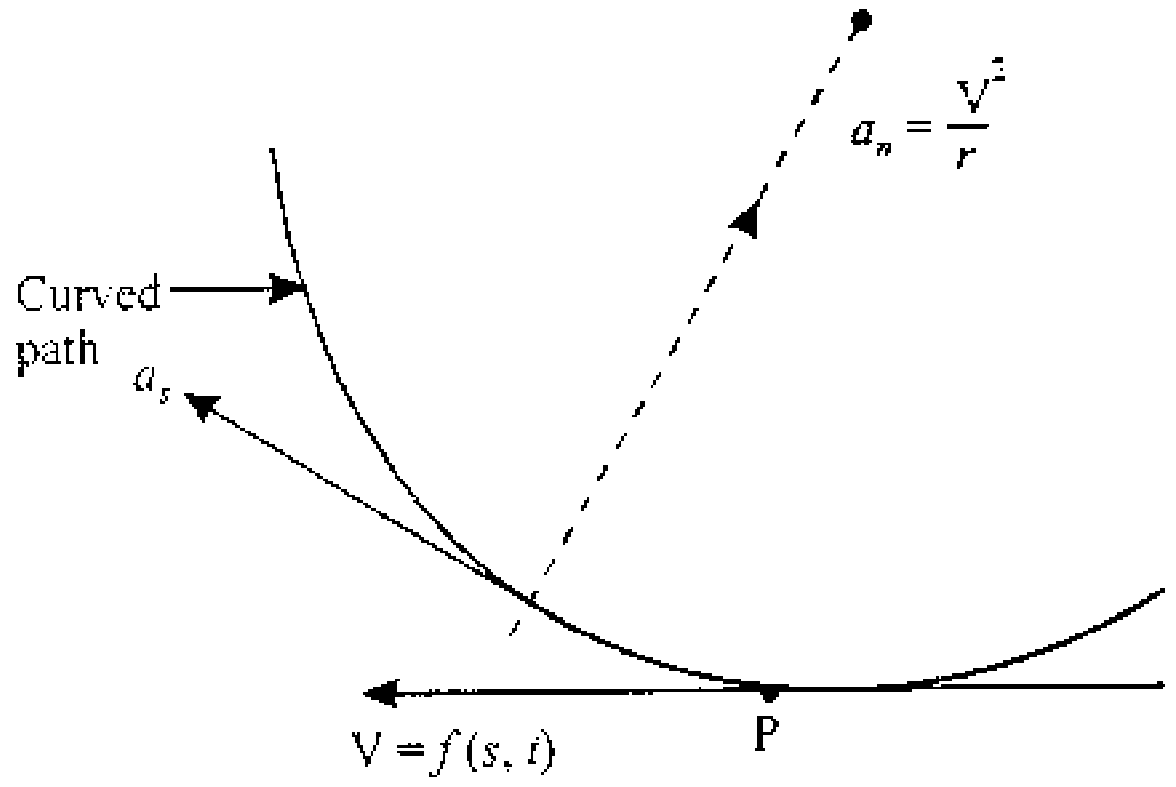

Fig. 5.1. Tangential and normal acceleration.

5.3.1. Steady and Unsteady Flows

Steady flow. The type of flow in which the fluid characteristics like velocity, pressure, density, etc. at a point *do not change* with time is called *steady flow*. Mathematically, we have

$$\left(\frac{\partial u}{\partial t}\right)_{x_0, y_0, z_0} = 0; \left(\frac{\partial v}{\partial t}\right)_{x_0, y_0, z_0} = 0; \left(\frac{\partial w}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

$$\left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0; \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0; \text{ and so on}$$

...re (x_0, y_0, z_0) is a fixed point in a fluid field where these variables are being measured *w.r.t.* ...e.

Example. *Flow through a prismatic or non-prismatic conduit at a constant flow rate Q m³/s is* ...*ady.*

(A prismatic conduit has a constant size shape and has a velocity equation in the form ...$= ax^2 + bx + c$ which is independent of time t).

Unsteady flow. It is that type of flow in which the velocity, pressure or density at a point ...*ge* w.r.t. time. Mathematically, we have

$$\left(\frac{\partial u}{\partial t}\right)_{x_0, y_0, z_0} \neq 0; \left(\frac{\partial v}{\partial t}\right)_{x_0, y_0, z_0} \neq 0; \left(\frac{\partial w}{\partial t}\right)_{x_0, y_0, z_0} \neq 0$$

$$\left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0; \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} \neq 0; \text{ and so on}$$

Example. *The flow in a pipe whose valve is being opened or closed gradually* (*velocity equation* ... *the form* $u = ax^2 + bxt$).

5.3.2. Uniform and Non-uniform Flows

Uniform flow. The type of flow, in which the velocity at any given time *does not change* with ...ect to space is called *uniform flow*. Mathematically, we have

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} = 0$$

where, ∂V = Change in velocity, and

∂s = Displacement in any direction.

Example. *Flow through a straight prismatic conduit* (*i.e. flow through a straight pipe of constant* ...*eter*).

Non-uniform flow. It is that type of flow in which the velocity at any given time *changes* with respect to space. Mathematically,

$$\left(\frac{\partial v}{\partial s}\right)_{t\,=\,\text{constant}} \neq 0$$

Example. (*i*) *Flow through a non-prismatic conduit.*

(*ii*) *Flow around a uniform diameter pipe-bend or a canal bend.*

5.3.3. One, Two and Three Dimensional Flows

One dimensional flow. It is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only. Mathematically,

$$u = f(x), v = 0 \text{ and } w = 0$$

where u, v and w are velocity components in x, y and z directions respectively.

Example. *Flow in a pipe where average flow parameters are considered for analysis.*

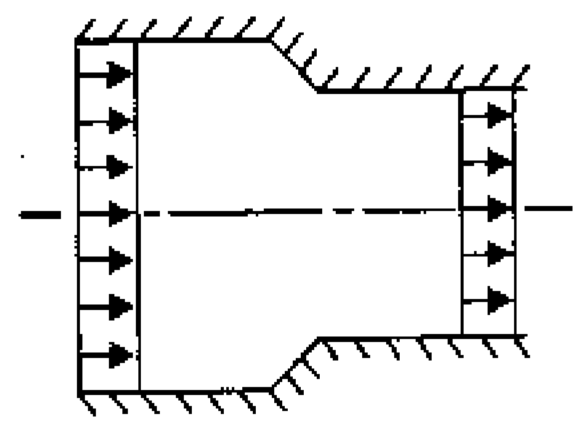

Fig. 5.2. One dimensional flow.

Two dimensional flow. The flow in which the velocity is a function of time and two rectangular space coordinates is called *two dimensional flow*. Mathematically,

$$u = f_1(x, y)$$
$$v = f_2(x, y)$$
$$w = 0$$

Examples. (*i*) *Flow between parallel plates of infinite extent.*

(*ii*) *Flow in the main stream of a wide river.*

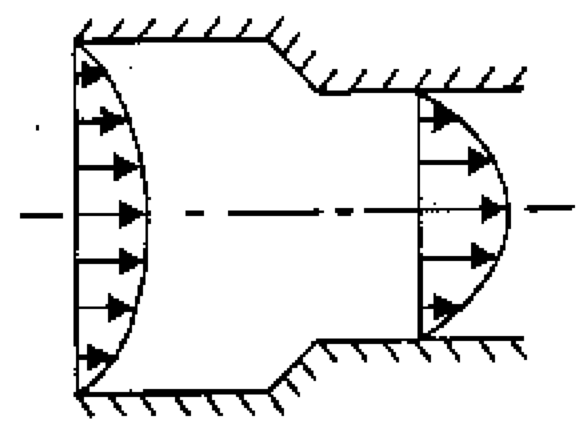

Fig. 5.3. Two dimensional flow.

Three dimensional flow. It is that type of flow in which the velocity is a function of time and three mutually perpendicular directions. Mathematically,

$$u = f_1(x, y, z)$$
$$v = f_2(x, y, z)$$
$$w = f_3(x, y, z)$$

Examples. (*i*) *Flow in a converging or diverging pipe or channel.*

(*ii*) *Flow in a prismatic open channel in which the width and the water depth are of the same order of magnitude.*

5.3.4. Rotational and Irrotational Flows

Rotational flow. A flow is said to be *rotational* if the fluid particles while moving in the direction of flow *rotate* about their mass centres. *Flow near the solid boundaries is rotational.*

Example. *Motion of liquid in a rotating tank.*

Irrotational flow. A flow is said to be *irrotational* if the fluid particles while moving in the direction of flow *do not rotate* about their mass centres. Flow outside the boundary layer is generally considered irrotational.

Example. *Flow above a drain hole of a stationary tank or a wash basin.*

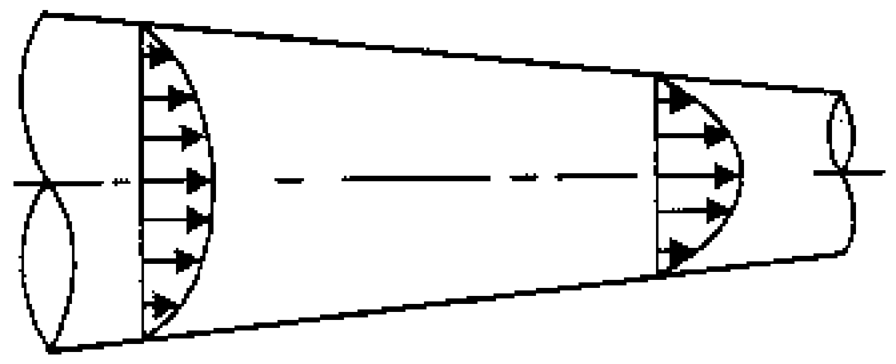

Fig. 5.4. Three dimensional flow.

Note. *If the flow is irrotational as well as steady*, it is known as ***Potential flow***.

5.3.5. Laminar and Turbulent Flows

Laminar flow. A laminar flow is one in which *paths taken by the individual particles do not cross one another and move along well defined paths* (*Fig.* 5.5), This type of flow is also called *stream-line flow or viscous flow.*

Examples. (*i*) *Flow through a capillary tube.*

(*ii*) *Flow of blood in veins and arteries.*

(*iii*) *Ground water flow.*

Turbulent flow. A turbulent flow is that flow in which fluid particles move in a *zig zag way* (*Fig* 5.6)

Example. *High velocity flow in a conduit of large size. Nearly all fluid flow problems encountered in engineering practice have a turbulent character.*

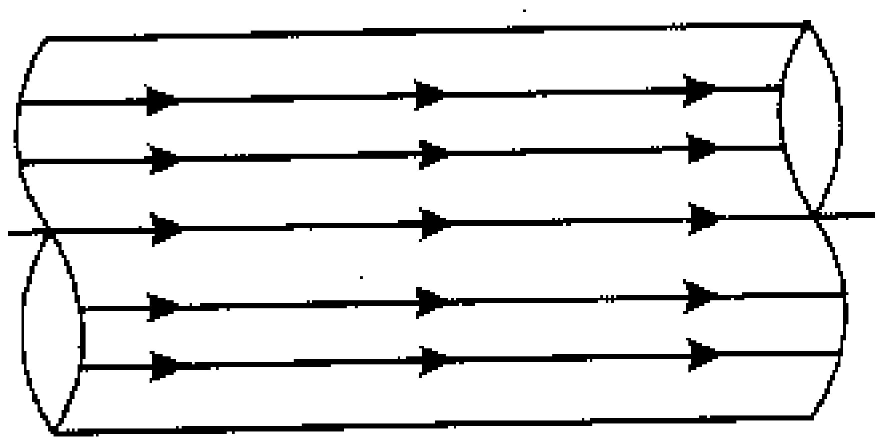

Fig. 5.5. Laminar flow.

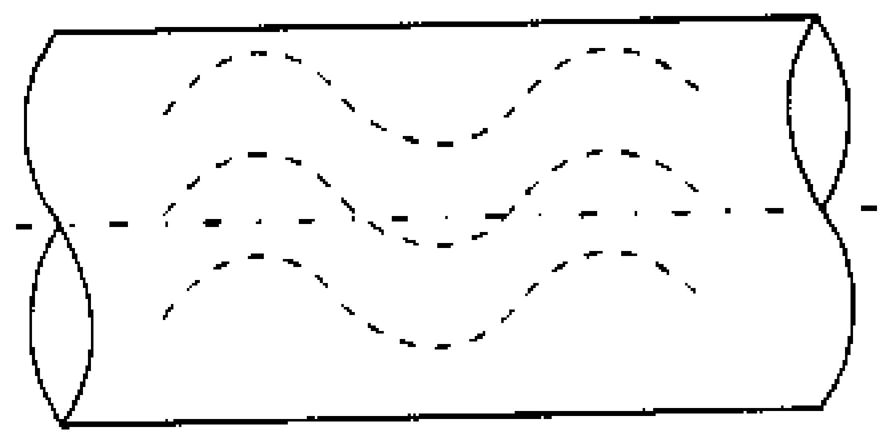

Fig. 5.6. Turbulent flow.

Laminar and turbulent flows are characterised on the basis of Reynolds number (refer chapter 10).

For Reynolds number (Re) < 2000 ... flow in pipes is *laminar*.

For Reynolds number (Re) > 4000 ... flow in pipes is *turbulent*

For R_e between 2000 and 4000 ... flow in pipes may be laminar or turbulent.

5.3.6. Compressible and Incompressible Flows

Compressible flow. It is that type of flow in which the *density* (ρ) *of the fluid changes* from [...]nt to point (or in other words *density is not constant for this flow*).

Mathematically, $\rho \neq$ constant.

Example. *Flow of gases through orifices, nozzles, gas turbines, etc.*

Incompressible flow. *It is that type of flow in which density is constant for the fluid flow. [...]ids are generally considered flowing incompressibly.*

Mathematically, ρ = constant.

Example. *Subsonic aerodynamics.*

Types of Flow Lines

Whenever a fluid is in motion, its innumerable particles move along certain lines depending [...] the conditions of flow. Although flow lines are of several types, yet some important from [...]ct point of view are given below:

5.4.1. Path line

A path line (Fig. 5.7) is the *path followed by a fluid particle in motion*. A path line shows the [...]ion of particular particle as it moves ahead. In general, this is the curve in three-dimensional [...]. However, if the conditions are such that [...] is two-dimensional the curve becomes [...]ensional.

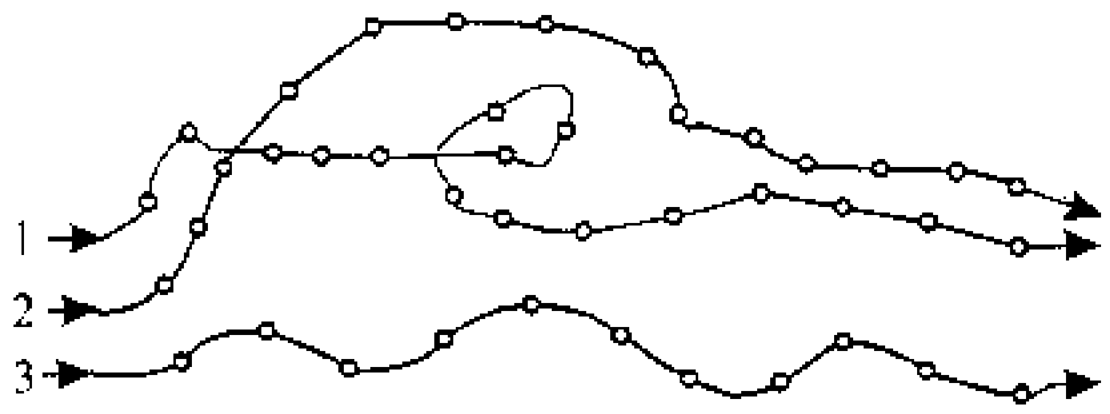

Fig. 5.7. Path lines.

5.4.2. Streamline

[...] *streamline* way be defined on as *an [...]ary line within the flow so that the [...] at any point on it indicates the velocity [...] point.*

[...]ation of a streamline in a three-dimensional flow is given as:

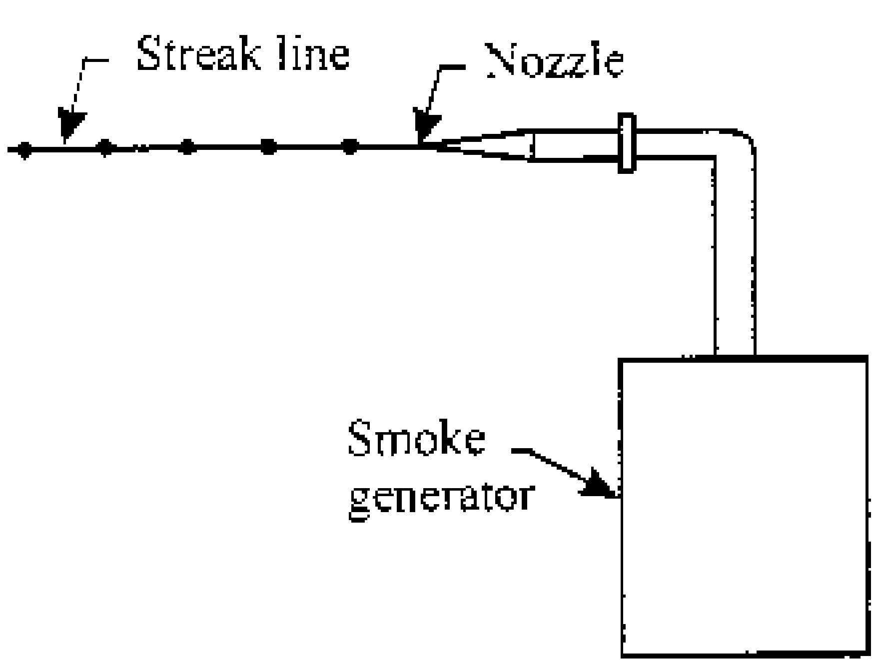

Fig. 5.10. Streak line of smoke issuing from a nozzle.

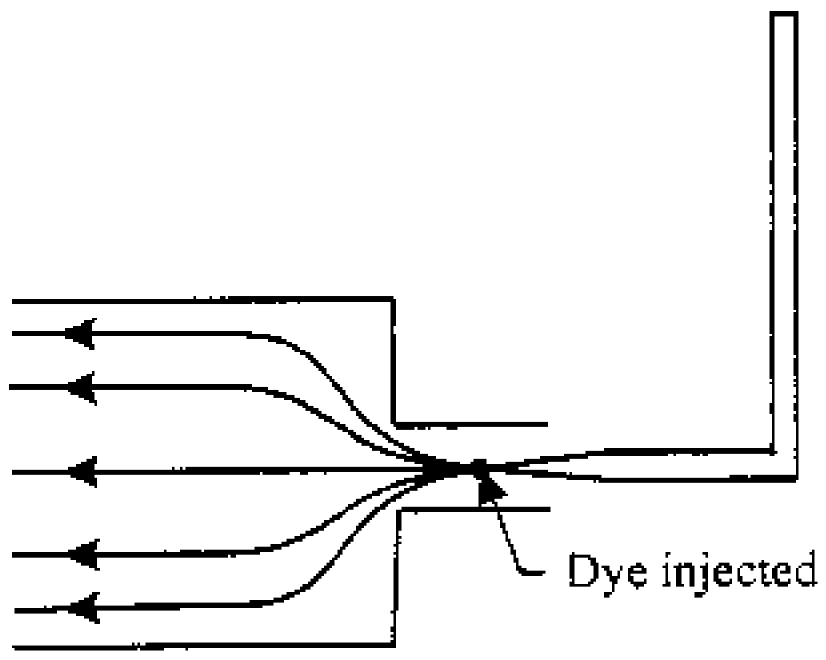

Fig. 5.11. Streak lines at $t = t_1$.

Example 5.1. *In a fluid, the velocity field is given by*

$$V = (3x + 2y)\, i + (2z + 3x^2)\, j + (2t - 3z)\, k$$

Determine:

(*i*) *The velocity components u, v, w at any point in the flow field*

(*ii*) *The speed at point (1, 1, 1);*

(*iii*) *The speed at time t = 2s at point (0, 0, 2).*

Also classify the velocity field as steady, or unsteady, uniform or non-uniform and one, two or three dimensional.

Solution. *Given*: Velocity, field, $V = (3x + 2y)\, i + (2z + 3x^2)\, j + (2t - 3z)\, k$

(*i*) **Velocity components:**

The velocity components are:

$$\mathbf{u = 3x + 2y,\ v = (2z + 3x^2),\ w = (2t - 3z)} \quad \textbf{(Ans.)}$$

(*ii*) **Speed point (1, 1, 1), $V_{(1,1,1)}$:**

Substituting $x = 1, y = 1, z = 1$ in the expressions for u, v and w, we have

$$u = (3 + 2) = 5,\ v = (2 + 3) = 5,\ w = (2t - 3)$$

$$\therefore \quad V^2 = u^2 + v^2 + w^2$$
$$= 5^2 + 5^2 + (2t - 3)^2$$
$$= 25 + 25 + 4t^2 - 12t + 9$$
$$= 4t^2 - 12t - 59$$

$$\therefore \quad V_{(1,1,1)} = \sqrt{4t^2 - 12t + 59} \quad \textbf{(Ans.)}$$

(*iii*) **Speed at $t = 2s$ at point (0, 0, 2):**

Substituting $t = 2, x = 0, y = 0, z = 2$ in the expressions for u, v and w, we get

$$u = 0,\ v = (2 \times 2 + 0) = 4,\ w = (2 \times 2 - 3 \times 2) = -2$$

$$\therefore \quad V^2 = u^2 + v^2 + w^2 = 0 + 4^2 + (-2)^2 = 20$$

or $\quad V_{(0,0,2)} = \sqrt{20} = $ **4.472 units (Ans.)**

Velocity field, type:

(*i*) Since V at given (x, y, z) *depends on t* it is **unsteady flow, (Ans.)**

(*ii*) Since at given t velocity changes in the X direction it is **non-uniform flow. (Ans.)**

(*iii*) Since V depends on x, y, z it is three **dimensional flow. (Ans.)**

Example 5.2 *Velocity for a two dimensional flow field is given by*

$$V = (3 + 2xy + 4t^2)\, i + (xy^2 - 3t)\, j$$

Find the velocity and acceleration at a point (1,2) *after* 2 *sec.*

Solution. *Given*: Velocity field: $V = (3 + 2xy + 4t^2)\, i + (xy^2 + 3t)\, j$

Velocity at (1, 2), $V_{(1,2)}$:

Substituting $x = 1, y = 2, t = 2$ in the expression of velocity field, we get

$$V = (3 + 2 \times 1 \times 2 + 4 \times 2^2)\, i + (1 \times 2^2 + 3 \times 2)\, j$$
$$= (3 + 4 + 16)\, i + (4 + 6)\, j$$
$$= 23i + 10j$$

$$\therefore \quad V_{(1,2)} = \sqrt{23^2 + 10^2} = \textbf{25.08 units (Ans.)}$$

Acceleration at point (1, 2), $a_{(1,2)}$:

we know, that, $$a = \frac{dV}{dt} = \left(u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} \right) + \frac{\partial V}{\partial t}$$

Also, $$V = (3 + 2xy + 4t^2)i + (xy^2 + 3t)j \qquad ...(Given)$$

$$\therefore \quad \frac{\partial V}{\partial x} = 2yi + y^2 j,$$

$$\frac{\partial V}{\partial y} = 2xi + 2xyj, \text{ and}$$

$$\frac{\partial V}{\partial t} = 8ti + 3j$$

$$\therefore \quad a = (3 + 2xy + 4t^2)(2yi + y^2 j) + (xy^2 + 3t)(2xi + 2xyi) + (8ti + 3j)$$

$$(\because u = 3 + 2xy + 4t^2 \text{ and } v = xy^2 + 3t)$$

Substituting the values, we get

$$a = (3 + 2 \times 1 \times 2 + 4 \times 2^2)(2 \times 2i + 2^2 j) + (1 \times 2^2 + 3 \times 2)$$
$$(2 \times 1i + 2 \times 1 \times 2j) + (8 \times 2 \times i + 3j)$$
$$= (3 + 4 + 16)(4i + 4j) + (4 + 6)(2i + 4j) + (16i + 3j)$$
$$= 23(4i + 4j) + 10(2i + 4j) + (16i + 3j)$$
$$= 92i + 92j + 20i + 40j + 16i + 3j$$
$$= 128\, i + 135\, j$$

$$\therefore \quad a_{(1,2)} = \sqrt{128^2 + 135^2} = \textbf{186.03 units (Ans.)}$$

Example 5.3. *Find the velocity and acceleration at a point (1, 2, 3) after 1 sec. for a three-dimensional flow given by $u = yz + t$, $v = xz - t$, $w = xy$, m/s*

Solution. *Given*: Three-dimensional flow field,

$$u = yz + t, \; v = xz - t, \; w = xy, \text{ m/s}$$

Velocity at a point 1, 2, 3 $V_{(1,2,3)}$:

Velocity at a point (1, 2, 3) after 1s is calculated as follows:

$$u = yz + t = 2 \times 3 + 1 = 7 \text{ m/s}, \; v = xz - t = 1 \times 3 - 1 = 2 \text{ m/s},$$
$$w = xy = 1 \times 2 = 2 \text{ m/s}.$$

$$\therefore \quad V_{(1,2,3)} = 7i + 2j + 2k$$
$$= \sqrt{7^2 + 2^2 + 2^2} = 7.55 \text{ m/s}$$

Hence $V_{(1,2,3)}$ **= 7.55 m/s (Ans.)**

Acceleration, $a_{(1,2,3)}$

Now, $$V = (yz + t)i + (xz - t)j + xy\, k, \text{ m/s}$$

Acceleration $a = \dfrac{dV}{dt} = \left(u\dfrac{\partial V}{\partial x} + v\dfrac{\partial V}{\partial y} + w\dfrac{\partial V}{\partial z}\right) + \dfrac{\partial V}{\partial t}$

$$a = (yz + t)(zi + yk) + (xz - t)(zi - xk) + xy(yi + xj) + (1i - 1j)$$

$$\therefore \quad a_{(1,2,3)} = 7(3j - 2k) + 2(3i - 1k) + 2(2i - 1j) + (1i - 1j)$$
$$= (21j + 14k + 6i + 2k + 4i + 2j) + (1i - 1j)$$
$$= (10i + 23j + 16k) + (1i - 1j)$$

The *convective acceleration components are*: (10, 23, 16) m/s^2

The *local acceleration components are*: (1, –1) m/s^2 along x and y directions.

The total acceleration of fluid particles at the points (1, 2, 3) is

$$a_{(1,2,3)} = \sqrt{(10+1)^2 + [23 + (-1)]^2 + 16^2}$$
$$= \sqrt{11^2 + 22^2 + 16^2} = 29.34 \text{ m/s}^2$$

Hence $a_{(1,2,3)}$ = **29.34 m/s^2 (Ans.)**

Example 5.4. *The velocity along the centreline of a nozzle of length l is given by*

$$V = 2t\left(1 - \frac{x}{2l}\right)^2$$

ere V = velocity in m/s, t = time in seconds from commencement of flow, x = distance from inlet nozzle. Calculate the convective acceleration, local acceleration and the total acceleration when = 6s, x = 1m and l = 1.6 m.

Solution. The velocity along the centreline of a nozzle, $V = 2t\left(1 - \dfrac{x}{2l}\right)^2$ ...(*Given*)

Local acceleration $= \dfrac{\partial V}{\partial t} = 2\left(1 - \dfrac{x}{2l}\right)^2$

At $t = 6$ s and $x = 1m$,

$$\frac{\partial V}{\partial t} = 2\left(1 - \frac{1}{2 \times 1.6}\right)^2 = \mathbf{0.945 \text{ m/s}^2 \text{ (Ans)}}$$

Convective acceleration $= V\dfrac{\partial V}{\partial x}$

$$= 2t\left(1 - \frac{x}{2l}\right)^2 \times 2t \times 2\left(1 - \frac{x}{2l}\right)\left(-\frac{1}{2l}\right)$$
$$= -\frac{4t^2}{l}\left(1 - \frac{x}{2l}\right)^3$$

At $t = 6s$ and $x = 1$ m,

Convective acceleration $= -\dfrac{4 \times 6^2}{1.6}\left(1 - \dfrac{1}{2 \times 1.6}\right)^3$

$$= \mathbf{-29.24 \text{ m/s}^2 \text{ (Ans)}}$$

Total acceleration = Local acceleration + convective acceleration

$$= 0.945 + (-29.24) = \mathbf{-28.295 \text{ m/s}^2}.$$

Example 5.5. *A conical pipe diverges uniformly from 100 mm to 200 m diameter over a length m. Determine the local and convective acceleration at the mid-section assuming*

(*i*) *Rate of flow is 0.12 m^3/s and it remains constant;*

(*ii*) *Rate of flow varies uniformly from 0.12 m^3/s to 0.24 m^3/s in 5 sec., at t = 2 sec.*

Solution. *Given:* Diameter at the inlet, $D_1 = 0.1 m$.

Diameter at the outlet, $D_2 = 0.2\ m$

Length $l = 1 m$

Diameter at any distance x metres from the inlet,

$$D_x = D_1 + \left(\frac{D_2 - D_1}{l}\right) \times x$$

$$= 0.1 + \left(\frac{0.2 - 0.1}{1}\right) \times x$$

$$= 0.1 + 0.1x = 0.1\ (1 + x)$$

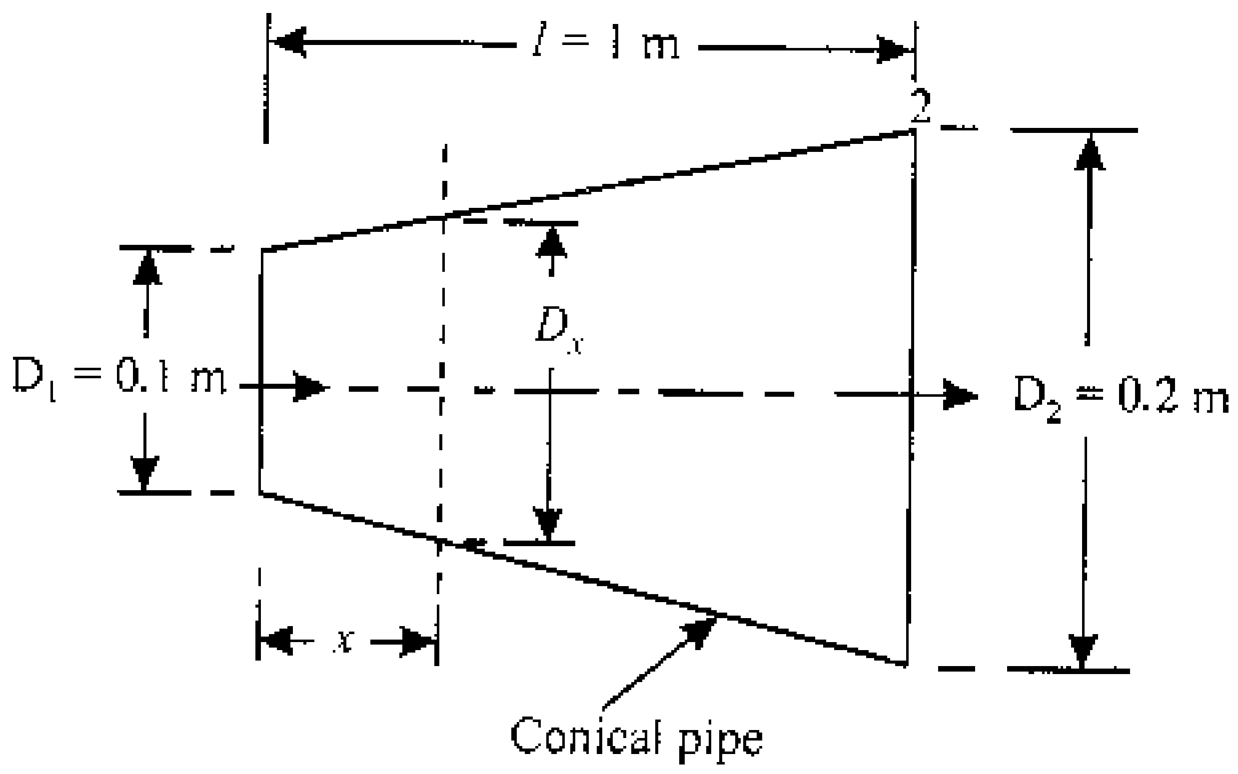

Fig. 5.12

∴ Cross-sectional area,

$$A_x = \frac{\pi}{4} \times D_x^2 = \frac{\pi}{4}\{0.1(1 + x)\}^2$$

$$= 0.00785\ (1 + x)^2$$

Velocity of flow, $u_x\ (= u) = \dfrac{Q}{A_x} = \dfrac{Q}{0.00785\,(1 + x)^2}$

Velocity gradient, $\dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x}\left[\dfrac{Q}{0.00785\ (1 + x)^2}\right] = \dfrac{-\,2Q}{0.00785\,(1 + x)^3}$

(*i*) **Discharge Q = 0.12 m^3/s = constant (at any section):**

Acceleration $= \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x}$

(*a*) **The local acceleration:**

The local acceleration

$$= \frac{\partial u}{\partial t} = 0, \text{ since the flow is } \textit{steady} \textbf{ (Ans.)}$$

(*b*) **The convective acceleration:**

The convective acceleration is

$$a_x = u\frac{\partial u}{\partial x} = \frac{Q}{0.00785\ (1 + x)^2} \times \frac{-\,2Q}{0.00785\,(1 + x)^3}$$

$$= \frac{-\,2Q^2}{(0.00785)^2\ (1 + x)^5}$$

∴ The *convective acceleration at mid-section*

$$(a_x)_{x = 0.5\text{m}} = \frac{-\,2 \times (0.12)^2}{(0.00785)^2\ (1 + 0.5)^5}$$

$$= -\,61{\cdot}5\ \text{m/s}^2 \textbf{ (Ans.)}$$

The -*ve* sign indicates *decrease in velocity* along the direction of flow (this is so as the cross-sectional area is increasing)

(*ii*) **Discharge *Q* varies w.r.t. time:**

The discharge Q varies from 0.12 m^3/s to 0.24 m^3/s in $5s$.

At $t = 2\ s$, the discharge is

$$Q = 0.12 + \frac{0.24 - 0.12}{5} \times 2 = 0.168\ \text{m}^3/\text{s}$$

(*a*) **The local acceleration is**

$$= \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[\frac{Q}{0.00785\,(1+x)^2}\right] = \frac{1}{0.00785\,(1+x)^2} \times \frac{\partial Q}{\partial t}$$

$$= \frac{1}{0.00785(1+0.5)^2} \times \left(\frac{0.168 - 0.12}{2}\right)$$

[since discharge changes 0.12 m^3/s to 0.168 m^3/s in 2*s*]

$= \mathbf{1.36\ m/s^2}$ **(Ans.)**

(*b*) **The convective acceleration at the mid-section**

$$(a_x)_{x=0.5} = \frac{-2Q^2}{(0.00785)^2\,(1+x)^5} = \frac{-2 \times 0.168^2}{(0.00785)^2\,(1+0.5)^5}$$

$= -\mathbf{120{\cdot}6\ m/s^2}$ **(Ans.)**

Total acceleration along the main flow is

$(a)\text{total} = (a)_{\text{local}} \div (a)_{\text{conv.}}$

$= 1.36 - 120.6 = -\mathbf{119.24\ m/s^2}$ **(Ans.)**

Example 5.6. *At entry to the pump intake the velocity is found to vary inversely as the square of radial distance from inlet to suction pipe. The velocity was found to be 0.6 m/s at a radial distance of 1.5 m. Calculate the acceleration of flow at radial distances of 0.5 m, 1.0 m and 1.5 m from the inlet. Consider the streamlines to be radial.*

Solution. The distribution of velocity is given by the relation

$$v = \frac{C}{r^2} \qquad ...(1)$$

(where r = radial distance from the intake)

Since the streamlines are *radial* normal acceleration is *zero*.

Acceleration $\quad a = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s}$

But $\frac{\partial v}{\partial t} = 0$ ... the flow being steady (as the velocity is dependent only on the radial distance from intake).

$\therefore \quad a = v.\frac{\partial v}{\partial s}$

Also $\quad v = \frac{C}{r^2}$ and $\frac{\partial v}{\partial s} = \frac{\partial v}{\partial r}$

(since r is measured along the streamline)

and $\quad \frac{\partial v}{\partial r} = -\frac{2C}{r^3}$

$$\therefore \quad a = v\frac{\partial v}{\partial s} = \frac{C}{r^2}\left(-\frac{2C}{r^3}\right) = -\frac{2C^2}{r^5} \qquad ...(2)$$

At $r = 1{\cdot}5$ m, $v = 0.6$ m/s

Substituting these values in eqn. (1), we get

$$0.6 = \frac{C}{1.5^2} \text{ or } C = 1.35\ m^3/s$$

Substituting, now, C = 1.35 in eqn. (2), we have

$$a = -\frac{2 \times 1.35^2}{r^5} = -\frac{3.645}{r^5}$$

$\therefore$ Acceleration of flow at $r = 0.5$ m $= -\frac{3.645}{(0.5)^5} = \mathbf{-116.64\ m/s^2}$ **(Ans.)**

(*ii*) Acceleration of flow at r = 1.0 m $= -\dfrac{3.645}{(1.0)^5}$ = **– 3.645 m/s² (Ans.)**

(*iii*) Acceleration of flow at r = 1.5 m $= -\dfrac{3.645}{(1.5)^5}$ = **– 0.48 m/s² (Ans.)**

Example 5.7. *For a three-dimensional flow the velocity distribution is given by* $u = -x$, $v = 3 - y$ *and* $w = 3 - z$. *What is the equation of a streamline passing through (1,2,2)?*

Solution. *Given:* $u = -x$, $v = 3 - y$, $w = 3 - z$ ...(velocity distribution)

Equation of a streamline passing through (1, 2, 2):

The streamlines are defined by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Substituting for u, v and w, we get

$$\underset{(i)}{\frac{dx}{-x}} = \underset{(ii)}{\frac{dy}{3-y}} = \underset{(iii)}{\frac{dz}{3-z}}$$

Considering the expressions (*i*) and (*ii*) and integrating, we get

$$\int \frac{dx}{-x} = \int \frac{dz}{(3-y)}$$

$$= -\log_e x = -\log_e (3 - y) + C_1$$

(where C_1 = constant of integration).

Since the streamline passes through $x = 1, y = 2$ $\therefore C_1 = 0$

$\therefore$ $(x)^{-1} = (3 - y)^{-1}$ or $x = (3 - y)$ ...(1)

Considering the expressions (*i*) and (*iii*), and integrating, we get

$$\int \frac{dx}{-x} = \int \frac{dy}{3-z}$$

or $-\log_e x = -\log (3 - z) + C_2$

(where C_2 = constant of integration)

Since the streamline passes through $x = 1, z = 2$ $\therefore C_2 = 0$

$\therefore$ $x^{-1} = (3 - z)^{-1}$

or $x = (3 - z)$ ...(2)

From (1) and (2), the equation of the streamline passing through (1, 2, 2) is given as

$$\boldsymbol{x = (3 - y) = (3 - z)} \textbf{ (Ans.)}$$

Example 5.8. *Obtain the equation to the streamlines for the velocity field given as:*

$$V = 2x^3 i - 6x^2 yj$$

Solution. *Given:* Velocity field, $V = 2x^3 i - 6x^2 yj$

Here $u = 2x^3$, $v = 6x^2 y$

The streamlines in two dimensions are defined by

$$\frac{dx}{u} = \frac{dy}{v}$$

or

$$\frac{dy}{dx} = \frac{v}{u} = \frac{-6x^2y}{2x^3} = \frac{-3y}{x}$$

Separating the variables, we have

$$\frac{dy}{y} = \frac{-3dx}{x}$$

Integrating, we get

$$\log_e y = -3\log_e x + C_1$$

or, $$\log_e y + 3\log_e x = C_1$$

or, $$yx^3 = \mathbf{C}\ \textbf{(Ans.)}$$

Note. The streamlines in the first quadrant can be sketched by giving different values for the constant $C\left(y = \dfrac{C}{x^3}\right)$.

Example 5.9. *For the following flows find the equation of the streamline passing through* (2,2):

(*i*) $V = 3\,xi - 3yj$

(*ii*) $V = -y^2 i - 6xj$

Solution. Equation of the streamline passing through (2, 2):

(*i*) $V = 3xi - 3yj$

$u = 3x$ and $v = -3y$

The equation of a streamline in two-dimensional flow is

$$\frac{dx}{u} = \frac{dy}{v}$$

or, $$\frac{dx}{3x} = -\frac{dy}{3y}$$

Integrating both sides, we get

$$\int \frac{dy}{3x} = -\int \frac{dy}{3y}$$

$$\frac{1}{3}\log_e x = -\frac{1}{3}\log_e y + \frac{1}{3}\log_e C$$

where, C is constant.

or, $\log_e xy = \log_e C$ or $xy = C$

For the streamline passing through (2, 2),

$$C = 2 \times 2 = 4$$

Hence, the required streamline equation is: $xy = 4$ **(Ans.)**

(*ii*) $V = -y^2 i - 6xj$

$u = -y^2$ and $v = -6x$

$$-\frac{dx}{y^2} = -\frac{dy}{6x} \quad \text{or} \quad 6x\,dx = y^2 dy$$

$$\int 6x\,dx = \int y^2\,dy$$

$$= \frac{6x^2}{2} = \frac{y^3}{3} + C$$

or, $$3x^2 - \frac{y^3}{3} = C$$

putting, $x = 2$, $y = 2$, we get

$$3 \times (2)^2 - \frac{(2)^3}{3} = C \quad \text{or} \quad C = \frac{28}{3}$$

Hence the equations of the required streamline is:

$$3x^2 - \frac{y^3}{3} = \frac{28}{3}$$

$$\mathbf{9x^2 - y^3 = 28}\ \textbf{(Ans.)}$$

Example 5.10. *The velocity vector in a flow is given by*

$$V = 3xi + 4yj - 7zk$$

Determine the equation passing through a point L(1, 2, 3)

Solution. The equation of a streamline is

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Here $u = 3x$, $v = 4y$ and $w = -7z$

$$\therefore \quad \frac{dx}{3x} = \frac{dy}{4y} = -\frac{dz}{7z}$$

Considering equations involving x and y, on integration

$$\frac{1}{3}\log_e x = \frac{1}{4}\log_e y + \log_e C_1' \text{ where } C'_1 = \text{a constant}$$

or, $$y = C_1 x^{4/3} \quad \ldots(i)$$

where C_1 is another constant:

Similarly, by considering equations with x and z and on integration

$$\frac{1}{3}\log_e x = -\frac{1}{7}\log_e z + \log_e C_2', \text{ where } C_2' = \text{ a constant}$$

or, $$z = \frac{C_2}{x^{7/3}} \quad \ldots(ii)$$

where C_2 is another constant.

Inserting the coordinates of the point L(1, 2, 3), we get

From eqn. (*i*) $$C_1 = \frac{y}{(x)^{4/3}} = \frac{2}{(1)^{4/3}} = 2$$

From eqn. (*ii*) $$C_2 = zx^{7/3} = 3 \times (1)^{7/3} = 3$$

Hence, the streamline passing through L is given by

$$y = 2x^{4/3} \text{ and } z = \frac{3}{x^{7/3}} \textbf{ (Ans.)}$$

5.5. Rate of Flow or Discharge

Rate of flow (or discharge) is defined as the *quantity of a liquid flowing per second through a section of pipe or a channel.* It is generally denoted by Q. Let us consider a liquid flowing through a pipe.

Let, A = Area of cross-section of the pipe, and

V = Average velocity of the liquid.

$\therefore$ Discharge, Q = Area × average velocity *i.e.*, $Q = A.V.$ ...(5.21)

If area is in m^2 and velocity is in m/s, then the discharge

$$Q = m^2 \times m/s = m^3/s = \text{cumecs}$$

5.6. Continuity Equation

The continuity equation is based on the *principle of conservation of mass*. It states as follows:

"If no fluid is added or removed from the pipe in any length then the mass passing across different sections shall be same."

Consider two cross-sections of a pipe as shown in Fig 5.13

Let, A_1 = Area of the pipe at section 1–1,

V_1 = Velocity of the fluid at section 1–1,

ρ_1 = Density of the fluid at section 1–1,

and, A_2, V_2, ρ_2 are corresponding values at sections 2–2.

The total quantity of fluid passing through section 1–1= $\rho_1 A_1 V_1$

and, the total quantity of fluid passing through section 2–2 = $\rho_2 A_2 V_2$

From the law of conservation of matter (theorem of continuity), we have

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad ...(5.22)$$

Eqn. (5.22) is applicable to the compressible as well as incompressible fluids and is called ***Continuity Equation.*** In case of *incompressible fluids*, $\rho_1 = \rho_2$ and the continuity eqn. (5.21) reduces to

$$A_1 V_1 = A_2 V_2 \qquad ...(5.23)$$

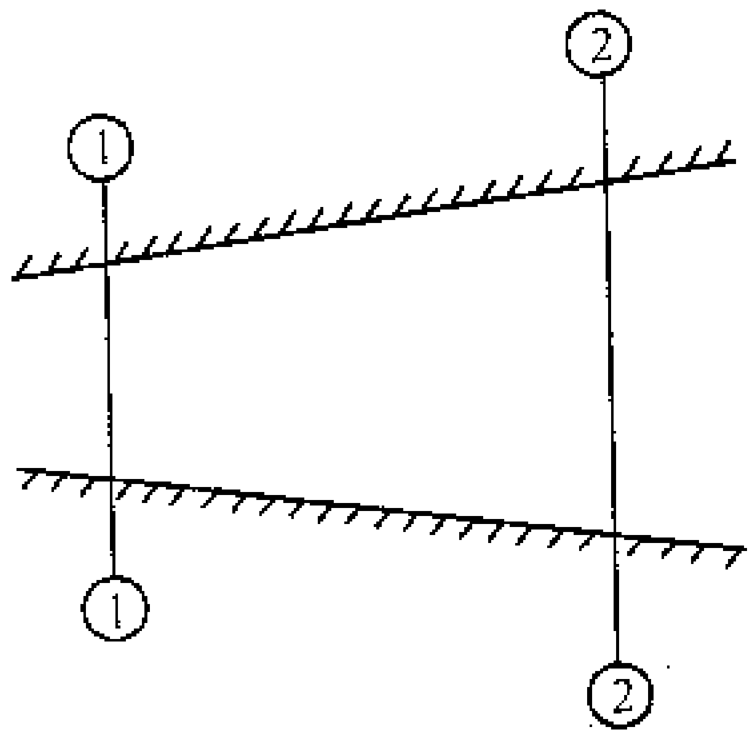

Fig. 5.13. Fluid flow through a pipe.

Example 5.11. *The diameters of a pipe at the sections 1-1 and 2-2 are 200 mm and 300 mm respectively. If the velocity of water flowing through the pipe at section 1-1 is 4m/s, find:*

(*i*) *Discharge through the pipe, and*

(*ii*) *Velocity of water at section 2-2*

Solution. Diameter of the pipe at *section 1-1*,

$$D_1 = 200 \text{ mm} = 0.2 \text{ m}$$

∴ Area $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2$

Velocity, $V_1 = 4$ m/s

Diameter of the pipe at *section 2-2*,

$$D_2 = 300 \text{ mm}$$

∴ Area, $A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ m}^2$

(*i*) **Discharge through the pipe, Q:**

Using the relation,

$Q = A_1 V_1$, we have

$Q = 0.0314 \times 4 =$ **0.1256 m³/s (Ans.)**

(*ii*) **Velocity of water at section 2-2, V_2:**

Using the relation,

$A_1 V_1 = A_2 V_2$, we have

$$V_2 = \frac{A_1 V_1}{A_2} = \frac{0.0314 \times 4}{0.0707}$$

= **1.77 m/s (Ans.)**

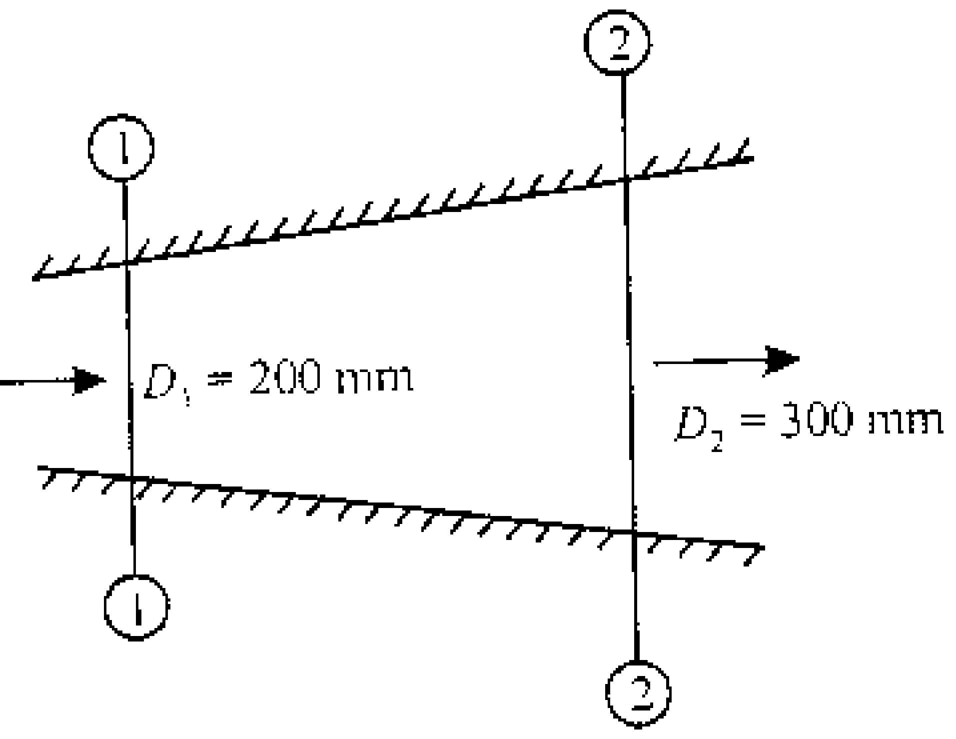

Fig. 5.14

Example 5.12. *A pipe (1) 450 mm in diameter branches into two pipes (2 and 3) of diameters 300 mm and 200 mm respectively as shown in Fig. 5.15. If the average velocity in 450 mm diameter pipe is 3 m/s find:*

(*i*) *Discharge through 450 mm diameter pipe;*

(*ii*) *Velocity in 200 mm diameter pipe if the average velocity in 300 mm pipe is 2.5 m/s.*

Solution. Diameter D_1 = 450 mm = 0.45 m

∴ Area $A_1 = \frac{\pi}{4} \times 0.45^2 = 0.159 \text{ m}^2$

Velocity $V_1 = 3$ m/s

Diameter $D_2 = 300$ mm = 0.3 m

∴ Area $A_2 = \frac{\pi}{4} \times 0.3^2 = 0.0707$ m²

Velocity $V_2 = 2.5$ m/s

Diameter $D_3 = 200$ mm = 0.2 m

Area $A_3 = \frac{\pi}{4} \times 0.2^2 = 0.0314$ m²

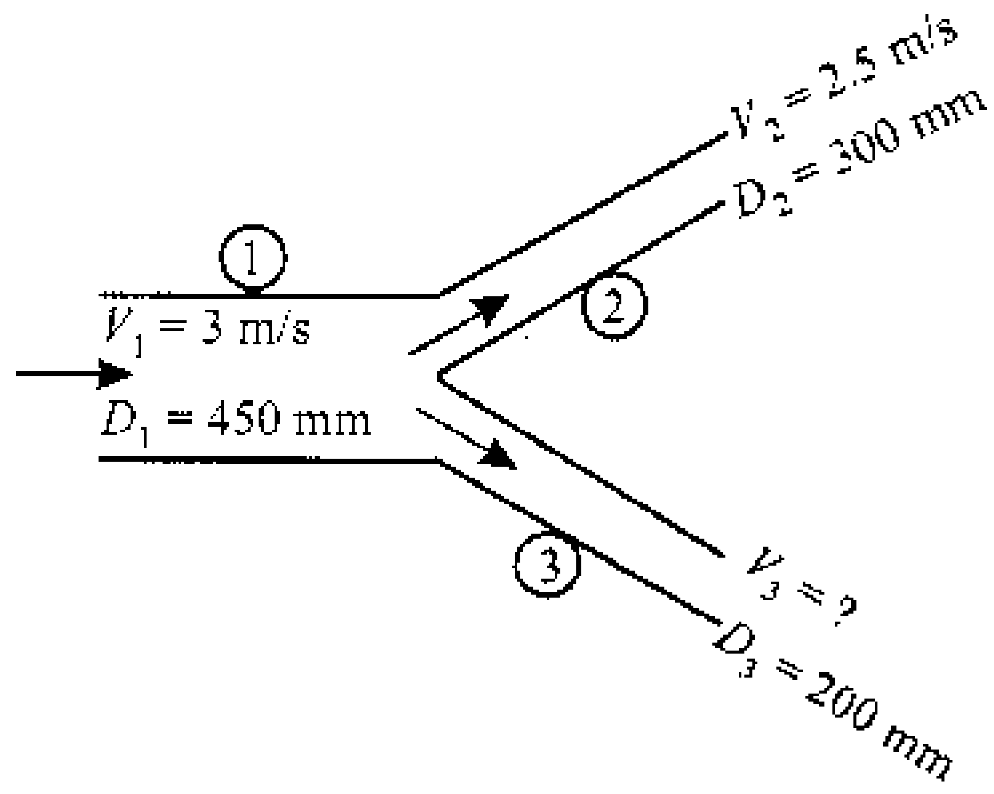

Fig. 5.15

(*i*) **Discharge through pipe (1) Q_1:**

Using the relation,

$$Q_1 = A_1 V_1 = 0.159 \times 3$$
$$= \mathbf{0{\cdot}477\ m^3/s\ (Ans.)}$$

(*ii*) **Velocity in pipe of diameter 200 mm i.e. V_3:**

Let Q_1, Q_2 and Q_3 be the discharge in pipes 1, 2 and 3 respectively.

Then, according to continuity equation

$$Q_1 = Q_2 + Q_3 \quad ...(i)$$

where $Q_1 = 0.447$ m³/s (calculated earlier)

and $Q_2 = A_2 V_2 = 0.0707 \times 2.5 = 0.1767$ m³/s

∴ $0.477 = 0.1767 + Q_2$ [from eq. (*i*)]

or $Q_3 = 0.477 - 0.1767 \simeq 0.3$ m³/s

But $Q_3 = A_3 V_3$

∴ $V_3 = \frac{Q_3}{A_3} = \frac{0.3}{0.0314} = 9.55$ m/s

i.e. V_3 = **9.55 m/s (Ans.)**

5.7. Continuity Equation in Cartesian Co-ordinates

Consider a fluid element (control volume) – parallelopiped with sides *dx*, *dy* and *dz* as shown in Fig. 5.16.

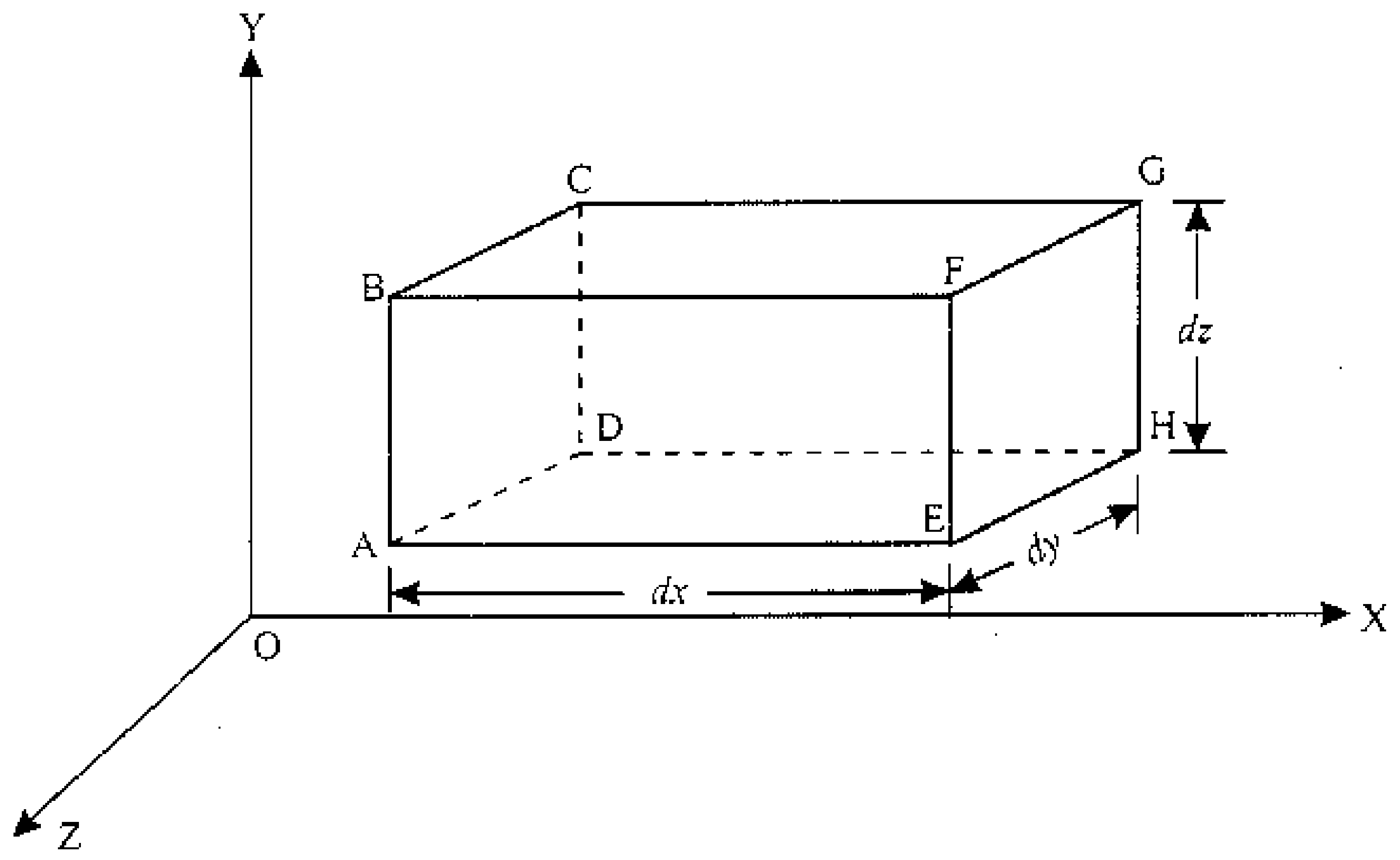

Fig. 5.16. Fluid element in three-dimensional flow.

Let, ρ = Mass density of the fluid at a particular instant;

u, v, w = Components of velocity of flow entering the three faces of the parallelopiped.

Rate of mass of fluid entering the face ABCD (*i.e.* fluid influx).

$$= \rho \times \text{velocity in } X\text{-direction} \times \text{area of ABCD}$$

$$= \rho\, u dy\, dz \quad \ldots(i)$$

Rate of mass of fluid leaving the face EFGH (*i.e.* fluid efflux).

$$= \rho\, u\, dy\, dz + \frac{\partial}{\partial x}(\rho u\, dy\, dz)\, dx \quad \ldots(ii)$$

The gain in mass per unit time due to flow in the X-direction is given by the difference between the fluid influx and fluid efflux.

$\therefore$ Mass accumulated per unit time due to flow in X-direction.

$$= \rho u\, dy\, dz - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy\, dz$$

$$= -\frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz \quad \ldots(iii)$$

Similarly, the gain in fluid mass per unit time in the parallelopiped due to flow in Y and Z-directions

$$= -\frac{\partial}{\partial y}(\rho v)\, dx\, dy\, dz \quad \text{(in Y-direction)} \quad \ldots(iv)$$

$$= -\frac{\partial}{\partial z}(\rho w)\, dx\, dy\, dz \quad \text{(in Z-direction)} \quad \ldots(v)$$

The total (or net) gain in fluid mass per unit for fluid along three co-ordinate axes

$$= -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx\, dy\, dz \quad \ldots(vi)$$

Rate of change of mass of the parallelopiped (control volume)

$$= \frac{\partial}{\partial t}(\rho\, dx\, dy\, dz) \quad \ldots(vii)$$

Equations (*vi*) and (*vii*), we get

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx\, dy\, dz = \frac{\partial}{\partial t}(\rho\, dx\, dy\, dz)$$

Simplification and rearrangement of terms would reduce the above expression to

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \ldots(5.24)$$

This eqn. (5.24) is the general equation of continuity in three-dimensions and is applicable to *any type of flow* and for any fluid whether *compressible or incompressible.*

For *steady flow* $\left(\frac{\partial \rho}{dt} = 0\right)$ *incompressible fluids* (ρ = constant) the equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} = 0 \quad \ldots(5.25)$$

For *two dimensional flow*, eqn. (5.25) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\because w = 0)$$

For *one dimensional flow*, say in X-direction, eqn (5.25) takes the form

$$\frac{\partial u}{\partial x} = 0 \qquad (\because v = 0, w = 0)$$

Integrating with respect to x, we get

$$u = \text{constant} \qquad ...(5.26)$$

If the area of flow is a then the rate of flow is

$$Q = a.u = \text{constant for } \textit{steady flow}$$

which is the same eqn. (5.23) and states that if area of flow a is constant the velocity of flow u will also be constant.

5.8. Equation of Continuity in Polar Coordinates

Consider a fluid element LMST as shown in Fig. 5.17. The sides of the element has the following dimensions.

LT = MS = dr; LM = $r d\theta$ and rdq and ST. = $(r + dr)d\theta$

Let, V_r = Component of the velocity in the radial direction, and

V_θ = Component of the velocity in the tangential direction.

Further, let thickness of the element perpendicular to the plane of paper be *unity*. As the third flows through the element, changes will place in its velocity as well as in the density.

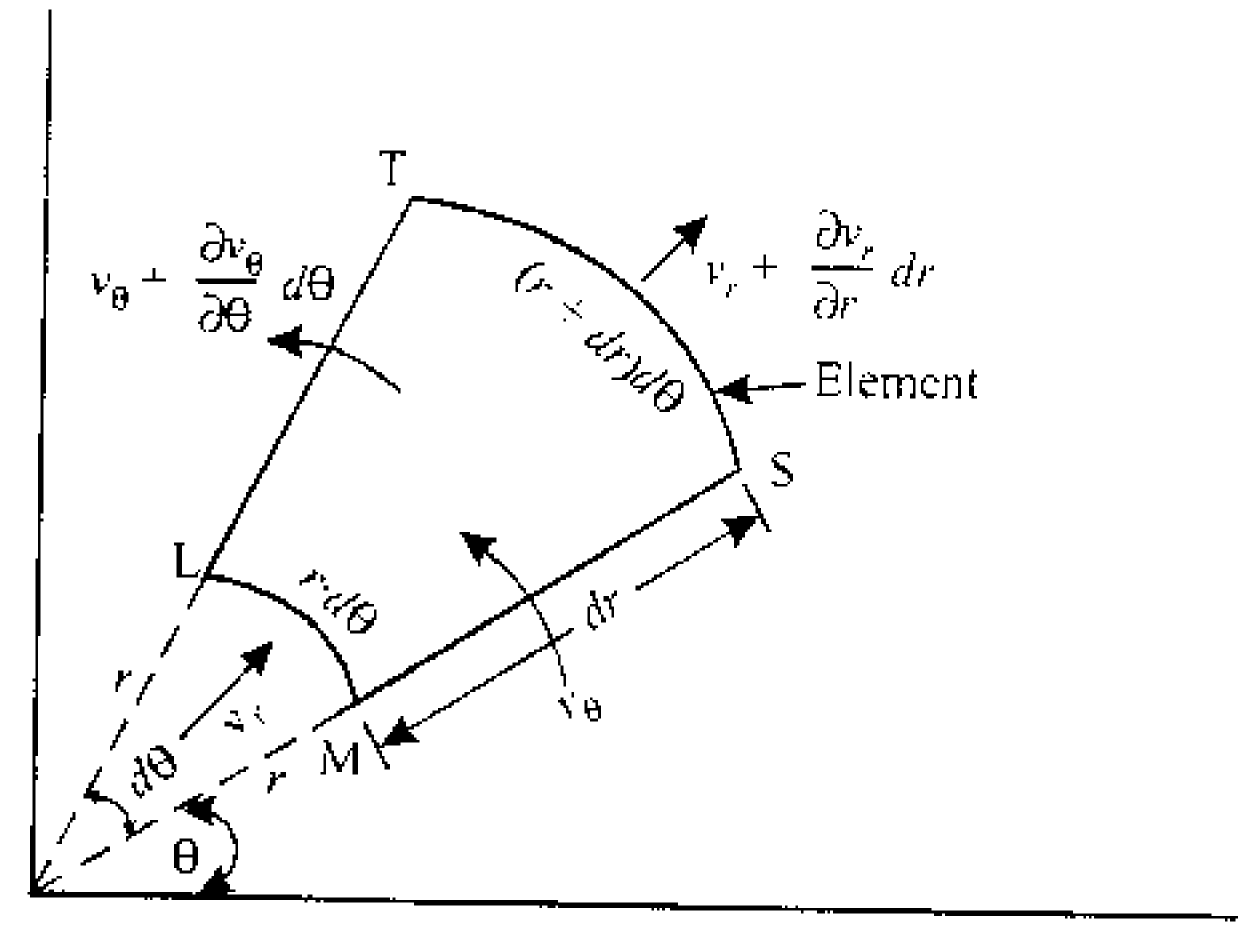

Fig. 5.17. Control volume for equation of continuity in polar coordinates.

Flow in radial direction:

Mass of fluid *entering the face LM* during time dt is given by:

Fluid *influx* = Density × (velocity × area) × time

$$= \rho \times (v_r \times r d\theta) \times dt$$

Mass of fluid *leaving the face ST* during the same time dt is given by:

$$\text{Fluid } \textit{efflux} = \left[\rho v_r + \frac{\partial}{\partial r}(\rho v_r) dr\right](r + dr)\, d\theta . dt$$

Mass accumulated in the element because of flow in radial direction

= Fluid inffux – fluid efflux

$$= \rho \times (v_r \times r d\theta) \times dt - \left[\rho v_r + \frac{\partial}{\partial r}(\rho v_r) dr\right](r + dr)\, d\theta\, dt$$

$$= -\left[\rho v_r\, dr\, d\theta + \frac{\partial}{\partial r}(\rho v_r)\, dr \cdot d\theta\right] dt$$

[Neglecting terms containing $(dr)^2$]

Flow in tangential direction:

The mass accumulated due to flow in the tangential direction (by a similar treatment as discussed [illegible]).

$$= \left[\rho v_\theta \, dr - \left\{ \rho v_\theta + \frac{\partial}{\partial \theta} (\rho v_\theta) \, d\theta \right\} dr \right] dt = - \frac{\partial}{\partial \theta} (\rho v_\theta) \, dr \, d\theta \, dt$$

$\therefore$ Total gain in fluid mass

$$= - \left[\rho v_r \, dr \, d\theta + \frac{\partial}{\partial r} (\rho v_r) \, dr \; r d\theta + \frac{\partial}{\partial \theta} (\rho v_\theta) \, dr \, d\theta \right] dt$$

$$= - \left[\rho v_r + \frac{\partial}{\partial r} (\rho v_r) r + \frac{\partial}{\partial \theta} (\rho v_\theta) \right] dr \, d\theta \, dt \qquad \text{...}(i)$$

Also, the rate of change of fluid mass in the element LMST

$$= \frac{\partial}{\partial t} (\text{Density} \times \text{volume}) \, dt$$

$$= \frac{\partial}{\partial t} \left[\rho \times \frac{r d\theta + (r + dr) d\theta}{2} dr \right] dt \simeq \frac{\partial}{\partial t} (\rho r d\theta \, dr) \, dt \qquad \text{...}(ii)$$

As per law of conservation of mass.

The total gain in mass = the rate of change of fluid mass in the element LMST

$$\therefore \qquad - \left[\rho v_r + \frac{\partial}{\partial r} (\rho v_r) r + \frac{\partial}{\partial \theta} (\rho v_\theta) \right] dr \, d\theta \, dt = \frac{\partial}{\partial t} (\rho r \, d\theta \, dr \, dt)$$

$$\text{or} \qquad \left[\rho v_r + \frac{\partial}{\partial r} (\rho v_r) r + \frac{\partial}{\partial \theta} (\rho v_\theta) \right] dr \, d\theta + \frac{\partial}{\partial t} (\rho r \, d\theta \, dr) = 0$$

For steady and compressible flow, $\frac{\partial}{\partial t} \rho r d\theta dr = 0$

$$\therefore \qquad \left[\rho v_r + \frac{\partial}{\partial r} (\rho v_r) r + \frac{\partial}{\partial \theta} (\rho v_\theta) \right] dr \, d\theta = 0 \qquad \text{...(5.27)}$$

Further, for *incompressible flow*, ρ = constant

$$\therefore \qquad v_r + \frac{\partial}{\partial r} (v_r) r + \frac{\partial}{\partial \theta} (v_\theta) = 0$$

$$\text{or} \qquad \frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{r \partial \theta} = 0 \qquad \text{...(5.28)}$$

Example 5.13. *Determine which of the velocity component sets given below satisfy the equation [illegible]tinuity:*

(*i*) $u = A \sin xy$

$v = - A \sin xy$

(*ii*) $u = x + y$

$v = x - y$

(*iii*) $u = 2x^2 + 3y$

$v = - 2xy + 3y^3 + 3zy$

$w = - \frac{3}{2} z^2 - 2xz - 6yz$

Solution. (*i*) $u = A \sin xy$; $v = - A \sin xy$

$$\frac{\partial u}{\partial x} = Ay \cos xy \; ; \; \frac{\partial v}{\partial y} = - Ax \cos xy$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = Ay \cos xy - Ax \cos xy \neq 0$$

i.e. Continuity equation is *not satisfied.* **(Ans.)**

(*ii*) $u = x + y;\ v = x - y$

$$\frac{\partial u}{\partial x} = 1\ ;\ \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$$

i.e. Continuity equation is *satisfied* **(Ans.)**

(*iii*) $u = 2x^2 + 3y;\ v = -2xy + 3y^3 + 3zy;\ w = -\frac{3}{2}z^2 - 2xz - 6yz$

$$\frac{\partial u}{\partial x} = 4x\ ;\ \frac{\partial v}{\partial y} = -2x + 9y^2 + 3z\ ;\ \frac{\partial w}{\partial z} = -3z - 2x - 6y$$

Hence,
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 4x - 2x + 9y^2 + 3z - 3z - 2x - 6y \neq 0$$

i.e. Continuity equation is *not satisfied* **(Ans.)**

Example 5.14. *Calculate the unknown velocity component in the following, so that the equation of continuity is satisfied.*

(*i*) $u = Ae^x$	(*ii*) $u = Aln\left(\frac{x}{l}\right)$	(*iii*) $u = ?$
$v = ?$	$v = ?$	$v = Axy$

Solution.

(*i*) $u = Ae^x;\ v = ?$

$$\frac{\partial u}{\partial x} = Ae^x = -\frac{\partial v}{\partial y}$$

$$v = \int -Ae^x\,dy = -Ae^x y + f(x)\ \textbf{(Ans.)}$$

(*ii*) $u = -A\,ln\left(\frac{x}{l}\right);\ v = ?$

$$\frac{\partial u}{\partial x} = -\frac{A}{(x/l)} \times \frac{1}{l} = -\frac{A}{x} = -\frac{\partial v}{\partial y}$$

$$v = \int \frac{A}{x}\,dy = \frac{Ay}{x} + f(x)\ \textbf{(Ans.)}$$

(*iii*) $u = ?;\ v = Axy$

$$\frac{\partial v}{\partial y} = Ax = -\frac{\partial u}{\partial x}$$

$$u = \int -Ax\,dx = -\frac{Ax^2}{2} + f(y)\ \textbf{(Ans.)}$$

Example 5.15. *In three-dimensional incompressible fluid flow, the velocity components in x and y-directions are:*

$$u = x^2 + y^2z^3;\ v = -(xy + yz + zx)$$

Use continuity equation to evaluate an expression for the velocity component w in the z-direction.

(Banglore University)

Solution. The continuity equation for a steady, three-dimensional incompressible fluid flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad ...(i)$$

$$u = x^2 + y^2z^3;\ v = -(xy + yz + zx)$$

$$\frac{\partial u}{\partial x} = 2x\ ;\ \frac{\partial v}{\partial y} = -(x + z)$$

Substituting these values in eqn. (*i*), we get.

$$2x - (x + z) + \frac{\partial w}{\partial z} = 0$$

or $$\frac{\partial w}{\partial z} = -x + z$$

Integrating *w.r.t.z* we have

$$w = -xz + \frac{z^2}{2} + C$$

Where C is a constant of integration which should be independent of z but may be function of x and / or y *i.e.* $C = f(x, y)$

$\therefore$ $$w = -x + \frac{z^2}{2} + f(x, y)\ \textbf{(Ans.)}$$

Example 5.16. *Given* $u = \log_e(y^2 + z^2)$ *and* $w = \log_e(x^2 + y^2)$. *What is the most general form of v so that the flow is possible for a steady three-dimensional incompressible flow?*

Solution. $u = \log_e(y^2 + z^2)$; $w = \log_e(x^2 - y^2)$

$$\frac{\partial u}{\partial x} = 0;\ \frac{\partial w}{\partial z} = 0$$

Substituting these values in continuity equation, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial v}{\partial y} = 0$$

Upon integration *w.r.t. z.*,

$$v = f(x, z)$$

By symmetry, one of the values of velocity component could be

$$\mathbf{v = \log_e(x^2 + z^2)\ (Ans.)}$$

Example 5.17. *For an incompressible fluid the velocity components are:* $u = x^3 - y^3 - z^2x$, $v = y^3 - z^3$, $w = -3x^2z - 3y^2z + \frac{z^3}{3}$. *Determine whether the continuity equation is satisfied.*

Solution. *Given:* $u = x^3 - y^3 - z^3x$, $v = y^3 - z^3$, $w = -3x^2z - 3y^2z + \frac{z^3}{3}$ ... velocity components

Now, $$\frac{\partial u}{\partial x} = 3x^2 - z^2$$

$$\frac{\partial v}{\partial y} = 3y^2$$

$$\frac{\partial w}{\partial z} = -3x^2 - 3y^2 + z^2$$

$\therefore$ $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = (3x^2 - z^2) + 3y^2 + (-3x^2 - 3y^2 + z^2) = 0$$

Hence **the continuity equation is satisfied. (Ans.)**

Example 5.18. *In a three-dimensional incompressible flow, the velocity components in y and z directions are $v = ax^3 - by^2 + cz^2$; $w = bx^3 - cy^2 + az^2x$. Determine the missing component of velocity distribution such that continuity equation is satisfied.*

Solution. *Given*: $v = ax^3 - by^2 + cz^2$, and

$$w = bx^3 - cy^2 + az^2x$$

Missing component, *u*:

The continuity equation for an incompressible fluid flow is given as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad ...(i)$$

From the given velocity components;

$$\frac{\partial v}{\partial y} = -2by\,;\ \frac{\partial w}{\partial z} = 2az\,x$$

Substituting these values in eqn. (*i*), we get

$$\frac{\partial u}{\partial x} - 2by + 2az\,x = 0$$

or
$$\frac{\partial u}{\partial x} = 2by - 2az\,x$$

Integrating w.r.t.*x*, we get

$$u = 2byx - 2az\frac{x^2}{2} + C \ \textbf{(Ans.)}$$

[where $C = f(y, z)$, the exact value will be known if the boundary conditions are known].

The constant of integration *C* is either a numerical constant or a function which is independent of *x*. If this constant is *omitted*, the velocity component may be expressed as

$$u = 2byx - azx^2 \ \textbf{(Ans.)}$$

Example 5.19. *The velocity components in x and y directions are given as $u = 2xy^3/3 - x^2y$ and $v = xy^2 - 2yx^3/3$. Indicate whether the given velocity distribution is:*

(*i*) *a possible field of flow.*

(*ii*) *not a possible field of flow.* **[UPSC Exam.]**

Solution. *Given*: $u = 2xy^3/3 - x^2y$, $v = xy^2 - 2yx^3/3$ ...Velocity components

A possible flow field (two-dimensional) must satisfy the continuity equation.

$$\therefore \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad ...(i)$$

Now,
$$\frac{\partial u}{\partial x} = \frac{2}{3}y^3 - 2xy, \frac{\partial v}{\partial y} = 2xy - \frac{2}{3}x^3$$

Substituting these values in eqn. (*i*), we get

$$\left(\frac{2}{3}y^3 - 2xy\right) + \left(2xy - \frac{2}{3}x^3\right) = \frac{2}{3}(y^3 - x^3)$$

Since the continuity equation is *not* satisfied, the given velocity components, therefore, **do not represent a possible flow field. (Ans.)**

Example 5.20. *In an incompressible flow, the velocity vector is given by*

$$V = (6xt + yz^2)i + (3t + xy^2)j + (xy - 2xyz - 6tz)k$$

(*i*) *Verify whether the continuity equation is satisfied.*

(*ii*) *Determine the acceleration vector at point L (2, 2, 2) at t = 2.0*

Solution. (*i*) $V = (6xt + yz^2)\,i + (3t + xy^2)j + (xy - 2xyz - 6tz)\,k$ ...(*Given*)

$$= ui + vj + wk$$

$$u = 6xt + yz^2, \quad \frac{\partial u}{\partial x} = 6t$$

$$v = 3t + xy^2, \quad \frac{\partial v}{\partial y} = 2xy$$

$$w = xy - 2xyz - 6tz, \quad \frac{\partial w}{\partial z} = -2xy - 6t$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 6t + 2xy - 2xy - 6t = 0$$

Hence the *continuity equation is satisfied* **(Ans.)**

Acceleration, $a = a_x i + a_y j + a_z k$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= 6x + (6xt + yz^2)(6t) + (3t + xy^2)(z^2) + (xy - 2xyz - 6tz)(2yz)$$

At point $L(2,2,2)$ and at $t = 2$,

$$a_x = 6 \times 2 + (6 \times 2 \times 2 + 2 \times 2^2)(6 \times 2) + (3 \times 2 + 2 \times 2^2)(2^2) + (2 \times 2 - 2 \times 2 \times 2 \times 2 - 6 \times 2 \times 2)(2 \times 2 \times 2)$$

$$= 12 + (32)(12) + (14)(4) + (-36)(8) = 164 \text{ units}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$= 3 + (6xt + yz^2)(y^2) + (3t + xy^2)(2xy) + (xy - 2xyz - 6tz)(0)$$

At point $L(2, 2, 2)$ and at $t = 2$

$$a_y = 3 + (6 \times 2 \times 2 + 2 \times 2^2)(2^2) + (3 \times 2 + 2 \times 2^2)(2 \times 2 \times 2) + (2 \times 2 - 2 \times 2 \times 2 - 6 \times 2 \times 2)(0)$$

$$= 3 + (32)(4) + (14)(8) = 243 \text{ units.}$$

Similarly, $$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$= -6z + (6xt + yz^2)(y - 2yz) + (3t + xy^2)(x - 2xz) + (xy - 2xyz - 6tz)(-2xy - 6t)$$

At point $L(2, 2, 2)$ and at $t = 2$,

$$a_z = -6 \times 2 + (6 \times 2 \times 2 + 2 \times 2^2)(2 - 2 \times 2 \times 2) + (3 \times 2 + 2 \times 2^2)(2 - 2 \times 2 \times 2) + (2 \times 2 - 2 \times 2 \times 2 \times 2 - 6 \times 2 \times 2)(-2 \times 2 \times 2 - 6 \times 2)$$

$$= -12 + (32)(-6) + (14)(-6) + (-36)(-20) = 432 \text{ units.}$$

Hence at $L(2, 2, 2)$ and at $t = 2$,

$$a = a_x i + a_y j + a_z k$$

or $$\mathbf{a = 164\,i + 243j + 432k} \textbf{ (Ans.)}$$

Example 5.21. *A two-dimensional incompressible flow in cylindrical polar coordinates is given by*

$$v_r = 2r \sin\theta \cos\theta; \quad v_\theta = -2r \sin^2\theta$$

Determine whether these velocity components represents a physically possible flow field.

Solution. The continuity equation, for a steady, two-dimensional incompressible flow is

$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{r\partial\theta} = 0 \qquad ...(5.28)$$

From the given velocity components,

$$\frac{\partial v_r}{\partial r} = \frac{\partial}{\partial r}(2r \sin\theta \cos\theta) = 2 \sin\theta \cos\theta$$

$$\frac{\partial v_\theta}{\partial \theta} = \frac{\partial}{\partial \theta}(-2r \sin^2\theta) = -4r \sin\theta \cos\theta$$

Inserting these values in the above equation, we get

$$\frac{2r \sin\theta \cos\theta}{r} + 2\sin\theta\cos\theta - \frac{4r\sin\theta\cos\theta}{r} = 0$$

or $$2\sin\theta \cos\theta + 2 \sin\theta \cos\theta - 4 \sin\theta \cos\theta = 0$$

i.e., L. H. S. = 0

Thus the *continuity equation is satisfied and hence the* **flow is physically possible. (Ans.)**

Example 5.22. *The tangential component of velocity in a two-dimensional flow of incompressible fluid is*

$$v_\theta = -\frac{C \sin\theta}{r^2}$$

where C is a constant.

(*i*) *Using continuity equation, determine the expression for radial velocity v.*

(*ii*) *Find the magnitude and direction of resultant velocity.*

Solution. *Given:* $v_\theta = -\frac{C \sin\theta}{r^2}$

(*i*) **Expression for v_r :**

The continuity equation for a two-dimensional, steady incompressible flow is

$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{r\partial\theta} = 0 \quad \text{[Eqn. (5.28)]}$$

or $$\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial \theta}(v_\theta) = 0 \quad ...(i)$$

For the given velocity component;

$$\frac{\partial v_\theta}{\partial \theta} = \frac{\partial}{\partial \theta}\left(-\frac{C\sin\theta}{r^2}\right) = -\frac{C}{r^2}\cos\theta \quad ...(ii)$$

From eqn. (*i*) and (*ii*), we have

$$\frac{\partial}{\partial r}(rv_r) = \frac{C}{r^2}\cos\theta$$

Integrating both sides w.r.t. *r* we have

$$rv_r = \int_0^r \frac{C}{r^2}\cos\theta\, dr = -\frac{C\cos\theta}{r}$$

∴ Radial component; $v_r = -\frac{C\cos\theta}{r^2}$ **(Ans.)**

(*ii*) **Resultant velocity:**

Resultant velocity $= \sqrt{v_r^2 + v_\theta^2}$

$$= \sqrt{\left(-\frac{C\cos\theta}{r^2}\right)^2 + \left(-\frac{C\sin\theta}{r^2}\right)^2}$$

$$= \frac{C}{r^2}\sqrt{(\cos^2\theta + \sin^2\theta)} = \frac{C}{r^2} \quad \textbf{(Ans.)}$$

5.9. Circulation and Vorticity

Let us consider a closed curve in a two-dimensional flow field shown in Fig. 5.18; the curve being cut by the stream lines. Let P be the point of intersection of the curve with one streamline, θ be the angle which the streamline makes with the curve. The component of velocity along the closed curve at the point of intersection is equal to $V\cos\theta$. **Circulation Γ** is defined mathematically *as the line integral of the tangential velocity about a closed path (contour)*.

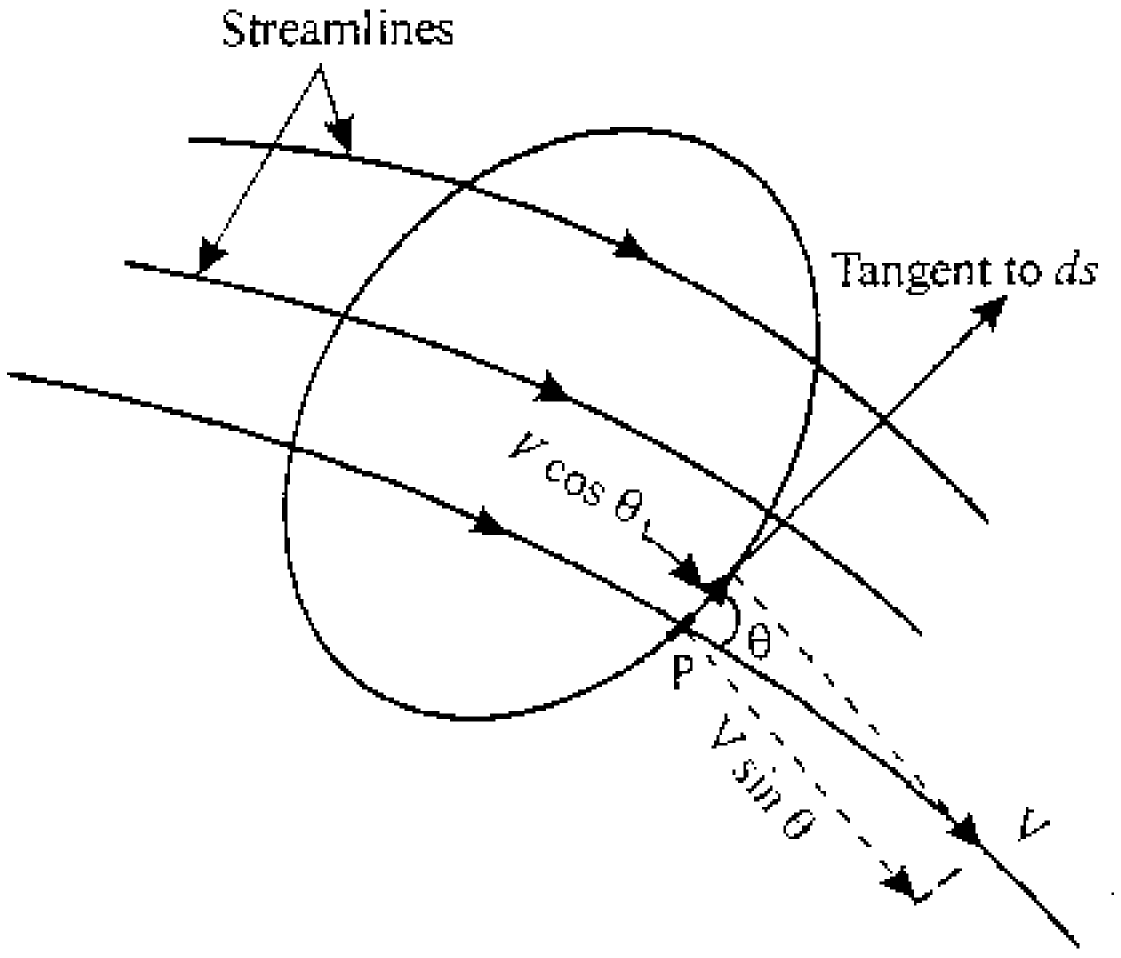

Fig. 5.18. Circulation in a two-dimensional flow.

Thus,

$$\Gamma = \oint V\cos\theta . ds$$

where, V = Velocity in the flow field at the element ds, and

θ = Angle between V and tangent to the path (in the positive anticlockwise direction along the path) at that point.

Circulation around regular curves can be obtained by integration. Let us consider the circulation around an elementary box (fluid element ABCD) shown in Fig. 5.19.

Starting from A and proceeding anticlockwise, we have

$$d\Gamma = u\,\Delta x + \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta y - \left(u + \frac{\partial u}{\partial y}\Delta y\right)\Delta x - v\,\Delta y$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x.\Delta y$$

The **vorticity** (Ω) is defined as the *circulation per unit of enclosed area,*

$$\Omega = \frac{\Gamma}{A}, \text{ Thus},$$

$$\Omega = \frac{d\Gamma}{\Delta x.\Delta y} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad ...(5.29)$$

If *a flow possesses vorticity, it is rotational.* Rotation ω (omega) is defined as one-half of the vorticity, or

$$\omega = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

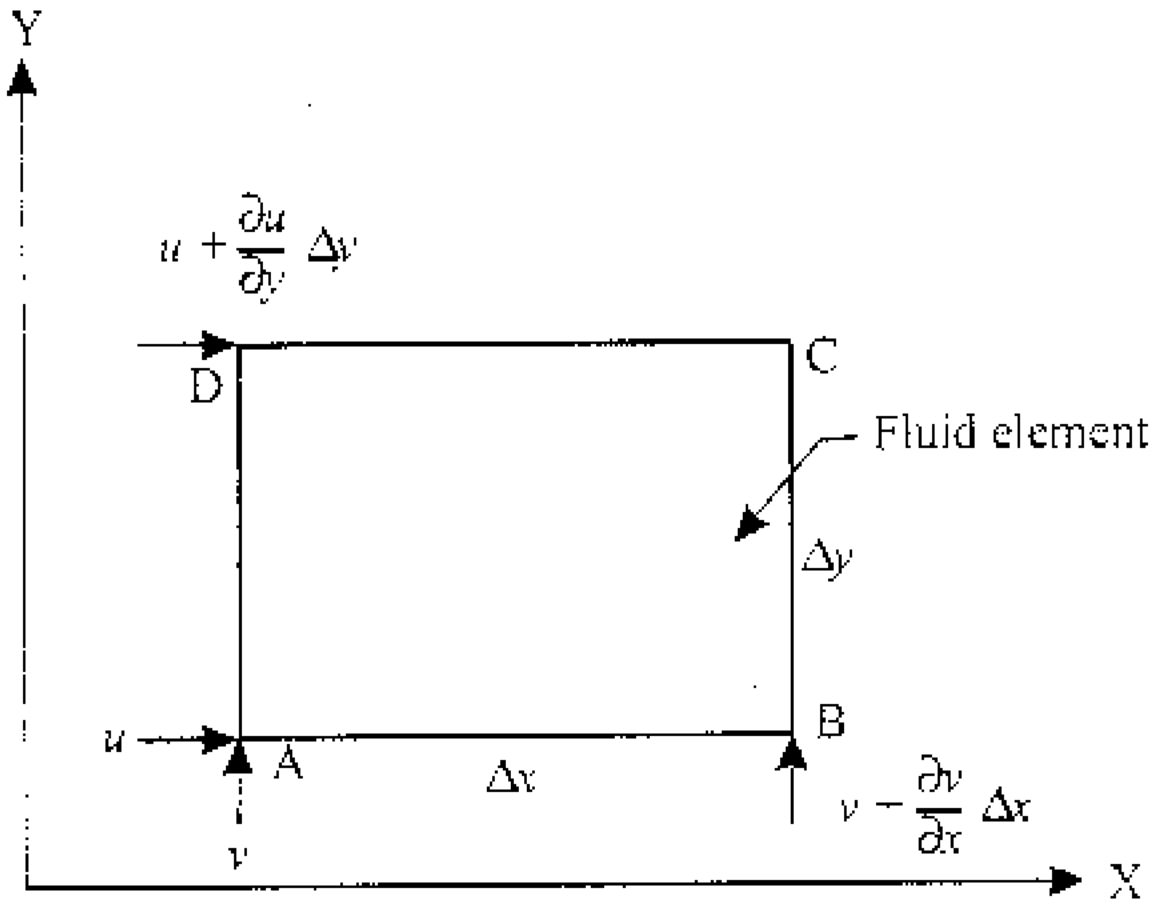

Fig. 5.19. Irrotational flow condition.

***The** flow is irrotational if rotation ω is zero.*

For a three-dimensional flow the rotation is possible about three axes. The expressions for rotation ω_z, ω_x *and* ω_y can be obtained in like manner.

$$\left.\begin{aligned} \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \end{aligned}\right\} \qquad ...(5.30)$$

In the vector notation, the above equation can be rewritten as

$$\omega = \frac{1}{2}[\omega_x i + \omega_y j + \omega_z k]$$

$$= \frac{1}{2}(\Delta \times V) \qquad ...(5.31)$$

The vector $(\Delta \times V)$ is the curl of velocity vector.

Vorticity in a fluid motion is taken numerically equal to *twice the value of rotation*

Vorticity, $\Omega = \text{curl } V = (\Delta \times V)$ ...(5.32)

Which may be expressed as:

$$\Omega = (\nabla \times V)\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \Omega_x i + \Omega_y j + \Omega_z k \qquad ...(5.33)$$

The vorticity components are separately given by:

$$\Omega_x = 2\omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\Omega_y = 2\omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\Omega_z = 2\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad ...(5.34)$$

The motion is described as **irrotational** *when the components of rotation or vorticity are 'zero' throughout certain portion of the fluid.*

When torque is applied to the fluid particle it will give rise to *rotation*; the torque is due to shear stress. Therefore, the rotation of fluid particle will always be associated with shear stress. As the shear stresses, in turn, depend upon the viscosity, the *rotational flow occurs where the viscosity effects are predominant.* However, in the cases where the viscosity effects are small, the flow is sometimes assumed to be irrotational. This simplifies analysis of problems of fluid flow.

Example 5.23. *Given that*

$$u = -4ax(x^2 - 3y^2)$$
$$v = 4ay(3x^2 - y^2)$$

Examine whether these velocity components represent a physically possible two-dimensional flow; if so whether the flow is rotational or irrotational?

Solution. *Given:* $u = -4ax(x^2 - 3y^2)$

$v = 4ay(3x^2 - y^2)$ ...Velocity components

A two-dimensional flow will be *continuous* if $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Now, $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[-4ax(x^2 - 3y^2)] = \frac{\partial}{\partial x}(-4ax^3 + 12axy^2) = -12ax^2 + 12ay^2$$

and $$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}[4ay(3x^2 - y^2)] = \frac{\partial}{\partial y}[12ayx^2 - 4ay^3] = 12ax^2 - 12ay^2$$

$\therefore$ $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (-12ax^2 + 12ay^2) + (12ax^2 - 12ay^2) = 0$$

Hence the given velocity components **represent a physically possible two-dimensional flow. (Ans.)**

The flow will be *irrotational* if,

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Now,
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[-4ax(x^2 - 3y^2)] = \frac{\partial}{\partial y}(-4ax^3 + 12axy^2) = 24\ axy$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}[4ay(3x^2 - y^2)] = \frac{\partial}{\partial x}[12ayx^2 - 4ay^3] = 24\ ayx$$

$\therefore$ $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$, hence the flow is **irrotational. (Ans.)**

Example 5.24. *Given that $u = xy$, $v = 2yz$. Examine whether these velocity components represent or three-dimensional incompressible flow; if three-dimensional, determine the third component.*

Solution. *Given:* $u = xy$, $v = 2yz$ ...velocity components

A two dimensional flow should satisfy the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

But
$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(xy) = y$$

and
$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2yz) = 2z$$

$\therefore$ $y + 2z \neq 0$

Hence the flow is **not two-dimensional.**

For the flow to be three-dimensional, it should satisfy the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

or
$$y + 2z + \frac{\partial w}{\partial z} = 0$$

or
$$\frac{\partial w}{\partial z} = -(y + 2z)$$

or
$$w = [-(y + 2z)]\,dz + f(x, y, t)$$

$$= -\left(yz + 2.\frac{z^2}{2}\right) + f(x, y, t)$$

$$= -yz + z^2 + f(x, y, t)$$

Hence the third components

$$\mathbf{w = -yz + z^2 + f(x, y, t)\ (Ans.)}$$

Example 5.25. *For a two-dimensional flow, the velocity components are $u = x/(x^2 + y^2)$, $v = y/(x^2 + y^2)$. Determine: (i) the acceleration components a_x and a_y; (ii) the rotation of w_z.*

Solution. *Given:* $a = x/(x^2 + y^2)$, $v = y/(x^2 + y^2)$ ... Velocity components

(i) **The acceleration components, a_x and a_y:**

We know that,

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \quad ...(i)$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \quad ...(ii)$$

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)=\frac{(x^2+y^2)\times 1-x(2x)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=\frac{\partial}{\partial y}[x(x^2+y^2)^{-1}]=x\times-(x^2+y^2)^{-2}\times 2y=-\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)=\frac{\partial}{\partial x}[y(x^2+y^2)^{-1}]=y\times-(x^2+y^2)^{-2}\times 2x=-\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{(x^2+y^2)\times 1-y\,(2y)}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

Substituting these values in eqns. (*i*) and (*ii*), we get

$$a_x=\frac{x}{(x^2+y^2)}\times\frac{(y^2-x^2)}{(x^2+y^2)^2}-\frac{y}{x^2+y^2}\times\frac{2xy}{(x^2+y^2)^2}$$

$$=\frac{xy^2-x^3}{(x^2+y^2)^3}-\frac{2xy^2}{(x^2+y^2)^3}=\frac{-x^3-xy^2}{(x^2+y^2)^3}=\frac{x(x^2+y^2)}{(x^2+y^2)^3}=-\frac{x}{(x^2+y^2)^2}$$

Hence $\mathbf{a_x=-\dfrac{x}{(x^2+y^2)^2}}$ **(Ans.)**

$$a_y=\left(\frac{x}{x^2+y^2}\right)\times\left[-\frac{2xy}{(x^2+y^2)^2}\right]+\frac{y}{(x^2+y^2)}\times\left(\frac{(x^2-y^2)}{(x^2+y^2)^2}\right)$$

$$=-\frac{2x^2y}{(x^2+y^2)^3}+\frac{y(x^2-y^2)}{(x^2+y^2)^3}=\frac{-2x^2y+x^2y-y^3}{(x^2+y^2)^3}=\frac{-y(x^2+y^2)}{(x^2+y^2)^3}=-\frac{y}{(x^2+y^2)^2}$$

Hence $\mathbf{a_y=-\dfrac{y}{(x^2+y^2)^2}}$ **(Ans.)**

(*ii*) **The rotation of ω_z:**

We know that,

$$\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=\frac{1}{2}\left[-\frac{2xy}{(x^2+y^2)^2}+\frac{2xy}{(x^2+y^2)^2}\right]=0$$

Hence the **flow is irrotational. (Ans.)**

Example 5.26. *If the velocity field is given by $u=(16y-8x)$, $v=(8y-7x)$ find the circulation around the closed curve defined by $x=4$, $y=2$, $x=8$, $y=8$.*

Solution. *Given:*

$$u=(16y-8x),\ v=(8y-7x) \qquad \text{...Velocity field}$$

Refer Fig 5.20

$$\Gamma_{ABCD}=\int_{ABCD}(udx+vdy)$$

$$=\int_{AB}(udx+vdy)+\int_{BC}(udx+vdy)+\int_{CD}(udx+vdy)+\int_{DA}(udx+vdy)$$

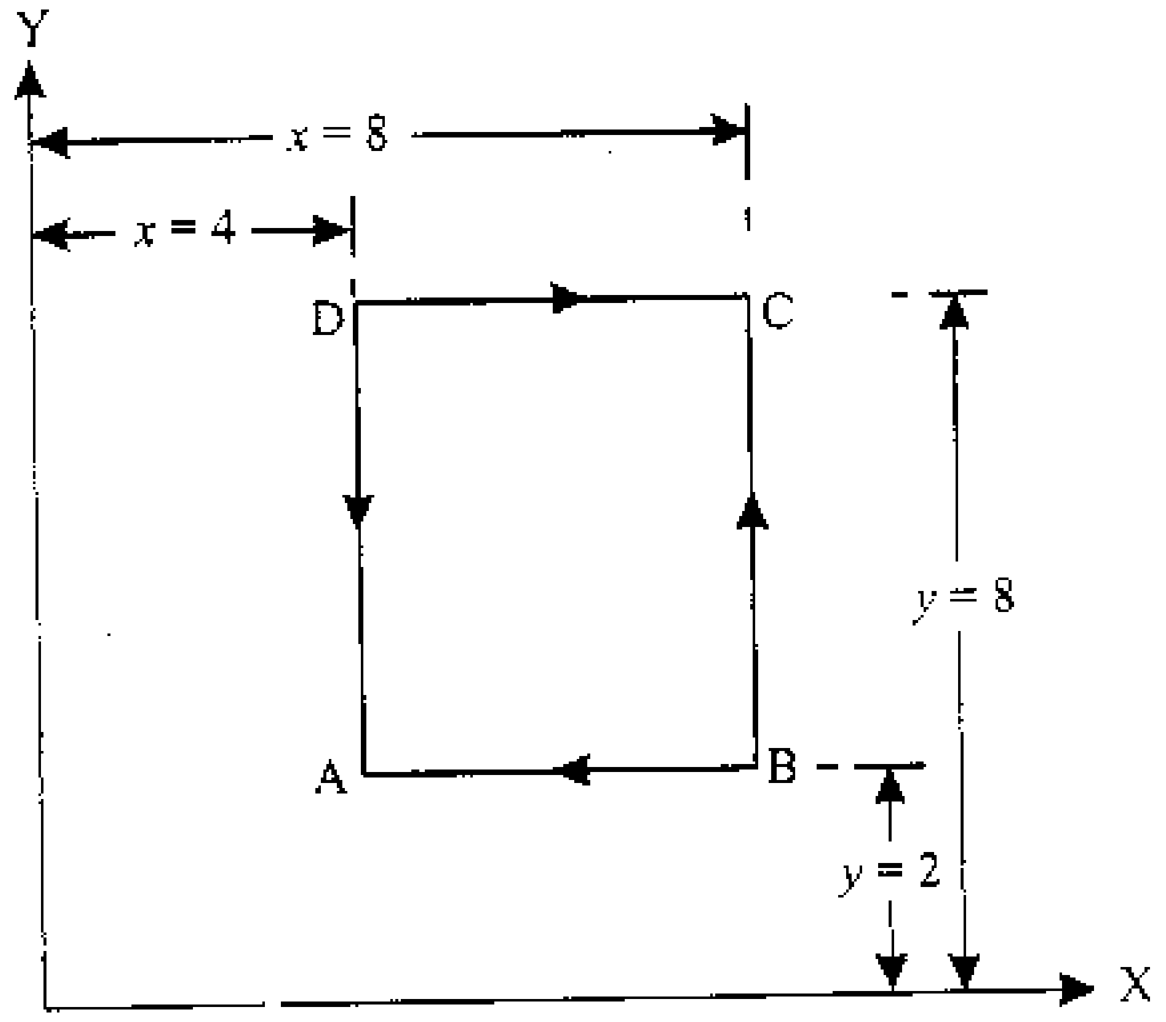

Fig. 5.20

$$= \int_4^8 (16y - 8x)\,dx + \int_2^8 (8y - 7x)\,dy + \int_8^4 (16y - 8x)\,dx + \int_8^2 (8y - 7x)\,dy$$

$$= \underset{(i)}{\left[16yx - 4x^2\right]_4^8} + \underset{(ii)}{\left[4y^2 - 7xy\right]_2^8} + \underset{(iii)}{\left[16yx - 4y^2\right]_8^4} + \underset{(iv)}{\left[4y^2 - 7xy\right]_8^2}$$

In integral (*i*): $y = 2$
In integral (*ii*): $x = 8$
In integral (*iii*): $y = 8$
In integral (*iv*): $x = 4$

Substituting these values, we have

$$\begin{aligned}\Gamma_{ABCD} &= [16 \times 2 \times 8 - 4 \times 8 \times 8 - 16 \times 2 \times 4 + 4 \times 4^2] \\ &+ [4 \times 8^2 - 2 \times 8 \times 8 - 4 \times 2^2 + 7 \times 8 \times 2] \\ &+ [16 \times 8 \times 4 - 4 \times 4^2 - 16 \times 8 \times 8 + 4 \times 8^2] \\ &+ [4 \times 2^2 - 7 \times 4 \times 2 - 4 \times 8^2 + 7 \times 4 \times 8] \\ &= [256 - 256 - 128 + 64] + [256 - 448 - 16 + 112] \\ &+ [512 - 64 - 1024 + 256] + [16 - 56 - 256 + 224] \\ &= -64 - 96 - 320 - 72 = -552\end{aligned}$$

Area of the curve $ABCD = (8 - 4) \times (8 - 2) = 24$

∴ Circulation per unit area $= -\dfrac{552}{24} = -23$ **(Ans.)**

Example 5.27. *A fluid flow is given by*

$$v_r = \left(1 - \frac{a}{r^2}\right)\cos\theta,\ v_t = -\left(1 + \frac{a}{r^2}\right)\sin\theta$$

(*i*) *Show that it represents a physically possible flow.*

(*ii*) *Determine whether the flow is rotational or irrotational.*

Solution. *Given:* $v_r = \left(1 - \dfrac{a}{r^2}\right)\cos\theta,\ v_\theta = -\left(1 + \dfrac{a}{r^2}\right)\sin\theta$ ...Velocity components (*i*) **Is the flow physically possible?**

The continuity equation for an incompressible fluid flow is given by

$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \qquad ...(1)$$

Now, $$\frac{\partial}{\partial r} v_r = \frac{\partial}{\partial r}\left(1 - \frac{a}{r^2}\right)\cos\theta = \left[-a\cos\theta[-2].\frac{1}{r^3}\right]$$

and $$\frac{\partial}{\partial \theta} v_\theta = \frac{\partial}{\partial \theta}\left[-\left(1 + \frac{a}{r^2}\right)\sin\theta\right] = \left[-\left(1 + \frac{a}{r^2}\right)\cos\theta\right]$$

Substituting these values in eqn. (1), we get

$$\frac{1}{r}\left(1 - \frac{a}{r^2}\right)\cos\theta + \left[-a\cos\theta + (-2).\frac{1}{r^2}\right] + \frac{1}{r}\left[-\left(1 + \frac{a}{r^2}\right)\cos\theta\right]$$

$$= \frac{\cos\theta}{r} - \frac{a\cos\theta}{r^3} + \frac{2a\cos\theta}{r^2} - \frac{\cos\theta}{r} - \frac{a\cos\theta}{r^3} = 0$$

Since the continuity equation is satisfied, therefore, **the flow is physically possible. (Ans.)**

(*ii*) **Flow–rotational or irrotational?**

Let us check for rotationality.

Vorticity is given by

$$\Omega = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \qquad ...(2)$$

Now $$\frac{\partial v_\theta}{\partial r} = \frac{\partial}{\partial r}(v_\theta) = \frac{\partial}{\partial r}\left[-\left(1 + \frac{a}{r^2}\right)\sin\theta\right] = \left[-a\sin\theta(-2)\frac{1}{r^3}\right]$$

and $$\frac{\partial v_r}{\partial \theta} = \frac{\partial}{\partial \theta}(v_r) = \frac{\partial}{\partial \theta}\left(1 - \frac{a}{r^2}\right)\cos\theta = \left[\left(1 - \frac{a}{r^2}\right)\times(-\sin\theta)\right]$$

Substituting these value in eqn. (2), we get

$$\Omega = \left[-a\sin\theta(-2).\frac{1}{r^3}\right] + \frac{1}{r}\left[-\left(1 + \frac{a}{r^2}\right)\sin\theta\right]$$

$$-\frac{1}{r}\left[\left(1 - \frac{a}{r^2}\right)\times(-\sin\theta)\right]$$

$$= \frac{2a\sin\theta}{r^3} - \frac{\sin\theta}{r} - \frac{a\sin\theta}{r^3} + \frac{\sin\theta}{r} - \frac{a\sin\theta}{r^3} = 0$$

Hence the **flow is irrotational. (Ans.)**

Example 5.28. *The velocity components for a fluid flow are:* $u = a + by - cz$, $v = d - bx - ez$, $w = f + cx - ey$ *where a, b, c, d, e and f are arbitrary constants.*

(*i*) *Show that it is a possible case of fluid flow.*

(*ii*) *Is the fluid flow irrotational? If not, determine the vorticity and rotation.*

[Allahabad University]

Solution. *Given:* $u = a + by - cz$, $v = d - bx - ez$, $w = f + cx - ey$ ...Velocity components.

(*i*) **Possible case of fluid flow?**

Continuity equation is given as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Now $\dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x}(a + by - cz) = 0$

$\dfrac{\partial v}{\partial y} = \dfrac{\partial}{\partial y}(d - bx - ez) = 0$, and $\dfrac{\partial w}{\partial z} = \dfrac{\partial}{\partial z}(f + cx - ey) = 0$

Since the equation of continuity is satisfied, therefore, the field is *possible case of fluid flow.* **(Ans.)**

(*ii*) **Is the flow field irrotational?**

For the flow to be irrotational, curl $V = O$ i.e. $(\Delta \times V) = 0$

$$\text{Now,}\quad (\nabla \times V) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Substituting velocity components, we have

$$(\nabla \times V) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (a + by - cz) & (d - bx - ez) & (f + cx - ey) \end{vmatrix}$$

$$\text{or}\quad (\nabla \times V) = i\left[\frac{\partial}{\partial y}(f + cx - ey) - \frac{\partial}{\partial z}(d - bx - ez)\right]$$

$$+ j\left[\frac{\partial}{\partial z}(a + by - cz) - \frac{\partial}{\partial x}(f + cx - ey)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(d - bx - ez) - \frac{\partial}{\partial y}(a + by - cz)\right]$$

$$= i(-e + e) + j(-c - c) + k(-b - b)$$

Since $(\nabla \times V) \neq 0$. the *flow is* **not irrotational (Ans.)**

Vorticity Ω:

Vorticity $\Omega = (\nabla \times V) = -2(cj + bk)$

$$= 2\sqrt{c^2 + b^2}$$

Hence $\boldsymbol{\Omega = 2\sqrt{c^2 + b^2}}$ **(Ans.)**

Rotation, ω:

we know, $\omega = \dfrac{\Omega}{2}$

$$= \frac{1}{2}[2\sqrt{c^2 + b^2}] = \sqrt{c^2 + b^2}$$

Hence rotation, $\boldsymbol{\omega = \sqrt{c^2 + b^2}}$ **(Ans.)**

Example 5.29. *Determine the components of rotation for the following velocity field pertaining the flow of an incompressible fluid:*

$$u = Cyz;\quad v = Czx;\ w = Cxy,\ \text{where } C = \text{constant}.$$

State whether the flow is rotational or irrotational.

Solution. *Given:* $u = Cyz;\ v = Czx;\ w = Cxy$ ... Velocity field

The components of rotation are:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}(Cx - Cx) = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(Cy - Cy) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(Cz - Cz) = 0$$

Since each of the rotation components is zero, the given flow field represents **irrotational flow** .**(Ans.)**

Example 5.30. *Determine the components of rotation about the various axes for the following flows:*

(*i*) $u = y^2,\ v = -3x$

(*ii*) $u = 3xy,\ v = \frac{3}{2}x^2 - \frac{3}{2}y^2$

(*iii*) $u = xy^3z,\ v = -y^2z^2,\ w = yz^2 - \frac{y^3z^2}{2}$

Solution. The components of rotation about the various axes are:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

(*i*) $u = y^2;\ v = -3x$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(-3 - 2y) \quad \textbf{(Ans.)}$$

As the flow is two-dimensional in x – y plane, $\omega_x = \omega_y = 0$ **(Ans.)**

(*ii*) $u = 3xy;\ v = \frac{3}{2}x^2 - \frac{3}{2}y^2$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(3x - 3x) = 0 \quad \textbf{(Ans.)}$$

As the flow is two-dimensional in the $x - y$ plane, $\omega_x = \omega_y = 0$

(*iii*) $u = xy^3z;\ v = -y^2z^2;\ w = yz^2 - \frac{y^3z^2}{2}$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(0 - 3xy^2z) = -\frac{3}{2}xy^2z \quad \textbf{(Ans.)}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left(z^2 - \frac{3y^2z^2}{2} + 2y^2z\right) \quad \textbf{(Ans.)}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(xy^3 - 0) = \frac{1}{2}xy^3 \quad \textbf{(Ans.)}$$

5.10 Velocity Potential and Stream Function

5.10.1. Velocity Potential

The **velocity potential** is *defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction.* It is denoted by ϕ (phi). Thus mathematically the velocity potential is defined as:

$$\phi = f(x, y, z, t) \quad \text{...for unsteady flow,}$$

and

$$\phi = f(x, y, z) \quad \text{...for steady flow; such that}$$

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\} \quad ...(5.35)$$

where u, v and w are the components of velocity in the x, y and z directions respectively.

The *negative sign* signifies that *ϕ decreases with an increase in the values of x, y and z.* In other words it indicates that the flow is always in the direction of decreasing ϕ.

For *an incompressible steady flow* the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

By substituting the values of u, v and w in terms of ϕ from eqn. 5.35, we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

$$\frac{d^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad ...(5.36)$$

This equation is known as **Laplace equation.**

Thus *any function ϕ that satisfies the Laplace equation will correspond to some case of fluid flow.*

The *rotational components* are given by [eqn. (5.30)]

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

By substituting the values of u, v and w in term of ϕ from eqn. (5.35), we get

$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y\,\partial z} + \frac{\partial^2 \phi}{\partial z\,\partial y}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z\,\partial x} + \frac{\partial^2 \phi}{\partial x\,\partial z}\right]$$

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

However, if ϕ is a continuous function then,

$$\frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y};\ \frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z};\ \text{and}\ \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$$

$\therefore\quad \omega_x = \omega_y = \omega_z = 0$ i.e. the flow is *irrotational*.

Thus if *velocity potential* (ϕ) *satisfies the Laplace equation, it represents the possible steady, incompressible, irrotational flow.* Often an *irrotational flow is known as* ***potential flow.***

Equipotential line:

An ***equipotential line*** *is one along which velocity potential ϕ is constant*

i.e. For equipotential line, ϕ = constant.

$$\therefore\quad d\phi = 0$$

But $\phi = f(x, y)$ for steady flow

$$\therefore\quad d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

But
$$\frac{\partial \phi}{\partial x} = -u \text{ and } \frac{\partial \phi}{\partial y} = -v$$

$$\therefore\quad d\phi = -udx - vdy = -(udx + vdy)$$

For equipotential line, $d\phi = 0$

or
$$-(udx + vdy) = 0$$

or
$$(udx + vdy) = 0$$

or
$$\frac{dy}{dx} = -\frac{u}{v} \qquad ...(5.37)$$

where $\frac{dy}{dx}$ = slope of equipotential line.

5.10.2. Stream Function

The ***stream function*** is *defined as a function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to this direction.* It is denoted by ψ (psi).

In case of two-dimensional flow, the stream function may be defined mathematically as

$$\psi = f(x, y, t) \qquad \text{...for unsteady flow, and}$$

$$\psi = f(x, y) \qquad \text{... for steady flow; such that}$$

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \qquad ...(5.38)$$

For two-dimensional flow the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the values of u and v from eqn. (5.38), we get

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

Hence **existence of ψ means a possible case of fluid flow.**

— The flow may be rotational or irrotational.

The rotational component ω_z is given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Substituting the values of u and v from eqn. (5.38), we get.

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)\right]$$

or

$$\omega_z = -\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right] \quad ...(5.39)$$

This equation is known as **Poisson's equation.** For an irrotational flow since $\omega_z = 0$, eqn.(5.39) becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \text{ i.e., } \Delta^2 \psi = 0$$

which is the *Laplace equation in* ψ.

In the polar co-ordinates:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \; v_\theta = \frac{\partial \psi}{\partial r}$$

— Let $\psi(x, y)$ represent the streamline L (See Fig. 5.21). The $\psi + d\psi$ represents the adjacent streamline M. The velocity vector V perpendicular to the line AB has components u and v in the direction of X-axis and Y-axis respectively. From continuity consideration, we have

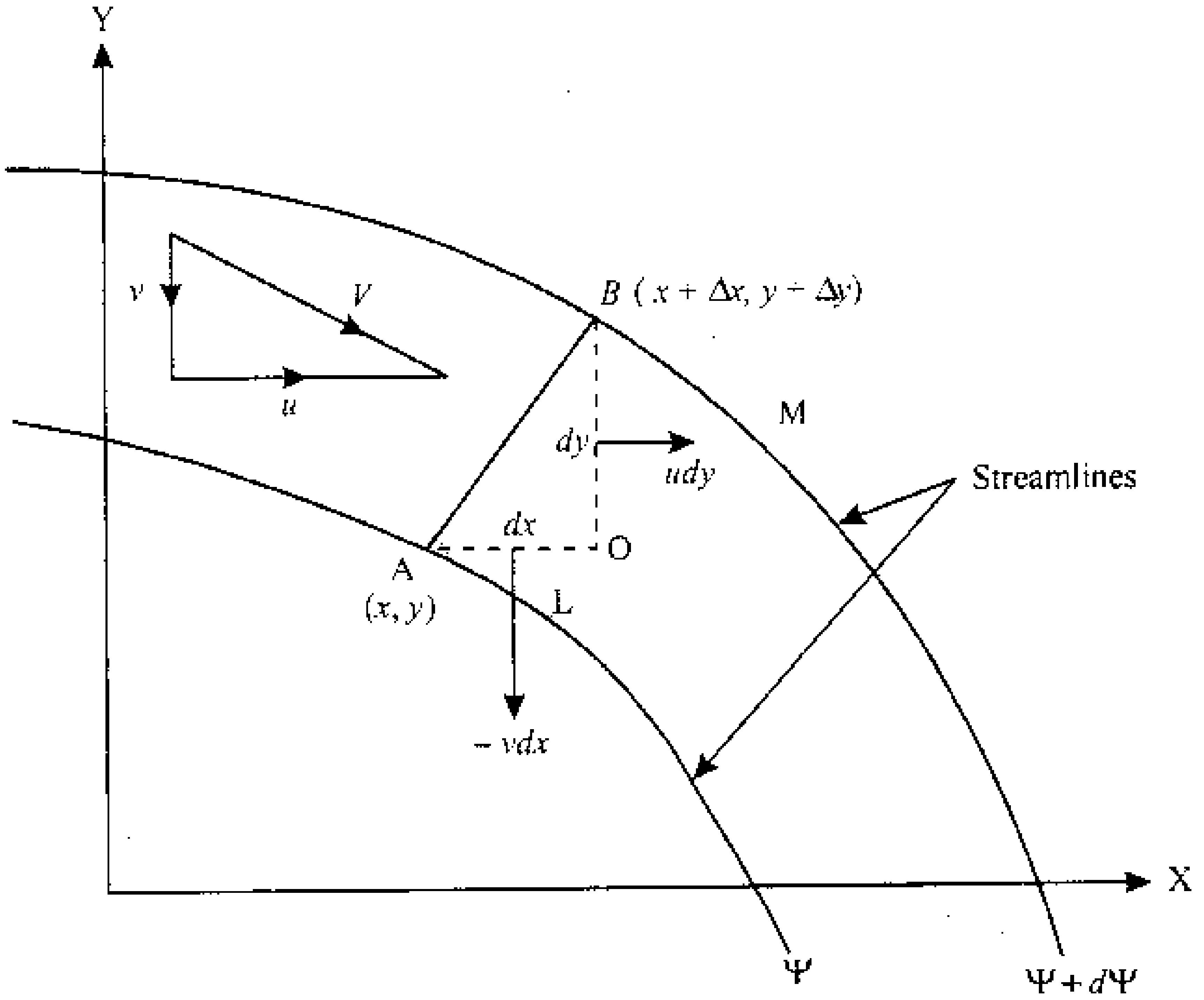

Fig. 5.21. Flow between two points and its relation to stream function.

Flow across AB = flow across AO + flow across OB

$$Vds = -vdx + udy$$

(The minus sign indicates that the velocity v is acting in the downward direction).

$$Vds = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = d\psi \quad ...(5.40)$$

i.e.

$$dq = d\psi \quad ...(5.41)$$

Obviously, the *stream function* can also be defined as the *flux or flow rate between two streamlines*. The units of ψ are m^3/s; discharge per unit thickness of flow.

Properties of stream function:

The properties of stream function are:

1. On any streamline, ψ is constant everywhere.

   [ψ = constant, represents the family of streamlines.
   ψ = constant, is a streamline equation.]

2. If the flow is continuous, the flow around any path in the fluid is zero.
3. The rate of change of ψ with distance in arbitrary direction is proportional to the component of velocity normal to that direction.
4. The algebraic sum of stream function for two incompressible flow patterns is the stream function for the flow resulting from the superimposition of these patterns,

i.e.

$$\frac{\partial \psi_1}{\partial s} + \frac{\partial \psi_2}{\partial s} = \frac{\partial (\psi_1 + \psi_2)}{\partial s} \quad ...(5.42)$$

Cauchy Riemann equations:

From the above discussion of velocity potential function and stream function we arrive at the following conclusions:

- *Potential function* (ϕ) *exists only for irrotational flow.*
- *Stream function* (ψ) *applies to both the rotational and irrotational flows* (which are steady and incompressible).
- In case of *irrotational flow, both the stream function and velocity function satisfy Laplace equation and as such they are interchangeable.*

For irrotational incompressible flow, the following relationship between ϕ and ψ holds good.

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad ...(5.43)$$

These equations, in hydrodynamics, are sometimes called *Cauchy Riemann equations.*

5.10.3. Relation between Stream Function and Velocity Potential

One of the properties of a stream function is that the *difference of its values at two points represent the flow across any line joining the points.* Thus if two lie on the same streamline, then, there being no flow across a streamline, the difference between the stream functions ψ_1 and $\psi_2 = 0$; this means the streamline is given by

$$\psi = \text{constant}.$$

Similarly ϕ = constant, represents a case for which the velocity potential is same at every point, and hence it represents an *equipotential line.*

Let two curves ϕ = constant and ψ = constant intersect each other at any point. At the point of intersection the slopes are:

For the curve ϕ = *constant*: Slope $= \dfrac{\partial y}{\partial x} = \dfrac{\left(\dfrac{\partial \phi}{\partial x}\right)}{\left(\dfrac{\partial \phi}{\partial y}\right)} = \dfrac{-u}{-v} = \dfrac{u}{v}$

For the curve ψ = *constant*: Slope $= \dfrac{\partial y}{\partial x} = \dfrac{\left(\dfrac{\partial \psi}{\partial x}\right)}{\left(\dfrac{\partial \psi}{\partial y}\right)} = \dfrac{-v}{+u} = -\dfrac{v}{u}$

Now, product of the slopes of these curves

$$= \frac{u}{v} \times -\frac{v}{u} = -1$$

It shows that these two sets of curves, viz *streamlines* and *equipotential lines intersect each other orthogonally at all points of intersection.*

5.11 Flow Nets

A grid obtained by drawing a series of streamlines and equipotential lines is known as a **flow net**. The flow net provides a simple graphical technique for studying two-dimensional irrotational flows especially in the cases where mathematical relations for stream function and velocity function are either not available or are rather difficult and cumbersome to solve.

5.11.1. Methods of Drawing Flow Nets

The following methods are used for drawing flow nets.

1. **Analytical method (or Mathematical analysis):**
   - Here, the equations corresponding to the curves ϕ and ψ are first obtained and the same are plotted to give the flow net pattern for the flow of fluid between the given boundary shape.
   - This method can be applied to problems with simple and ideal boundary conditions.
2. **Graphical method:**
   - A graphical method consists of drawing streamlines and equipotential lines such that they cut orthogonally and form curvilinear squares.
   - This method consumes lot of time and requires lot of erasing to get the proper shape of a flow net.
3. **Electrical analogy method:**
   - This method is a practical method of drawing a flow net for a particular set of boundaries.
   - It is based on the fact that the flow of fluids and flow of electricity through a conductor are analogous. These two systems are similar in the respect that electric potential is analogous to the velocity potential, the electric current is analogous to the velocity of flow, and the homogeneous conductor is analogous to the homogeneous fluid.
4. **Hydraulic models:**
   - Streamlines can be traced by injecting a dye in a seepage model or Heleshaw apparatus.
   - Then by drawing equipotential lines the flow net is completed.

Fig 5.22. shows some typical flow nets.

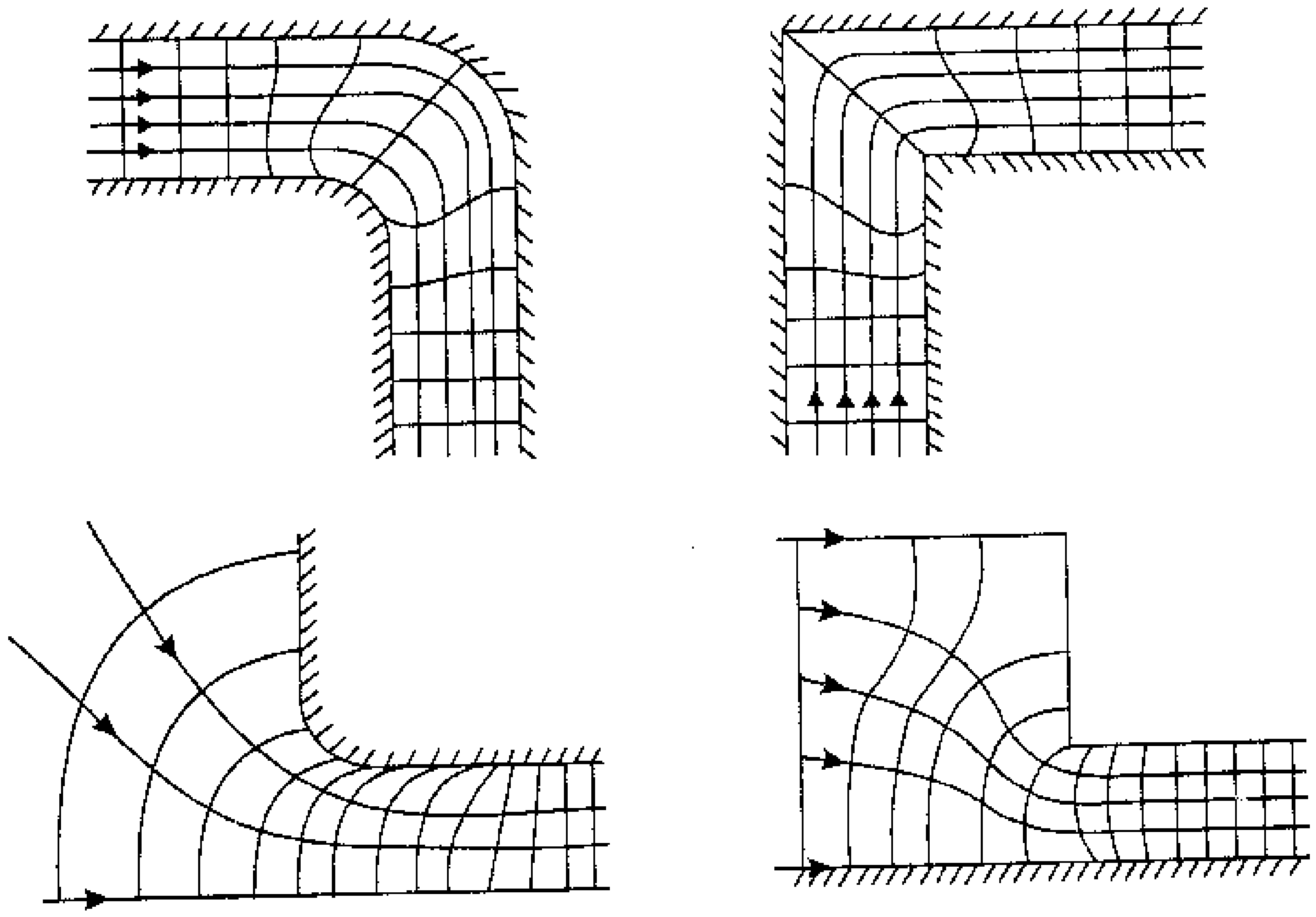

Fig. 5.22. Typical flow nets.

5.11.2. Uses and Limitations of Flow Nets

Use of flow nets:

The following are the *uses* of flow-net analysis:

1. To determine the streamlines and equipotential lines.
2. To determine quantity of seepage and upward lift pressure below hydraulic structure.
3. To determine the velocity and pressure distribution, for given boundaries of flow (provided the velocity distribution and pressure at any reference section are known).
4. To determine the design of the outlets for their streamlining.

Limitations of flow nets:

The following are the *limitations* of flow net:

1. The flow net analysis cannot be applied in the region close to the boundary where the effects of viscosity are predominant.
2. In case of a flow of a fluid past a solid body, while the flow net gives a fairly accurate picture of the flow pattern for the upstream part of the solid body, it can give little information concerning the flow conditions at the rear because of separation and eddies.

Example 5.31. *Verify whether the following functions are valid potential functions:*

(*i*) $\phi = A(x^2 - y^2)$ (*ii*) $\phi = A \cos x$

Solution. (*i*) $\phi = A\,(x^2 - y^2)$

$$\frac{\partial \phi}{\partial x} = 2Ax\,; \qquad \frac{\partial \phi}{\partial y} = -2Ay$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2A\,; \qquad \frac{\partial^2 \phi}{\partial y^2} = -2A$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2A + (-2A) = 0$$

Hence $\phi = A\,(x^2 - y^2)$ is a **valid potential function (Ans.)**

(*ii*) $\phi = A \cos x$

$$\frac{\partial \phi}{\partial x} = -\,A \sin x\,;\ \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\,A \cos x,\ \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\,A \cos x \neq 0$$

Hence $\phi = A \cos x$ is **not a valid function (Ans.)**

Example 5.32. *Which of the following functions represent possible irrotational flow?*

(*i*) $\psi = A(x^2 - y^2)$ (*ii*) $\psi = xy$

(*iii*) $\phi = \left(r - \frac{2}{r}\right)\sin\theta$ (*iv*) $\phi = Ur\cos\theta + \frac{U}{r}\cos\theta$

Solution. *For an irrotational fluid flow phenomenon* ϕ *as well* ψ *satisfy Laplace equation.*

(*i*) $\psi = A(x^2 - y^2)$

$$\frac{\partial \psi}{\partial x} = 2Ax\,;\ \frac{\partial \psi}{\partial y} = -\,2Ay$$

$$\frac{\partial^2 \psi}{\partial x^2} = 2A;\ \frac{\partial^2 \psi}{\partial y^2} = -\,2A$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2A - 2A = 0$$

Hence $\psi = A(x^2 - y^2)$ represents a **possible irrotational flow (Ans.)**

(*ii*) $\psi = xy$

$$\frac{\partial \psi}{\partial x} = y\,;\quad \frac{\partial \psi}{\partial y} = x$$

$$\frac{\partial^2 \psi}{\partial x^2} = 0\,;\quad \frac{\partial \psi}{\partial y} = 0$$

$$\therefore \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Hence $\psi = xy$ represents a **possible irrotational flow. (Ans.)**

(*iii*) $\phi = \left(r - \frac{2}{r}\right)\sin\theta$

Laplace equation in radial coordinates (r, θ):

$$\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$\frac{\partial \phi}{\partial r} = \left(1 + \frac{2}{r^2}\right)\sin\theta$$

$$\frac{\partial^2 \phi}{\partial r^2} = -\,\frac{2}{r^3}\sin\theta$$

$$\frac{\partial \phi}{\partial \theta} = \left(r - \frac{2}{r}\right)\cos\theta$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = -\left(r - \frac{2}{r}\right)\sin\theta = \left(\frac{2}{r} - r\right)\sin\theta$$

L.H.S. of Laplace equation is

$$\frac{1}{r} \times \left(1 + \frac{2}{r^2}\right)\sin\theta - \frac{2}{r^3}\sin\theta + \frac{1}{r^2}\left(\frac{2}{r} - r\right)\sin\theta$$

$$= \sin\theta\left(\frac{1}{r} + \frac{2}{r^3} - \frac{2}{r^3} + \frac{2}{r^3} - \frac{1}{r}\right)$$

$$= \sin\theta\left(\frac{2}{r^3}\right) \neq 0$$

Hence, the given function **doesnot represent any possible irrotational flow. (Ans.)**

(*iv*)

$$\phi = Ur\cos\theta + \frac{U}{r}\cos\theta$$

$$\frac{\partial \phi}{\partial r} = U\cos\theta - \frac{U}{r^2}\cos\theta$$

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{2U}{r^3}\cos\theta$$

$$\frac{\partial \phi}{\partial \theta} = -Ur\sin\theta - \frac{U}{r}\sin\theta$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = -Ur\cos\theta - \frac{U}{r}\cos\theta$$

The Laplace equation, in radial coordinates (r,θ) is

$$\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Substituting for L.H.S. terms, we get

$$= \frac{1}{r}\left(U\cos\theta - \frac{U}{r^2}\cos\theta\right) + \frac{2U}{r^3}\cos\theta + \frac{1}{r^2}\left(-Ur\cos\theta - \frac{U}{r}\cos\theta\right)$$

$$U\cos\theta\left(\frac{1}{r} - \frac{1}{r^3} + \frac{2}{r^3} - \frac{1}{r} - \frac{1}{r^3}\right) = 0$$

The Laplace equation is satisfied and hence the given function ϕ **represents a possible irrotational flow**.

Example 5.33. *The velocity components in a fluid flow are given by*

$$u = 2xy;\ v = a^2 + x^2 - y^2$$

(*i*) *Show that the flow is possible.*

(*ii*) *Derive the relative stream function.*

Solution. *Given*: $u = 2xy$; $v = a^2 + x^2 - y^2$ ...Velocity components

(*i*) $\frac{\partial u}{\partial x} = 2y$; $\frac{\partial v}{\partial y} = -2y$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2y + (-2y) = 0$$

The *continuity equation for steady, incompressible flow is satisfied.* Hence, **flow is possible. (Ans.)**

(*ii*) The stream function ψ is related to u and v as:

$$u = \frac{\partial \psi}{\partial y} = 2xy$$

or $\quad \psi = \int 2xy\, dy = xy^2 + f(x)$ ...(*i*)

$$-\frac{\partial \psi}{\partial x} = -y^2 - f'(x) = v = a^2 + x^2 - y^2$$

Hence $\quad f'(x) = -(a^2 + x^2)$

or $\quad f(x) = -a^2 x - \frac{x^3}{3} + \text{constant}$

Inserting for $f(x)$ in eqn. (*i*) we get

$$\psi = xy^2 - ax^2 - \frac{x^3}{3} + \text{constant}$$

Thus the relative $\boldsymbol{\psi} = xy^2 - a^2x - \frac{x^3}{3} + \text{constant}$ **(Ans.)**

Example 5.34. *A stream function is given by:*

$$\psi = 3x^2y + (3 + t)y^2$$

Find the flow rates across the faces of the triangular prism having a thickness of 2.5 m in the Z-direction at the time instant t = 3 seconds.

Solution. Refer Fig. 5.23.

$\psi = 3x^2y + (3 + t)\,y^2$ ...Given

The coordinates of point M are: (0,1)

$\therefore \quad \psi_M = 0 + (3 + 3) \times 1^2 = 6$

The coordinates of point L are: (1.5, 0)

$\therefore \quad \psi_L = 3 \times 1.5^2 \times 0 + (3 + 3) \times 0 = 0$

The coordinates of point L are: (0,0)

$\therefore$ *Flow rate across face* MO = 2.5 $(\psi_M - \psi_O)$

= 2.5 (6 – 0) = **15 m³/s (Ans.)**

Flow rate across face LO = 2.5 $(\psi_L - \psi_O)$

= 2.5 (0 – 0) = 0

$\therefore$ *Flow rate across face* LM = 2.5 $(\psi_M - \psi_L)$

= 2.5 (6 – 0) = **15 m³/s (Ans.)**

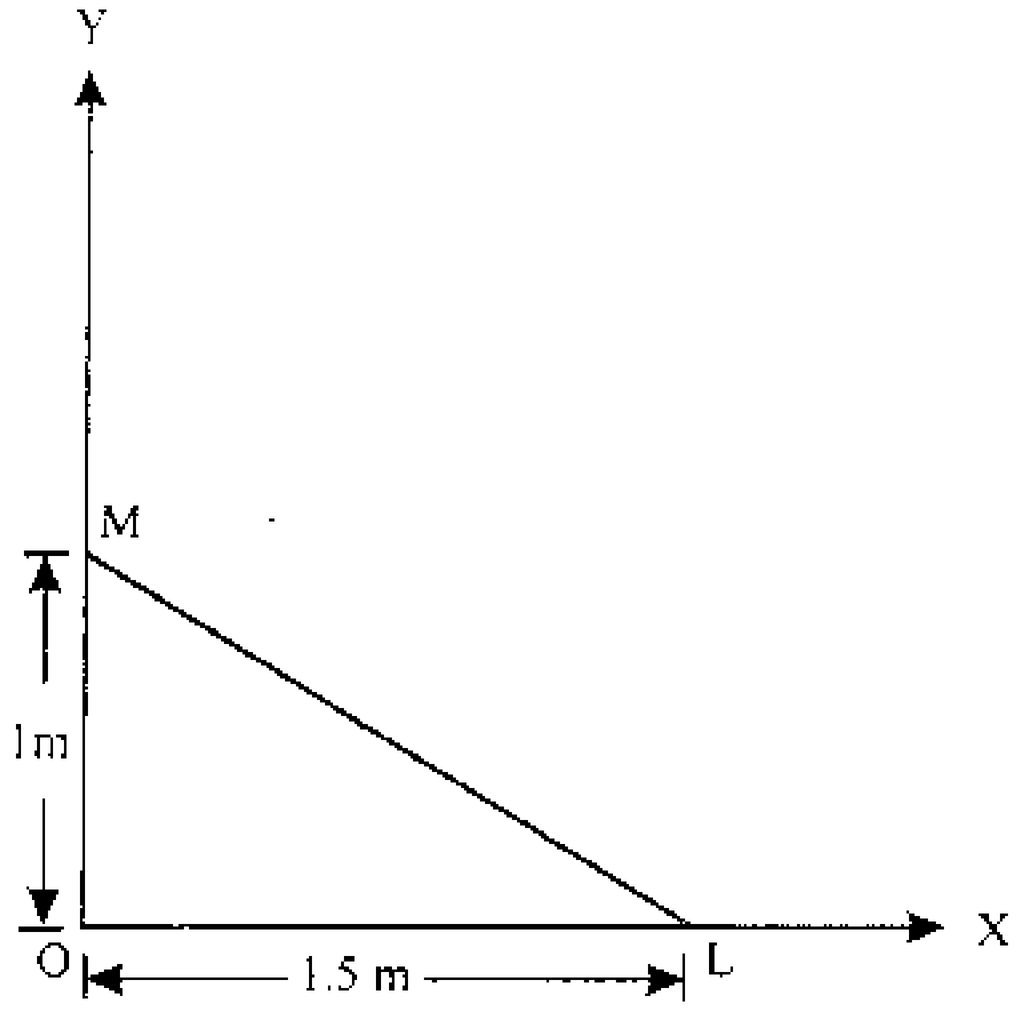

Fig. 5.23

Example 5.35. *A flow is described by the stream function ψ = 4xy. Locate the point at which the velocity vector has a magnitude 7 units and makes an angle of 150° with X-axis.*

Solution. Stream function, $\psi = 4xy$ ...Given

The velocity components for the given flow field are:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(4xy) = 4x$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(4xy) = -4y$$

$$V = \sqrt{u^2 + v^2} \quad \text{or} \quad 7 = \sqrt{(4x)^2 + (-4y)^2} = 4\sqrt{x^2 + y^2} \qquad \text{...(i)}$$

$$\tan\theta = \frac{v}{u} \quad \text{or} \quad \tan 150° = \frac{-4y}{4x} = -\frac{y}{x}$$

or $\quad -0.577 = -\frac{y}{x} \quad$ or $\quad y = 0.577\,x$

Substituting, for y in eqn. (*i*), we get.

$$7 = 4\sqrt{x^2 + (0.577x)^2} = 4.62\ x$$

$$\therefore \quad x = \frac{7}{4.62} = \mathbf{1{\cdot}515\ Ans.}$$

Example. 5.36. *Find a relevant stream function to each of the following sets of velocity components of steady, incompressible flow:*

(*i*) $u = 2cx;\ v = -2cy$

(*ii*) $u = -cx/y;\ v = c \log_e xy$

(*iii*) $u = x + y\ ;\ v = x - y$

Solution. (*i*) $u = 2cx;\ v = -2cy$

$$\frac{\partial u}{\partial x} = 2c\ ;\ \frac{\partial v}{\partial x} = -2c$$

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = 2c + (-2c) = 0$$. Hence the *flow is possible and* ψ *exists.*

$$u = \frac{\partial \psi}{\partial y} = 2cx$$

$$\psi = 2cxy - f(x)$$

$$-\frac{\partial \psi}{\partial x} = -2cy - f'(x) = v = -2cy$$

Hence $f'(x) = 0$ and $f(x) = c_1 = a$ constant

$$\therefore \quad \boldsymbol{\psi = 2cxy + c_1}\ \textbf{(Ans.)}$$

(*ii*) $u = -cx/y;\ v = c \log_e xy$

$$\frac{\partial u}{\partial x} = -c/y\ ;\ \frac{\partial v}{\partial y} = c/y$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -c/y + c/y = 0$$

Hence the *flow is possible and* ψ *exists.*

$$u = \frac{\partial \psi}{\partial y} = -cx/y$$

$$\psi = -cx \log_e y + f(x)$$

$$-\frac{\partial \psi}{\partial x} = c\log_e y - f'(x) = v = c\log_e xy = c\log_e x + c\log_e y$$

or $\quad f'(x) = -c\log_e x$

$$f(x) = -\int c\log_e x.dx = -c(x\log_e x - x) + c_2$$

where c_2 = a constant.

Hence the stream function representing this flow is

$$\therefore \quad \psi = -cx\log_e y - c\,(x\log_e x - x) + c_2$$
$$= -cx\log_e y - cx\log_e x + cx + c_2$$

or $\quad \boldsymbol{\psi = -cx\log_e xy + cx + c_2}$ **(Ans.)**

(*iii*) $u = x - y;\ v = x - y$

$$\frac{\partial u}{\partial x} = 1\ ;\ \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 + (-1) = 0$$. Hence the *flow is possible and* ψ *exists.*

$$u = \frac{\partial \psi}{\partial y} = x + y$$

$$\psi = xy - \frac{y^2}{2} + f(x)$$

$$-\frac{\partial \psi}{\partial x} = -y - f'(x) = v = x - y$$

or $$f'(x) = -x$$

and $$f(x) = \int -x\,dx = -\frac{x^2}{2} + c \text{, where } c = \text{constant}$$

$\therefore$ $$\psi = xy + \frac{y^2}{2} - \frac{x^2}{2} - c$$

or $$\psi = \frac{1}{2}(y^2 - x^2) + xy + c \text{ (Ans.)}$$

Example 5.37. *For the following stream functions calculate velocity at a point (1, 2):*

(*i*) $\psi = 3xy$ (*ii*) $y = 3x^2y - y^3$

Solution. (*i*) $\psi = 3xy$ ... (Given)

$$u = \frac{\partial \psi}{\partial y} = 3x$$

$$v = -\frac{\partial \psi}{\partial x} = -3y$$

At (1,2): $u = 3 \times 1 = 3$

$$v = -3 \times 2 = -6$$

$\therefore$ $$\mathbf{V} = \sqrt{u^2 + v^2} = \sqrt{(3)^2 + (-6)^2} = \sqrt{45} \text{ units. (Ans.)}$$

(*ii*) $\psi = 3x^2y - y^3$ ...(Given)

$$u = \frac{\partial \psi}{\partial y} = 3x^2 - 3y^2$$

$$v = -\frac{\partial \psi}{\partial x} = -6xy$$

At (1,2): $u = 3 \times (1)^2 - 3 \times (2)^2 = -9$

$$v = -6 \times 1 \times 2 = -12$$

$$\mathbf{V} = \sqrt{u^2 + v^2} = \sqrt{(-9)^2 + (-12)^2} = 15 \text{ (Ans.)}$$

Example 5.38. *What is the irrotational velocity field associated with the potential* $\phi = 3x^2 - 3x$ $3y^2 + 16t^2 + 12zt$. *Does the flow field satisfy the incompressible continuity equation?*

[UPSC Exam. CES, Fluid Mechanics]

Solution. *Given:* $\phi = 3x^2 - 3x + 3y^2 + 16t^2 - 12zt$

The velocity field is represented by

$$u = -\frac{\partial \phi}{\partial x}, v = -\frac{\partial \phi}{\partial y}$$

$\therefore$ $$u = -\frac{\partial}{\partial x}(3x^2 - 3x + 3y^2 + 16t^2 + 12zt) = -6x + 3$$

and $$v = -\frac{\partial}{\partial y}(3x^2 - 3x - 3y^2 + 16t^2 + 12zt) = -6y$$

Also $$\frac{\partial u}{\partial x} = \frac{\partial}{dx}(-6x + 3) = -6$$

and $$\frac{\partial v}{\partial y} = \frac{\partial}{dy}(-6y) = -6$$

The continuity equation for an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{dy} = 0$$

Substituting the values, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{dy} = -6 - 6 = -12$$

This shows that the given velocity field **does not satisfy the continuity equation. (Ans.)**

Example 5.39. *The velocity potential function for a two-dimensional flow is $\phi = x(2y - 1)$. At a point $P(4, 5)$ determine:*

(*i*) *The velocity, and*

(*ii*) *The value of stream function.* **[Panjab University]**

Solution. *Given*: $\phi = x(2y - 1)$ ...Velocity potential function.

(*i*) **The velocity at a point P(4, 5).**

The velocity components in x and y directions are:

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{dx}[x(2y - 1)] = -2y + 1$$

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}[x(2y - 1)] = -2x$$

∴ Resultant velocity,

$$V = \sqrt{u^2 + v^2} = \sqrt{(-2y + 1)^2 + (-2x)^2}$$

∴ At the point $P(4.5)$ when $x = 4$ and $y = 5$

$$V = \sqrt{(-2 \times 5 + 1)^2 + (-2 \times 4)^2} = \sqrt{9^2 + 8^2} = \textbf{12.04 units (Ans.)}$$

(*ii*) **The value of stream function at the point (4, 5):**

For stream function,

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

or, $$d\psi = -vdx + udy$$

or, $$d\psi = +2xdx + (-2y + 1)dy$$

Integrating both sides, we get

$$\psi = +2 \times \frac{x^2}{2} + \left(-2 \times \frac{y^2}{2} + y\right) + C$$

(where C = constant of integration)

For $\psi = 0$ at the origin, the constant $C = 0$

∴ $$\psi = +x^2 - y^2 + y$$

At the point $P(4,5)$

$$\psi = +(4)^2 - (5)^2 + 5 = -\textbf{4 units (Ans.)}$$

Example 5.40. *For a two-dimensional flow the velocity function is given by the expression, $\phi = x^2 - y^2$.*

(*i*) *Determine velocity components in x and y directions.*

(*ii*) *Show that the velocity components satisfy the conditions of flow continuity and irrotationality.*

(*iii*) *Determine stream function and the flow rate between the streamlines (2, 0) and (2, 2).*

(*iv*) *Show that the streamlines and potential lines intersect orthogonally at the point (2, 2).*

Solution. *Given*: $\phi = x^2 - y^2$ ...Velocity function.

(*i*) **Velocity components in x and y directions:**

The velocity components in x and y directions are:

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}(x^2 - y^2) = -2x \text{ (Ans.)}$$

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}(x^2 - y^2) = +2y \text{ (Ans.)}$$

(*ii*) **Continuity, irrotationality = ?**

From the velocity components, we have

$$\frac{\partial u}{\partial x} = -2, \frac{\partial v}{\partial y} = +2$$

Conditions of flow continuity will be satisfied if

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the values, we get $-2 + 2 = 0$

Hence the **velocity components satisfy the flow continuity conditions (Ans.)**

Now $$\nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x & +2y & 0 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(+2y)\right] + j\left[\frac{\partial}{\partial z}(-2x) - \frac{\partial}{\partial x}(0)\right]$$

$$+ \left[\frac{\partial}{\partial x}(+2y) - \frac{\partial}{\partial y}(-2x)\right]$$

Since curl V is zero hence the flow is **irrotational.** **...(Proved)**

(*iii*) **Stream function and flow rate:**

The differential $d\psi$ for the stream function is (eqn. 5.40)

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy$$

$$= -(+2y)\, dx + (-2x)\, dy = -2d(xy)$$

Integrating, we get

$$\psi = -2xy + C \text{ (Ans.)}$$

(where C = constant of integration)

Now $\psi(2, 0) = 2 \times 2 \times 0 = 0$

and $\psi(2, 2) = 2 \times 2 \times 2 = 8$

Hence flow between the streamlines through (2, 0) and (2, 2)

$$= 8 - 0 = 8 \text{ m}^3\text{/s (Ans.)}$$

(*iv*) **Intersection of streamlines and potential lines orthogonally at point (2, 2) = ?**

Slope of streamline,

$$\left(\frac{\partial y}{\partial x}\right)_{\psi = \text{const.}} = -\frac{v}{u} = -\frac{+2y}{-2x} = 1 \text{ at } (2, 2)$$

Slope of potential line,

$$\left(\frac{\partial y}{\partial x}\right)_{\phi = \text{const.}} = \frac{u}{v} = \left(\frac{-2x}{+2y}\right) = -1 \text{ at } (2, 2)$$

Thus, $$\left(\frac{\partial y}{\partial x}\right)_{\psi = \text{const.}} \times \left(\frac{\partial y}{\partial x}\right)_{\phi = \text{cosnt.}} = 1 \times (-1) = -1$$

which shows that the streamlines and the potential lines intersect *orthogonally* ....**(Proved)**

Example 5.41. *A two-dimensional flow field is given by $\phi = 3xy$, determine:*

(*i*) *The stream function*

(*ii*) *The velocity at L(2, 6) and M (6,6) and the pressure difference between the points L and M.*

(*iii*) *The discharge between the streamlines passing through the points L and M.*

Solution. *Given*, $\phi = 3xy$ ...Flow field

(*i*) **The stream function ψ:**

We know, $$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}(3xy) = -3y$$

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}(3xy) = -3x$$

Also $$u = \frac{\partial \psi}{\partial y} = -3y, \text{ and } v = -\frac{\partial \psi}{\partial x} = -3x$$

Again $$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \quad \text{or} \quad \partial\psi = 3x dx + (-3y)\, dy$$

Integrating both sides, we get

$$\psi = \int 3x\, dx + \int (-3y)\, dy$$

$$= 3 \times \frac{x^2}{2} - 3 \times \frac{y^2}{2} + C = \frac{3}{2}(x^2 - y^2) + C$$

(where C = constant of integration.)

For $\psi = 0$ at the origin, the constant $C = 0$

$$\therefore \quad \psi = \frac{3}{2}(x^2 - y^2) \text{ (Ans.)}$$

(*ii*) **Velocities at L and M:**

At $L(2, 6)$: $u = -3 \times 6 = -18$, $v = -3 \times 2 = -6$

$$\therefore \quad \mathbf{V_L} = \sqrt{u^2 + v^2} = \sqrt{(-18)^2 + (-6)^2} = \mathbf{18.97 \text{ units (Ans.)}}$$

At M(6, 6): $u = -3 \times 6 = -18$, $v = -3 \times 6 = -18$

$$\therefore \quad \mathbf{V_M} = \sqrt{u^2 + v^2} = \sqrt{(-18)^2 + (-18)^2} = \mathbf{25.45 \text{ units (Ans.)}}$$

Pressure difference between *L* and *M*:

For two-dimensional plane flow

$$\frac{p_L}{w} + \frac{V_L^2}{2g} = \frac{p_M}{w} + \frac{V_M^2}{2g}$$

$$\therefore \quad \frac{p_L - p_M}{w} = \frac{1}{2g}(V_M^2 - V_L^2) = \frac{648 - 360}{2 \times 9.81} = \mathbf{14.68 \text{ units (Ans.)}}$$

(*iii*) **The discharge between the streamlines, q:**

$$\psi = \frac{3}{2}(x^2 - y^2)$$

$$\psi_{L(2,6)} = \frac{3}{2}(2^2 - 6^2) = -48 \text{ units.}$$

$$\psi_{M(6,6)} = \frac{3}{2}(6^2 - 6^2) = 0$$

$\therefore$ $q = \psi_M - \psi_L = 0 - (-48) = 48$ **units (Ans.)**

Example 5.42. *If $\phi = 3xy$, find x and y components of velocity at (1, 3) and (3, 3). Determine the discharge passing between streamlines passing through these points.* **[Roorkee University]**

Solution. *Given*: $\phi = 3xy$ ...Velocity potential function.

The velocity components in terms of ϕ are given by

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}$$

But $$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(3xy) = 3y, \text{ and}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(3xy) = 3x$$

$\therefore$ $u = -3y$ and $v = -3x$

Hence the velocity components at (1, 3) and (3, 3) are:

At (1,3): $\left.\begin{aligned} u &= -3 \times 3 = -9 \\ v &= -3 \times 1 = -3 \end{aligned}\right\}$ **(Ans)**

At (3,3): $\left.\begin{aligned} u &= -3 \times 3 = -9 \\ v &= -3 \times 3 = -9 \end{aligned}\right\}$ **(Ans.)**

Discharge between the streamlines:

The total derivative ψ may be written as

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

But $$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

$\therefore$ $\partial\psi = -vdx + udy$

or $\partial\psi = 3xdx - 3ydy$

Integrating, we get $\psi = \frac{3}{2}x^2 - \frac{3}{2}y^2 + C$

(where C = constant of integration)

Discharge between the streamlines passing through (1, 3) and (3, 3)

$$= \psi_{(1,3)} - \psi_{(3,3)} = \frac{3}{2}(1-9) - \frac{3}{2}(9-9) = -12 \text{ units (Ans.)}$$

Example 5.43. *The streamlines are represented by*

(*a*) $\psi = x^2 - y^2$ (*b*) $\psi = x^2 + y^2$

(*i*) *Determine the velocity and its direction at (2, 2).*

(*ii*) *Sketch the streamlines and show the direction of flow in each case.*

Solution. In a two-dimensional steady flow the velocity components in terms of ψ are given as

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

Case (*a*) $\psi = x^2 - y^2$:

(*i*) **Velocity and its direction at (2, 2):**

$$\psi = x^2 - y^2 \quad ...(1)$$

$\therefore$ $$\frac{\partial \psi}{\partial y} = -2y \text{ and } \frac{\partial \psi}{\partial x} = 2x$$

$\therefore$ $$u = -2y \text{ and } v = -2x$$

$\therefore$ $$V = \sqrt{u^2 + v^2} = \sqrt{(-2y)^2 + (-2x)^2} = 2\sqrt{x^2 + y^2} \quad ...(2)$$

i.e. $$V = 2\sqrt{x^2 + y^2}$$

$\therefore$ Velocity at (2, 2) = $2\sqrt{2^2 + 2^2} = 4\sqrt{2}$ **units (Ans.)**

Its direction has a slope, $\frac{\partial y}{\partial x} = -\frac{v}{u} = -\frac{-2x}{-2y} = -1$ ...(3)

$\therefore$ Velocity vector is inclined at **45° to *x*-axis (Ans.)**

(*ii*) **Streamlines-sketch:**

The streamlines are lines of constant ψ, and for constant ψ, eqn. (1) represents hyperbola, which may be plotted for different values of ψ as shown in the table given below:

$$\begin{bmatrix} \psi = x^2 - y^2 \\ x = \pm\sqrt{y^2 - \psi} \end{bmatrix}$$

	y	0	1	2	3
$\psi = 1$	$x = \pm\sqrt{y^2 - \psi}$	± 1	$\pm\sqrt{2}$	$\pm\sqrt{5}$	$\pm\sqrt{10}$
$\psi = 2$	$x = \pm\sqrt{y^2 + \psi}$	$\pm\sqrt{2}$	$\pm\sqrt{3}$	$\pm\sqrt{6}$	$\pm\sqrt{11}$
$\psi = 3$	$x = \pm\sqrt{y^2 - \psi}$	$\pm\sqrt{3}$	$\pm\sqrt{4}$	$\pm\sqrt{7}$	$\pm\sqrt{12}$

Fig. 5.24 shows the pattern of streamlines.

Case (b) $\psi = x^2 + y^2$:

(*i*) Velocity and its direction at (2, 2)

$$\psi = x^2 + y^2 \quad ...(4)$$

Now, $\frac{\partial \psi}{\partial y} = 2y$ and $\frac{\partial \psi}{\partial x} = 2x$

Hence $u = 2y$ and $v = -2x$

The resultant velocity,

$$V = \sqrt{u^2 + v^2} = \sqrt{(2y)^2 + (-2x)^2}$$
$$= 2\sqrt{x^2 + y^2}$$

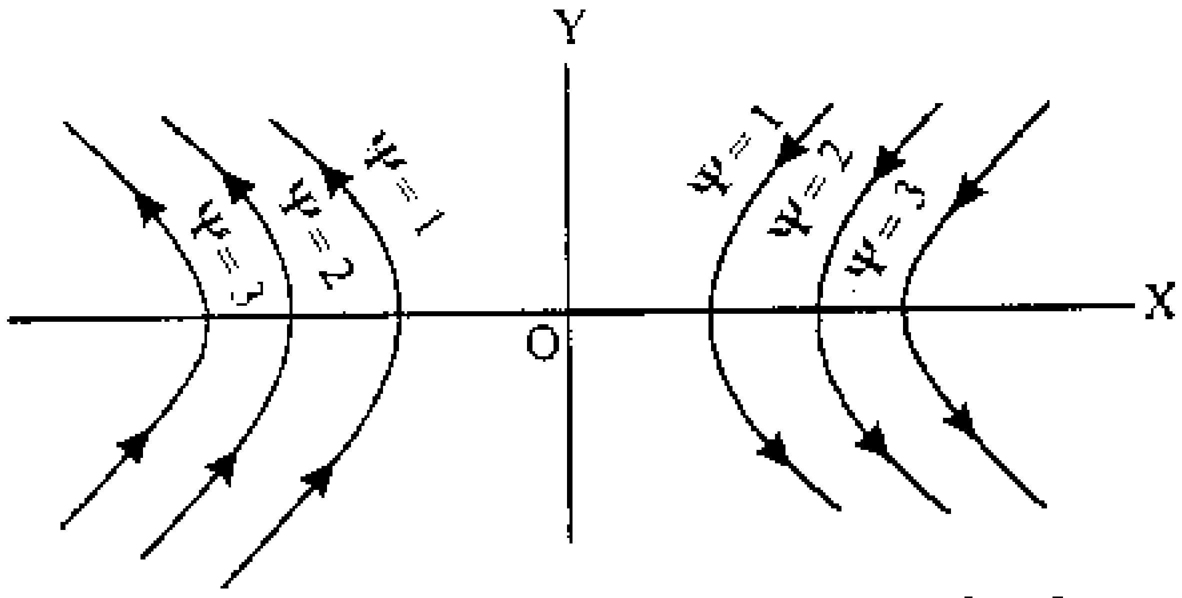

Fig. 5.24. Pattern of streamlines for $\psi = x^2 - y^2$.

$\therefore$ Velocity at (2, 2) = $2\sqrt{2^2 + 2^2} = 4\sqrt{2}$ **units (Ans.)**

Its direction has a slope,

$$\frac{\partial y}{\partial x} = -\frac{v}{u} = -\frac{-2x}{2y} = \frac{x}{y} = \frac{2}{2} = 1$$

i.e velocity makes an angle θ with the axis shown 5.25.

$\tan\theta = 1$ or $\theta = 45°$

(*ii*) **Streamlines-sketch:**

For a streamlines ψ = constant, and for non-zero value of ψ, eqn. (4) represents *concentric circles* with centre at the origin (0, 0) and radius $\sqrt{\psi}$. In the 1st (i.e. positive) quadrant. (x, y both positive), $u = 2y$ and $v = -2x$ is negative. Therefore, streamlines have clockwise direction as shown in Fig 5.26.

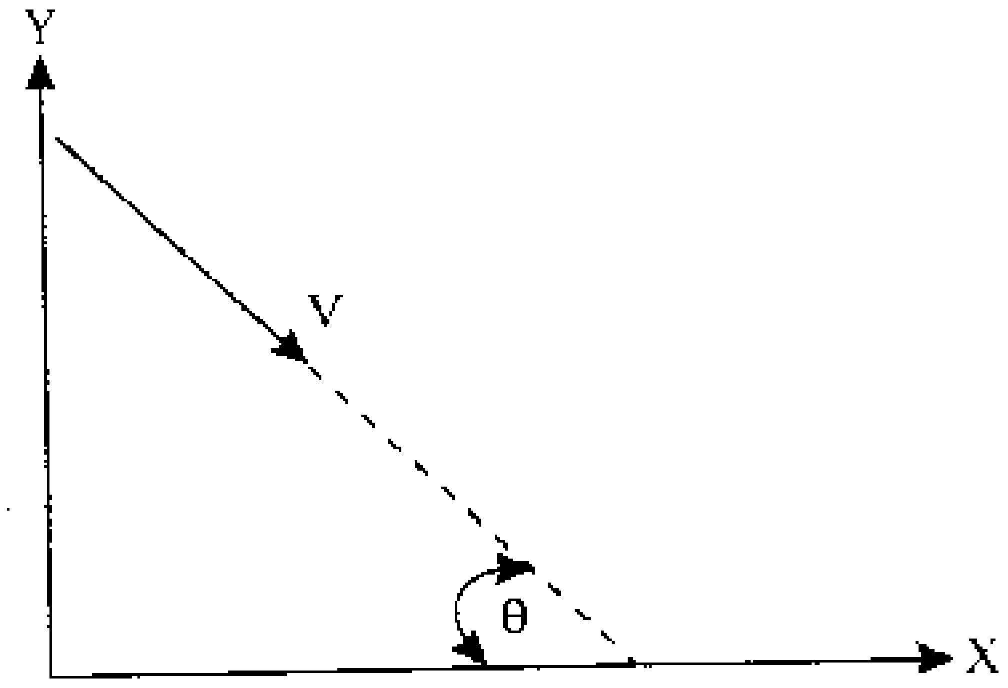

Fig. 5.25

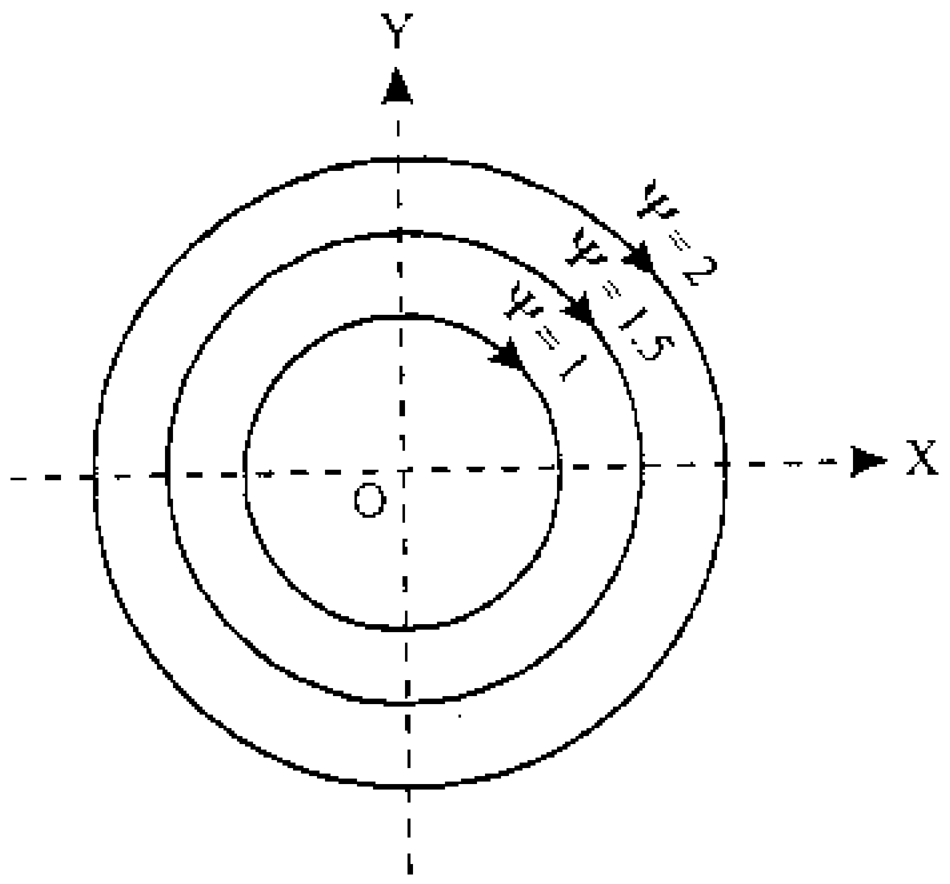

Fig. 5.26. Pattern of streamlines $\psi = x^2 + y^2$.

Example 5.44. *If the expression for stream function is described by* $\psi = x^3 - 3xy^2$, *determine whether flow is rotational or irrotational. If the flow is irrotational, then indicate the correct value of the velocity potential.*

(*a*) $\phi = y^3 - 3xy^2$ (*b*) $\phi = -3x^2y$ **[UPSC Exam. CES, Fluid mechanics]**

Solution. *Given*: $\psi = x^3 - 3xy^2$ ...Stream function

A two-dimensional flow in $x - y$ plane will be irrotational if the vorticity vector in the z-direction is zero.

i.e., $$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad ...(1)$$

We know, $$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(x^3 - 3xy^2\right) = -6xy\text{, and}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(x^3 - 3xy^2\right) = -\left(3x^2 - 3y^2\right) = -3(x^2 - y^2)$$

$$\therefore \quad \frac{\partial u}{\partial y} = -6x\text{, and } \frac{\partial v}{\partial x} = -6x$$

Substituting these value in eqn. (1). we get

$$\Omega_z = -6x - (-6x) = 0$$

Hence the **flow is irrotational. (Ans.)**

For an irrotational flow Laplace equation in ϕ must be satisfied *i.e.* $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Let us check the validity for each expression for ϕ:

(*a*) $\phi = y^3 - 3xy^2$

$$\frac{\partial^2 \phi}{\partial x^2} = -6y \text{ and } \frac{\partial^2 \phi}{\partial y^2} = 6y$$

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -6y + 6y = 0$$

(*b*) $\phi = -3x^2y$

$$\frac{\partial^2 \phi}{\partial x^2} = -6y \text{ and } \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y} \neq 0$$

Hence the correct value of $\phi = y^3 - 3x^2y$ **(Ans.)**

Example 5.45. *In a two-dimensional incompressible flow, the fluid velocity components a[re] given by* $u = x - 4y$ *and* $v = -y - 4x$. *Show that velocity potential exists and determine its form [as] well as stream function.* **[AMIE Winter, 20[..]**

Solution. *Given:* $u = x - 4y$ and $v = -y - 4x$ ...Velocity compone[nts]

The velocity potential will exist if flow is *irrotational*. Therefore, the *vorticity component i[n] the Z-direction must be zero.*

i.e. $$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

But, $$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(-y - 4x) = -4, \text{ and}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x - 4y) = -4$$

$$\therefore \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -4 - (-4) = 0$$

Since the *vorticity is zero,* the *flow is irrotational*; hence the **velocity potential exists. (Ans.)**

Total change in velocity potential

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$$= -udx - vdy = -(x - 4y)dx - (-y - 4x)dy$$

or $$d\phi = -xdx + 4ydx + ydy + 4xdy$$

Integrating, we get

$$\phi = -\frac{x^2}{2} + 4xy + \frac{y^2}{2} + 4xy + C$$

$$= \frac{1}{2}(y^2 - x^2) + 8xy + C$$

(where C = constant of integration).

For $\phi = 0$ at the origin, the constant $C = 0$

$$\therefore \quad \phi = \frac{1}{2}(y^2 - x^2) + 8xy \quad \textbf{(Ans.)}$$

$$a_x = 8\,(2 - 2 \times 1 \times 3) + 2.67\,(3^2 - 1^2) = -\,10.64 \text{ m/s}^2$$

Further,
$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

$$= \left(2x - xy^2 + \frac{y^3}{3}\right)(y^2 - x^2) + \left(xy^2 - xy - \frac{x^3}{3}\right)(2xy - 2)$$

$$= 8\,(3^2 - 1^2) + 2.67\,(2 \times 1 \times 3 - 2) = 74.68 \text{ m/s}^2$$

Resultant acceleration, $a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-10.64)^2 + (74.68)^2} =$ **75.43 m/s² (Ans.)**

(*ii*) **Is the flow physically possible:**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (2 - 2xy) + (2xy - 2) = 0$$

As the continuity equation is satisfied, hence the **flow is physically possible.**

Expression for stream function:

The differential $d\psi$ for stream function is,

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$$

$$= -\,vdx + udy$$

$$= -\left(xy^2 - 2y - \frac{x^3}{3}\right)dx + \left(2x - x^2y + \frac{y^3}{3}\right)dy$$

$$= \left(-\,xy^2 + 2y + \frac{x^3}{3}\right)dx + \left(2x - x^2y + \frac{y^3}{3}\right)dy$$

or
$$d\psi = \frac{x^3}{3}dx + \frac{y^3}{3}dy + 2d(xy) - d\left(\frac{x^2y^2}{2}\right)$$

On integration, we get

$$\psi = \frac{x^4}{12} + \frac{y^4}{12} + 2xy - \frac{x^2y^2}{2}$$

(*iii*) **Discharge between streamlines passing through (1, 3) and (2, 3):**

$$\psi_{(1,3)} = \frac{1^4}{12} + \frac{3^4}{12} + 2 \times 1 \times 3 - \frac{1^2 \times 3^2}{2} = 8.33 \text{ m}^2\text{/s}$$

$$\psi_{(2,3)} = \frac{2^4}{12} + \frac{3^4}{12} + 2 \times 2 \times 3 - \frac{2^2 \times 3^2}{2} = 2.08 \text{ m}^3\text{/s}$$

Hence discharge between the streamlines

$\psi_{(1,\,3)} - \psi_{(2,\,3)} = 8.33 - 2.08 =$ **6.25 m³/s (Ans.)**

(*iv*) **Is the flow irrotational?**

Rotation (angular velocity), $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$$= \frac{1}{2}\left[\left(y^2 - x^2\right) - \left(y^2 - x^2\right)\right] = 0$$

As the rotation is zero, the **flow is irrotational** *and the potential function does exist:*

Velocity potential:

Here $\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - \frac{\partial}{\partial y}(Cy) = -C$

$A = \pi a^2$

$\therefore$ $\Gamma = -C \times \pi a^2 = \mathbf{-C\pi a^2}$ **m²/s (Ans.)**

Second method:

Refer Fig. 5.27

We know, $\Gamma = \oint V_\theta . ds$

(where V_θ is the tangential velocity)

$= \oint u.dx = \oint C.y\,dx$

From Fig. 5.27

$x = a\cos\theta;\ y = a + a\sin\theta = a(1+\sin\theta)$

$\therefore\ dx = -a\sin\theta\, d\theta$

$\therefore\ \Gamma = \int_0^{2\pi} Ca(1+\sin\theta).(-a\sin\theta d\theta)$

$= -Ca^2 \int_0^{2\pi} (\sin\theta + \sin^2\theta)\, d\theta$

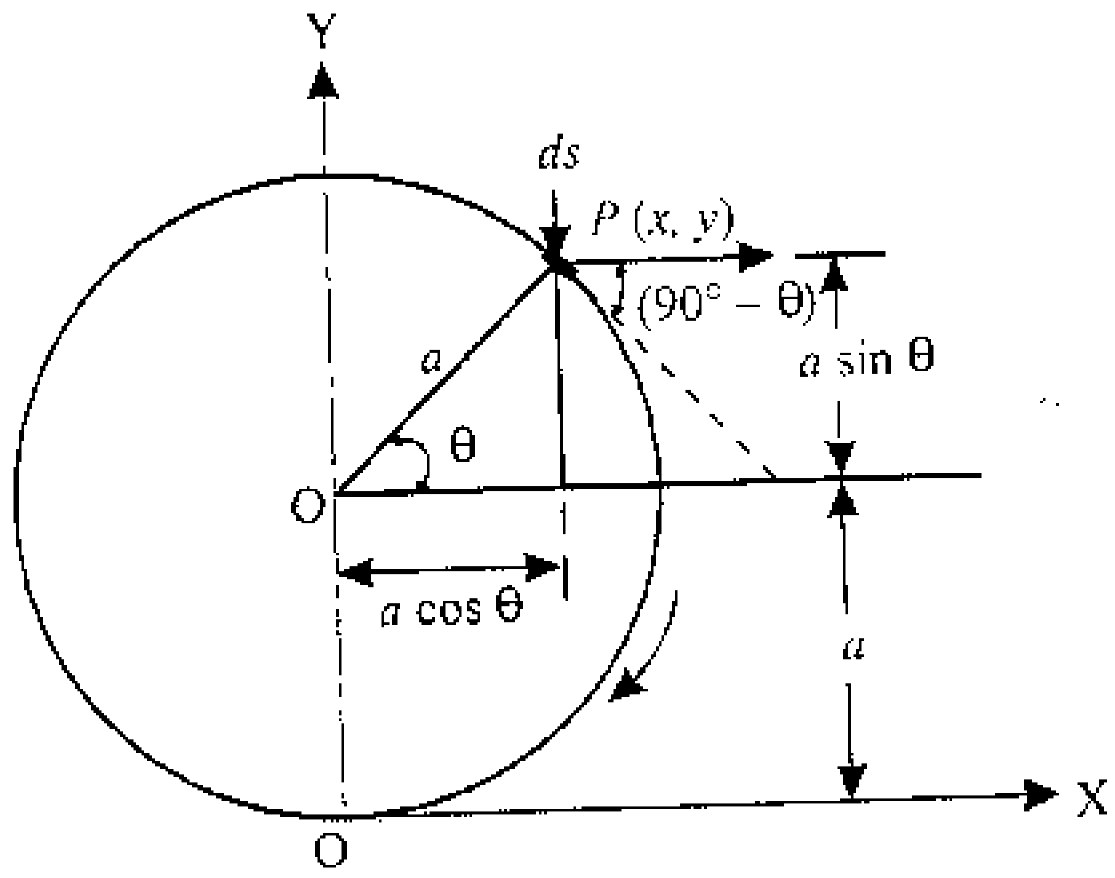

Fig. 5.27. Circulation about a circle.

$$= -Ca^2 \int_0^{2\pi} \left(\sin\theta + \frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= -\frac{Ca^2}{2} \int_0^{2\pi} (2\sin\theta + 1 - \cos 2\theta)\, d\theta$$

$$= -\frac{Ca^2}{2}\left[-2\cos\theta + \theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= -\frac{Ca^2}{2}[(-2 + 2\pi - 0) - (-2 + 0 - 0)]$$

$$= -\frac{Ca^2}{2} \times 2\pi = -C\pi a^2 \ \text{m}^2/\text{s}$$

i.e. $\mathbf{\Gamma = -C\pi a^2}$ **m²/s (Ans.)**

Example 5.48. *The flow field of a fluid is given by* $V = xyi + 2yzj - (yz + z^2)k$

(*i*) *Show that it represent a possible three-dimensional steady incompressible continuous flow.*

(*ii*) *Is this flow rotational or irrotational?*

If rotational, determine at point A (2, 4, 6):

(*a*) *Angular velocity,*

(*b*) *Vorticity,*

(*c*) *Shear strain, and*

(*d*) *Dilatency.*

Solution. *Given*: $V = xyi + 2yzj - (yz + z^2)\,k$ ... Fluid flow field

Here, $u = xy,\ v = 2yz,\ w = -(yz + z^2)$

$$\frac{\partial u}{\partial x} = y,\quad \frac{\partial v}{\partial y} = 2z,\quad \frac{\partial w}{\partial z} = -(y + 2z)$$

$$e_y = \frac{\partial v}{\partial y} = 2z = 2 \times 6 = \mathbf{12} \quad \textbf{(Ans.)}$$

$$e_z = \frac{\partial w}{\partial z} = -(y + 2z) = -(4 + 2 \times 6) = \mathbf{-16} \textbf{ (Ans.)}$$

Example 5.49. *The stream function $\psi = 4xy$, in which ψ is in cm²/second and x and y are in metres, describe the incompressible flow between the boundaries shown below (Fig. 5.28).*

Calculate:

(*i*) *Velocity at* B,

(*ii*) *Convective acceleration at B, and*

(*iii*) *Flow rate per unit width across AB.* **(UPSC, CES, Fluid Mechanic**

Solution. Given: $\psi = 4xy$ ... Stream functi

(where ψ is in cm²/s and x and y in metres)

Co-ordinates of A are: (3, 0)

Co-ordinates of B are:

$$x = 3 \text{ m}, \; y = \frac{3}{x} = \frac{3}{3} = 1 \text{ m}$$

$$\left(\because xy = 3, \therefore y = \frac{3}{x}\right)$$

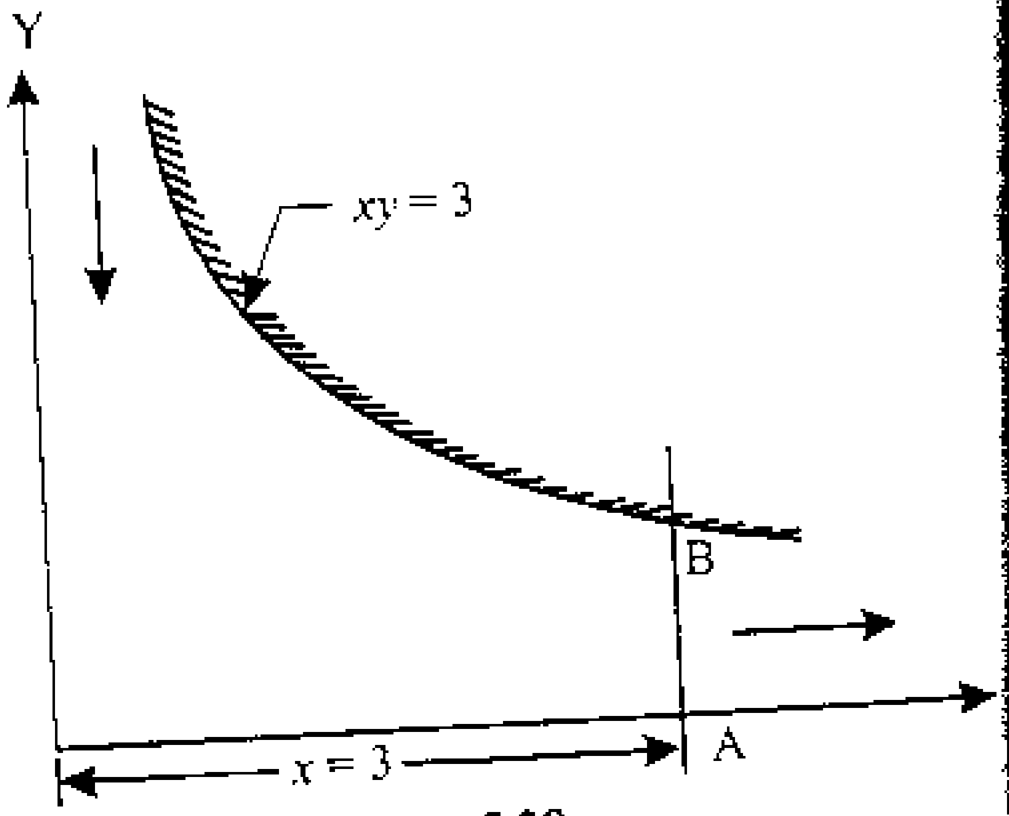

Fig. 5.28

Hence co-ordinates of B are: (3, 1)

(*i*) **Velocity at B:**

Velocity components are given by

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(4xy) = 4x, \text{ and}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(4xy) = -4y$$

At point B: $u = 4 \times 3 = 12$ cm/s, and

$v = -4 \times 1 = -4$ cm/s

$\therefore$ Velocity at B, $V_B = \sqrt{u^2 + v^2} = \sqrt{12^2 + (-4)^2}$ = **12.65 cm/s (Ans.)**

(*ii*) **Convective acceleration at B, a_B:**

Convective acceleration, $a = \sqrt{a_x^2 + a_y^2}$

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}, \text{ and } a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

But, $u = 4x$ and $v = -4y$

$$\therefore \quad \frac{\partial u}{\partial x} = 4, \; \frac{\partial u}{\partial y} = 0; \; \frac{\partial v}{\partial x} = 0, \; \frac{\partial v}{\partial y} = -4$$

$$\therefore \quad a_x = (4x)(4) + (-4y)(0) = 16x \text{ and}$$

$$a_y = (4x)(0) + (-4y)(-4) = 16y$$

Also $a = \sqrt{a_x^2 + a_y^2} = \sqrt{(16x)^2 + (16y)^2}$

$\therefore$ Convective acceleration at B (3,1) is

$$a_B = \sqrt{(16 \times 3)^2 + (16 \times 1)^2} = 16\sqrt{3^2 + 1} = \mathbf{50.6 \text{ cm/s}^2} \textbf{ (Ans.)}$$

(*iii*) **Flow rate per unit width across AB, q_{AB}:**

$$q_{AB} = \psi_B - \psi_A = \psi_{(3,\,1)} - \psi_{(3,\,0)} = 4 \times 3 \times 1 - 4 \times 3 \times 0 = 12 \text{ cm}^3\text{/s/cm}$$

i.e. $\mathbf{q_{AB} = 12}$ **cm³/s/cm (Ans.)**

HIGHLIGHTS

1. *Fluid kinematics* is a branch of fluid mechanics which deals with the study of velocity and acceleration of the particles of fluids in motion and their distribution in space without considering any forces or energy involved.
2. The motion of fluid particles may be described by the following methods:
   (*i*) *Langrangian method.* In this method, the observer concentrates on the movement of a single particle. The path taken by the particle and the changes in its velocity and acceleration are studied.
   (*ii*) *Eulerian method.* In Eulerian method, the observer concentrates on a point in the fluid system. Velocity, acceleration and other characteristics of the fluid at that particular point are studied.
   The components of acceleration of the fluid particle are given by:

$$a_x = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \frac{\partial u}{\partial t}$$

$$a_y = \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \frac{\partial v}{\partial t}$$

$$a_z = \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \frac{\partial w}{\partial t}$$

Resultant velocity, $V = \sqrt{u^2 + v^2 + w^2}$

$(V = ui + vj + wk)$ ... in vector notation.

Resultant acceleration, $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$(a = a_x i + a_y j + a_z k)$ ... in vector notation.

Also, $a = V\dfrac{\partial V}{\partial s} + \dfrac{\partial V}{\partial t}$

$\dfrac{V\partial V}{\partial s}$ ...is called *convective acceleration.*

$\dfrac{\partial V}{\partial t}$ ... is called *local acceleration.*

3. *Types of fluid flow*:
   (*i*) Steady and unsteady flows
   (*ii*) Uniform and non-uniform flows
   (*iii*) One, two and three-dimensional flows
   (*iv*) Rotational and irrotational flows
   (*v*) Laminar and turbulent flows
   (*vi*) Compressible and incompressible flows.

4. *Types of flow lines*:
   (*i*) *Path line.* It is the path followed by a fluid particle in motion.
   (*ii*) *Stream line.* It is an imaginary line within the flow so that the tangent at any point on it indicates the velocity at that point.
   (*iii*) *Stream tube.* It is a fluid mass bounded by a group of streamlines.
   (*iv*) *Streak line.* It is a curve which gives an instantaneous picture of the location of the fluid particles which have passed through a given point.

5. The *continuity equation* based on the principle of conservation of mass is stated as follows: "If no fluid is added or removed from the pipe in any length then the mass passing across different sections shall be same".

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad \text{...in case of compressible fluids}$$

$$A_1 V_1 = A_2 V_2 \quad \text{...in case of incompressible fluids.}$$

The continuity equation in three dimensions in *cartesian co-ordinates* is given as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{...for steady flow of incompressible fluid } (\rho = \text{constant})$$

The continuity equation in *polar co-ordinates* is given as:

$$\frac{1}{r}(\rho v_r) + \frac{\partial}{\partial r}(\rho v_r) + \frac{\partial}{r\partial\theta}(\rho v_\theta) = 0 \quad \text{... for compressible flo}$$

$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{r\partial\theta} = 0 \quad \text{...for incompressible flo}$$

where, v_r = Velocity component in radial direction, and
v_θ = Velocity component in tangential direction.

6. *Circulation* (Γ) is defined mathematically as the line integral of the tangential veloc about a closed path (contour).

$$\Gamma = \oint V \cos\theta . ds$$

where, V = Velocity in the flow field, and
θ = Angle between V and the tangent to the path (in the positi anticlockwise direction along the path) at that point.

Vorticity (Ω) is defined as the circulation per unit of enclosed area $\left(i.e.\ \Omega = \frac{\Gamma}{A} \right)$.

If *a flow possesses vorticity, it is rotational.* The flow is irrotational if rotation (ω) is ze

The expression for rotation are:

$$w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right); \quad w_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right); \quad w_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

7. *Velocity potential* (ϕ) is defined as a scalar function of space and time such that its negat derivative with respect to any direction gives the fluid velocity in that direction.

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}$$

Stream function (Ψ) is defined as a scalar function of space and time, such that its par derivative with respect to any direction gives the velocity components at right angles the counterclockwise direction) to this direction.

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}$$

Existence of Ψ means a possible case of flow.

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} = \frac{\partial \Psi}{\partial y} \\ v &= -\frac{\partial \phi}{\partial y} = -\frac{\partial \Psi}{\partial x} \end{aligned}\right\}$$ ... are known as Cauchy-Reimann equations.

8. *Flow net.* A grid obtained by drawing a series of streamlines and equipotential lines is known as a flow net.

OBJECTIVE TYPE QUESTIONS

Choose the Correct Answer:

1. The motion of fluid particles may be described by which of the following methods?
   (*a*) Langrangian method
   (*b*) Eulcrain method
   (*c*) both (*a*) and (*b*)
   (*d*) none of the above.
2. In which of the following methods, the observer concentrates on a point in the fluid system?
   (*a*) Langrangian method
   (*b*) Eulerian method
   (*c*) Any of the above
   (*d*) None of the above.
3. Normal acceleration in fluid-flow situation exists only when
   (*a*) the flow is unsteady
   (*b*) the flow is two-dimensional
   (*c*) the streamlines are straight and parallel
   (*d*) the streamlines are curved.
4. In a steady flow the velocity
   (*a*) does not change from place to place
   (*b*) at a given point does not change with time
   (*c*) may change its direction but the magnitude remains unchanged
   (*d*) none of the above.
5. The flow in a pipe whose valve is being opened or closed gradually is an example of
   (*a*) steady flow (*b*) unsteady flow
   (*c*) rotational flow (*d*) compressible flow.
6. The type of flow in which the velocity at any given time does not change with respect to space is called
   (*a*) steady flow (*b*) compressible flow
   (*c*) uniform flow (*d*) rotational flow.
7. Flow in a pipe where average flow parameters are considered for analysis is an example of
   (*a*) incompressible flow
   (*b*) one-dimensional flow
   (*c*) two-dimensional flow
   (*d*) three-dimensional flow.
8. The flow in a river during the period of heavy rainfall is
   (*a*) steady, non-uniform and three-dimensional
   (*b*) steady, uniform, two-dimensional
   (*c*) unsteady, uniform, three-dimensional
   (*d*) unsteady, non-uniform and three-dimensional.
9. Flow between parallel plates of infinite extent is an example of
   (*a*) one-dimensional flow
   (*b*) two-dimensional flow
   (*c*) three-dimensional flow
   (*d*) compressible flow.
10. If the flow is irrotational as well as steady it is known as
    (*a*) non-uniform flow
    (*b*) one-dimensional flow
    (*c*) potential flow
    (*d*) none of the above.
11. High velocity flow in a conduit of large size is known as
    (*a*) laminar flow
    (*b*) turbulent flow
    (*b*) either of the above
    (*d*) none of the above.
12. If the Reynolds number is less than 2000, the flow in a pipe is
    (*a*) laminar flow (*b*) turbulent flow
    (*c*) transition flow (*d*) none of the above.
13. The path followed by fluid particle in motion is called a
    (*a*) streamline (*b*) path line
    (*c*) streak line (*d*) none of the above.