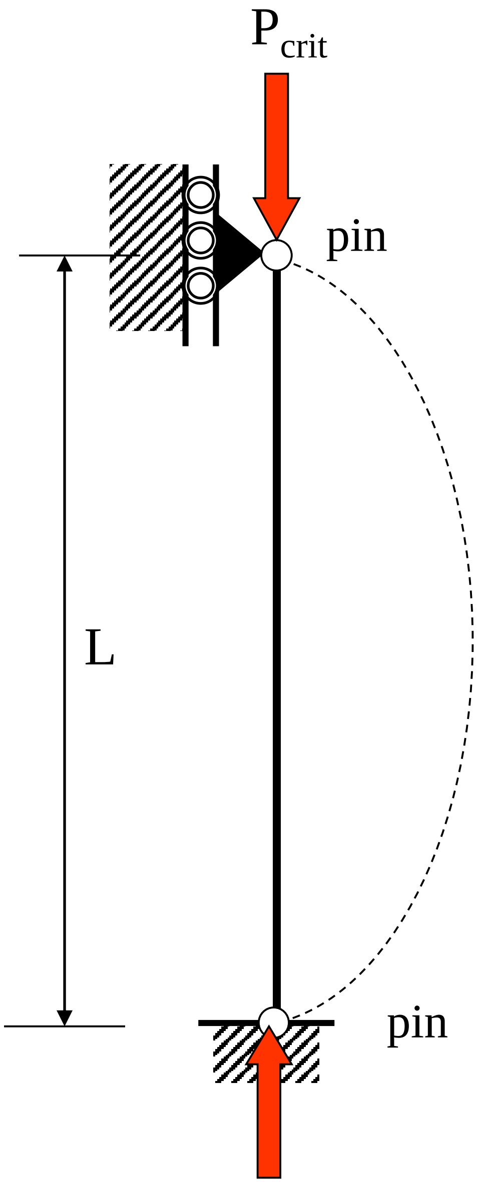
ELASTIC BUCKLING OF COLUMNS



COLUMNS CRITICAL LOAD

- Long slender members subjected to an axial compressive force are called columns, and the lateral deflection that occurs is called buckling.
- The maximum axial load that a column can support when it is on the verge of buckling is called the ***critical load (P_{cr})***

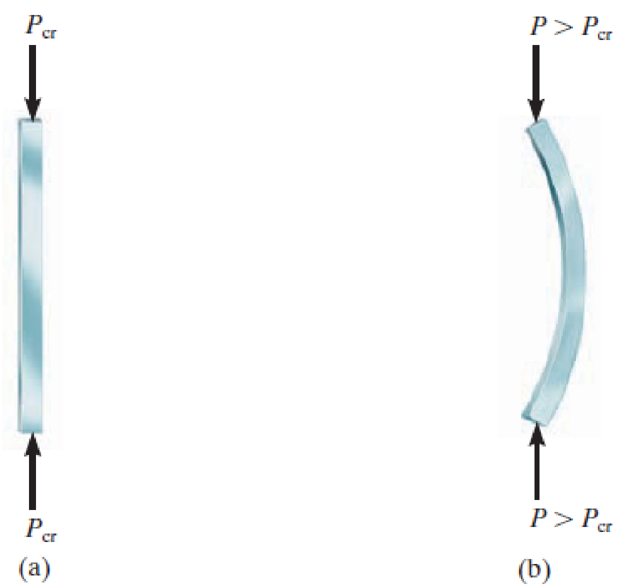


Fig 1

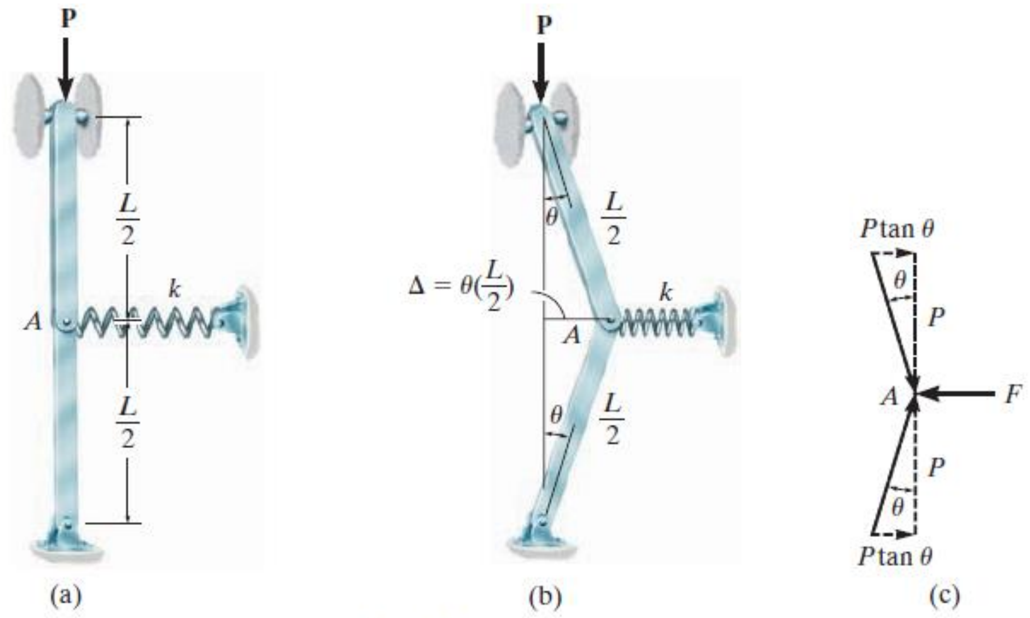
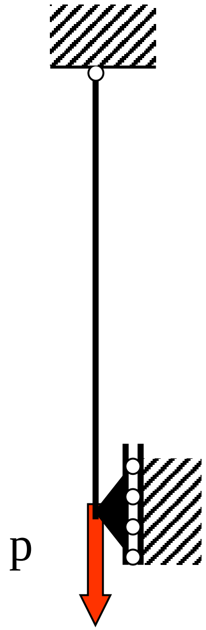
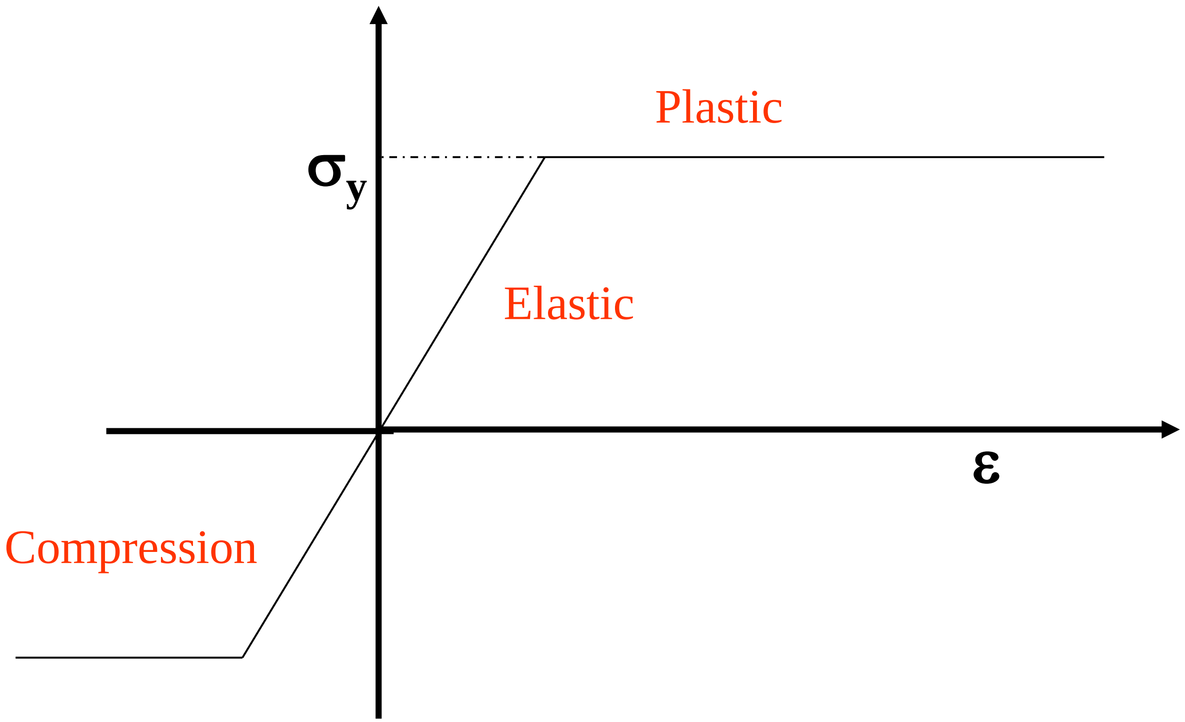


Fig 2

TENSION

- Tension stresses. No buckling
- Failure by yielding of ductile material (steel, Aluminium)





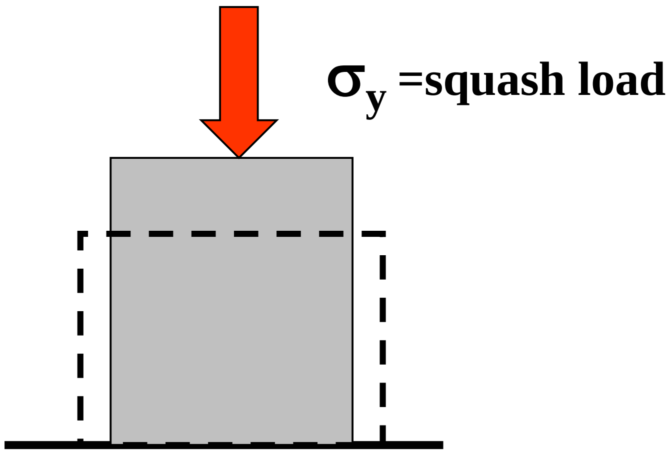
COMPRESSION

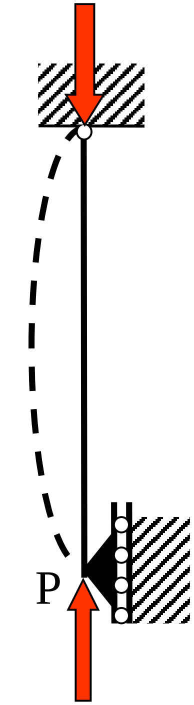
STOCKY or SHORT STRUT

FAILURE BY YIELDING

SLENDER STRUT

FAILURE BY BUCKLING



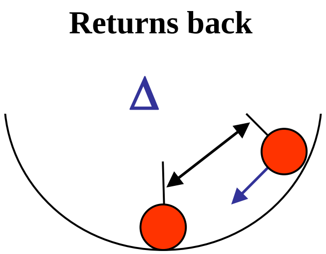


Stocky column

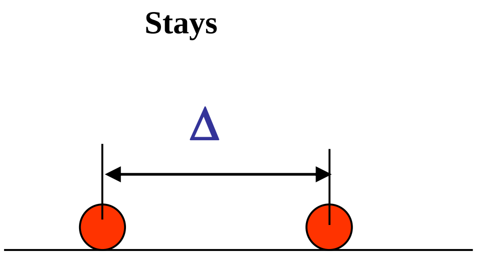
Slender column

STABILITY

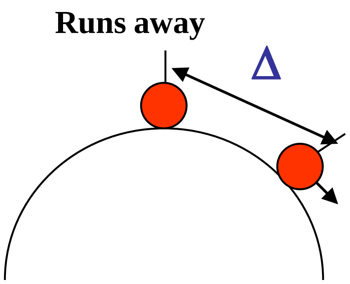
3 Types of equilibrium



STABLE



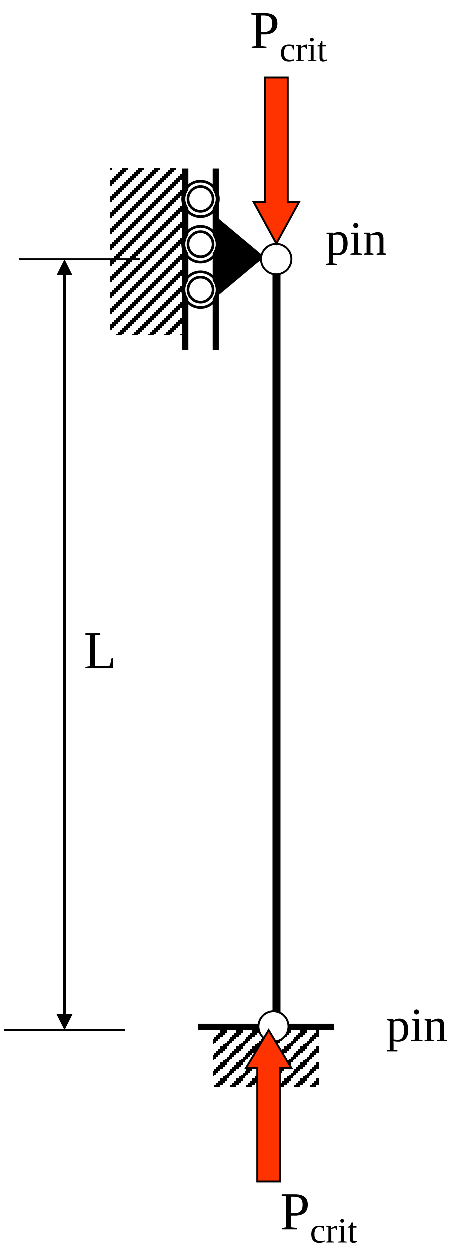
NEUTRAL



UNSTABLE

THE SIMPLE STRUT

DISPLACEMENT METHOD



PERFECT STRUT

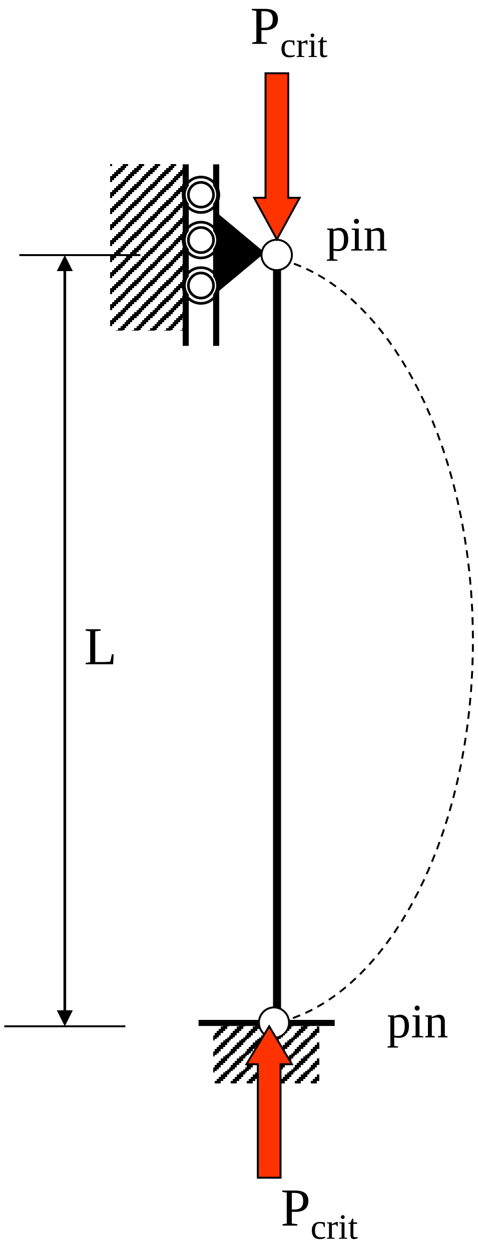
Conditions:

1. Perfectly straight (no initial deformation in the strut)
2. Homogenous (no residual stresses)
3. No eccentric loads (Loads are applied exactly through the centroid of the section)

NO INITIAL IMPERFECTIONS

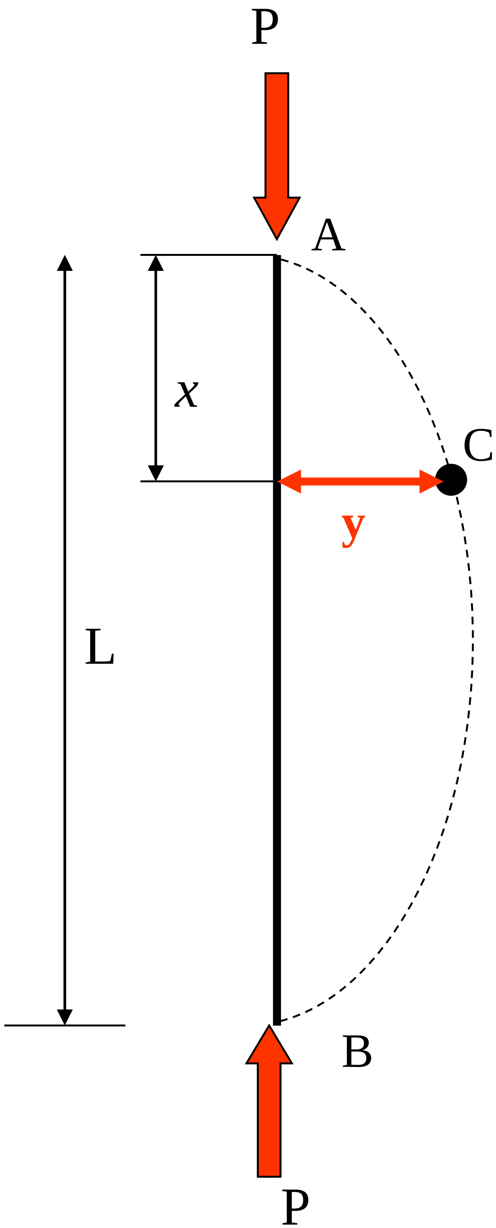
PERFECT STRUT FAILS BY COLLAPSING IMMEDIATELY AFTER REACHING THE CRITICAL LOAD

NORMAL STRUT



- ✓Has imperfections like initial curvature or eccentricity
- ✓This strut tends to bow as load increased until reaching a run away point where failure by buckling takes place
- ✓Failure occurs by combined bending and direct stresses
- ✓Almost all columns in practice have imperfections, i.e. they are not perfect struts

GOVERNING DEFERENTIAL EQUATION



Internal moment $M_I = -EI\dfrac{d^2y}{dx^2}$ (- hogging)

External Moment $M_E = P.y$

Equilibrium of point C:

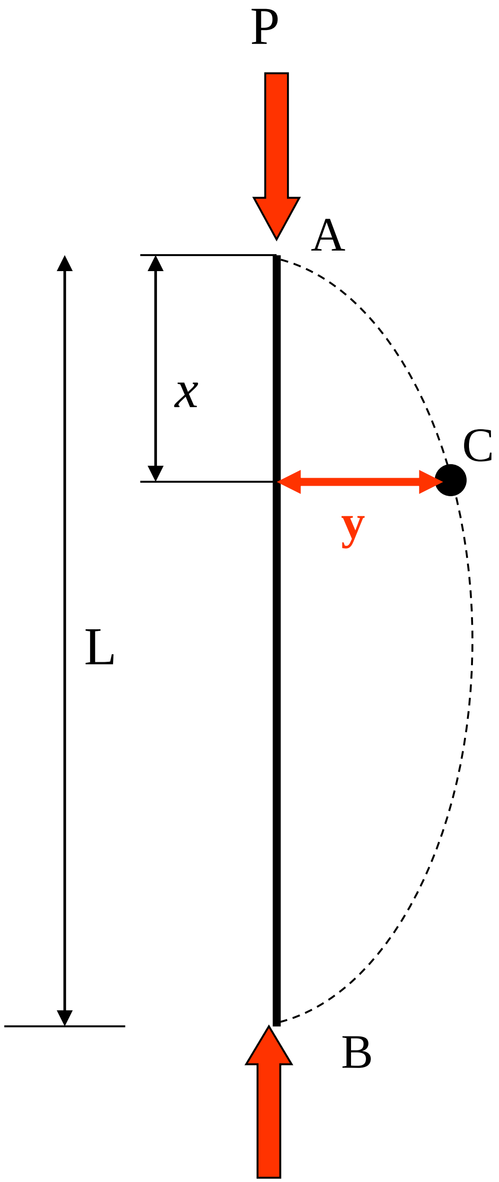
$M_E = M_I$

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \bullet y = 0$$

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

where

$$\omega^2 = \frac{P}{EI}$$



SOLUTION OF THE GOVERNING EQUATION

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

where

$$\omega^2 = \frac{P}{EI}$$

SOLUTION

$$y = A \sin \omega x + B \cos \omega x$$

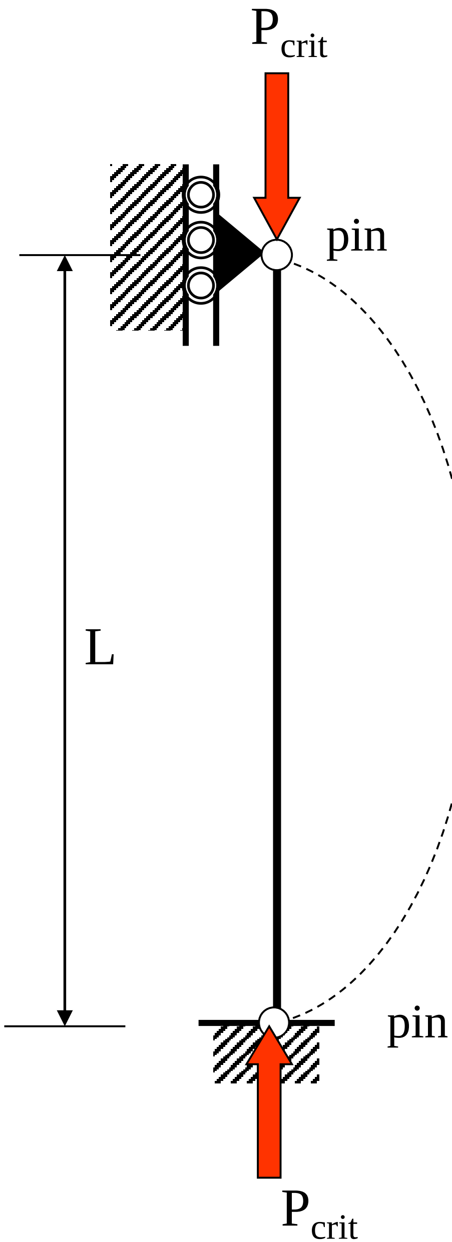
2 unknowns A & B require 2 boundary conditions

- At $x = 0$, $y = 0$; $B = 0$
- At $x = L$, $y = 0$; $0 = A \sin\omega L$

$A \sin(\omega L) = 0;$ $\omega L = n\pi$

$$P = \frac{n^2 \pi^2 EI}{L^2} \text{ where } n = 1,2,3,...$$

SOLUTIONS OF GOVERNING EQUATION FOR DIFFERENT CASES



Basic case:

Smallest value of P (also known as EULER LOAD) is when n = 1, thus

$$P_{CRIT} = P_E = \frac{\pi^2 EI}{L^2}$$

Case 1. Simple Strut. Both ends pinned

$\omega = n^2\pi/L^2 = P/EI$

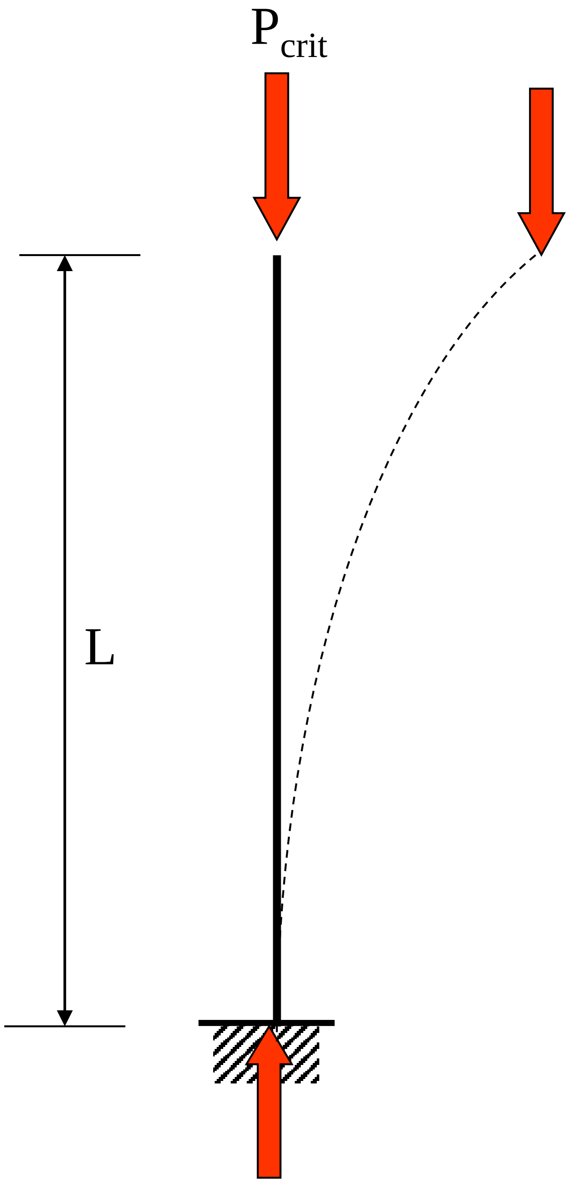
$$P_E = n^2 \frac{\pi^2 EI}{L^2}$$

Le = L

IDEAL COLUMN (cont)

- Smallest value of P (also known as EULER LOAD) is when n = 1, thus $$P_{cr} = \frac{\pi^2 EI}{L^2}$$
- column will buckle about the principal axis of the cross section having the ***least moment of inertia***
- For design purposes, the above equation can also be written in a more useful form by expressing $I = Ar^2$
- Corresponding stress is $\sigma_{cr} = \dfrac{\pi^2 E}{(L/r)^2}$; $\quad where \quad \sigma_{cr} \leq \sigma_Y$
- Where r = √ (I/A) is called ‘radius of gyration’
- (L/r) is called the ‘**slenderness ratio**’

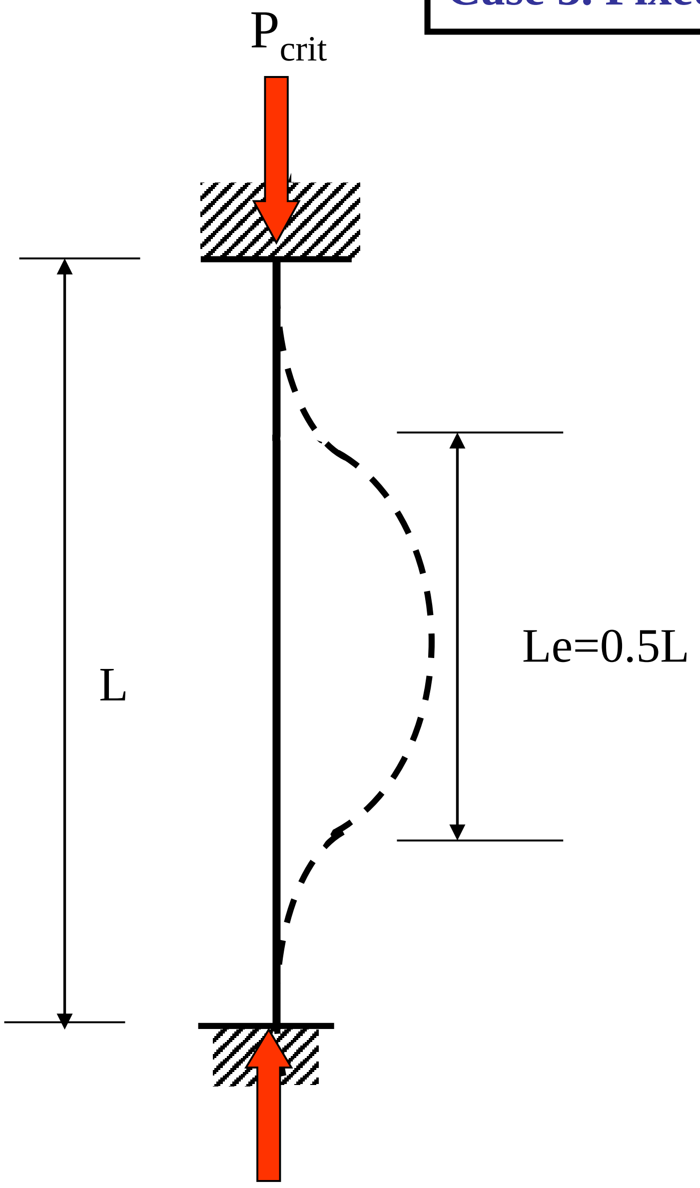
Case 2. Cantilever Strut. One end fixed and the other free



$$P_E = \frac{\pi^2 EI}{4L^2}$$

Le=2L

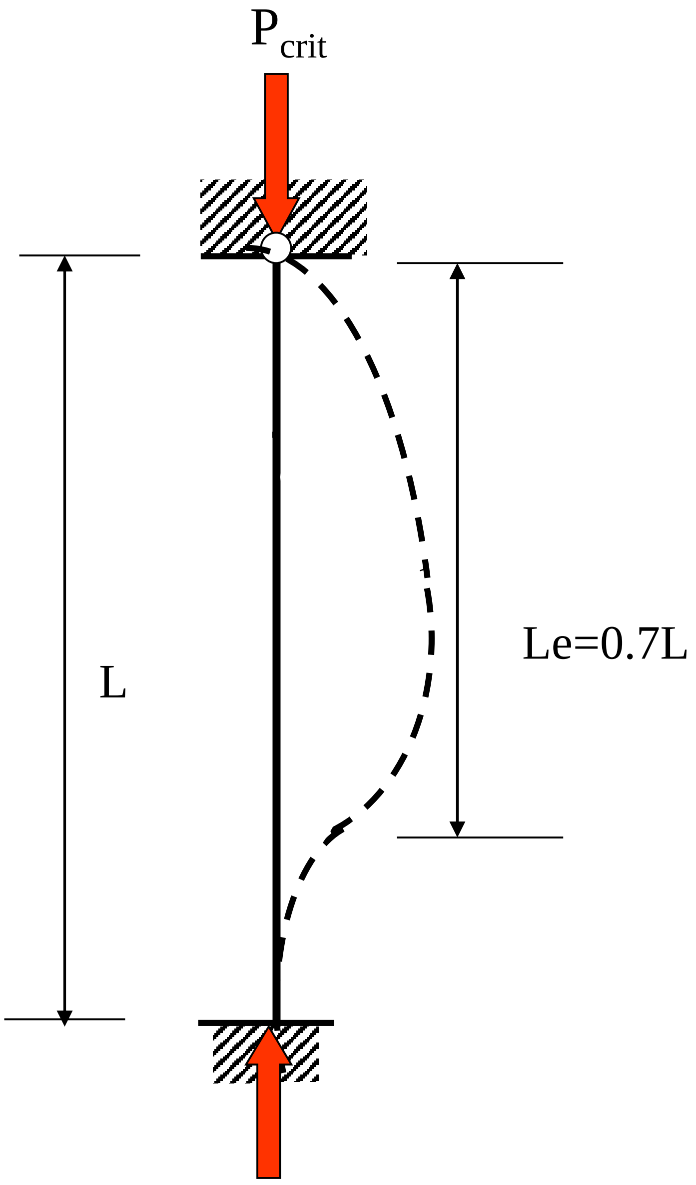
Case 3. Fixed at both ends



$$P_E = \frac{4\pi^2 EI}{L^2}$$

Le=0.5L

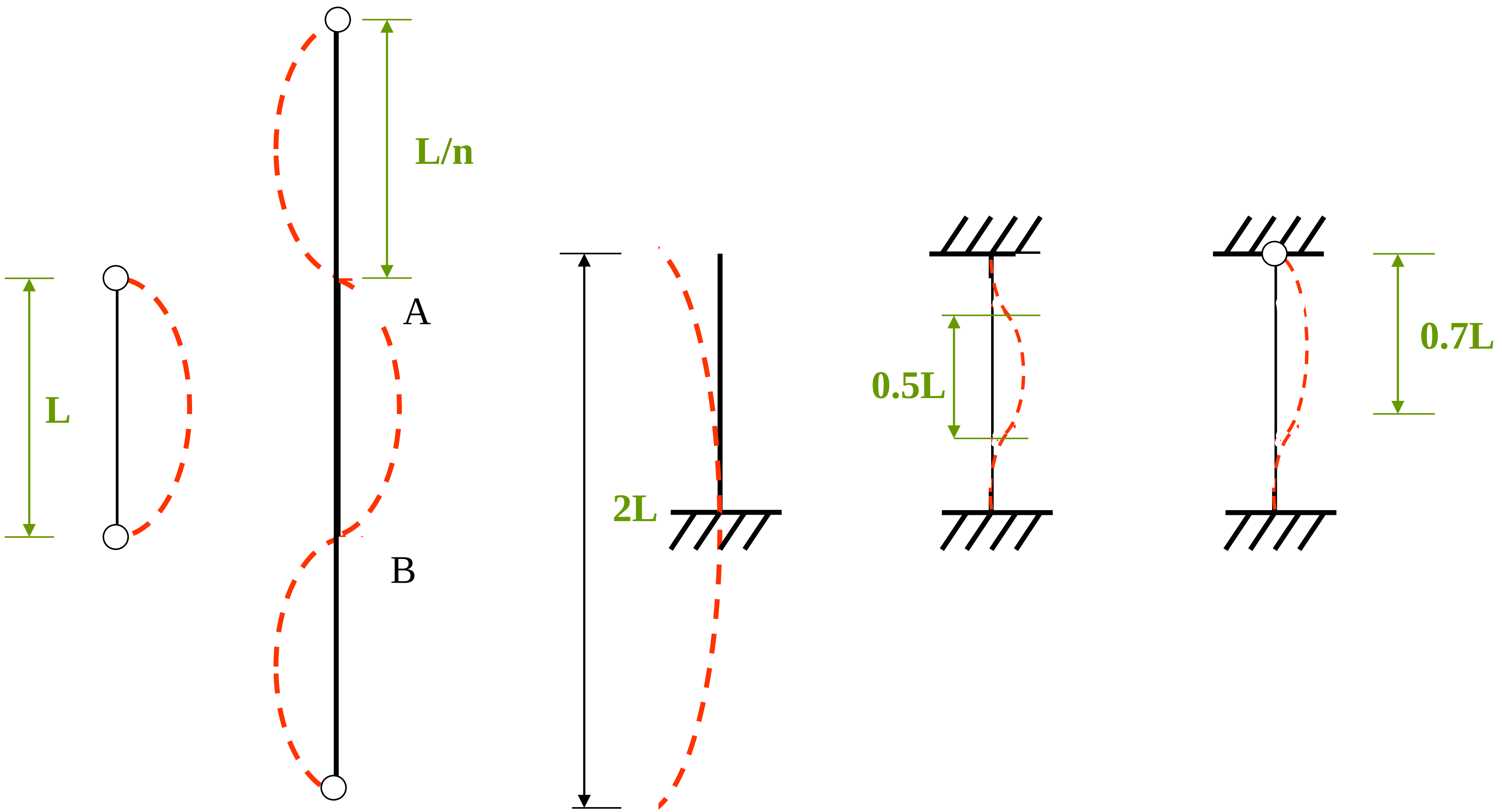
Case 4. One end fixed the other pinned



$$P_E = \frac{2.042\,\pi^2 EI}{L^2}$$

Le=0.7L

Theoretical Effective Lengths



Le=L

Le=L/n

Le=2L

Le=0.5L

Le=0.7L

CRITICAL STRESS

$$P_E = \frac{\pi^2 EI}{L^2}$$

λ - Slenderness ratio , depends on geometrical parameters $\lambda = \frac{L_e}{r}$

It is a measure of the column's flexibility, and serves to classify columns as long, intermediate, or short.

Le = Effective length of the strut

r = radius of gyration, $r = \sqrt{\frac{I}{A}}$

$$\sigma_E = \frac{\pi^2 E}{\lambda^2}$$

Column will buckle about the principal axis of the cross section having the ***least moment of inertia***

Example 1

Calculate the capacity of a 4 m steel column for 2 cases:

1.Fixed supports at both ends

2.Simply supported at both ends

Given I = 9500x10^4 mm^4 and E = 210 kN/mm^2

Solution

1.Fixed supports at both ends

$$Le = \frac{L}{2} = \frac{4}{2} = 2$$

$$F = \frac{\pi^2 EI}{L^2}$$

$$F = \frac{\pi^2 \times 210 \times 9500 \times 10^4}{2000^2}$$

$$= \mathbf{49224.65kN}$$

2. Simply Supported

$$Le = L = 4$$

$$F = \frac{\pi^2 EI}{L^2}$$

$$F = \frac{\pi^2 \times 210 \times 9500 \times 10^4}{4000^2}$$

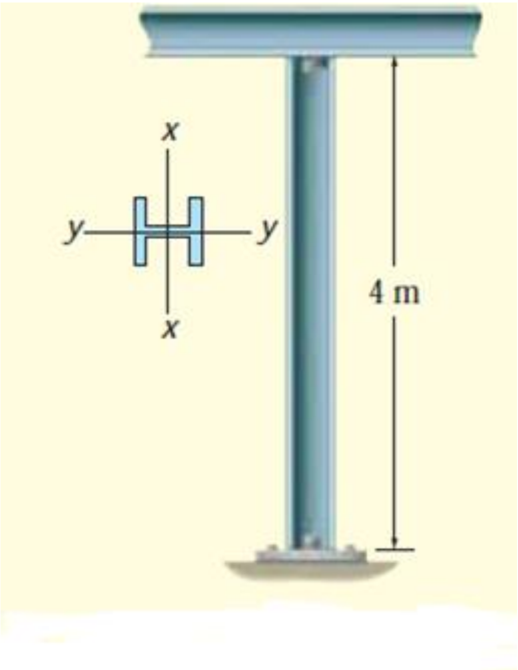
$$= \mathbf{12306.16kN}$$

EXAMPLE 2

The A-36 steel W200 X 46 member shown below is to be used as a pin-connected column. Determine the largest axial load it can support before it either begins to buckle or the steel yields.

Given $A = 5890 \text{ mm}^2, I_x = 45.5 \times 10^6 \text{ mm}^4, I_y = 15.3 \times 10^6 \text{ mm}^4$

$\sigma_Y = 250MPa \qquad E = 200MPa$



EXAMPLE 2 (cont)

Solutions

- By inspection, buckling will occur about the *y–y* axis.

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \left(200\times10^9\right)\left(15.3\times10^6\right)\left(1/1000\right)^4}{4^2} = 1887.6 \text{ kN}$$

- When fully loaded, the average compressive stress in the column is

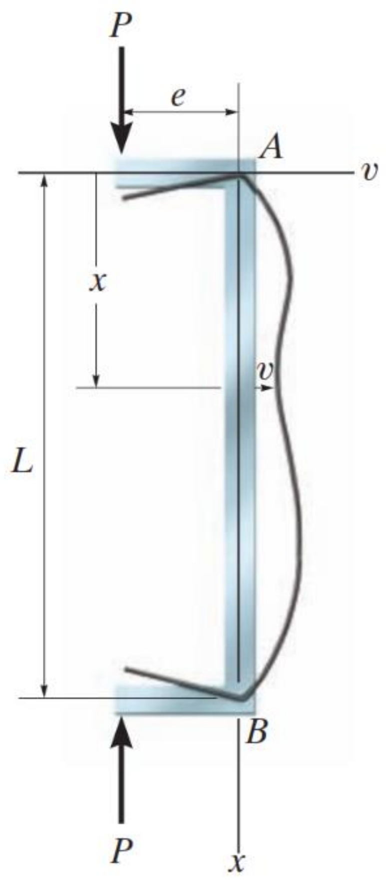
$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{1887.6\times1000}{5890} = 320.5 \text{ N/mm}^2$$

- Since this stress exceeds the yield stress,

$$250 = \frac{P}{5890} \Rightarrow P = 1472.5 \text{ kN (Ans)}$$

In actual practice, a factor of safety would be placed on this loading

ECCENTRIC LOADING OF COLUMNS



ECCENTRIC LOAD LOADING – THE SECANT FORMULA

- For design of a column subjected to eccentric load, consider the moment-curvature equation

$$EI\frac{d^2v}{dx^2} = M = -P(e+v)$$

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = \frac{P}{EI}e$$

- or $\frac{d^2v}{dx^2} + \lambda^2 v = -\lambda^2 e$ where $\lambda^2 = \frac{P}{EI}$

The solution is $v = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) - e$

- Since $v = 0$ at $x = 0$, so $C_2 = e$;
- at $x = L, v = 0$
- So, $C_1 = \frac{e[1-\cos(\lambda L)]}{\sin \lambda L}$

ECCENTRIC LOAD LOADING – THE SECANT FORMULA

Since

$$1 - \cos(\lambda) = 2\sin^{2}\left(\frac{\lambda L}{2}\right)$$

and

$$\sin(\lambda) = 2\sin\left(\frac{\lambda L}{2}\right)\cos\left(\frac{\lambda L}{2}\right)$$

We have;

$$C_1 = e\tan\left(\frac{\lambda L}{2}\right)$$

Hence;

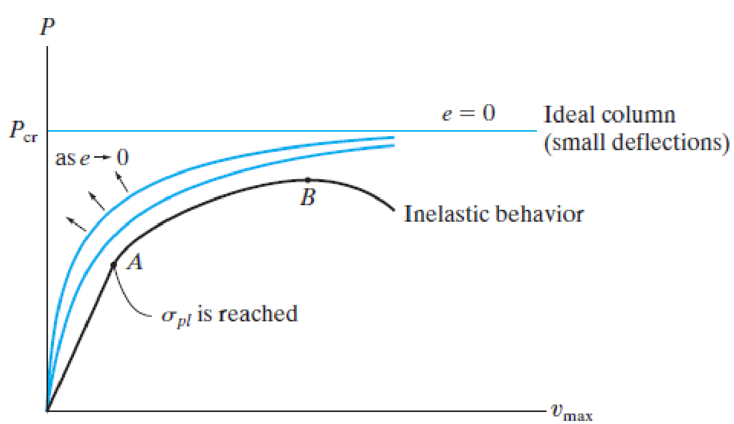
$$v = e\left[\tan\left(\frac{\lambda L}{2}\right)\sin(\lambda x) + \cos(\lambda x) - 1\right]$$

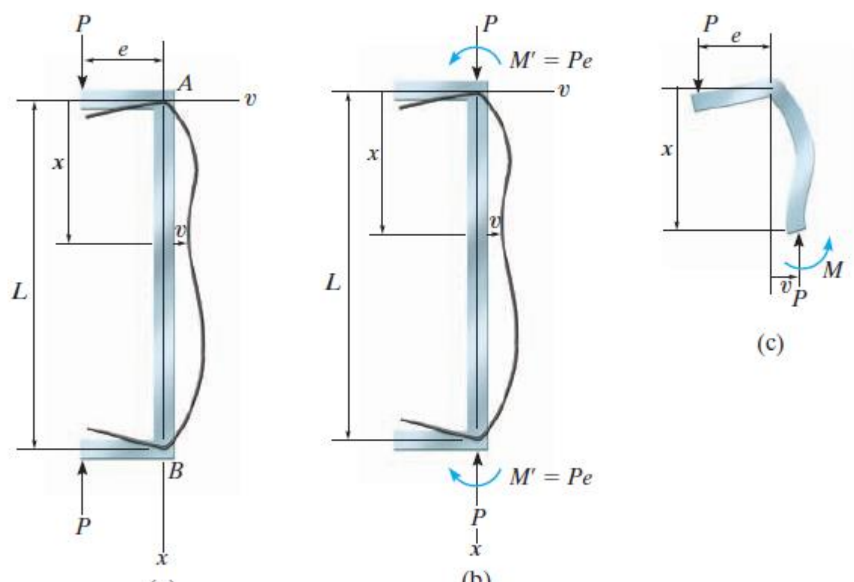
SECANT FORMULA (cont)

Due to symmetry of loading, both the maximum deflection & maximum stress occur at the column’s midpoint. Therefore, at $x = L/2$

$$v_{\max|x=L/2} = e\left[\sec\left(L\lambda/2\right) - 1\right]$$

Note: A nonlinear relationship occurs between the load P and the defection v. As a result, the principle of superposition does not apply here.





SECANT FORMULA (cont)

Notice that if e approaches zero, then v_{max} approaches zero. However, if the terms in the brackets approach infinity as e approaches zero, then v_{max} will have a nonzero value. Mathematically, this would represent the behavior of an axially loaded column at failure when subjected to the critical load P_{cr} Therefore, to find P_{cr} we require

$$\sec\left(\sqrt{\frac{P_{cr}}{EI}}\right)\left(\frac{L}{2}\right) = \infty$$

$$\sqrt{\frac{P_{cr}}{EI}}\left(\frac{L}{2}\right) = \frac{\pi}{2}$$

$$\boldsymbol{P_{cr} = \frac{\pi^2 EI}{L^2}}$$

SECANT FORMULA (cont)

Maximum moment occurs at the column's midpoint, i.e

$$M = \left| P\left(e + v_{\max}\right) \right| \text{ or } M = Pe\left(\frac{\lambda L}{2}\right)$$

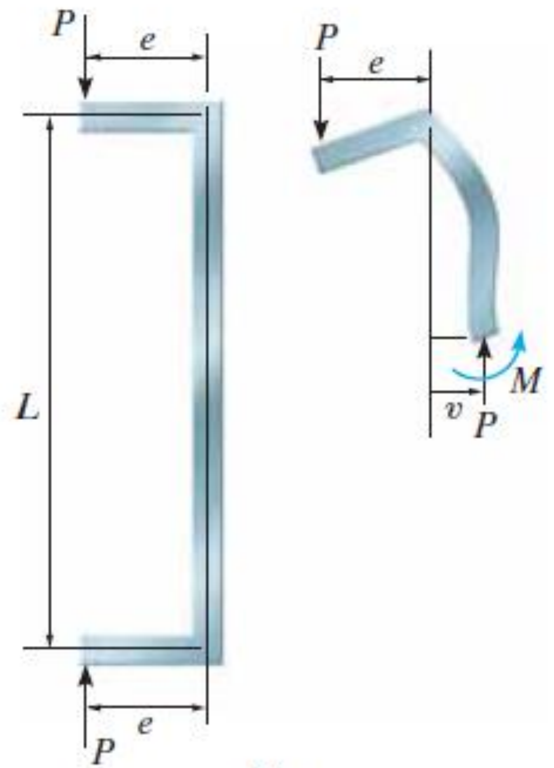
$$\sigma_{\max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Pe \cdot c}{I}\sec\left(\frac{\lambda L}{2}\right)$$

$$\textit{noting that } r^2 = \frac{I}{A}$$

$$\text{Or, } \sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\left(\frac{P}{EA}\right)}\right)\right]$$

This expression is referred to as the *secant formula*

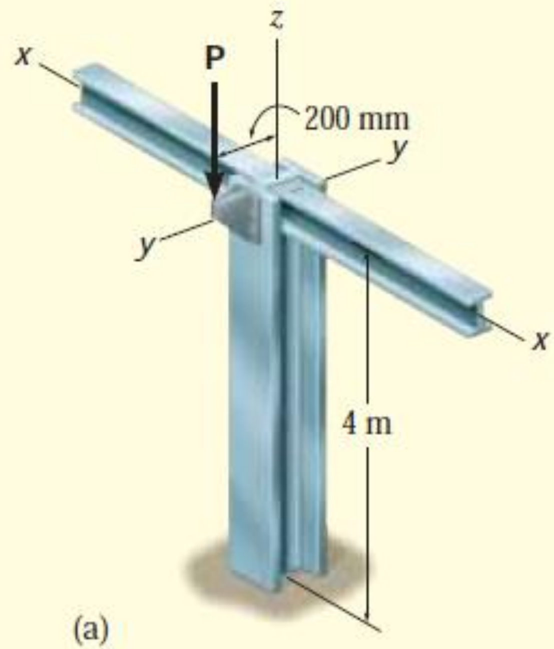
$\frac{ec}{r^2}$ = *eccentricity ratio*



(a)

EXAMPLE 3

The W200 x 59 A-36 steel column shown is fixed at its base and braced at the top so that it is fixed from displacement, yet free to rotate about the *y–y* axis. Also, it can sway to the side in the *y–z* plane. Determine the maximum eccentric load the column can support before it either begins to buckle or the steel yields. σ_Y=250MPa



(a)

Wide-Flange Sections or W Shapes SI Units

Designation	Area A	Depth d	Web thickness t_w	Flange width b_f	Flange thickness t_f	x–x axis I	x–x axis S	x–x axis r	y–y axis I	y–y axis S	y–y axis r
mm × kg/m	mm^2	mm	mm	mm	mm	$10^6\ mm^4$	$10^3\ mm^3$	mm	$10^6\ mm^4$	$10^3\ mm^3$	mm
W310 × 129	16 500	318	13.10	308.0	20.6	308	1940	137	100	649	77.8
W310 × 74	9 480	310	9.40	205.0	16.3	165	1060	132	23.4	228	49.7
W310 × 67	8 530	306	8.51	204.0	14.6	145	948	130	20.7	203	49.3
W310 × 39	4 930	310	5.84	165.0	9.7	84.8	547	131	7.23	87.6	38.3
W310 × 33	4 180	313	6.60	102.0	10.8	65.0	415	125	1.92	37.6	21.4
W310 × 24	3 040	305	5.59	101.0	6.7	42.8	281	119	1.16	23.0	19.5
W310 × 21	2 680	303	5.08	101.0	5.7	37.0	244	117	0.986	19.5	19.2
W250 × 149	19 000	282	17.30	263.0	28.4	259	1840	117	86.2	656	67.4
W250 × 80	10 200	256	9.40	255.0	15.6	126	984	111	43.1	338	65.0
W250 × 67	8 560	257	8.89	204.0	15.7	104	809	110	22.2	218	50.9
W250 × 58	7 400	252	8.00	203.0	13.5	87.3	693	109	18.8	185	50.4
W250 × 45	5 700	266	7.62	148.0	13.0	71.1	535	112	7.03	95	35.1
W250 × 28	3 620	260	6.35	102.0	10.0	39.9	307	105	1.78	34.9	22.2
W250 × 22	2 850	254	5.84	102.0	6.9	28.8	227	101	1.22	23.9	20.7
W250 × 18	2 280	251	4.83	101.0	5.3	22.5	179	99.3	0.919	18.2	20.1
W200 × 100	12 700	229	14.50	210.0	23.7	113	987	94.3	36.6	349	53.7
W200 × 86	11 000	222	13.00	209.0	20.6	94.7	853	92.8	31.4	300	53.4
W200 × 71	9 100	216	10.20	206.0	17.4	76.6	709	91.7	25.4	247	52.8
W200 × 59	7 580	210	9.14	205.0	14.2	61.2	583	89.9	20.4	199	51.9
W200 × 46	5 890	203	7.24	203.0	11.0	45.5	448	87.9	15.3	151	51.0
W200 × 36	4 570	201	6.22	165.0	10.2	34.4	342	86.8	7.64	92.6	40.9
W200 × 22	2 860	206	6.22	102.0	8.0	20.0	194	83.6	1.42	27.8	22.3

EXAMPLE 3 (cont)

Solutions

- For *y–y* axis buckling, it is subjected to an axial load *P*.

the effective length factor is for fixed-pinned ends is L_e =KL = 0.7X4000

- so

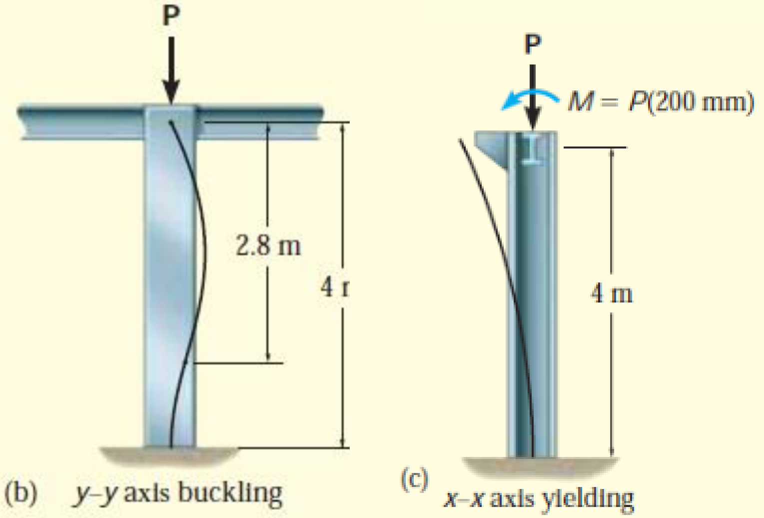
$$(P_{cr})_y = \frac{\pi^2 E I_y}{(L_e)_y^2} = \frac{\pi^2(200\times10^3)(20.4\times10^6)}{2800^2} = 5136\text{ kN}$$

- For *x–x* axis yielding, it is subjected to an axial load *P* and moment M.

$$\sigma_Y = \frac{P_x}{A}\left[1+\frac{ec}{r_x^2}\sec\left(\frac{(KL)_x}{2r_x}\sqrt{\frac{P_x}{EA}}\right)\right]$$

$$1.895\times10^6 = P_x\left[1+2.598\sec\left(1.143\times10^{-3}\sqrt{P_x}\right)\right]$$

$$P_x = 419368\text{ N} = 419.4\text{ kN}\quad\text{(Ans)}$$



Since this value (419.4kN) is less than 5136kN, failure will occur about the *x–x* axis.