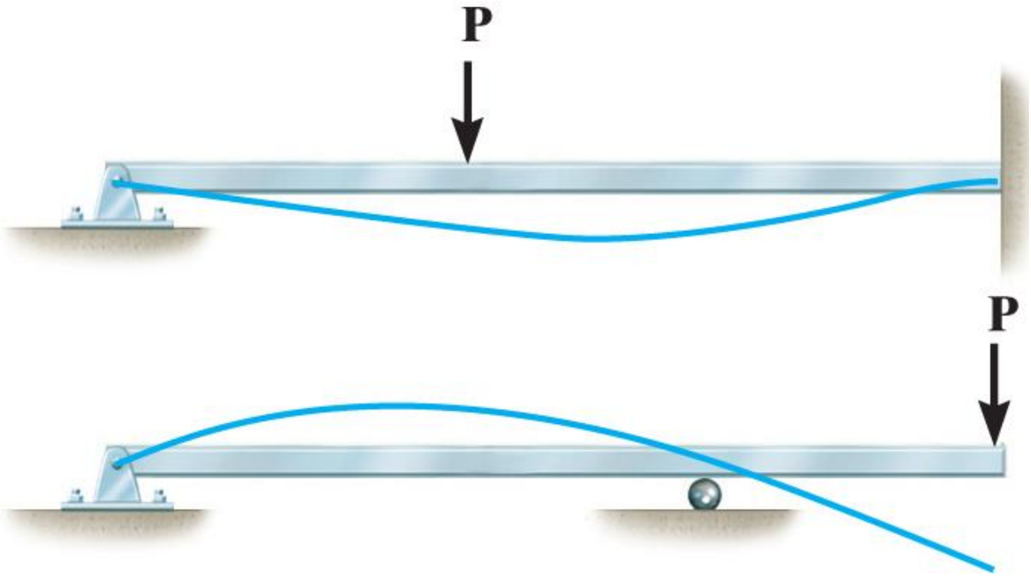
DEFLECTION OF BEAMS

- ✓ Determine the deflection and slope at specific points on beams and shafts, using various analytical methods including:
  - ➢ The integration method
  - ➢ The use of discontinuity functions
  - ➢ The method of superposition
- ✓ Determine the same, using a semi-graphical technique, called the moment-area method.

APPLICATIONS

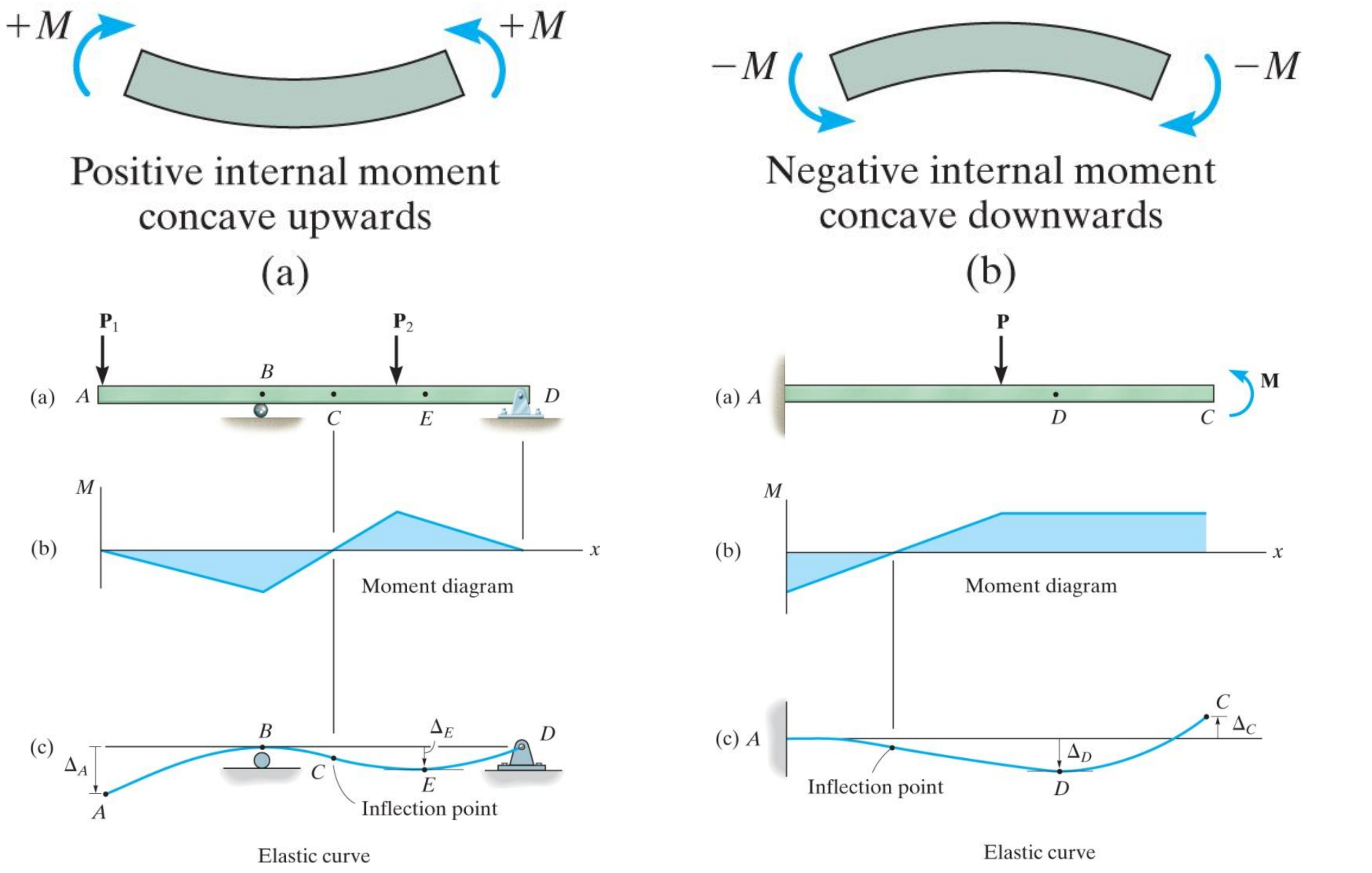
ELASTIC CURVE

- The deflection diagram of the longitudinal axis that passes through the centroid of each cross-sectional area of the beam is called the elastic curve, which is characterized by the deflection and slope along the curve



ELASTIC CURVE (cont)

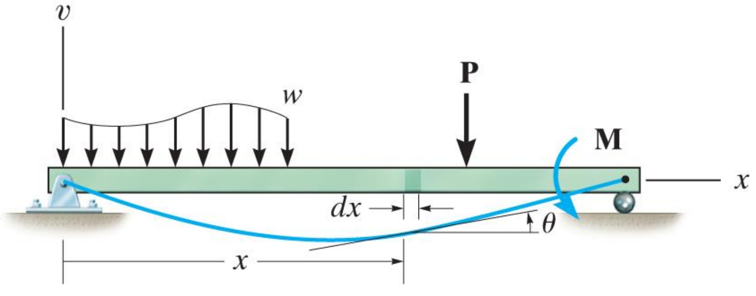
- Moment-curvature relationship:
  - Sign convention:



ELASTIC CURVE (cont)

- Consider a segment of width dx, the strain in fibre are ds, located at a position y from the neutral axis is ε = (ds' – ds)/ds. However, ds = dx = ρdθ and ds' = (ρ-y) dθ, and so ε = [(ρ – y) dθ – ρdθ

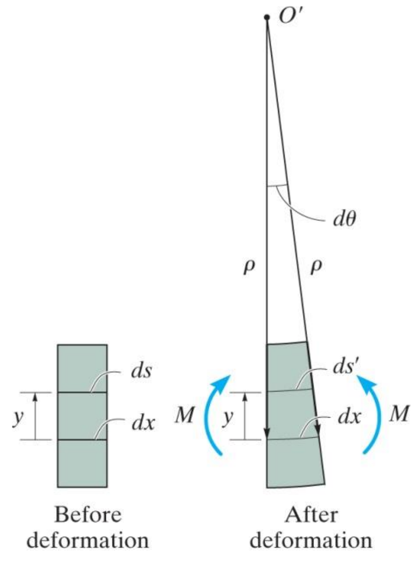
$$\frac{1}{\rho} = -\frac{\varepsilon}{y}$$



(a)

- Comparing with the Hooke's Law ε = σ / E and the flexure formula σ = -My/I

$$\frac{1}{\rho} = \frac{M}{EI} \text{ or } \frac{1}{\rho} = -\frac{\sigma}{Ey}$$



(b)

SLOPE AND DISPLACEMENT BY INTEGRATION

- The equation of the elastic curve is defined by the coordinates v and x. To compute the deflection $v = f(x)$, we must be able to represent the curvature $(1/\rho)$ in terms of v and x.

- Kinematic relationship between radius of curvature ρ and location x:

$$\frac{1}{\rho} = -\frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

- Then using the moment curvature equation, we have

$$\frac{M}{EI} = \frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}} \approx \frac{d^2v}{dx^2}$$

SLOPE AND DISPLACEMENT BY INTEGRATION

$$\frac{M}{EI} = \frac{d^2v}{dx^2}$$

- The equation can also be written in two alternative forms

- Differentiate each side with respect to x and substitute $V = dM/dx$

$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x)$$

- Differentiating again, using $w = dV/dx$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = w(x)$$

SLOPE AND DISPLACEMENT BY INTEGRATION

- Flexural rigidity (EI) is constant along beam, thus

$$EI\frac{d^4v}{dx^4} = w(x)$$

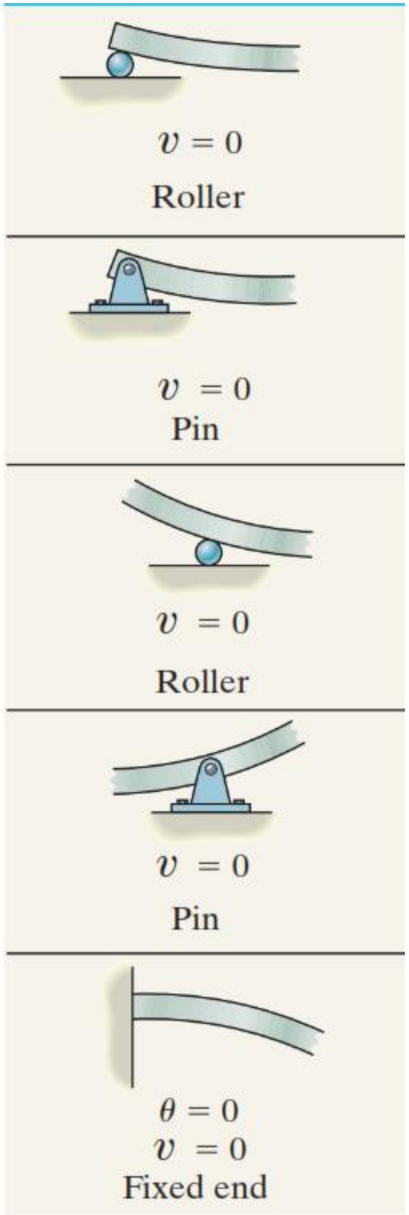
$$EI\frac{d^3v}{dx^3} = V(x)$$

$$EI\frac{d^2v}{dx^2} = M(x)$$

- Solution of any of these equations requires successive integrations to obtain v.

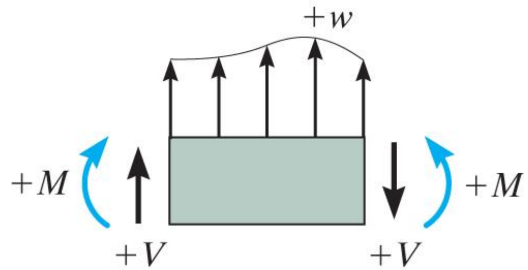
SLOPE AND DISPLACEMENT BY INTEGRATION (cont)

- Boundary Conditions:
  - The integration constants can be determined by imposing the boundary conditions, or
  - Continuity condition at specific locations
- Note, if the beam is supported by a *roller* or *pin*, then it is required that the displacement be *zero* at these points.
- At the fixed support, the *slope* and *displacement* are both *zero*.



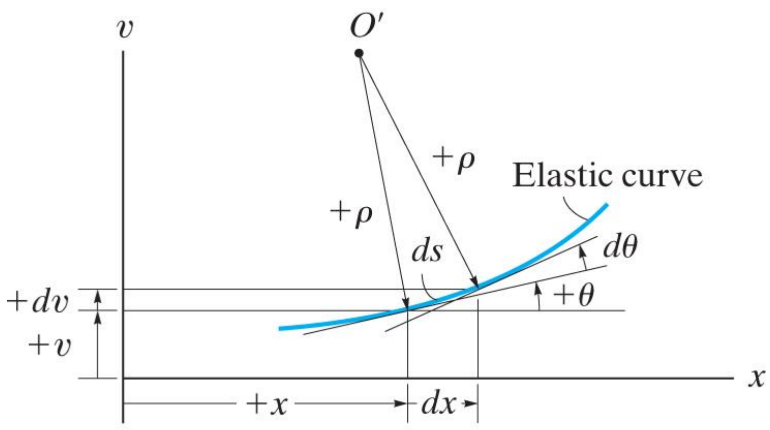
SLOPE AND DISPLACEMENT BY INTEGRATION (cont)

- Sign convention:



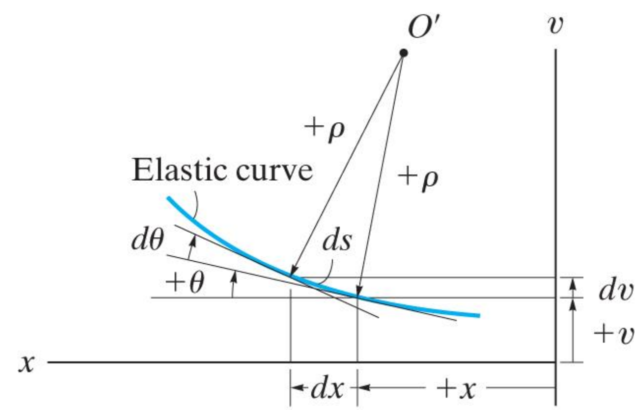
Positive sign convention

(a)



Positive sign convention

(b)

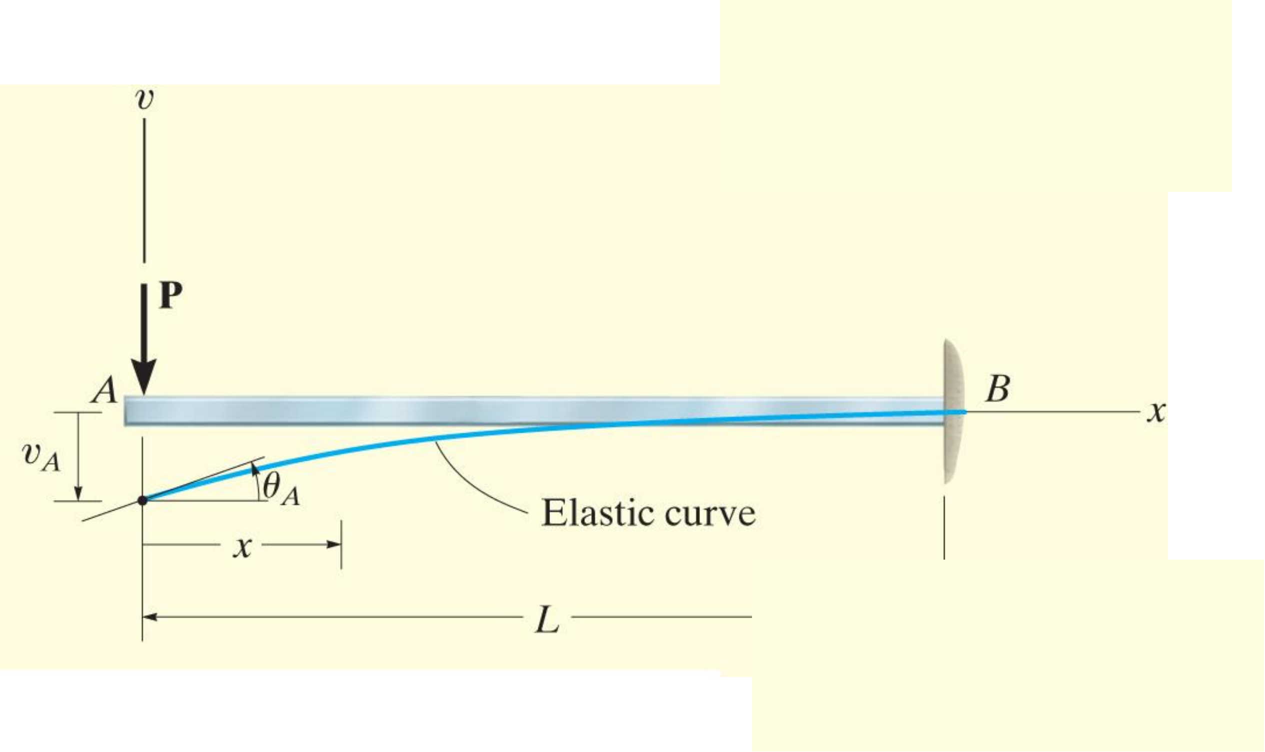


Positive sign convention

(c)

EXAMPLE 1

The cantilevered beam shown below is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. EI is constant.



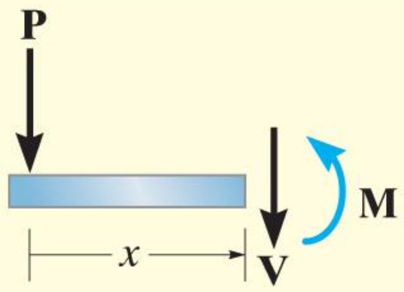
EXAMPLE 1 (cont)

Solutions

- Elastic Curve: shown in the Question figure

- Moment Function: From the free-body diagram, with **M** acting in the *positive direction*, we have

$$M = -Px$$



(b)

- Slope and Elastic Curve

- Applying $EI\frac{d^2v}{dx^2} = M(x)$ and integrating twice yields

$$EI\frac{d^2v}{dx^2} = -Px \qquad (1)$$

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \qquad (2)$$

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \qquad (3)$$

EXAMPLE 1 (cont)

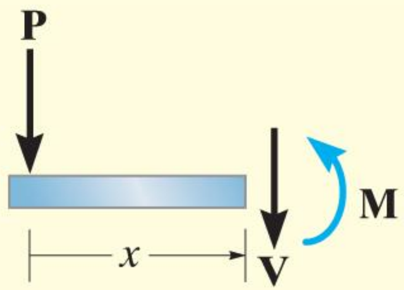
Solutions

- Using the boundary conditions $dv/dx = 0$ at $x = L$ and $v = 0$ at $x = L$, equations 2 and 3 become

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1 L + C_2$$

$$\Rightarrow C_1 = \frac{PL^2}{2} \text{ and } C_2 = -\frac{PL^3}{3}$$



(b)

- Substituting these results, with $\theta = dv/dx$, we get

$$\theta = \frac{P}{2EI}\left(L^2 - x^2\right)$$

Elastic Curve Eqn: $$v = \frac{P}{6EI}\left(-x^3 + 3L^2 x - 2L^3\right) \quad \text{(Ans)}$$

EXAMPLE 1 (cont)

Solutions

- Maximum slope and displacement occur at for which $A(x = 0)$,

$$\theta_A = \frac{PL^2}{2EI} \quad (4)$$

$$v_A = -\frac{PL^3}{3EI} \quad (5)$$

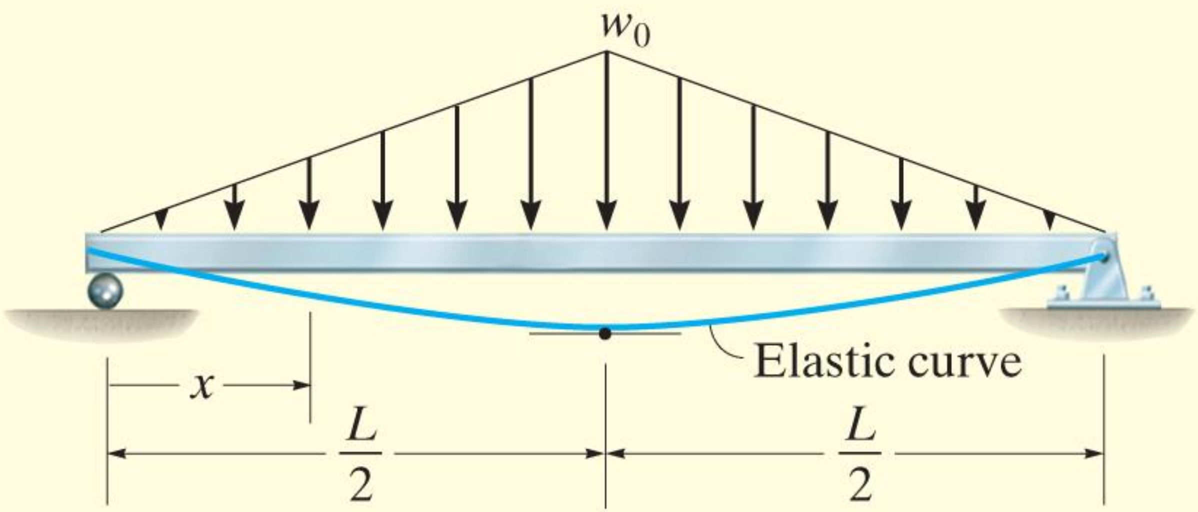
- If this beam ($L = 5$ m; Load; $P = 30$ kN) was designed without a factor of safety by assuming the allowable normal stress is equal to the yield stress is 250 MPa; then a W310 x 39 would be found to be adequate ($I = 84.4(10^6)\text{mm}^4$)

$$\theta_A = \frac{30(5)^2(1000)^2}{2[200][84.4(10^6)]} = 0.0222\,\text{rad}$$

$$v_A = -\frac{30(5)^2(1000)^2}{3[200][84.4(10^6)]} = -74.1\,\text{mm}$$

EXAMPLE 2

The simply supported beam shown below supports the triangular distributed loading. Determine its maximum deflection. *EI* is constant.



(a)

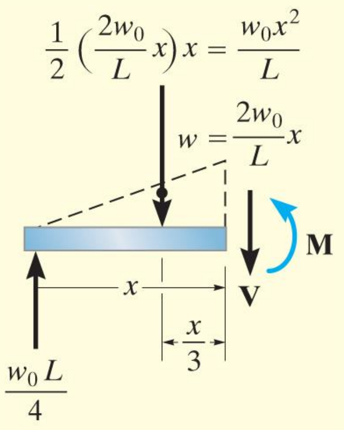
EXAMPLE 2 (cont)

Solutions

- Due to symmetry only one *x* coordinate is needed for the solution,

$$0 \le x \le L/2$$

- The equation for the distributed loading is $w = \dfrac{2w_0}{L}x$.



(b)

- Hence

$$+\sum M_{NA} = 0; \qquad M + \frac{w_0x^2}{L}\left(\frac{x}{3}\right) - \frac{w_0L}{4}(x) = 0$$

$$M = -\frac{w_0x^2}{3L} + \frac{w_0L}{4}x$$

EXAMPLE 2 (cont)

Solutions

- Integrating twice, we have

$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x$$

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 + C_1$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 + C_1x + C_2$$

- For boundary condition,

  - $v = 0, x = 0$ and
  - $dv/dx = 0, x = L/2$

$$C_1 = -\frac{5w_0L^3}{192}$$

$$C_2 = 0$$

EXAMPLE 2 (cont)

Solutions

- Hence

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0 L}{24}x^3 - \frac{5w_0 L^3}{192}x$$

- For maximum deflection at $x = L/2$,

$$v_{\max} = -\frac{w_0 L^4}{120EI} \qquad \text{(Ans)}$$

DISCONTINUITY FUNCTIONS

- A simplified method for finding the eqn of the elastic curve for a multiple loaded beam using a single expression, formulated from the loading on the beam , $w = w(x)$, or the beam's internal moment, $M = M(x)$ is discussed below.

  Discontinuity functions

  Macaulay functions

- Such functions can be used to describe distributed loadings, written generally as

$$\langle x - a \rangle^n = \begin{cases} 0 & for\ x < a \\ (x-a)^n & for\ x \geq a \end{cases}$$

$$n \geq 0$$

DISCONTINUITY FUNCTIONS

Macaulay functions

- x represents the coordinate position of a point along the beam
- a is the location on the beam where a “discontinuity” occurs, or the point where a distributed loading begins

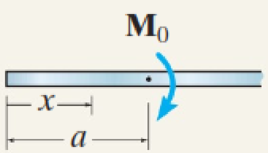
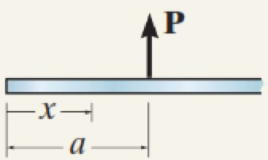
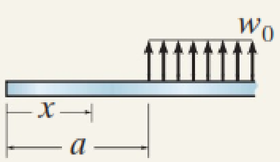
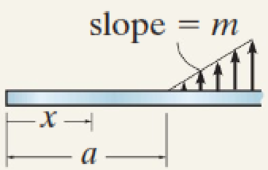
- Integrating Macaulay functions, we get

$$\int \langle x - a \rangle^n dx = \frac{\langle x - a \rangle^{n+1}}{n+1} + C$$

- The functions describe both uniform load and triangular load.

USE OF CONTINUOUS FUNCTIONS

- *Macaulay* functions

Loading	Loading Function $w = w(x)$	Shear $V = \int w(x)dx$	Moment $M = \int V dx$
	$w = M_0 \langle x-a \rangle^{-2}$	$V = M_0 \langle x-a \rangle^{-1}$	$M = M_0 \langle x-a \rangle^{0}$
	$w = P \langle x-a \rangle^{-1}$	$V = P \langle x-a \rangle^{0}$	$M = P \langle x-a \rangle^{1}$
	$w = w_0 \langle x-a \rangle^{0}$	$V = w_0 \langle x-a \rangle^{1}$	$M = \frac{w_0}{2} \langle x-a \rangle^{2}$
	$w = m \langle x-a \rangle^{1}$	$V = \frac{m}{2} \langle x-a \rangle^{2}$	$M = \frac{m}{6} \langle x-a \rangle^{3}$

USE OF CONTINUOUS FUNCTIONS

- *Macaulay* functions

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases}$$

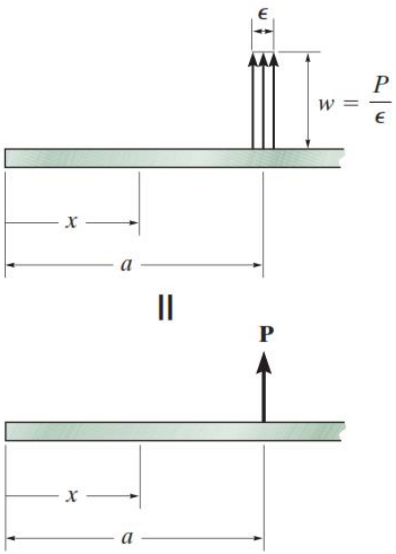
$$n \geq a$$

- Integration of *Macaulay* functions:

$$\int \langle x-a\rangle^n dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C$$

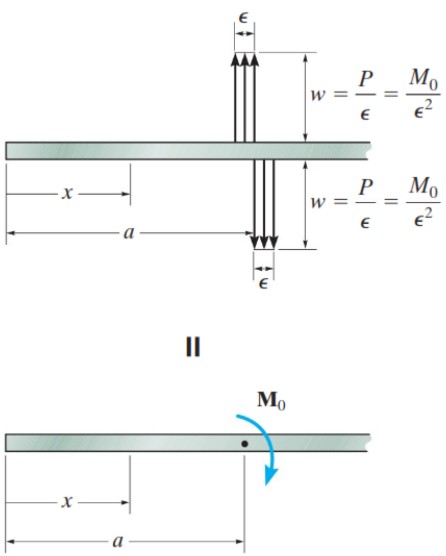
USE OF CONTINUOUS FUNCTIONS (cont)

- Singularity Functions:



- Concentrated force **P** (+ve upwards) can be considered a special case of a distributed loading having an intensity of $w = P/\epsilon$ when its length $\epsilon \to 0$

$$w = P\langle x-a\rangle^{-1} = \begin{cases} 0 & \text{for } x \neq a \\ P & \text{for } x = a \end{cases}$$



- Similarly, a couple moment $\boldsymbol{M_0}$, considered *positive clockwise*, is a limit as $\epsilon \to 0$ of two distributed loadings

$$w = M_0\langle x-a\rangle^{-2} = \begin{cases} 0 & \text{for } x \neq a \\ M_0 & \text{for } x = a \end{cases}$$

USE OF CONTINUOUS FUNCTIONS (cont)

- Note: Integration of these two singularity functions yields results that are different from those of *Macaulay* functions. Specifically,

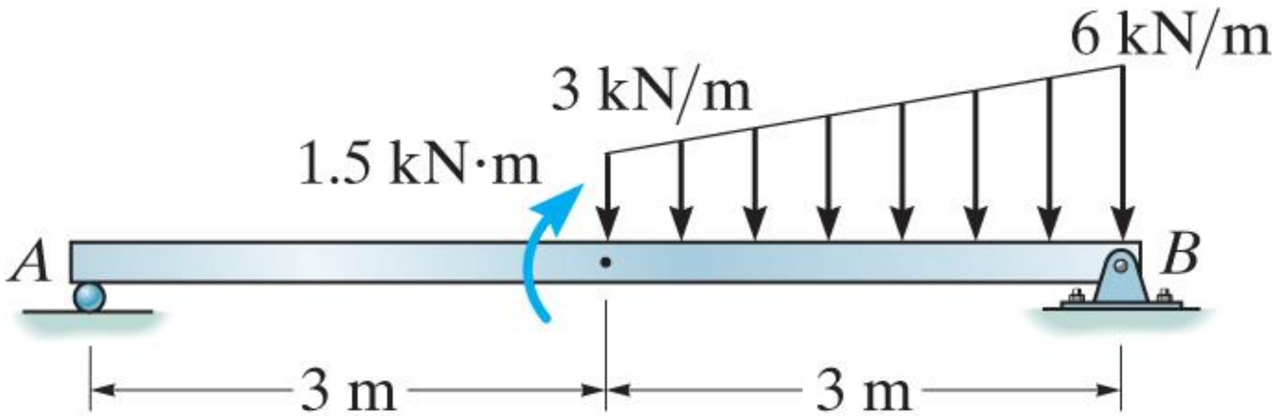
$$\int \langle x-a \rangle^{n} dx = \langle x-a \rangle^{n+1}, n = -1, -2$$

- :

:

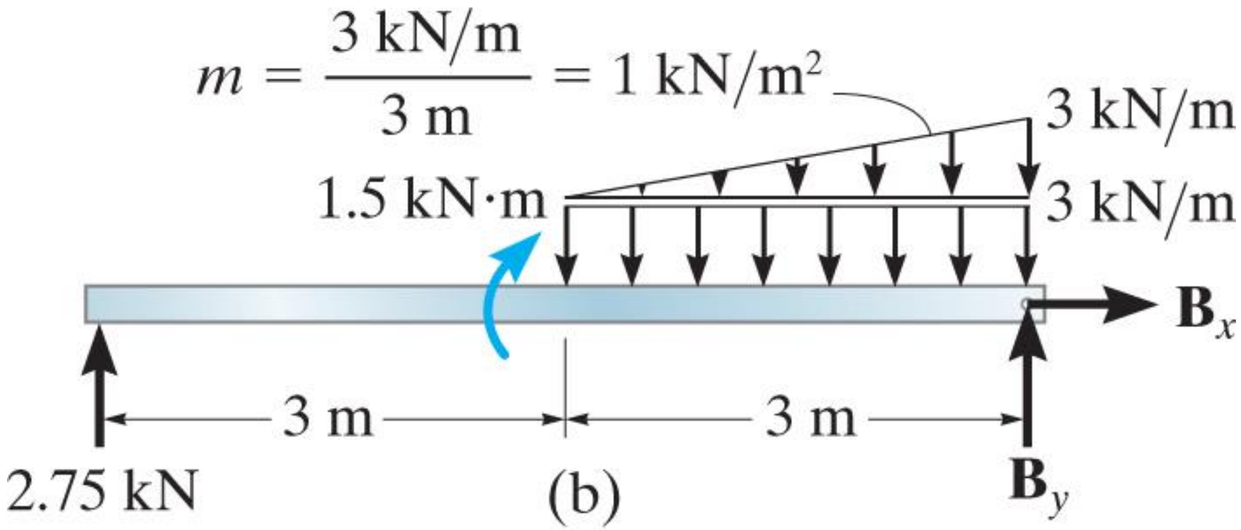
EXAMPLE

The following example shows us of how to use discontinuity functions to describe the loading or internal moment in a beam



EXAMPLE

- Reactive force is 2.75-kN on roller acts upwards
- 1.5-kN/m couple moment is also positive since it acts clockwise.
- Trapezoidal loading is negative and by superposition has been separated into triangular and uniform loadings.



EXAMPLE

- The loading at any point x *on the beam is therefore*

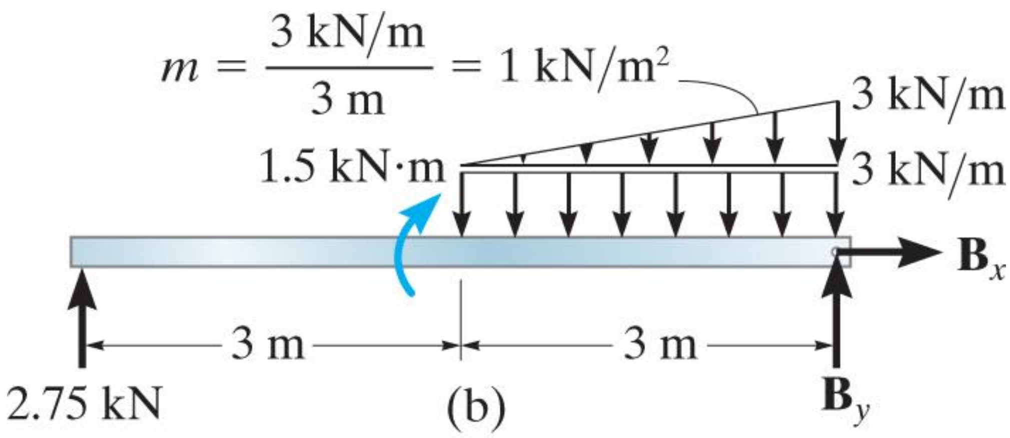
$$w = 2.75\ \text{kN}\langle x - 0\rangle^{-1} + 1.5\ \text{kN}\cdot\text{m}\langle x - 3\ \text{m}\rangle^{-2} - 3\ \text{kN/m}\langle x - 3\ \text{m}\rangle^{0} - 1\ \text{kN/m}^2\langle x - 3\ \text{m}\rangle^{1}$$

$$w = P\langle x-a\rangle^{-1}$$

$$w = M_0\langle x-a\rangle^{-2}$$

$$w = w_0\langle x-a\rangle^{0}$$

$$w = m\langle x-a\rangle^{1}$$



EXAMPLE

- Moment expression can also formulated in similar manner

$$M = 2.75\text{ kN}\langle x - 0\rangle^1 + 1.5\text{ kN}\cdot\text{m}\langle x - 3\text{ m}\rangle^0 - \frac{3\text{ kN/m}}{2}\langle x - 3\text{ m}\rangle^2 - \frac{1\text{ kN/m}^2}{6}\langle x - 3\text{ m}\rangle^3$$

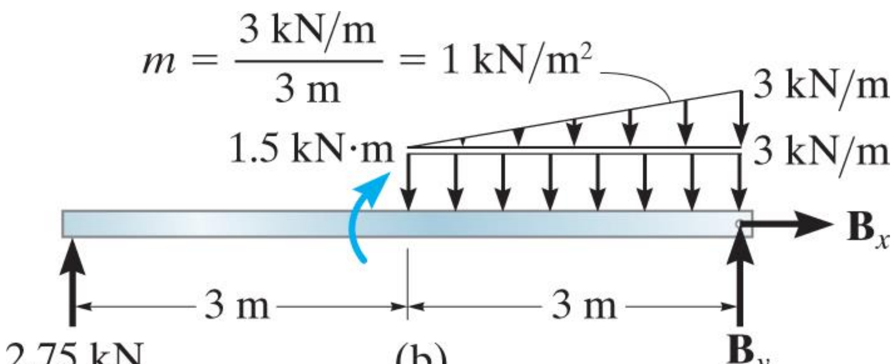
$$M = P\langle x-a\rangle^1$$

$$M = M_0\langle x-a\rangle^0$$

$$M = \frac{w_0}{2}\langle x-a\rangle^2$$

$$M = \frac{m}{6}\langle x-a\rangle^3$$

$$M = 2.75x + 1.5\langle x - 3\rangle^0 - 1.5\langle x - 3\rangle^2 - \frac{1}{6}\langle x - 3\rangle^3$$



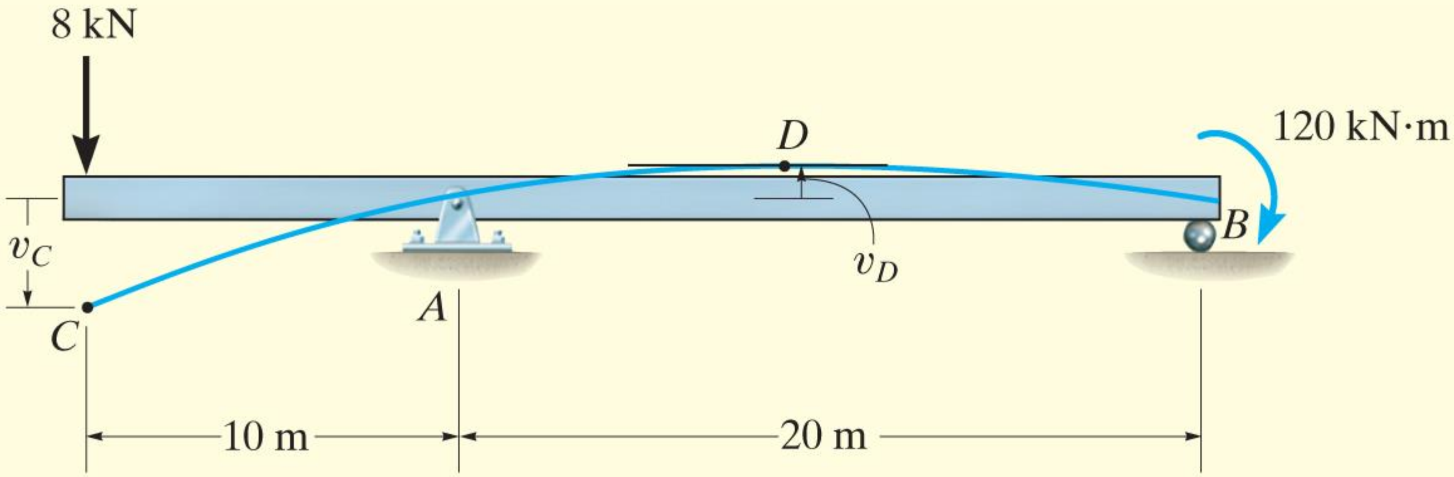
(b)

EXAMPLE

- The deflection of the beam can now be determined after this equation is integrated two successive times
- Constants of integration are evaluated using the boundary conditions of zero displacement at *A* and *B*.

EXAMPLE 3

Determine the maximum deflection of the beam shown in Fig. below. *EI* is constant.

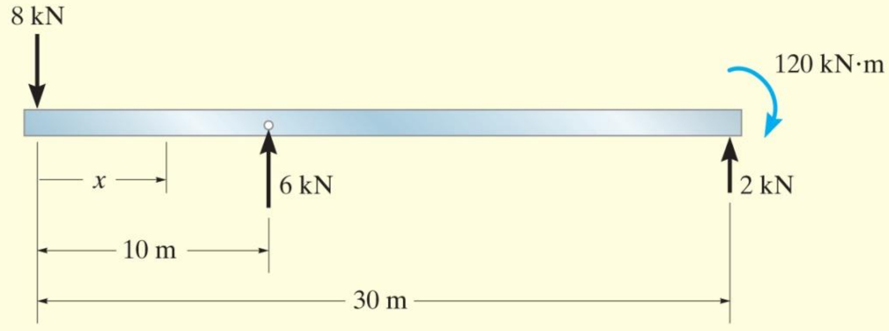


(a)

EXAMPLE 3 (cont)

Solutions

- The beam deflects as shown below. The boundary conditions require zero displacement at *A* and *B*.



(b)

- The loading function for the beam can be written as

$$w = -8\langle x - 0\rangle^{-1} + 6\langle x - 10\rangle^{-1}$$

EXAMPLE 3 (cont)

Solutions

- Integrating, we have

$$V = -8\langle x-0\rangle^0 + 6\langle x-10\rangle^0$$

- In a similar manner,

$$M = -8\langle x-0\rangle^1 + 6\langle x-10\rangle^1$$
$$= \left(-8x + 6\langle x-10\rangle^1\right)\text{kN}\cdot\text{m}$$

- Integrating twice yields

$$EI\frac{d^2v}{dx^2} = -8x + 6\langle x-10\rangle^1$$

$$EI\frac{dv}{dx} = -4x^2 + 3\langle x-10\rangle^2 + C_1$$

$$EIv = -\frac{4}{3}x^3 + \langle x-10\rangle^3 + C_1x + C_2 \quad (1)$$

EXAMPLE 3 (cont)

Solutions

- From Eq. 1, the boundary condition $v = 0$ at $x = 10\ m$ and at $x = 30$ m gives

$$0 = -1333 + (10-10)^3 + C_1(10) + C_2$$

$$0 = -36000 + (30-10)^3 + C_1(30) + C_2$$

$$\Rightarrow C_1 = 1333 \text{ and } C_2 = -12000$$

- Thus,

$$EI\frac{dv}{dx} = -4x^2 + 3\langle x-10\rangle^2 + 1333 \quad (2)$$

$$EIv = -\frac{4}{3}x^3 + \langle x-10\rangle^3 + 1333x - 12000 \quad (3)$$

EXAMPLE 3 (cont)

Solutions

- To obtain the displacement of *C*, set $x = 0$ in Eq. 3.

$$v_C = -\frac{13000}{EI}\ \mathrm{kN \cdot m^3} \quad \text{(Ans)}$$

- The *negative* sign indicates that the displacement is *downward*

- To locate point *D*, use $Eq.2$ with $x > 10$ and $dv/dx = 0$,

$$0 = -x_D^2 + 3\langle x_D - 10\rangle^2 + 1333$$

$$x_D^2 + 60x_D - 1633 = 0$$

$$\text{Solving for the positive root, } x_D = 20.3\ \mathrm{m}$$

EXAMPLE 3 (cont)

Solutions

- Hence, from Eq. 3,

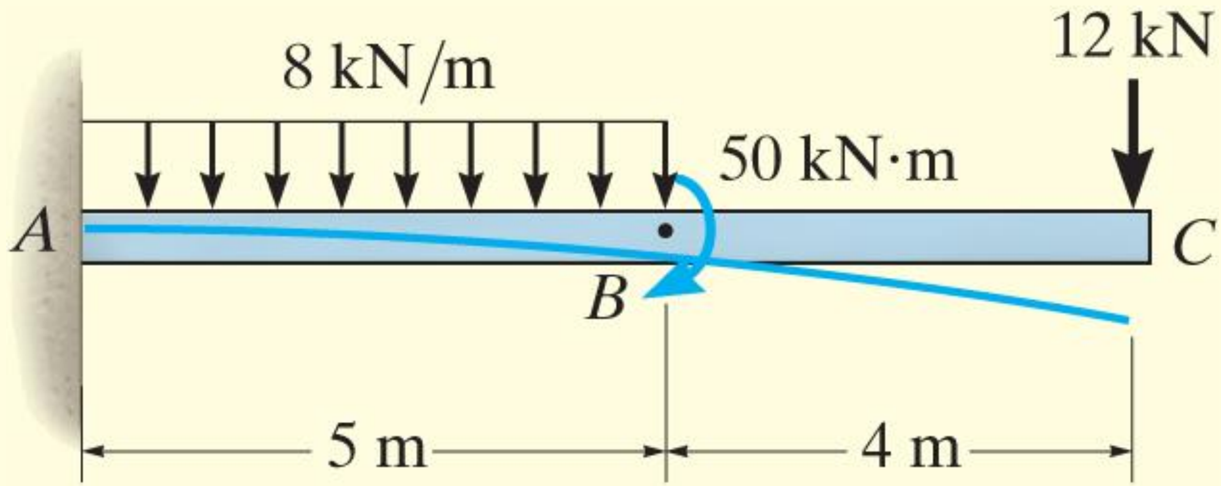
$$EIv_D = -\frac{4}{3}(20.3)^3 + (20.3-10)^3 + 1333(20.3) - 12000$$

$$v_D = \frac{5006}{EI}\,\text{kN}\cdot\text{m}^3$$

- Comparing this value with v_C, we see that $v_{max} = v_C$.

EXAMPLE 4

Determine the equation of the elastic curve for the cantilevered beam shown below. *EI* is constant.

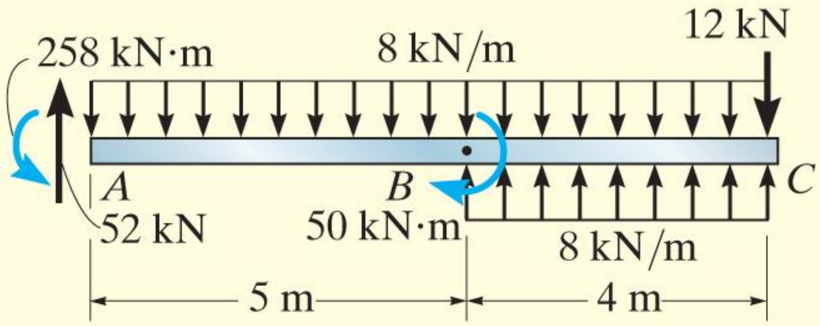


(a)

EXAMPLE 4 (cont)

Solutions

- The boundary conditions require zero slope and displacement at *A*.
- The support diagram reactions at *A* have been calculated by statics and are shown on the free-body,



(b)

$$w = 52\langle x - 0\rangle^{-1} - 258\langle x - 0\rangle^{-2} - 8\langle x - 0\rangle^{-0} + 50\langle x - 5\rangle^{-2} + 8\langle x - 5\rangle^{0}$$

EXAMPLE 4 (cont)

Solutions

- Since $dV/dx = -w(x)$ and $dM/dx = V$

$$V = 52\langle x-0\rangle^0 - 258\langle x-0\rangle^{-1} - 8\langle x-0\rangle^1 + 50\langle x-5\rangle^{-1} + 8\langle x-5\rangle^1$$

$$M = -258\langle x-0\rangle^0 + 52\langle x-0\rangle^1 - \frac{1}{2}(8)\langle x-0\rangle^2 + 50\langle x-5\rangle^0 + \frac{1}{2}(8)\langle x-5\rangle^2$$

$$= \left(-258 + 52x - 4x^2 + 50\langle x-5\rangle^0\right) + 4\langle x-5\rangle^2 \text{ kN}\cdot\text{m}$$

- Integrating twice, we have

$$EI\frac{d^2v}{dx^2} = -258 + 52x - 4x^2 + 50\langle x-5\rangle^0 + 4\langle x-5\rangle^2$$

$$EI\frac{dv}{dx} = -258x + 26x^2 - \frac{4}{3}x^3 + 50\langle x-5\rangle^1 + \frac{4}{3}\langle x-5\rangle^3 + C_1$$

$$EIv = -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^2 + \frac{1}{3}\langle x-5\rangle^4 + C_1x + C_2$$

EXAMPLE 4 (cont)

Solutions

- Since $dv/dx = 0$, $x = 0$, $C_1 = 0$; and $v = 0$, $C_2 = 0$. Thus

$$v = \frac{1}{EI}\left(-129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^2 + \frac{1}{3}\langle x-5\rangle^4\right)\text{m} \quad \text{(Ans)}$$

METHOD OF SUPERPOSITION

- Necessary conditions to be satisfied:
  1. The load $w(x)$ is linearly related to the deflection $v(x)$,
  2. The load is assumed not to change significantly the original geometry of the beam.

- Then, it is possible to find the slope and displacement at a point on a beam subjected to several different loadings by algebraically adding the effects of its various component parts.

MOMENT AREA METHOD

Theorem 1:

- The angle between the tangents at any two points on the elastic curve equals the area under the M/EI diagram between these two points.

$$EI\frac{d^2v}{dx^2} = EI\frac{d}{dx}\left(\frac{dv}{dx}\right) = M$$

- Since $\theta \approx dv/dx$, so $d\theta = \left(\dfrac{M}{EI}\right)dx$

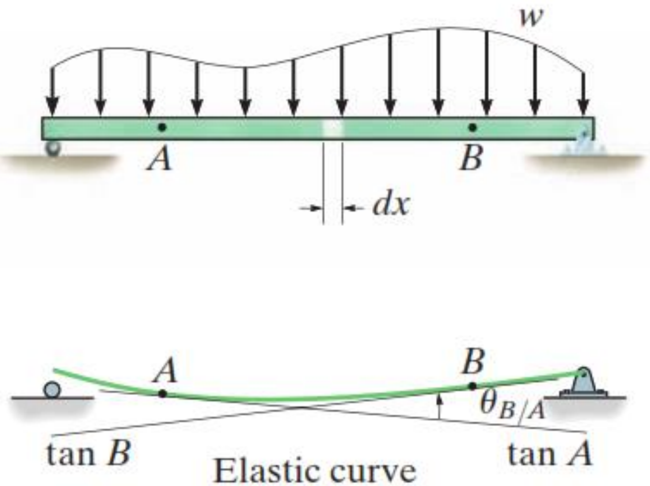
- Therefore,

$$\theta_{B/A} = \int_A^B \frac{M}{EI}dx$$

MOMENT AREA METHOD (cont)

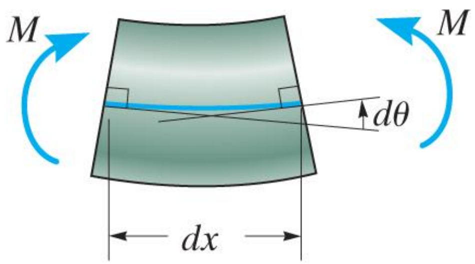
Theorem 1 (cont):

- This equation forms the basis for the first moment-area theorem

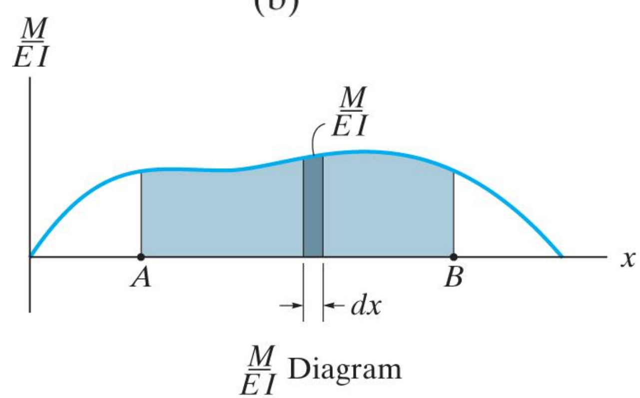


(a)

$$\theta_{B/A} = \int_A^B \frac{M}{EI} dx$$



(b)



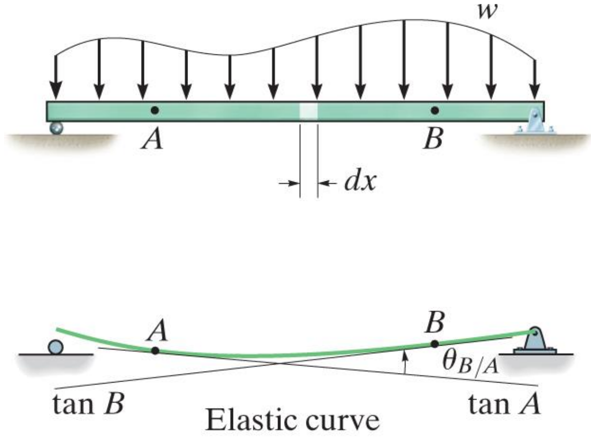
(c)

The angle, measured in radians, between the tangents at any two points on the elastic curve equals the area under the M/EI diagram between these two points.

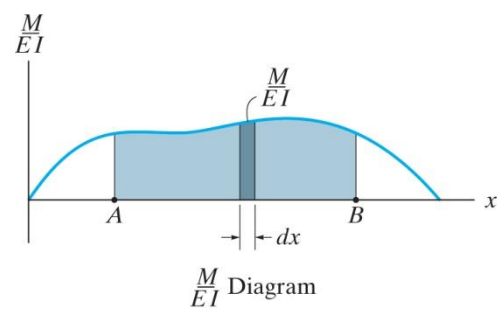
MOMENT AREA METHOD (cont)

Theorem 2:

- The vertical deviation of the tangent at a point (A) on the elastic curve with respect to the tangent extended from another point (B) equals the moment of the area under the M/EI diagram between these two points (A and B). This moment is computed about point (A) where the vertical deviation ($t_{A/B}$) is to be determined



(a)



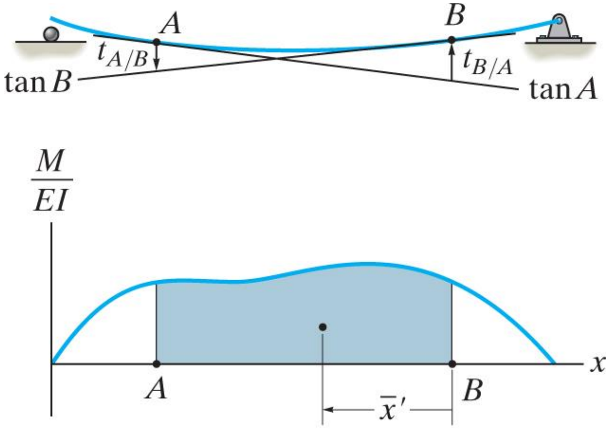
(c)

MOMENT AREA METHOD (cont)

Theorem 2 (cont):

- The vertical deviation of the tangent at A with respect to the tangent at B is

$$t_{A/B} = \int_A^B x \frac{M}{EI} dx$$



(c)

- Then,

$$t_{A/B} = \bar{x} \int_A^B \frac{M}{EI} dx$$

- Where $\bar{x}$ is the location of the centroid of the shaded area $\int (M/EI)\, dx$ between A and B

MOMENT AREA METHOD - SUMMARY

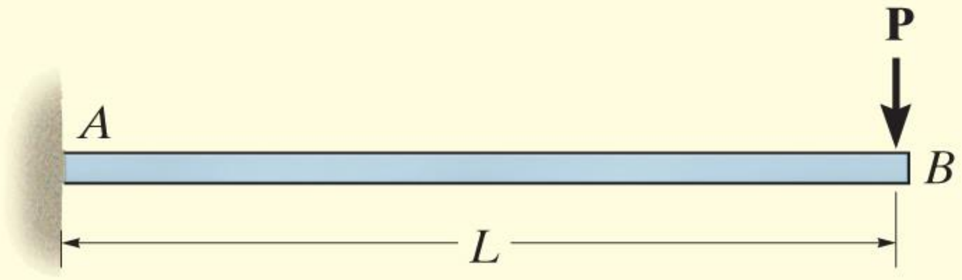
- Theorem 1 is used to determine the *angle* between any two tangents on the elastic curve and Theorem 2 to determine the vertical distance between the tangents

- A *positive* $\theta_{B/A}$ represents a *counterclockwise* rotation of the tangent at *B* with respect to the tangent at *A*

- A *positive* $t_{B/A}$ indicates that *B* on the elastic curve lies *above* the extended tangent from *A*.

EXAMPLE 5

Determine the slope of the beam shown in Fig. below at point *B*. *EI* is constant.

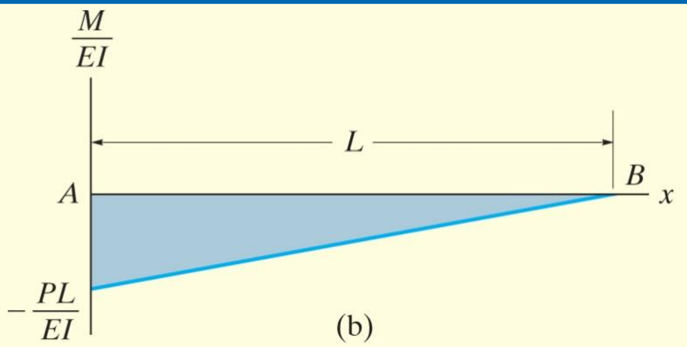


(a)

EXAMPLE 5 (cont)

Solutions

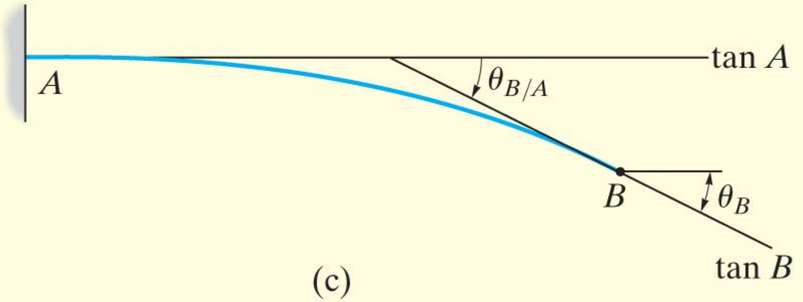
- M/EI diagram will be drawn first (Fig. b).



- The force **P** causes the beam to deflect as shown in Fig. c.

- By the construction, the angle between tan *A* and tan *B* is equivalent to $\theta_{B/A}$, where

$$\theta_B = \theta_{B/A}$$



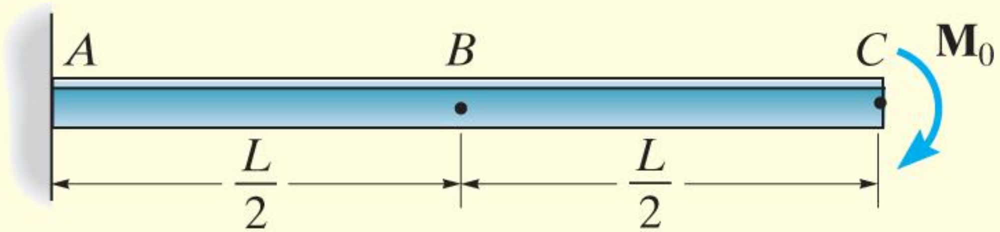
- Using moment-area theorem,

$$\theta_B = \frac{1}{2}\left(-\frac{PL}{EI}\right)L = -\frac{PL^2}{2EI} \quad (\text{Ans})$$

- The *negative sign* indicates that the angle measured from the tangent at *A* to the tangent at *B* is *clockwise*. This checks, since the beam slopes downwards at *B*.

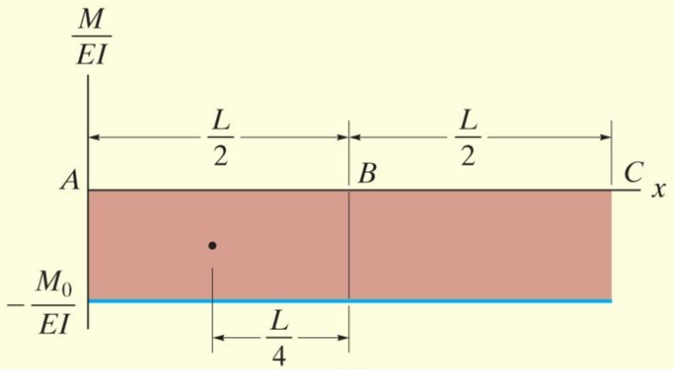
EXAMPLE 6

Determine the displacement of points *B* and *C* of the beam shown in Fig. below. *EI* is constant.

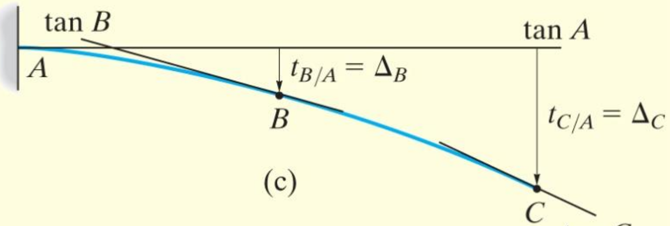


(a)

Solutions



(b)



(c)

EXAMPLE 6 (cont)

Solutions

- The required displacements can be related directly to the vertical distance between the tangents at *B* and *A* and *C* and *A*.

$$v_B = t_{B/A} \qquad v_C = t_{C/A}$$

- Applying Theorem 2,

$$v_B = t_{B/A} = \left(\frac{L}{4}\right)\left[\left(-\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\right] = -\frac{M_0 L^2}{8EI} \quad \text{(Ans)}$$

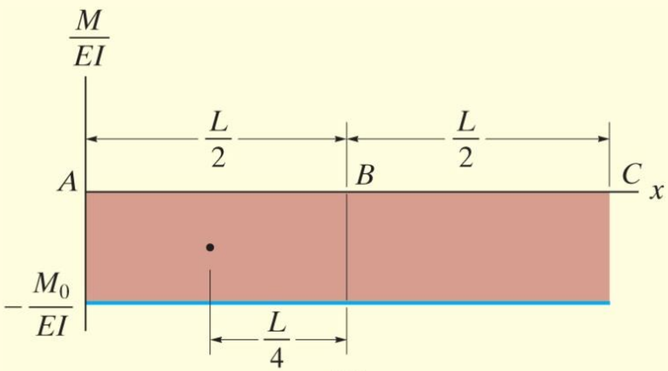
- We must determine the moment of the area under the *entire* $\frac{M}{EI}$ diagram from *A* to *C* about point *C* (the point on the elastic curve).

$$v_C = t_{C/A} = \left(\frac{L}{2}\right)\left[\left(-\frac{M_0}{EI}\right)(L)\right] = -\frac{M_0 L^2}{2EI} \quad \text{(Ans)}$$

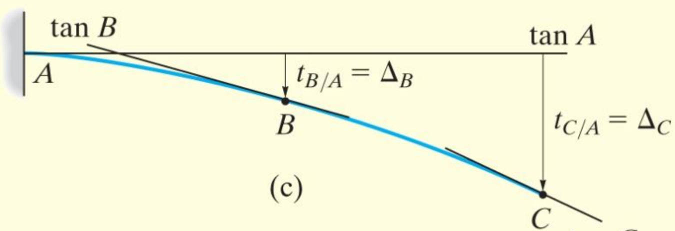
EXAMPLE 6 (cont)

Solutions

- Since both answers are *negative*, they indicate that points *B* and *C* lie *below* the tangent at *A*.
- This checks with Fig. *c*.



(b)



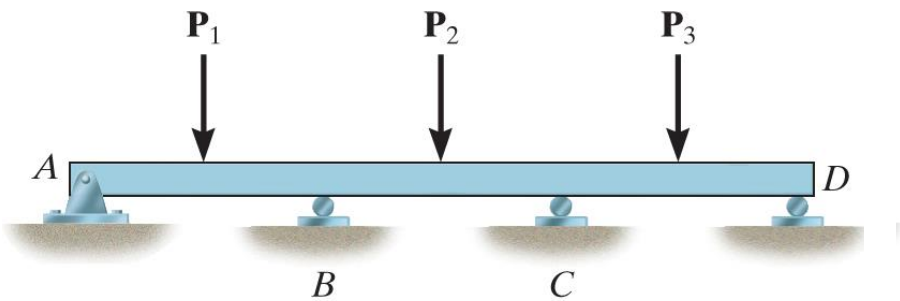
(c)

$$v_B = t_{B/A} \quad ; \quad v_C = t_{C/A}$$

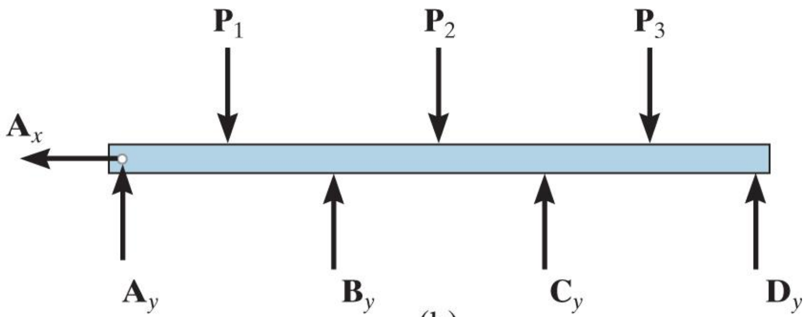
STATISTICALLY INDETERMINATE BEAMS AND SHAFT

- **Definition**:

  A member of any type is classified *statically indeterminate* if the number of unknown reactions **exceeds** the available number of equilibrium equations, e.g. a continuous beam having 4 supports



(a)



(b)

STATISTICALLY INDETERMINATE BEAMS (cont)

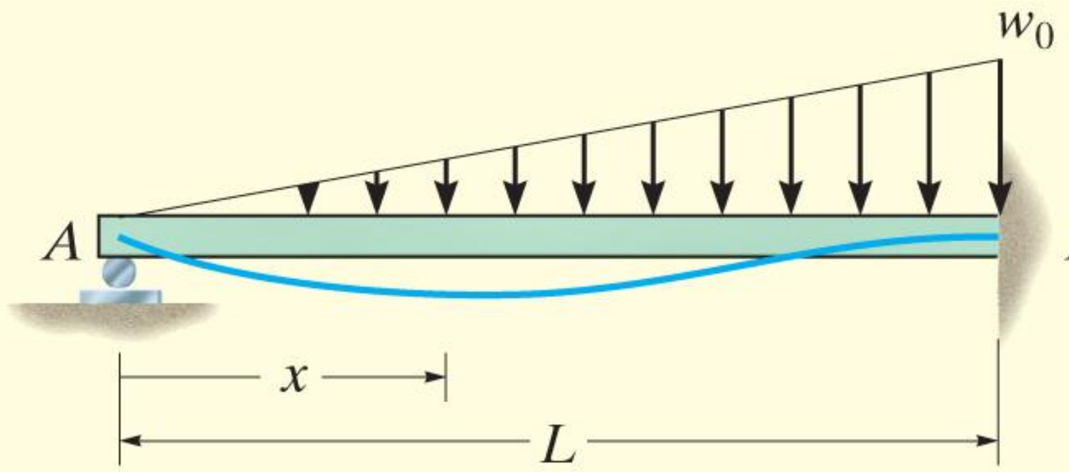
Strategy:

- The additional support reactions on the beam or shaft that are not needed to keep it in stable equilibrium are called *redundants*. It is first necessary to specify those redundant from conditions of geometry known as compatibility conditions.

- Once determined, the redundants are then applied to the beam, and the remaining reactions are determined from the equations of equilibrium.

EXAMPLE 1 – USE OF THE INTEGRATRION METHOD

The beam is subjected to the distributed loading shown in Fig.*a*. Determine the reaction at *A*. *EI* is constant.



(a)

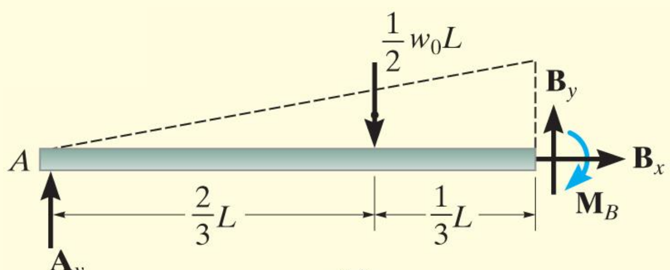
EXAMPLE 1 (cont)

Solutions

- The beam deflects as shown in Fig. *a*. Only one coordinate *x* is needed.

- The beam is indeterminate to the first degree as indicated from the free-body diagram, Fig. *b* using a redundant force A_y

$$M = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$



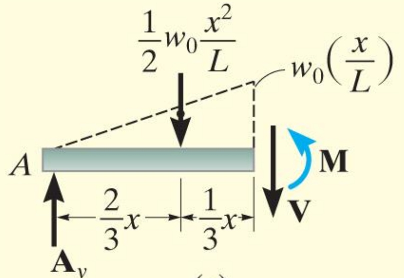
(b)

- Applying Moment function equation:

$$EI \frac{d^2 v}{dx^2} = A_y x - \frac{1}{6} w_o \frac{x^3}{L}$$

$$EI \frac{dv}{dx} = \frac{1}{2} A_y x^2 - \frac{1}{24} w_o \frac{x^4}{L} + C_1$$

$$EIv = \frac{1}{6} A_y x^3 - \frac{1}{120} w_o \frac{x^5}{L} + C_1 x + C_2$$



(c)

EXAMPLE 1 (cont)

Solutions

- The 3 unknowns A_y, C_1 and C_2 are determined from the boundary conditions

  i. $x = 0, \ v = 0$

  ii. $x = L, \ dv/dx = 0$

  iii. $x = L, \ v = 0$

a. $EI\frac{d^2v}{dx^2} = A_y x - \frac{1}{6}w_o\frac{x^3}{L}$

b. $EI\frac{dv}{dx} = \frac{1}{2}A_y x^2 - \frac{1}{24}w_o\frac{x^4}{L} + C_1$

c. $EIv = \frac{1}{6}A_y x^3 - \frac{1}{120}w_o\frac{x^5}{L} + C_1 x + C_2$

- Applying these conditions yields,

$$x = 0, \ v = 0; \qquad 0 = 0 - 0 + 0 + C_2$$

$$x = L, \ \frac{dv}{dx} = 0; \qquad 0 = \frac{1}{2}A_y L^2 - \frac{1}{24}w_0 L^3 + C_1$$

$$x = L, \ v = 0; \qquad 0 = \frac{1}{6}A_y L^3 - \frac{1}{120}w_0 L^4 + C_1 L + C_2$$

- Solving,

$$A_y = \frac{1}{10}w_0 L \quad \text{(Ans)}$$

$$C_1 = -\frac{1}{120}w_0 L^3 \qquad C_2 = 0$$

USE OF THE METHOD OF SUPERPOSITION

Procedures:

Elastic Curve

- Specify the unknown redundant forces or moments that must be removed from the beam in order to make it statistically determinate and stable.

- Using the principle of superposition, draw the statistically indeterminate beam and show it equal to a sequence of corresponding *statistically determinate* beams.

USE OF THE METHOD OF SUPERPOSITION (cont)

Procedures:

Elastic Curve (cont)

- The first of these beams, the primary beam, supports the same external loads as the statistically indeterminate beam, and each of the other beams “added” to the primary beam shows the beam loaded with a separate redundant force or moment.
- Sketch the deflection curve for each beam and indicate symbolically the displacement or slope at the point of each redundant force or moment.

USE OF THE METHOD OF SUPERPOSITION (cont)

Procedures:

Compatibility Equations

- Write a compatibility equation for the displacement or slope at each point where there is a redundant force or moment.

- Determine all the displacements or slopes using an appropriate method (integration, Macaulay, or Moment Area method).

USE OF THE METHOD OF SUPERPOSITION (cont)

Procedures:

Compatibility Equations (cont)

- Substitute the results into the compatibility equations and solve for the unknown redundant.

- If the numerical value for a redundant is *positive*, it has the same *sense of direction* as originally assumed. Similarly, a negative numerical value indicates the redundant acts *opposite* to its assumed *sense of direction.*

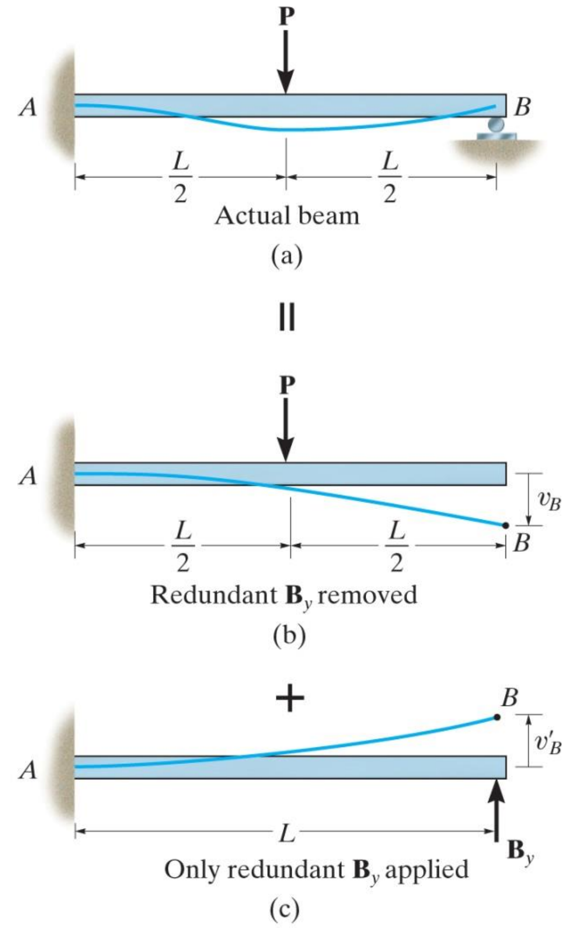
USE OF THE METHOD OF SUPERPOSITION (cont)

Procedures:

Equilibrium Equations

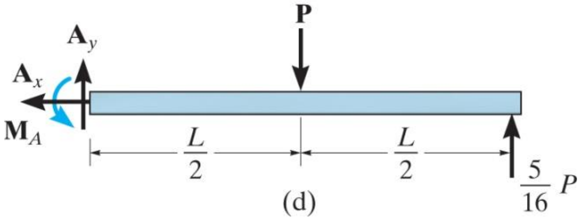
- Once the redundant forces and/or moments have been determined, the remaining unknown reactions can be found from the equations of equilibrium applied to the loadings shown on the beam's free body diagram.

$$0 = -v_B + v'_B$$



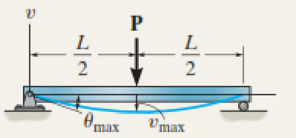
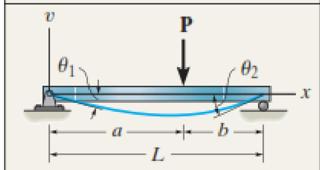
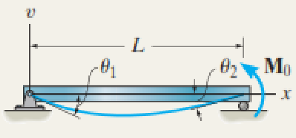
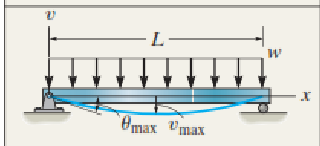
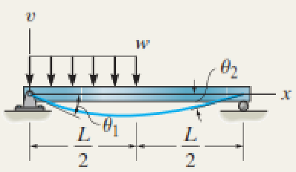
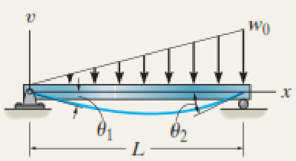
From Literature; $v_B = \dfrac{5PL^3}{48EI}$ $\quad v'_B = \dfrac{B_y L^3}{3EI}$

$B_y = \dfrac{5}{16}P$; Thus $A_x = 0$; $A_y = \dfrac{11}{16}P$; $M_A = \dfrac{3}{16}PL$



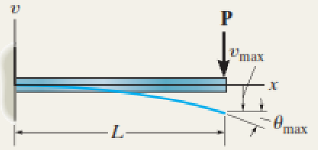
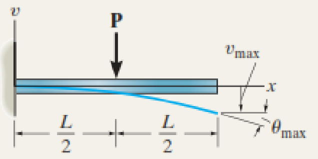
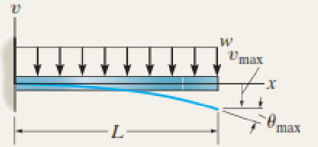
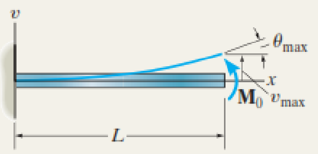
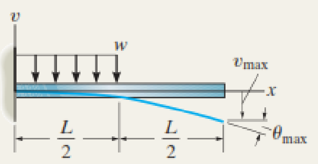
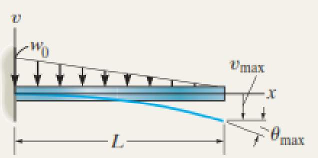
DEFLECTIONS TABLES OF STANDARD BEAMS

Simply Supported Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{16EI}$	$v_{max} = \dfrac{-PL^3}{48EI}$	$v = \dfrac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \dfrac{-Pab(L+b)}{6EIL}$ $\theta_2 = \dfrac{Pab(L+a)}{6EIL}$	$v\Big\rvert_{x=a} = \dfrac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \dfrac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \dfrac{-M_0L}{6EI}$ $\theta_2 = \dfrac{M_0L}{3EI}$	$v_{max} = \dfrac{-M_0L^2}{\sqrt{243}EI}$ at $x = 0.5774L$	$v = \dfrac{-M_0x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \dfrac{-wL^3}{24EI}$	$v_{max} = \dfrac{-5wL^4}{384EI}$	$v = \dfrac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \dfrac{-3wL^3}{128EI}$ $\theta_2 = \dfrac{7wL^3}{384EI}$	$v\Big\rvert_{x=L/2} = \dfrac{-5wL^4}{768EI}$ $v_{max} = -0.006563\dfrac{wL^4}{EI}$ at $x = 0.4598L$	$v = \dfrac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \dfrac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \dfrac{-7w_0L^3}{360EI}$ $\theta_2 = \dfrac{w_0L^3}{45EI}$	$v_{max} = -0.00652\dfrac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \dfrac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

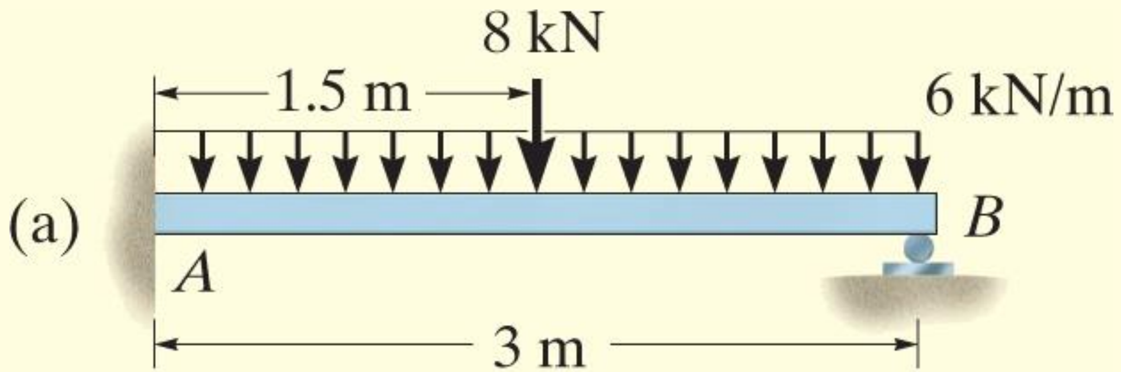
DEFLECTIONS TABLES OF STANDARD BEAMS

Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{2EI}$	$v_{max} = \dfrac{-PL^3}{3EI}$	$v = \dfrac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \dfrac{-PL^2}{8EI}$	$v_{max} = \dfrac{-5PL^3}{48EI}$	$v = \dfrac{-Px^2}{6EI}\left(\frac{3}{2}L - x\right) \quad 0 \le x \le L/2$ $v = \dfrac{-PL^2}{24EI}\left(3x - \frac{1}{2}L\right) \quad L/2 \le x \le L$
	$\theta_{max} = \dfrac{-wL^3}{6EI}$	$v_{max} = \dfrac{-wL^4}{8EI}$	$v = \dfrac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \dfrac{M_0L}{EI}$	$v_{max} = \dfrac{M_0L^2}{2EI}$	$v = \dfrac{M_0x^2}{2EI}$
	$\theta_{max} = \dfrac{-wL^3}{48EI}$	$v_{max} = \dfrac{-7wL^4}{384EI}$	$v = \dfrac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right)$ $0 \le x \le L/2$ $v = \dfrac{-wL^3}{192EI}(4x - L/2)$ $L/2 \le x \le L$
	$\theta_{max} = \dfrac{-w_0L^3}{24EI}$	$v_{max} = \dfrac{-w_0L^4}{30EI}$	$v = \dfrac{-w_0x^2}{120EIL}(10L^3 - 10L^2x + 5Lx^2 - x^3)$

EXAMPLE 2

Determine the reactions at the roller support *B* of the beam shown in Fig. below, then draw the shear and moment diagrams. *EI* is constant.



Actual beam

EXAMPLE 2 (cont)

Solutions

- By inspection, the beam is statically indeterminate to the first degree.

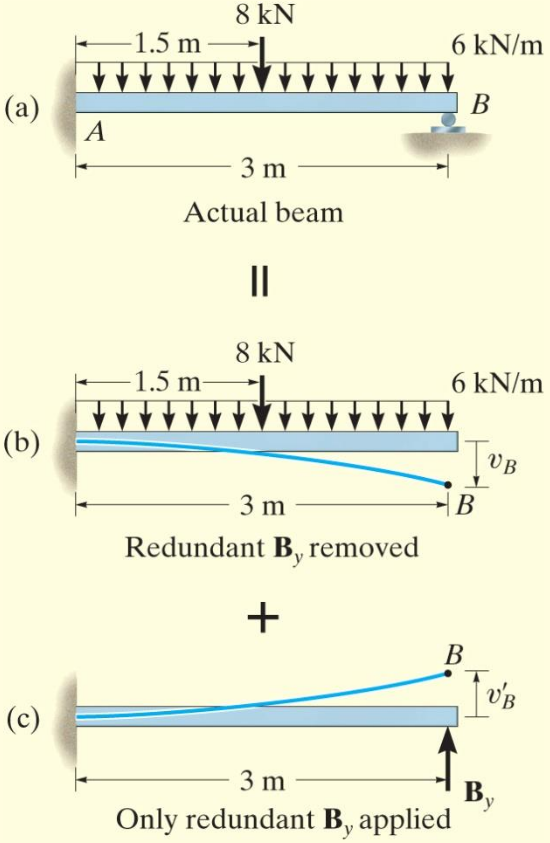
- Taking positive displacement as downward, the compatibility equation at *B* is

$$(+\downarrow) \qquad 0 = v_B - v'_B \qquad (1)$$

- Displacements can be obtained from tables.

$$v_B = \frac{wL^4}{8EI} + \frac{5PL^3}{48EI} = \frac{83.25\ \text{kN}\cdot\text{m}^3}{\text{EI}}\downarrow$$

$$v'_B = \frac{PL^3}{3EI} = \frac{(9\ \text{m}^3)B_y}{EI}\uparrow$$



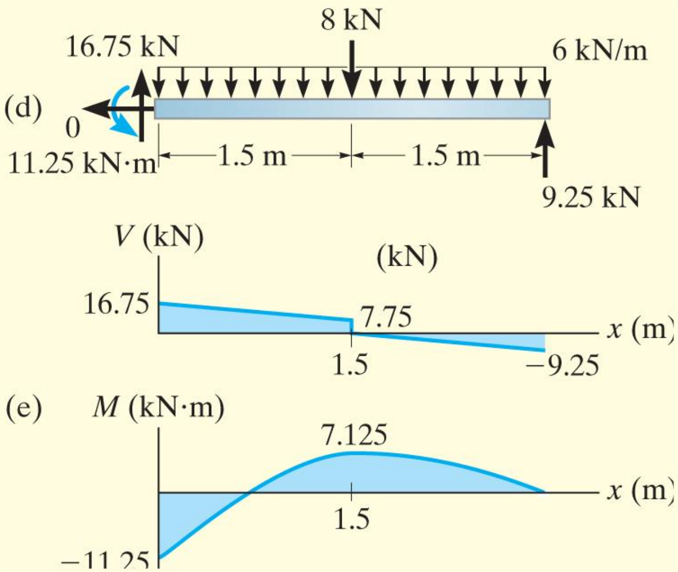
EXAMPLE 2 (cont)

Solutions

- Substituting into Eq. 1 and solving yields

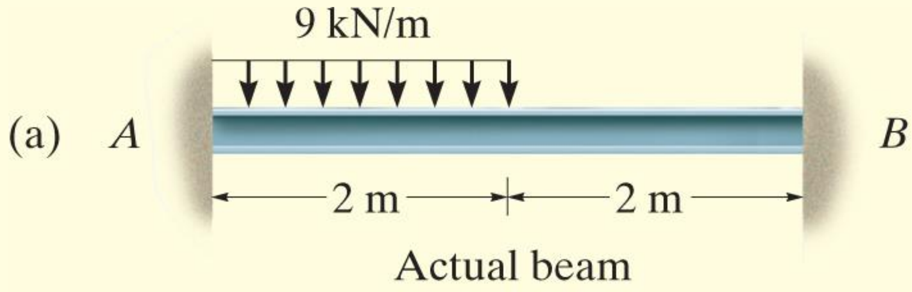
$$0 = \frac{83.25}{EI} - \frac{9B_y}{EI}$$

$$B_y = 9.25\,\text{kN}$$



EXAMPLE 3

Determine the moment at *B* for the beam shown in Fig. below. *EI* is constant. Neglect the effects of axial load.



EXAMPLE 3 (cont)

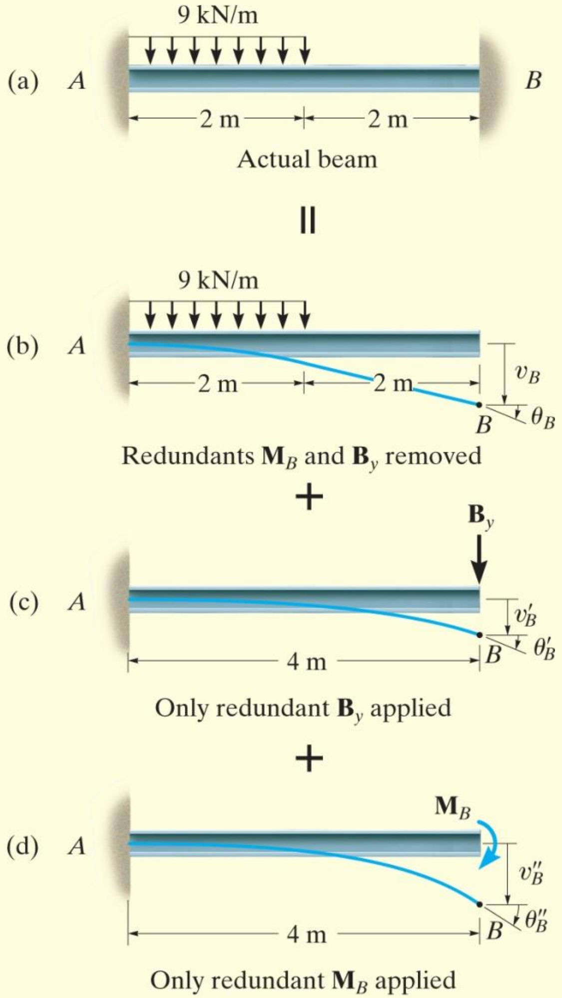
Solutions

- Since the axial load on the beam is neglected, there will be a vertical force and moment at *A* and *B*.

- Referring to the displacement and slope at *B*, we require

$$\circlearrowleft (+) \quad 0 = \theta_B + \theta'_B + \theta''_B \qquad (1)$$

$$(+\downarrow) \quad 0 = v_B + v'_B + v''_B \qquad (2)$$



EXAMPLE 3 (cont)

Solutions

- Use standard beam deflections to calculate slopes and displacements,

$$\theta_B = \frac{wL^3}{48EI} = \frac{12\ \mathrm{kN \cdot m^3}}{\mathrm{EI}} \curvearrowright$$

$$v_B = \frac{7wL^4}{384EI} = \frac{42\ \mathrm{kN \cdot m^3}}{EI} \downarrow$$

$$\theta'_B = \frac{PL^2}{2EI} = \frac{8B_y}{EI} \curvearrowright$$

$$v'_B = \frac{PL^3}{3EI} = \frac{21.33B_y}{EI} \downarrow$$

$$\theta''_B = \frac{ML}{EI} = \frac{4M_B}{EI} \curvearrowright$$

$$v''_B = \frac{ML^2}{2EI} = \frac{8M_B}{EI} \downarrow$$

EXAMPLE 3 (cont)

Solutions

- Substituting these values into Eqs. 1 and 2 and cancelling out the common factor *EI*, we get

$$\circlearrowleft (+) \qquad 0 = 12 + 8B_y + 4M_B$$

$$(+\downarrow) \qquad 0 = 42 + 21.33B_y + 8M_B$$

- Solving these equations simultaneously gives

$$B_y = 3.375\,\text{kN}$$

$$M_B = 3.75\,\text{kN}\cdot\text{m} \quad \text{(Ans)}$$

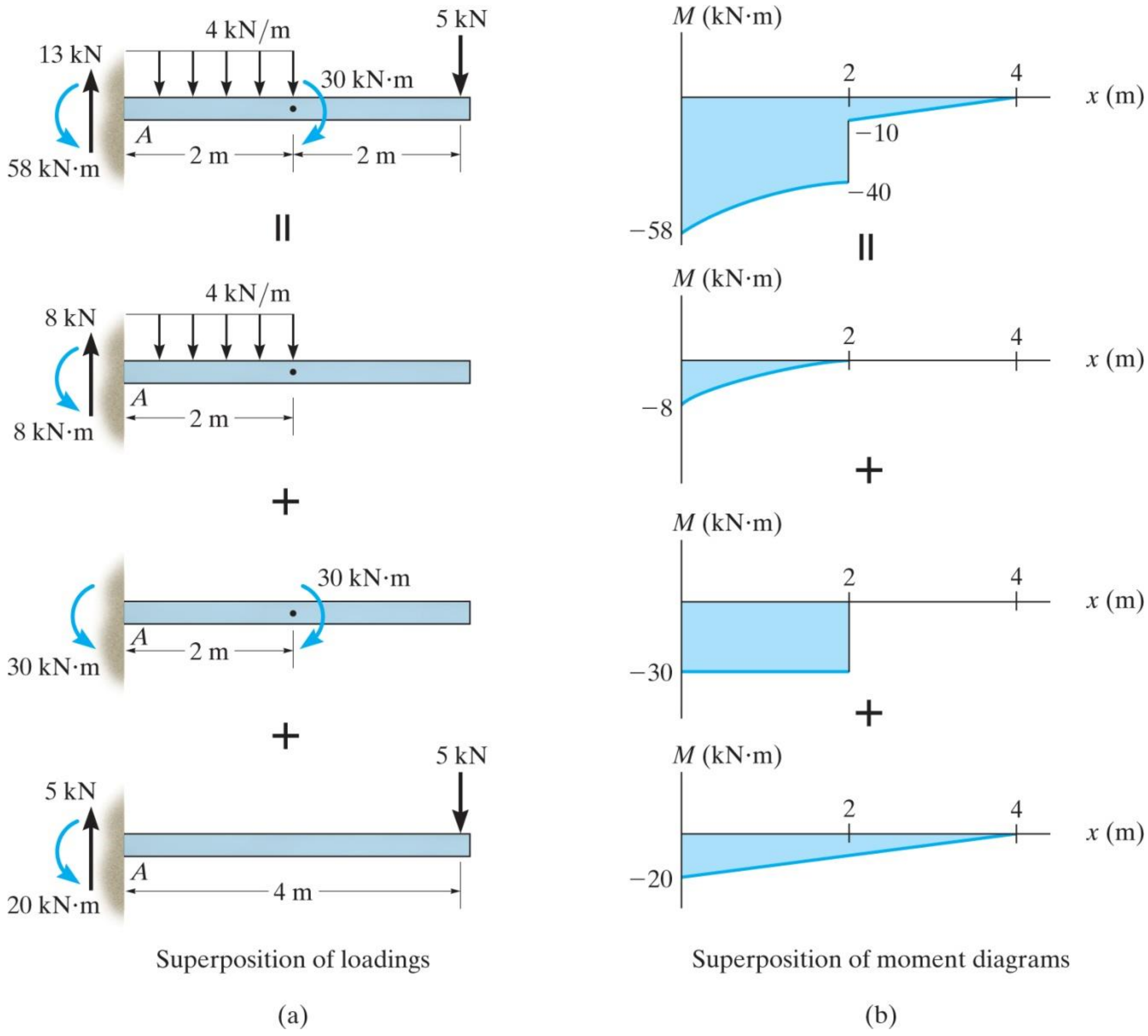
USE OF THE MOMENT-AREA METHOD

Procedures:

- Construct separately the M/EI diagrams for each applied force or moment, and each redundant as well.
- Then use the method of superposition and apply the two moment area theorems to obtain the proper relationship between the tangents on the elastic curve in order to meet the conditions of displacement and/or slope at the supports of the beam or shaft.
- Note: If the two end-moments 20kN.m are redundants, they remain as unknown in the M/EI diagram

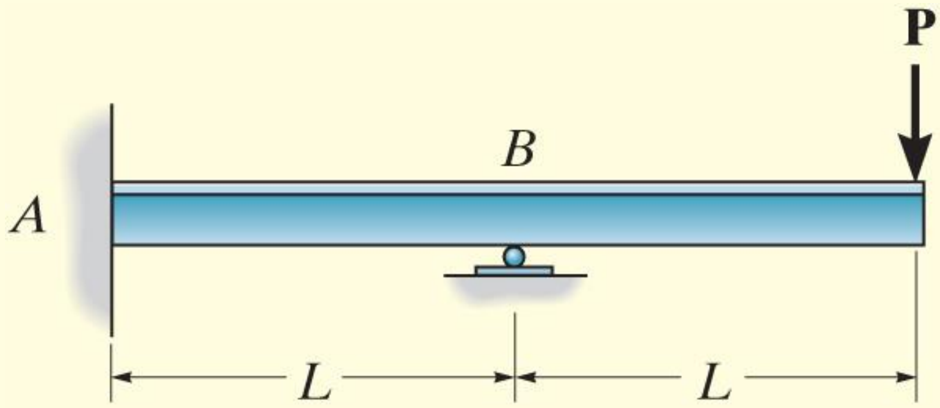
USE OF THE MOMENT-AREA METHOD (cont)

Procedures:



EXAMPLE 4

The beam is subjected to the concentrated force shown in Fig. below. Determine the reactions at the supports. *EI* is constant.

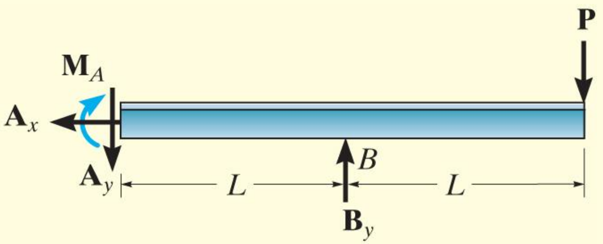


(a)

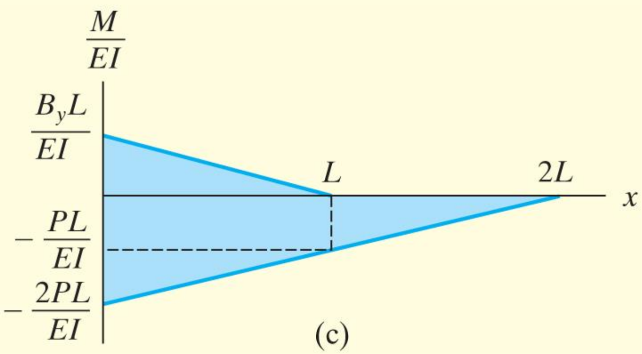
EXAMPLE 4 (cont)

Solutions

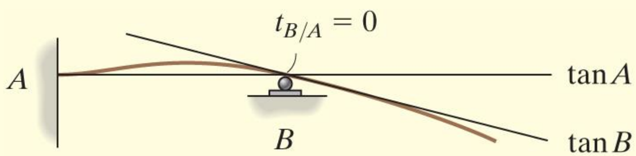
- The free-body diagram is shown in Fig. *b*.
- Using the method of superposition, the separate M/EI diagrams for the redundant reaction B_y and the load **P** are shown in Fig. *c*.
- The elastic curve for the beam is shown in Fig. *d*.



(b)



(c)



(d)

EXAMPLE 4 (cont)

Solutions

- Applying Theorem 2, we have

$$t_{B/A} = \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{B_y L}{EI}\right)L\right] + \left(\frac{L}{2}\right)\left[\frac{-PL}{EI}(L)\right] + \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(-\frac{PL}{EI}\right)(L)\right] = 0$$

$$B_y = 2.5P \qquad (\text{Ans})$$

- Using this result, the reactions at *A* on the free-body diagram, Fig. *b*, are

$$+\rightarrow \sum F_x = 0; \qquad A_x = 0 \quad (\text{Ans})$$

$$+\uparrow \sum F_y = 0; \qquad -A_y + 2.5P - P = 0 \Rightarrow A_y = 1.5P \quad (\text{Ans})$$

$$\circlearrowleft + M_A = 0; \qquad -M_A + 2.5P(L) = P(2L) = 0 \Rightarrow M_A = 0.5PL \quad (\text{Ans})$$