# DEFLECTION OF BEAMS

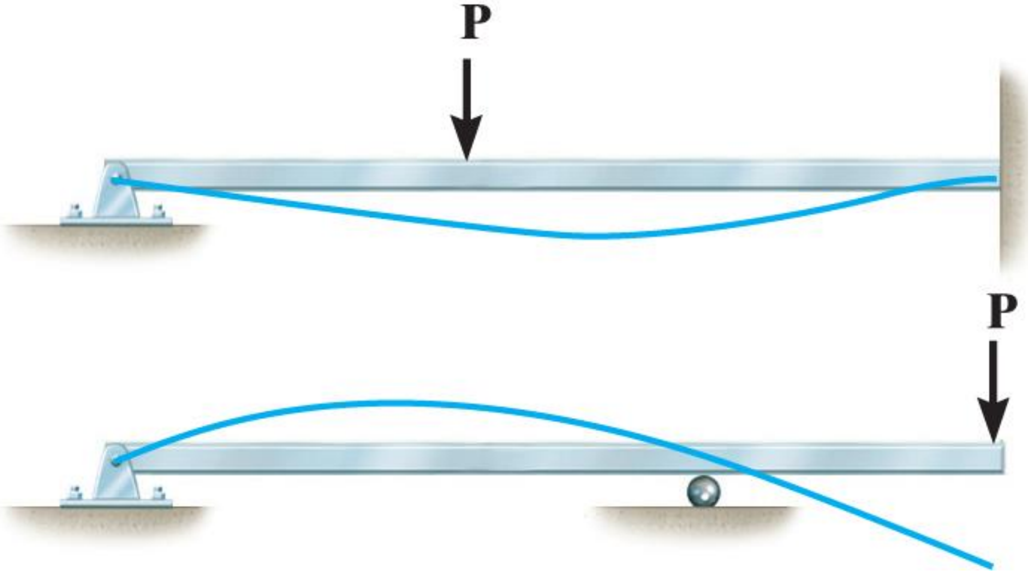
- ✓ Determine the deflection and slope at specific points on beams and shafts, using various analytical methods including:
  - ➢ The integration method
  - ➢ The use of discontinuity functions
  - ➢ The method of superposition

- ✓ Determine the same, using a semi-graphical technique, called the moment-area method.

# APPLICATIONS

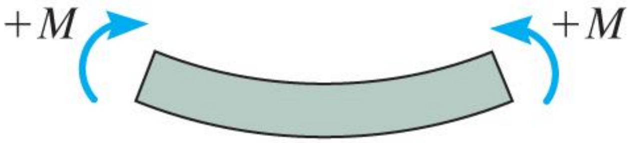
# ELASTIC CURVE

- The deflection diagram of the longitudinal axis that passes through the centroid of each cross-sectional area of the beam is called the elastic curve, which is characterized by the deflection and slope along the curve

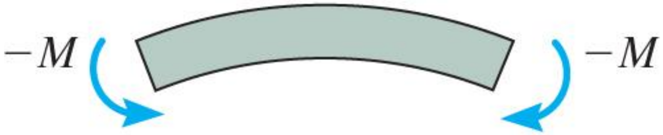


# ELASTIC CURVE (cont)

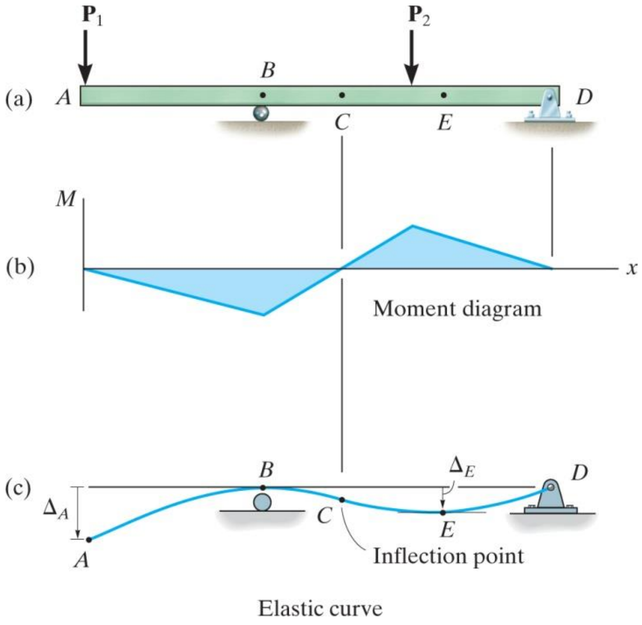
- Moment-curvature relationship:
  - Sign convention:

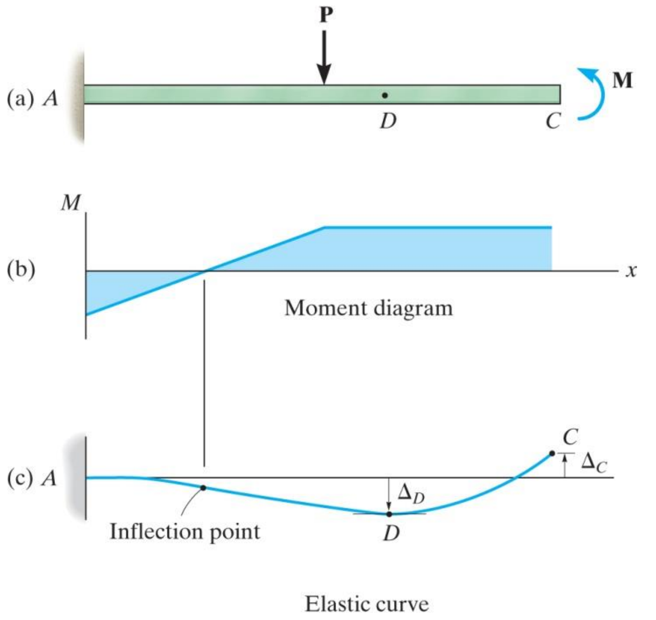


Positive internal moment
concave upwards
(a)



Negative internal moment
concave downwards
(b)

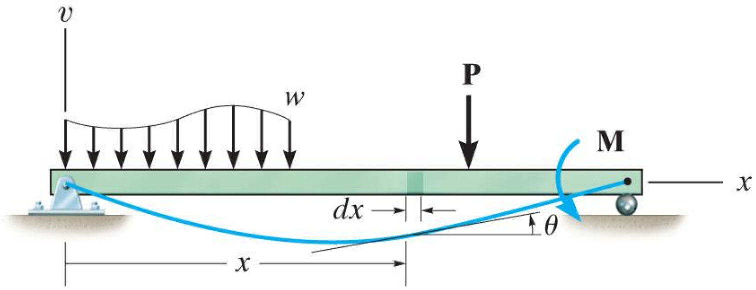




# ELASTIC CURVE (cont)

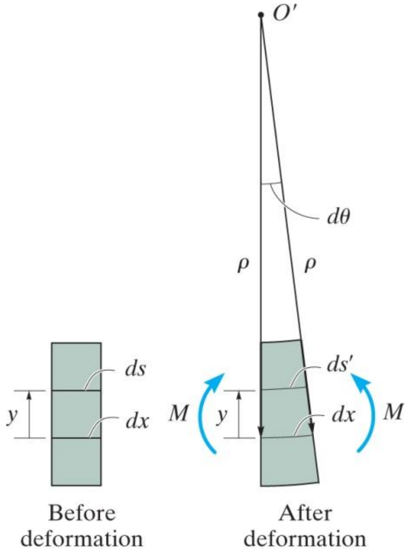
- Consider a segment of width dx, the strain in fibre are ds, located at a position y from the neutral axis is ε = (ds’ – ds)/ds. However, ds = dx = ρdθ and ds’ = (ρ-y) dθ, and so ε = [(ρ – y) dθ – ρdθ ] / (ρdθ), or

$$\frac{1}{\rho} = -\frac{\varepsilon}{y}$$



- Comparing with the Hooke’s Law ε = σ / E and the flexure formula σ = -My/I

$$\frac{1}{\rho} = \frac{M}{EI} \text{ or } \frac{1}{\rho} = -\frac{\sigma}{Ey}$$



(b)

# SLOPE AND DISPLACEMENT BY INTEGRATION

- The equation of the elastic curve is defined by the coordinates $v$ and $x$. To compute the deflection $v = f(x)$, we must be able to represent the curvature $(1/\rho)$ in terms of $v$ and $x$.

- Kinematic relationship between radius of curvature $\rho$ and location $x$:

$$\frac{1}{\rho} = -\frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

- Then using the moment curvature equation, we have

$$\frac{M}{EI} = \frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1+\left(dv/dx\right)^2\right]^{3/2}} \approx \frac{d^2v}{dx^2}$$

# SLOPE AND DISPLACEMENT BY INTEGRATION

$$\frac{M}{EI} = \frac{d^2v}{dx^2}$$

- The equation can also be written in two alternative forms

- Differentiate each side with respect to $x$ and substitute $V = dM/dx$

$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x)$$

- Differentiating again, using $w = dV/dx$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = w(x)$$

# SLOPE AND DISPLACEMENT BY INTEGRATION

- Flexural rigidity ($EI$) is constant along beam, thus

$$EI\frac{d^4v}{dx^4} = w(x)$$

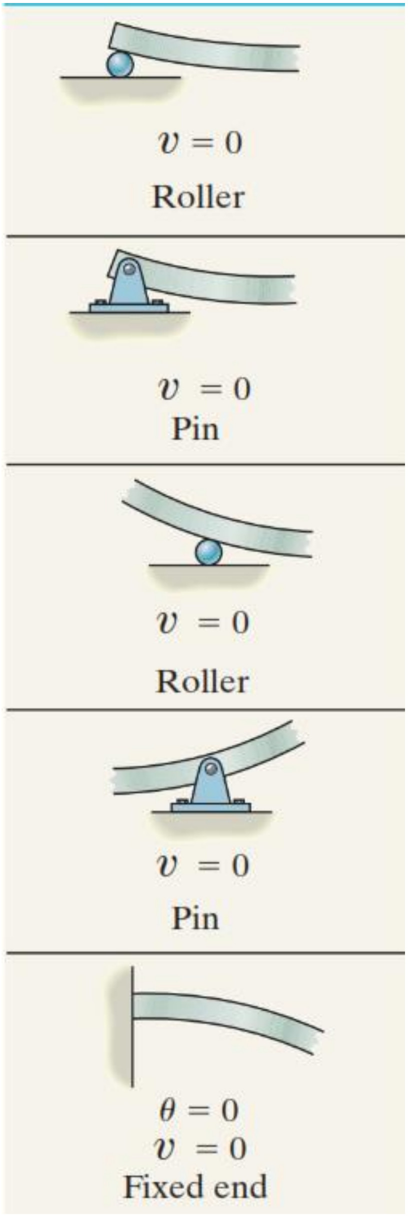
$$EI\frac{d^3v}{dx^3} = V(x)$$

$$EI\frac{d^2v}{dx^2} = M(x)$$

- Solution of any of these equations requires successive integrations to obtain $v$.

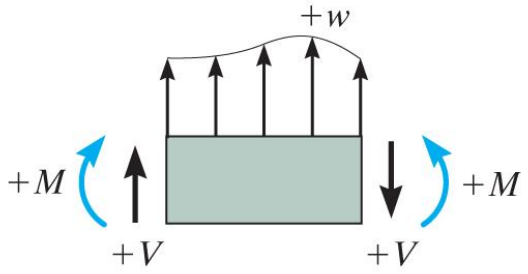
# SLOPE AND DISPLACEMENT BY INTEGRATION (cont)

- Boundary Conditions:
  - The integration constants can be determined by imposing the boundary conditions, or
  - Continuity condition at specific locations
- Note, if the beam is supported by a *roller* or *pin*, then it is required that the displacement be *zero* at these points.
- At the fixed support, the *slope* and *displacement* are both *zero*.



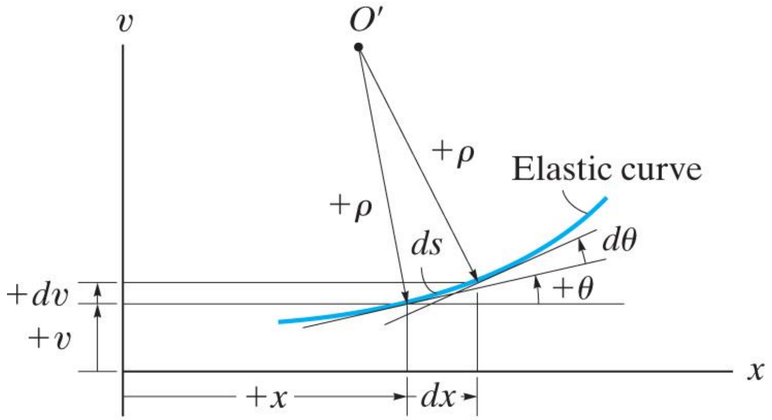
# SLOPE AND DISPLACEMENT BY INTEGRATION (cont)

- Sign convention:



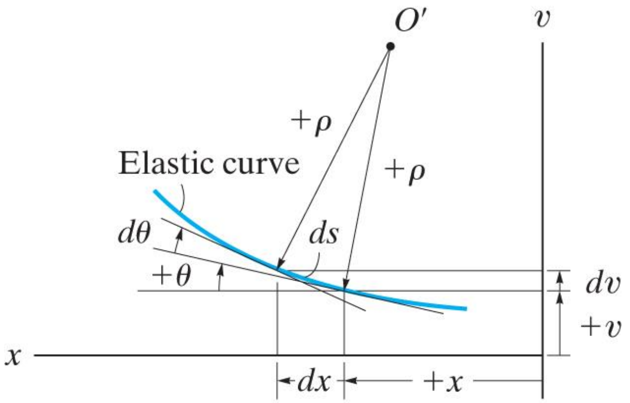
Positive sign convention

(a)



Positive sign convention

(b)

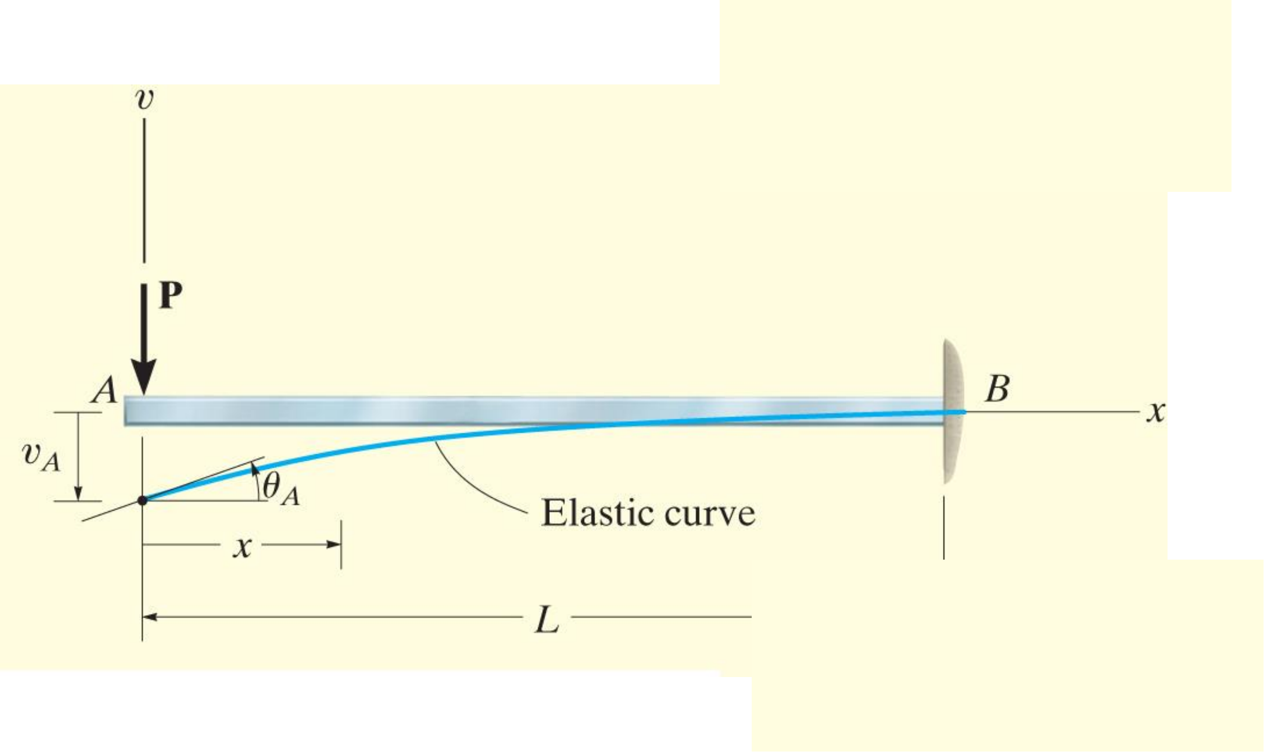


Positive sign convention

(c)

# EXAMPLE 1

The cantilevered beam shown below is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. $EI$ is constant.



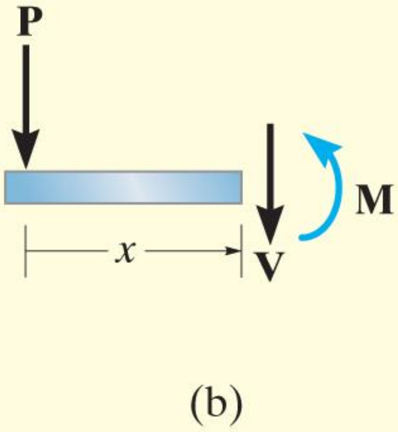
# EXAMPLE 1 (cont)

## Solutions

- Elastic Curve: shown in the Question figure

- Moment Function: From the free-body diagram, with **M** acting in the *positive direction*, we have

$$M = -Px$$



- Slope and Elastic Curve

- Applying $EI\frac{d^2v}{dx^2} = M(x)$ and integrating twice yields

$$EI\frac{d^2v}{dx^2} = -Px \qquad (1)$$

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \qquad (2)$$

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \qquad (3)$$

# EXAMPLE 1 (cont)

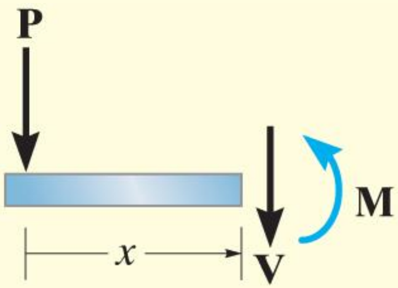
## Solutions

- Using the boundary conditions $dv/dx = 0$ at $x = L$ and $v = 0$ at $x = L$, equations 2 and 3 become

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1 L + C_2$$

$$\Rightarrow C_1 = \frac{PL^2}{2} \text{ and } C_2 = -\frac{PL^3}{3}$$



(b)

- Substituting these results, with $\theta = dv/dx$, we get

$$\theta = \frac{P}{2EI}\left(L^2 - x^2\right)$$

Elastic Curve Eqn: $$v = \frac{P}{6EI}\left(-x^3 + 3L^2 x - 2L^3\right) \quad \text{(Ans)}$$

# EXAMPLE 1 (cont)

## Solutions

- Maximum slope and displacement occur at for which $A(x = 0)$,

$$\theta_A = \frac{PL^2}{2EI} \quad (4)$$

$$v_A = -\frac{PL^3}{3EI} \quad (5)$$

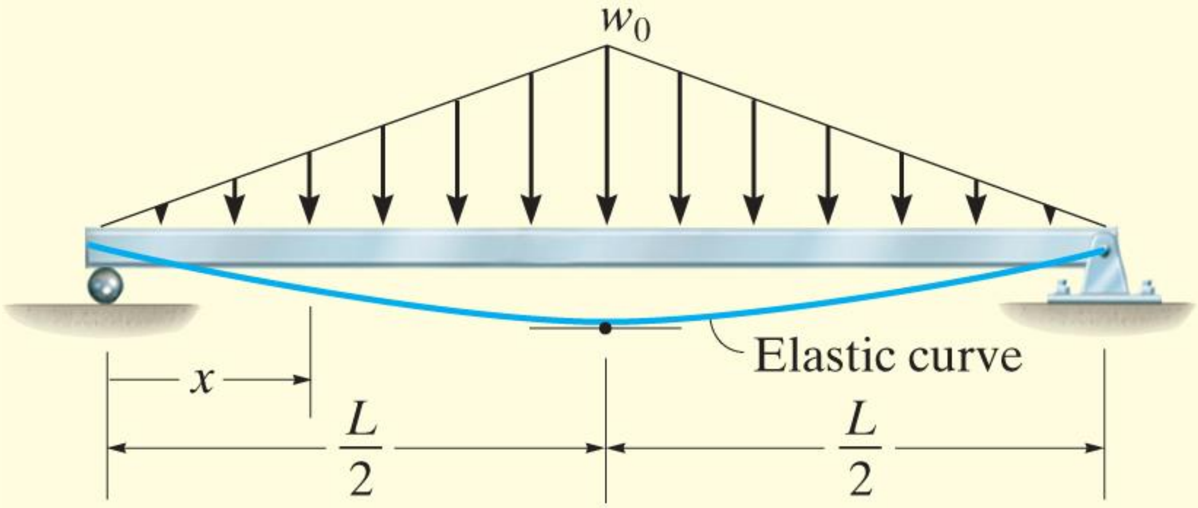
- If this beam ($L = 5$ m; Load; $P = 30$ kN) was designed without a factor of safety by assuming the allowable normal stress is equal to the yield stress is 250 MPa; then a W310 x 39 would be found to be adequate ($I = 84.4(10^6)\text{mm}^4$)

$$\theta_A = \frac{30(5)^2(1000)^2}{2[200][84.4(10^6)]} = 0.0222\,\text{rad}$$

$$v_A = -\frac{30(5)^2(1000)^2}{3[200][84.4(10^6)]} = -74.1\,\text{mm}$$

# EXAMPLE 2

The simply supported beam shown below supports the triangular distributed loading. Determine its maximum deflection. *EI* is constant.



(a)

# EXAMPLE 2 (cont)

## Solutions

- Due to symmetry only one *x* coordinate is needed for the solution,

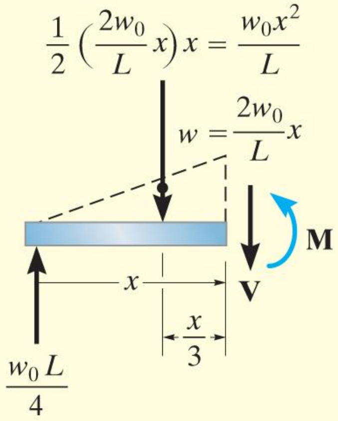
$$0 \le x \le L/2$$

- The equation for the distributed loading is $w = \dfrac{2w_0}{L} x$.

- Hence

$$+\sum M_{NA} = 0; \qquad M + \frac{w_0 x^2}{L}\left(\frac{x}{3}\right) - \frac{w_0 L}{4}(x) = 0$$

$$M = -\frac{w_0 x^2}{3L} + \frac{w_0 L}{4} x$$



(b)

# EXAMPLE 2 (cont)

## Solutions

- Integrating twice, we have

$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x$$

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 + C_1$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 + C_1x + C_2$$

- For boundary condition,
  - $v = 0, x = 0$ and
  - $dv/dx = 0, x = L/2$

$$C_1 = -\frac{5w_0L^3}{192}$$

$$C_2 = 0$$

# EXAMPLE 2 (cont)

## Solutions

- Hence

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0 L}{24}x^3 - \frac{5w_0 L^3}{192}x$$

- For maximum deflection at $x = L/2$,

$$v_{\max} = -\frac{w_0 L^4}{120EI} \qquad \text{(Ans)}$$