Deflections Using Energy Methods

Method of Virtual Work: Trusses

External Work & Strain Energy

- For more complicated loadings or for structures such as trusses & frames, it is suggested that energy methods be used for the computations

- Most energy methods are based on the conservation of energy principles

- Work done by all external forces acting on a structure, U_e is transformed into internal work or strain energy U_i
  - $U_e = Ui$

External Work & Strain Energy

- If the material's elastic limit is not exceeded, the elastic strain energy will return the structure to its undeformed state when the loads are removed

- When a force F undergoes a disp dx in the same direction as the force, the work done is
  - $dU_e = F\,dx$

- If the total disp is x, the work becomes:

$$U_e = \int_0^x F\,dx$$

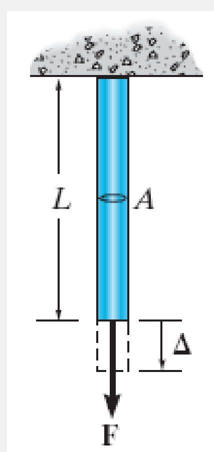
External Work & Strain Energy

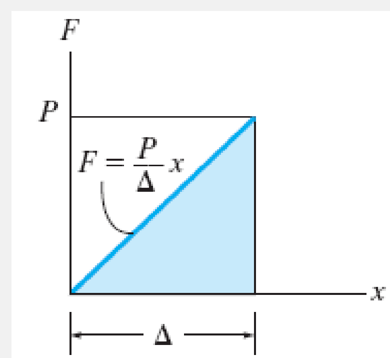
- Consider the effect caused by an axial force applied to the end of a bar

- F is gradually increased from 0 to some limiting value $F = P$

- The final elongation of the bar becomes Δ

- If the material has a linear elastic response, then $F = (P/\quad)x$

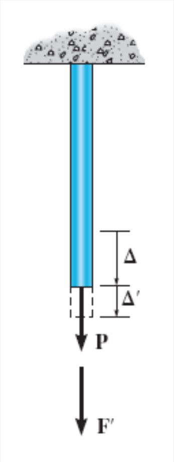


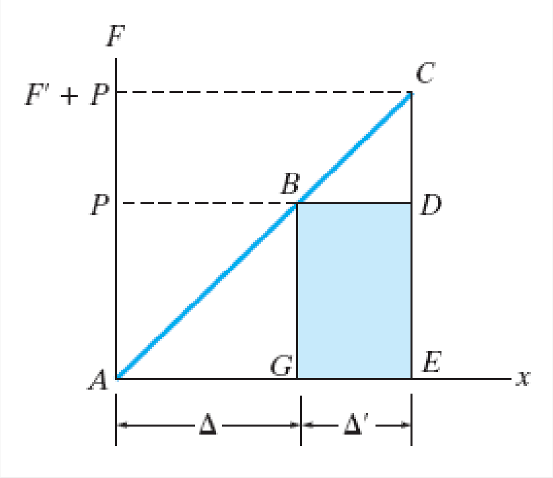


External Work & Strain Energy

- By integrating we have: $$U_e = \frac{1}{2} \mathrm{P}\Delta$$

- Suppose P is already applied to the bar & that another force F' is now applied, so that the bar deflects further by an amount Δ'





External Work & Strain Energy

- The work done by P when the bar undergoes the further deflection is then
  - $d\,U_e' = P\Delta'$
- Here the work rep the shaded rectangular area
- In this case, P does not change its magnitude since Δ' is caused only by F'
  - Work = force x disp

External Work & Strain Energy

- When a force P is applied to the bar, followed by an application of the force F', the total work done by both forces is rep by the triangular area ACE

- The triangular area ABG rep the work of P that is caused by disp Δ

- The triangular area BCD rep the work of F' since this force causes a displacement Δ'

- Lastly the shaded rectangular area BDEG rep the additional work done by P

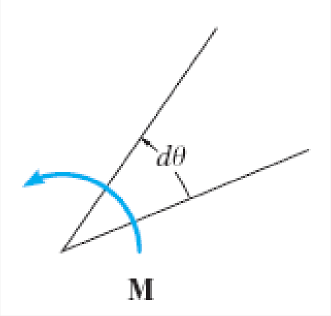
External Work & Strain Energy

- The work of a moment = magnitude of the moment $(M)x$ the angle $(d\theta)$ through which it rotates

  - $dU_e = Md\theta$

- If the total angle of rotation is θ rad, the work becomes

$$Ue = \int_0^{\theta} Md\theta$$



External Work & Strain Energy

- If the moment is applied gradually to a structure having a linear elastic response from 0 to M, then the work done is

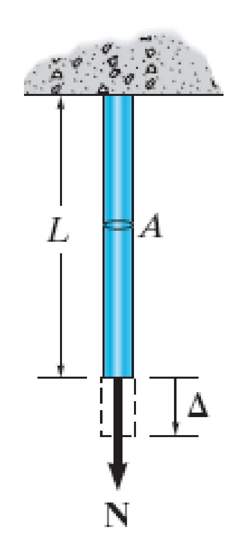
$$Ue = \frac{1}{2} M\theta$$

- However, if the moment is already applied to the structure & other loadings further distort the structure an amount θ', then M rotates θ' & the work done is

$$Ue' = M\theta'$$

External Work & Strain Energy

- When an axial force N is applied gradually to the bar, it will strain the material such that the external work done by N will be converted into strain energy
- Provided the material is linearly elastic, Hooke's Law is valid
  - $\sigma = E\varepsilon$
- If the bar has a constant x-sectional area A and length L



External Work & Strain Energy

- The normal stress is $\sigma = N/A$
- The final strain is $\varepsilon = \Delta/L$
- Consequently, $N/A = E(\Delta/L)$
- Final deflection:

$$\Delta = \frac{NL}{AE}$$

- Substituting with P = N,

$$U_i = \frac{N^2 L}{2AE}$$

External Work & Strain Energy

- Consider the beam, P & w are gradually applied

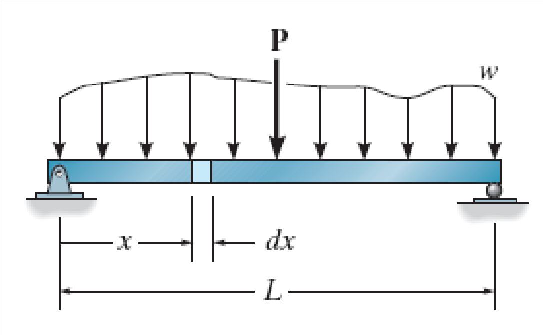
- These loads create an internal moment M in the beam at a section located a distance x from the left support

- Consequently, the strain energy or work stored in the element can be determined since the internal moment is gradually developed

External Work & Strain Energy

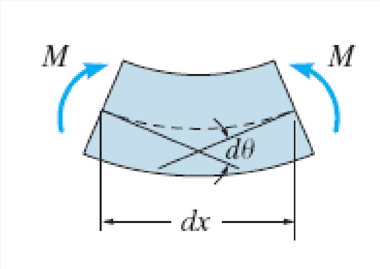
- Hence,

$$dU_i = \frac{M^2 dx}{2EI}$$



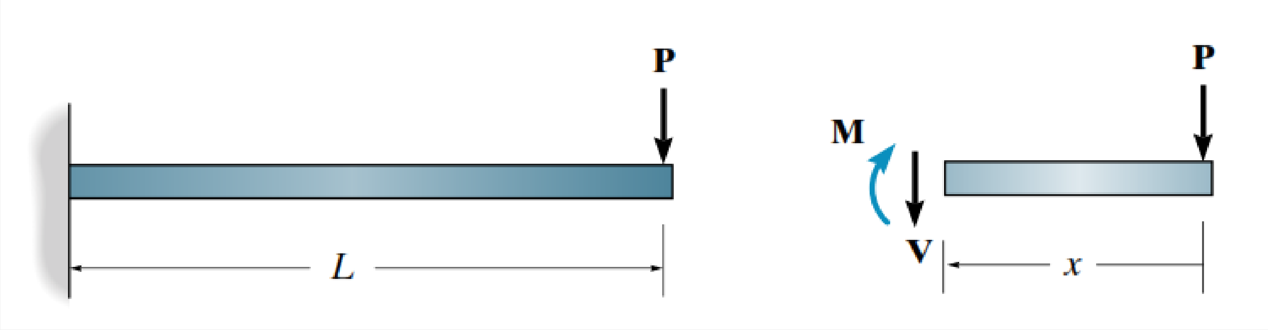
- The strain energy for the beam is determined by integrating this result over the beam's length

$$U_i = \int_0^L \frac{M^2 dx}{2EI}$$



Principle of Work & Energy

- Consider finding the displacement at a point where the force P is applied to the cantilever beam



- *The external work*: $$U_e = \frac{1}{2} P\Delta$$

- To obtain the resulting strain energy, we must first determine the internal moment as a function of position x in the beam

Principle of Work & Energy

- In this case, $M = -Px$ so that:

$$U_i = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{(-Px)^2 dx}{2EI} = \frac{1}{6}\frac{P^2L^3}{EI}$$

- Equating the external work to internal strain energy & solving for the unknown disp, we have:

$$U_e = U_i$$

$$\frac{1}{2}P\Delta = \frac{1}{6}\frac{P^2L^3}{EI}$$

$$\Delta = \frac{PL^3}{3EI}$$

Principle of Work & Energy

- Limitations
  - It will be noted that only one load may be applied to the structure
  - Only the displacement under the force can be obtained

Principle of Work & Energy

- If we take a deformable structure of any shape or size & apply a series of external loads P to it, it will cause internal loads u at points throughout the structure

- As a consequence of these loadings, external disp Δ will occur at the P loads & internal disp δ will occur at each point of internal loads u

- In general, these displacements do not have to be elastic, & they may not be related to the loads

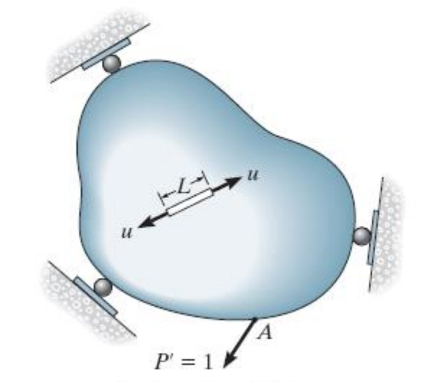
Principle of Virtual Work

- In general, the principle states that:

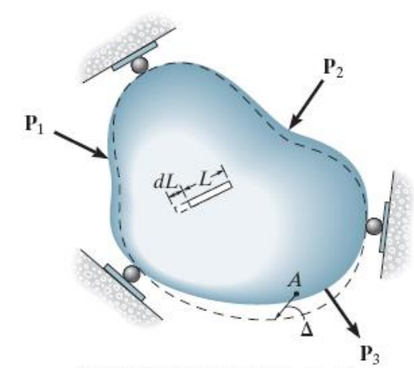
$$\sum P\Delta \quad = \quad \sum u\delta$$

Work of Ext loads = Work of Int loads

- Consider the structure (or body) to be of arbitrary shape
- Suppose it is necessary to determine the disp Δ of point A on the body caused by the "real loads" P_1, P_2 and P_3



Apply virtual load **P′** first.



Then apply real loads **P**1, **P**2, **P**3.

(b)

Principle of Virtual Work

- It is to be understood that these loads cause no movement of the supports
- They can strain the material beyond the elastic limit
- Since no external load acts on the body at A and in the direction of Δ, the disp Δ, the disp can be determined by first placing on the body a "**virtual**" load such that this force P' acts in the same direction as Δ

Principle of Virtual Work

- We will choose P' to have a unit magnitude, $P' = 1$

- Once the virtual loadings are applied, then the body is subjected to the real loads P_1, P_2 and P_3,

- Point A will be displaced an amount Δ causing the element to deform an amount dL

Principle of Virtual Work

- As a result, the external virtual force P' & internal load u “ride along” by Δ and dL respectively & therefore, perform external virtual work of $1.\Delta$ on the body and internal virtual work of $u.dL$ on the element

- By choosing $P' = 1$, it can be seen from the solution for Δ follows directly since $(1)\Delta = \Sigma u dL$

$$1.\Delta = \sum u.dL$$

- A virtual couple moment M’ having a unit magnitude is applied at this point

Principle of Virtual Work

- This couple moment causes a virtual load u_θ in one of the elements of the body
- Assuming that the real loads deform the element an amount dL, the rotation θ can be found from the virtual –work eqn

$$1.\theta = \sum u_\theta .dL$$

$$1 \cdot \Delta = \Sigma u \cdot dL$$

virtual loadings: 1 and u; real displacements: Δ and dL

$$1 \cdot \theta = \Sigma u_\theta \cdot dL$$

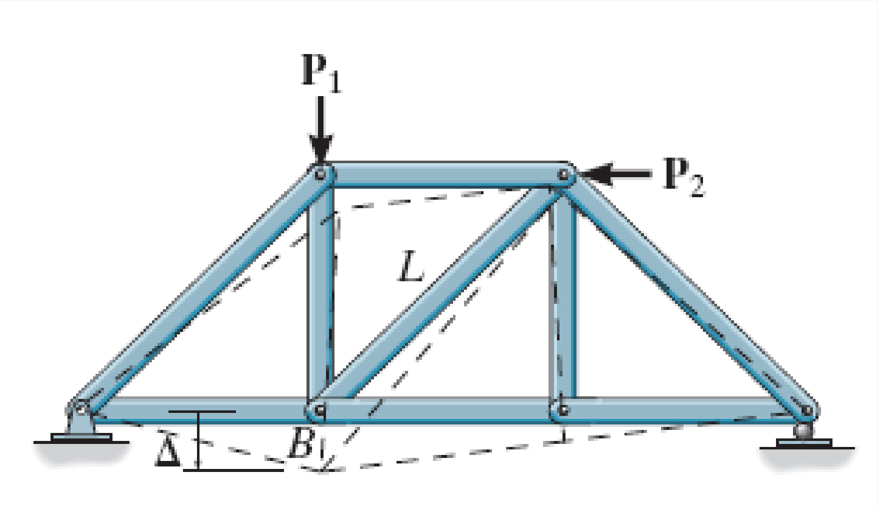
virtual loadings: 1 and u_θ; real displacements: θ and dL

This method for applying the principle of virtual work is often referred to as the ***method of virtual forces***, since a virtual force is applied resulting in the calculation of a *real displacement*.

Method of virtual work: Trusses

- External loading
  - Consider the vertical displacement Δ of joint B
  - If the applied loadings P_1 & P_2 cause a linear elastic material response, the element will deform as

$$\Delta L = NL / AE$$



Method of virtual work: Trusses

- External loading
  - The virtual work equation of the truss is:

$$1.\Delta = \sum \frac{nNL}{AE}$$

$1 =$ ext virtual unit load acting on the truss joint in the stated direction of Δ

$n =$ int virtual normal force in a truss member caused by the ext virtual unit load

$\Delta =$ ext joint disp caused by the real loads on the truss

$N =$ internal normal force in a truss member caused by the real load

$L =$ length of the member

$A =$ cross-sectional area of member

$E =$ modulus elasticity of a member

Method of virtual work: Trusses

- External loading
  - The external virtual load creates internal virtual forces n in each of the members
  - The real loads caused the truss joints to be displaced in the same direction as the virtual unit load
  - Each member is disp NL/AE in the same direction as its respective n force
  - Hence, ext virtual work = sum of int. (virtual) strain energy stored in truss members

Method of virtual work: Trusses

- Temperature
  - In some cases, truss members may change their length due to temperature
  - The disp of a selected truss joint may be written as

$$\Delta L = \alpha \Delta T L$$

$$1 . \Delta = \sum n \alpha \Delta T L$$

$\Delta =$ ext joint disp caused by temperature change
$\alpha =$ coefficient of thermal expansion of member
$\Delta T =$ change in temperature of a member

Method of virtual work: Trusses

- Fabrication errors & camber

  - Errors in fabricating the lengths of the members of a truss may occur

  - Truss members may also be made slightly longer or shorter in order to give the truss a camber

  - Camber is often built into bridge truss so that the bottom cord will curve upward by the same amount equivalent to the downward deflection when subjected to the bridge's full dead weight

Method of virtual work: Trusses

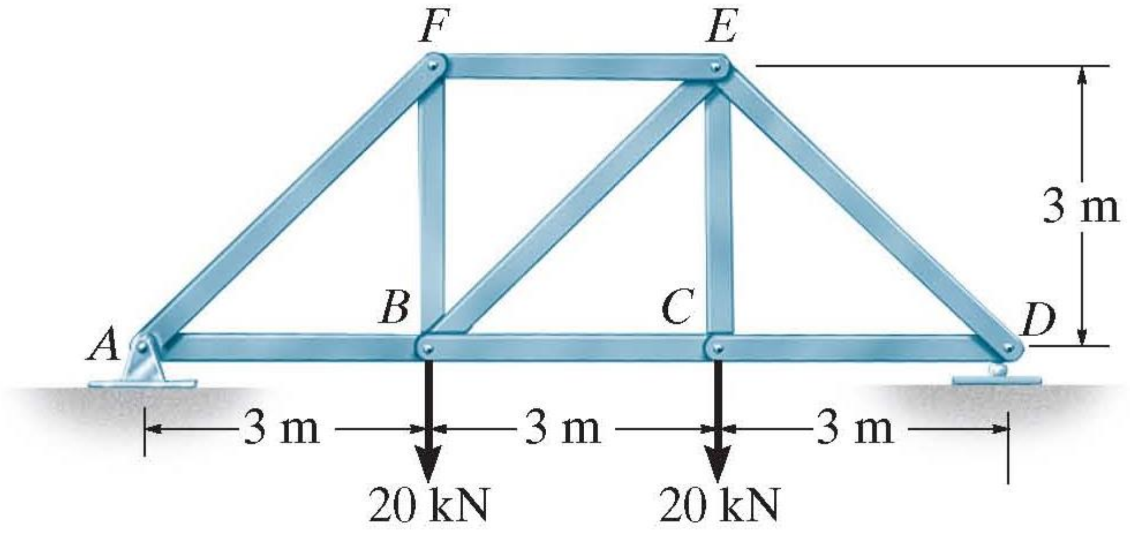
- Fabrication errors & camber
  - The disp of a truss joint from its expected position can be written as:

$$1.\Delta = \sum n\Delta L$$

Δ = ext joint disp caused by fabrication errors
ΔL = difference in length of the member from its intended size as caused by fabrication error

Example 1

Determine the vertical disp of joint C of the steel truss. The cross-sectional area of each member = 300mm^2 and E = 200GPa.

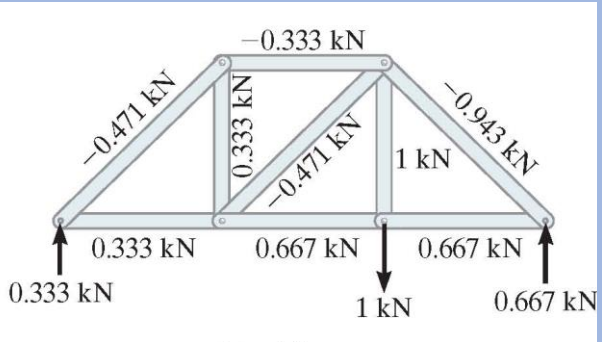


Solution

Virtual force

Only a vertical 1kN load is placed at joint C

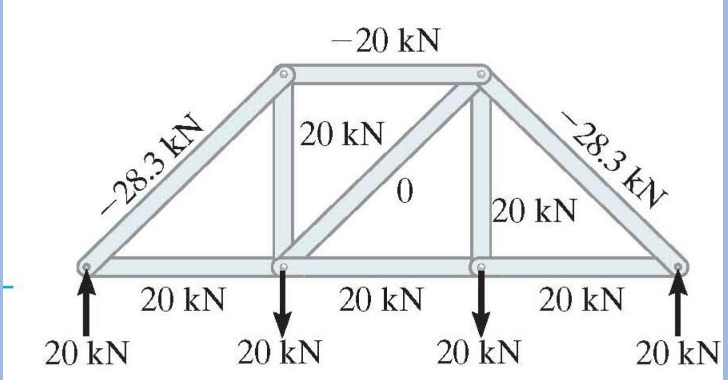
The force in each member is calculated using method of joints



virtual forces **n**

Real forces

The real forces are calculated using method of joints



real forces **N**

Solution

Virtual work eqn

Arranging the data in tabular form,

Member	n (kN)	N (kN)	L (m)	nNL ($kN^2 \cdot m$)
AB	0.333	20	3	20
BC	0.667	20	3	40
CD	0.667	20	3	40
DE	–0.943	–28.3	4.24	113
FE	–0.333	–20	3	20
EB	–0.471	0	4.24	0
BF	0.333	20	3	20
AF	–0.471	–28.3	4.24	56.6
CE	1	20	3	60
				Σ369.6

Solution

Virtual work eqn

$$1kN.\Delta_{c_v} = \sum \frac{nNL}{AE} = \frac{369.6kN^2m}{AE}$$

$$1kN.\Delta_{c_v} = \frac{369.6kN^2m}{(300(10^{-6})m^2)[200(10^6)kN/m^2]}$$

$$\Delta_{c_v} = 0.00616m = 6.16mm$$

Example 2

The cross-sectional area of each member has

$A = 400mm^2$

$E = 200GPa$

- Determine the vertical displacement of joint *C* if a 4kN force is applied to the truss at *C*.
- If no loads act on the truss, what would be the vertical displacement of joint *C* if member *AB* were 5 mm too short?

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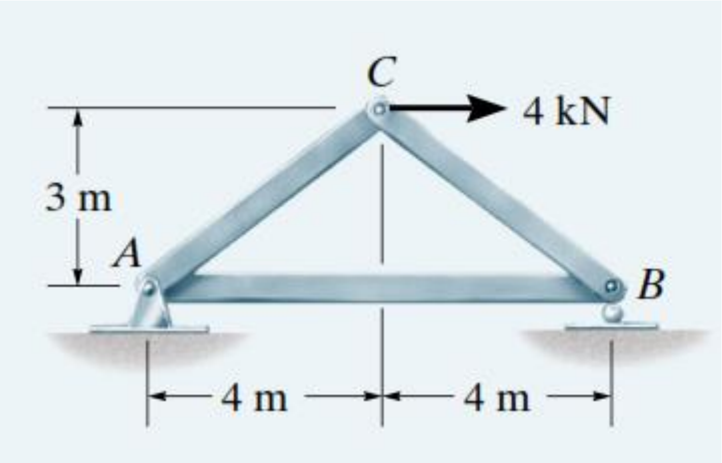
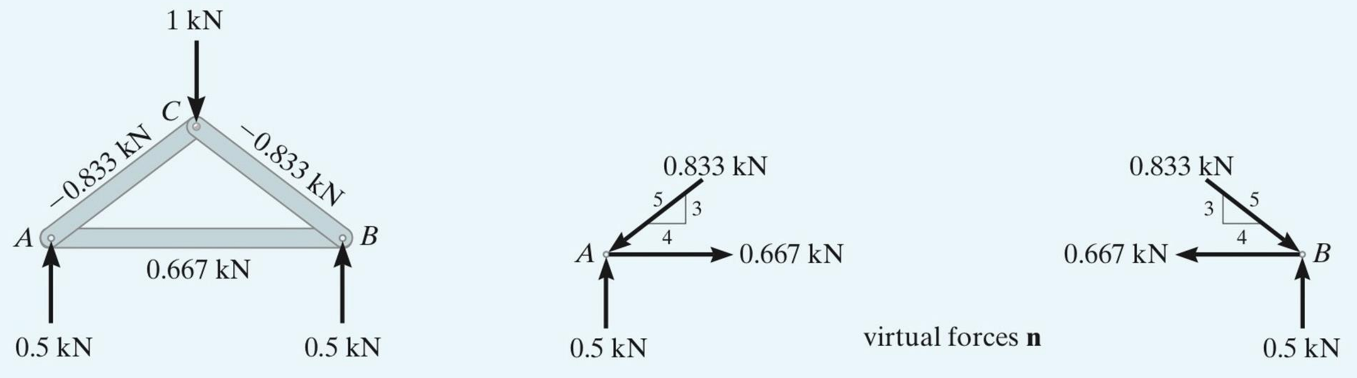


EXAMPLE 2 (continued)

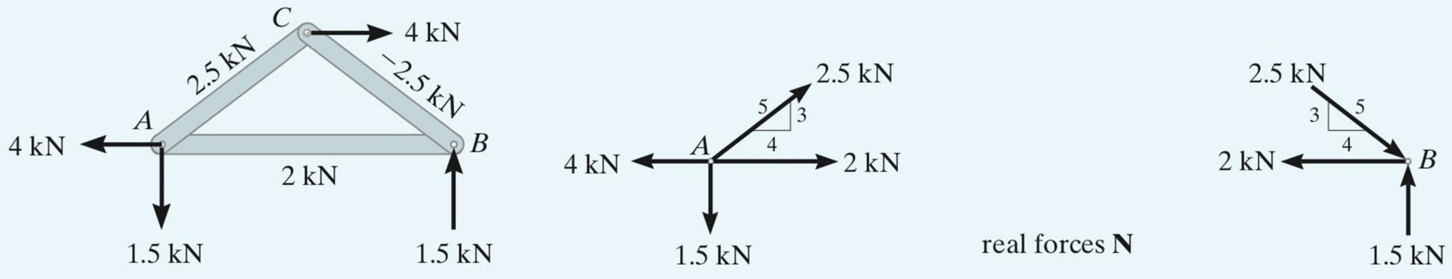
Virtual Forces *n*. *Since the vertical displacement of joint C is to be* \determined, a virtual force of 1 kN is applied at *C in the vertical* direction. The units of this force are the *same as those of the real* loading. The support reactions at *A and B are calculated and the* **n force in each member is determined by the method of joints as** shown on the free-body diagrams of joints *A and B, Fig. b*



(b)

Real Forces *N*. *The joint analysis of A and B when the real load of* 4 kN is applied to the truss is given in Fig. *c*

EXAMPLE 2 (continued)



(c)

Virtual-Work Equation. Since AE is constant, each of the terms nNL can be arranged in tabular form and computed. Here positive numbers indicate tensile forces and negative numbers indicate compressive forces.

Member	n (kN)	N (kN)	L (m)	$n\ NL$ ($kN^2 \cdot m$)
AB	0.667	2	8	10.67
AC	−0.833	2.5	5	−10.41
CB	−0.833	−2.5	5	10.41
				$\Sigma 10.67$

Thus,

$$1\ \text{kN} \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{10.67\ \text{kN}^2 \cdot \text{m}}{AE}$$

EXAMPLE 2 (continued)

Substituting the values $A = 400 \text{ mm}^2 = 400(10^{-6}) \text{ m}^2$, $E = 200 \text{ GPa} = 200(10^6) \text{ kN/m}^2$, we have

$$1 \text{ kN} \cdot \Delta_{C_v} = \frac{10.67 \text{ kN}^2 \cdot \text{m}}{400(10^{-6}) \text{ m}^2 (200(10^6) \text{ kN/m}^2)}$$

$$\Delta_{C_v} = 0.000133 \text{ m} = 0.133 \text{ mm} \qquad \textit{Ans.}$$

Here we must apply $1.\Delta = \sum n\Delta L$. Since the vertical displacement of *C is to be determined, we can use the results of* Fig. *b. Only member AB undergoes a change in length, namely,* of *L = -0.005 m.*
Thus,

$$1 \cdot \Delta = \Sigma n \, \Delta L$$

$$1 \text{ kN} \cdot \Delta_{C_v} = (0.667 \text{ kN})(-0.005 \text{ m})$$

$$\Delta_{C_v} = -0.00333 \text{ m} = -3.33 \text{ mm} \qquad \textit{Ans.}$$

The negative sign indicates joint *C* is displaced *upward,* opposite to the 1-kN vertical load. Note that if the 4-kN load and fabrication error are both accounted for, the resultant displacement is then $\Delta_{C_v} = 0.133 - 3.33 = -3.20$ mm (upward).