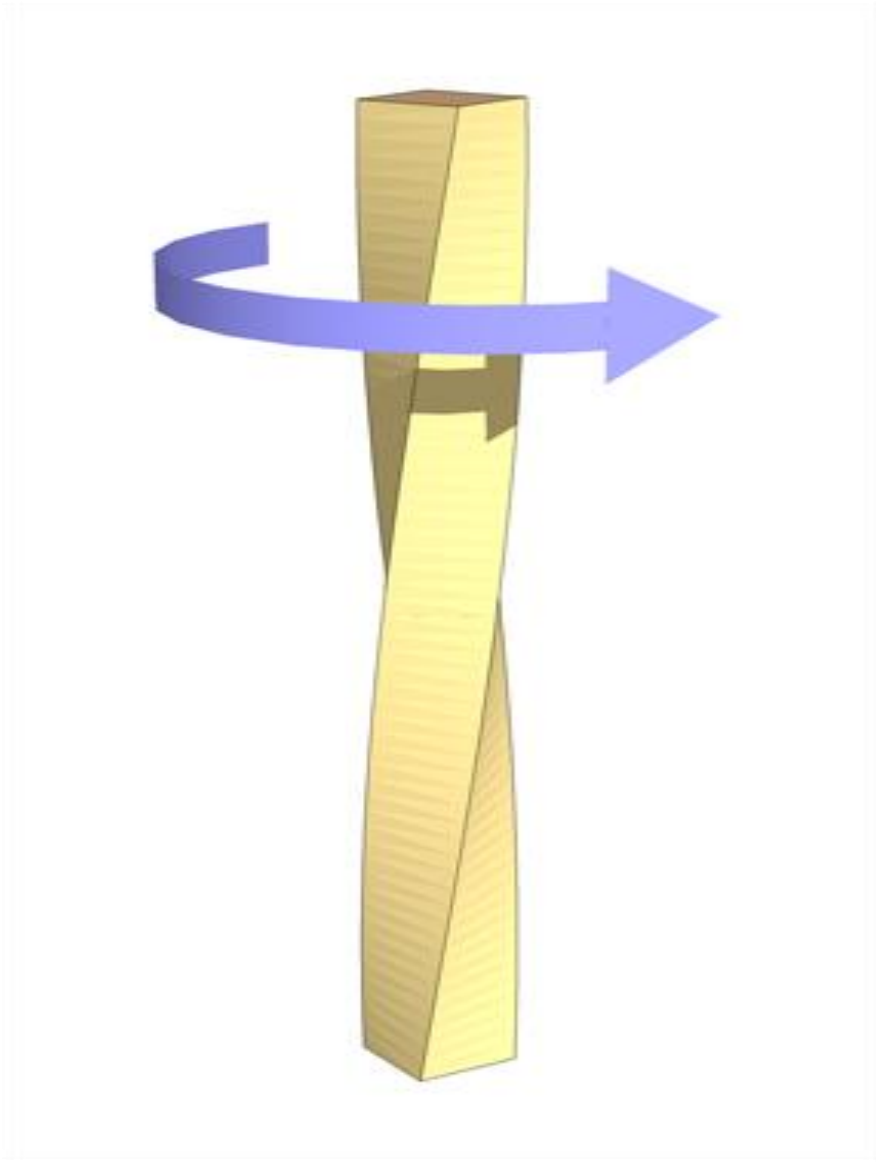
# TORSION



# What will be covered

**1** Determine the shear stresses in a circular shaft due to torsion

**2** Determine the angle of twist

**3** Analyze statically indeterminate torque-loaded members

**4** Deal with solid non-circular shafts and thin-walled tubes

# INTRODUCTION

- Torque is a moment that tends to twist a member about its longitudinal axis.

- In the design of machinery (and some structures), the problem of transmitting a torque from one plane to a parallel plane is frequently encountered

- The simplest device for accomplishing this function is called a shaft

- The torsion problem is concerned with the determination of stresses in shaft and the deformation of the shaft.

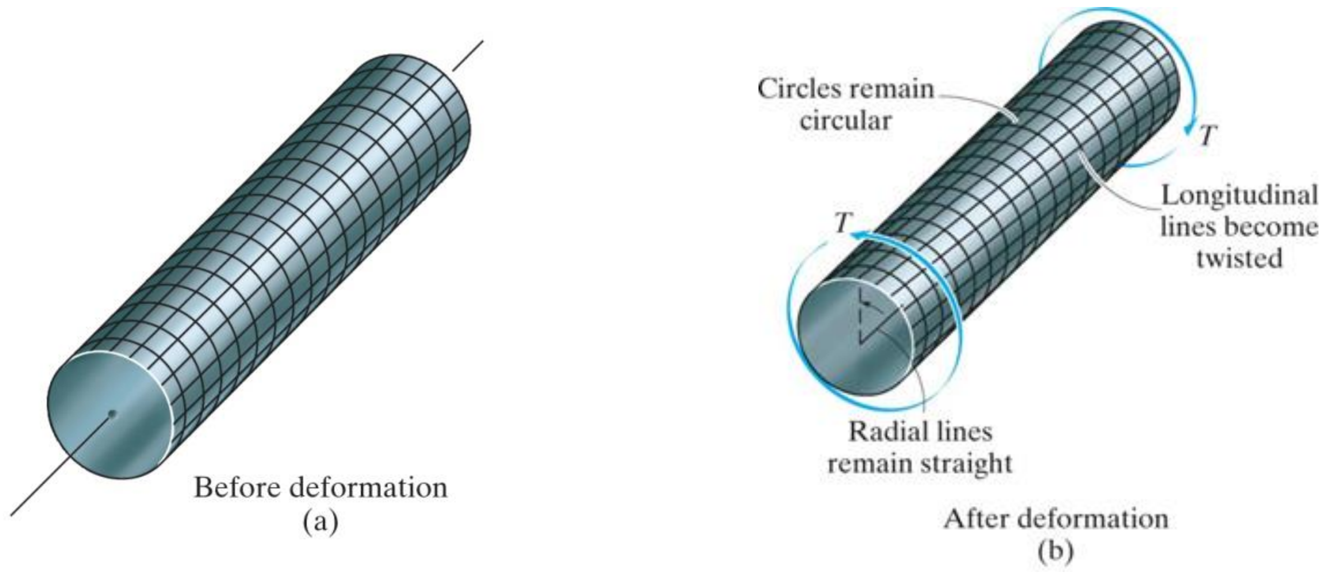
# Comparisons between Axial Deformation and Torsion

| Axial Deformation | Torsion |
|---|---|
| Axial Force ($F$) | Torque ($T$) |
| Elongation ($e$) | Twist angle ($\phi$) |
| Normal stress ($\sigma$) | Shear stress ($\tau$) |
| Extensional strain ($\epsilon$) | Shear strain ($\gamma$) |
| Modulus of elasticity ($E$) | Shear modulus ($G$) |

# Torsion Formula (1 of 4)

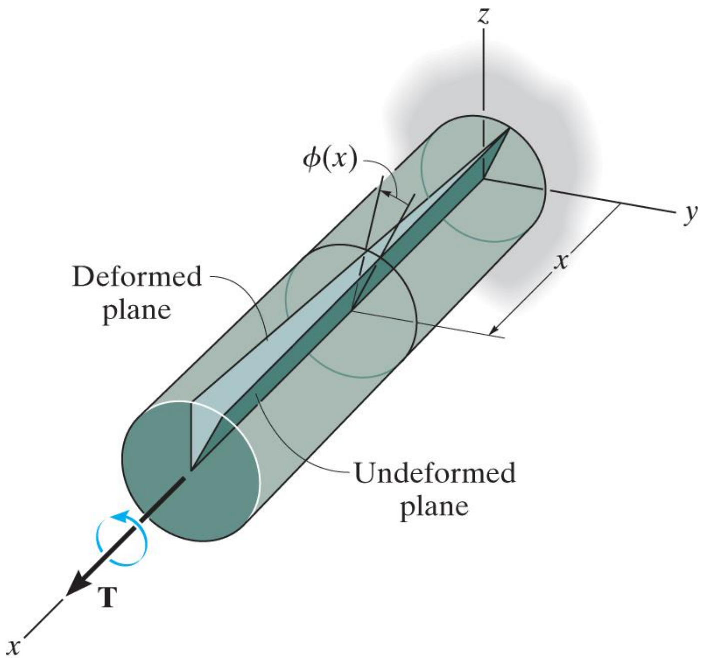
- Assumptions:
  - Linear and elastic deformation
  - Plane section remains plane and undistorted

**Figure 1**



# Torsion Formula (2 of 4)

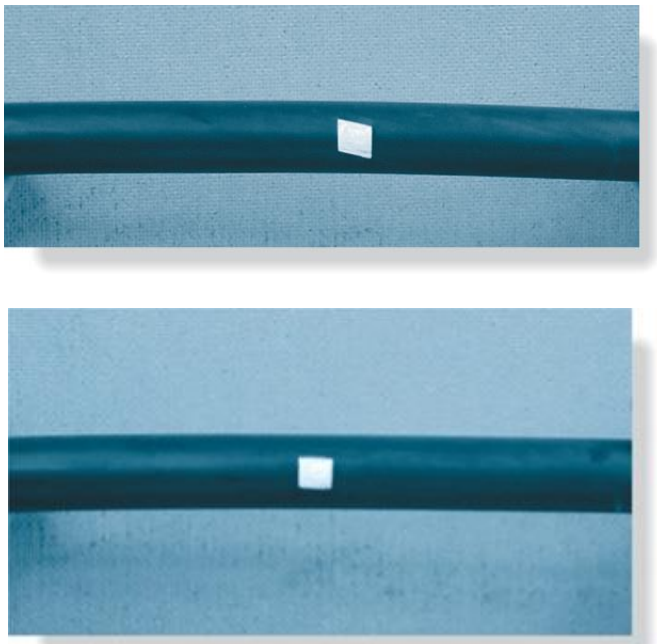
**Figure 2**




The angle of twist $\phi(x)$ increases as $x$ increases.

**Figure 3**

Notice the deformation of the rectangular element when this rubber bar is subjected to a torque.



# Torsion Formula (3 of 4)

- Linear distribution of stress: $\tau = \dfrac{\rho}{c}\tau_{\max}$
- Torsion – shear relationship:

$$T = \int_A \rho(\tau)\,dA = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{\max}\,dA$$

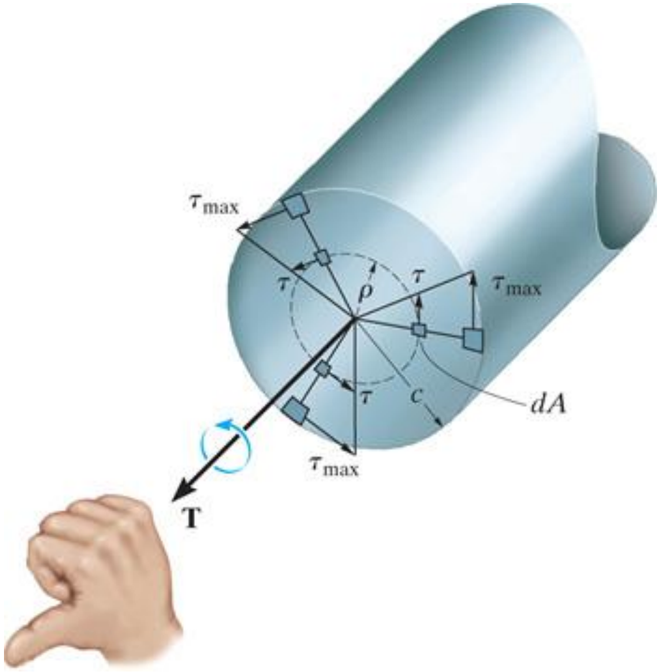
$$T = \frac{\tau_{\max}}{c}\int_A \rho^2\,dA$$

$$\tau_{\max} = \frac{Tc}{J}$$

Similarly, $\tau = \dfrac{T\rho}{J}$

**Figure 4**

Shear stress varies linearly along radial line of the cross section.



# Torsion Formula (4 of 4)

- Polar moment of inertia
  - For solid shaft: $$J = \int_A \rho^2 dA = \int_0^C \rho^2 (2\pi\rho d\rho) = 2\pi \int_0^C \rho^3 d\rho = 2\pi \left(\frac{1}{4}\right)\rho^4 \Big|_0^C$$
    $$J = \frac{\pi}{2} c^4$$
  - For tubular shaft: $$J = \frac{\pi}{2}\left(c_o^4 - c_i^4\right)$$

**Figure 5**

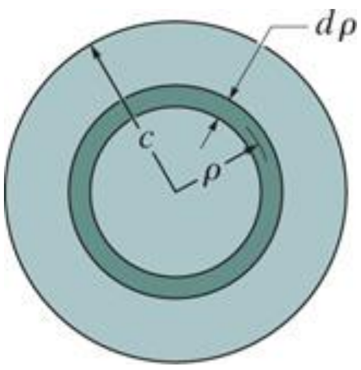
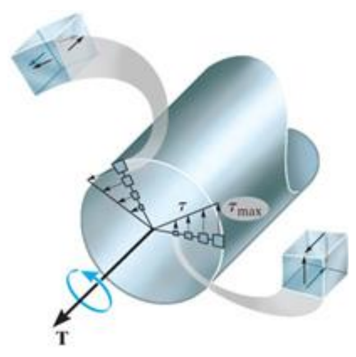
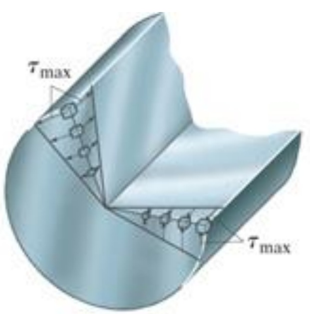



**Figure 6a**




(a)

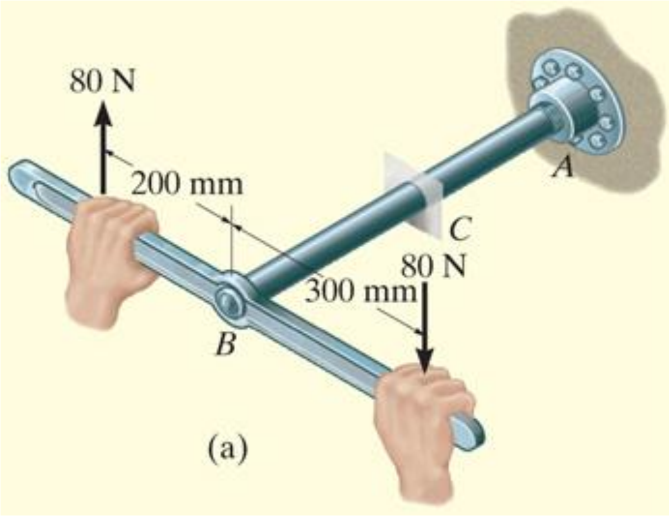
**Figure 6b**




Shear stress varies linearly along each radial line of the cross section.
(b)

# Example 1 (1 of 4)

The pipe shown in below has an inner diameter of 80 mm and an outer diameter of 100 mm. If its end is tightened against the support at *A* using a torque wrench at *B*, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the 80-N forces are applied to the wrench.



# Example 1 (2 of 4)

## Solutions

- The only unknown at the section is the internal torque **T**

$$\sum M_y = 0;$$

$$80(0.3) + 80(0.2 - T) = 0$$

$$T = 40 \text{ N} \cdot \text{m}$$

- The polar moment of inertia for the pipe's cross-sectional area is

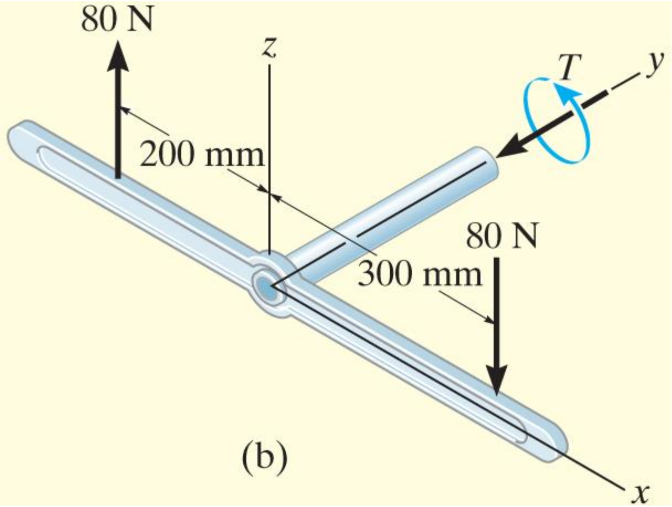
$$J = \frac{\pi}{2}\left[(0.05)^4 - (0.04)^4\right] = 5.796\left(10^{-6}\right) \text{ m}^4$$

# Example 1 (3 of 4)

- For any point lying on the outside surface of the pipe,

$$\rho = c_0 = 0.05 \text{ m}$$

$$\tau_0 = \frac{Tc_0}{J} = \frac{40(0.05)}{5.796(10^{-6})} = 0.345 \text{ MPa} \quad \text{(Ans)}$$



(b)

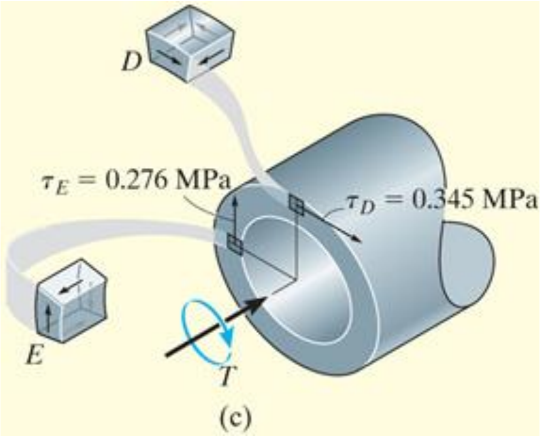
# Example 1 (4 of 4)

- And for any point located on the inside surface, $\rho = c_i = 0.04 \text{ m}$

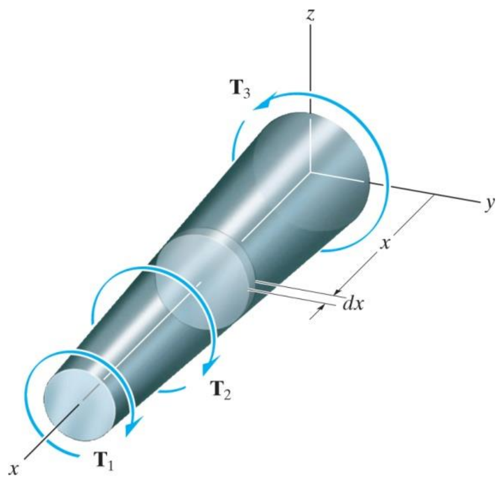
$$\tau_i = \frac{Tc_i}{J} = \frac{40(0.04)}{5.796(10^{-6})} = 0.276 \text{ MPa} \quad \text{(Ans)}$$

- The resultant internal torque is equal but opposite.

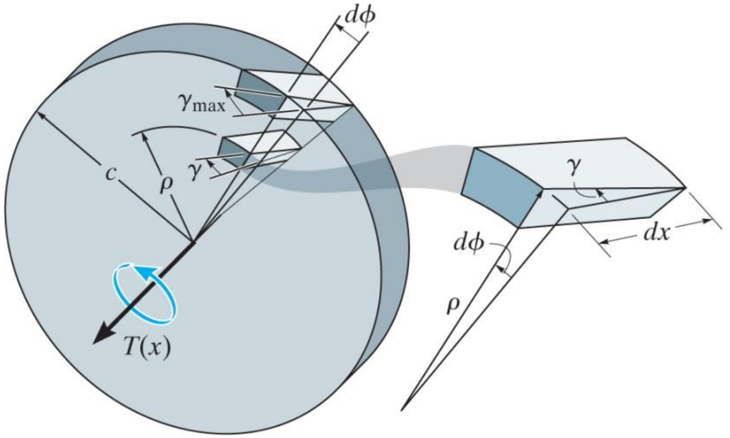
**Figure 7**



# Angle of Twist (1 of 2)




(a)




(b)

$$d\phi = \gamma \frac{dx}{\rho}$$

$$d\phi = \frac{T(x)}{J(x)G} dx$$

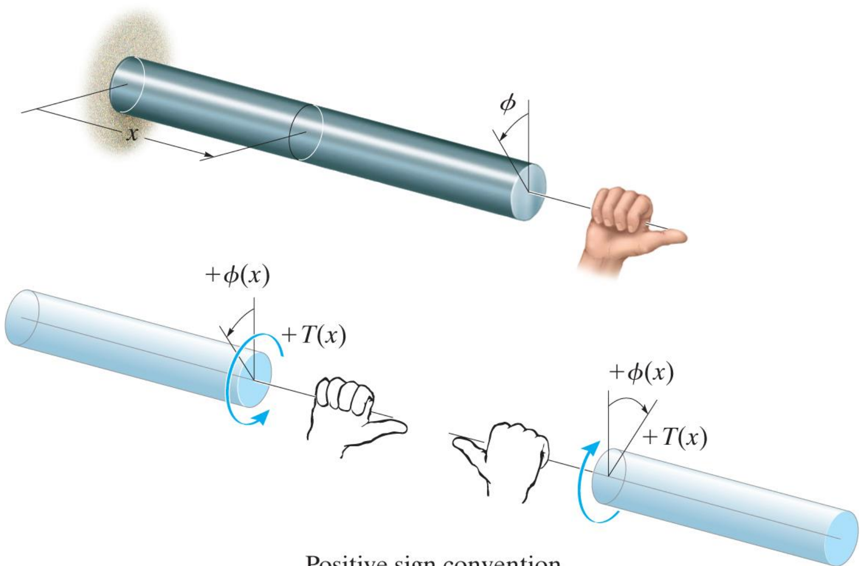
- For constant torque and cross-sectional area:

$$\phi = \frac{TL}{JG}$$

# Angle of Twist (2 of 2)

- Sign convention for both torque and angle of twist
  - positive if (right hand) thumb directs outward from the shaft

**Figure 8**

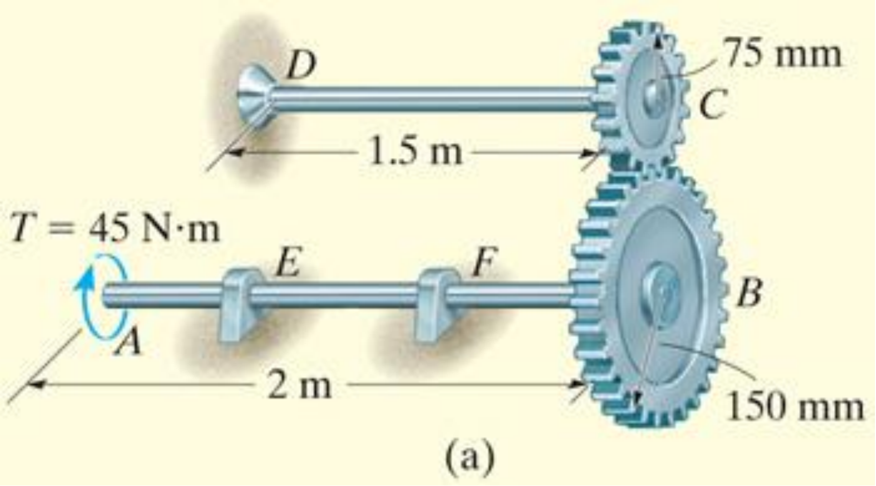



Positive sign convention for $T$ and $\phi$.

# Example 2 (1 of 6)

The two solid steel shafts are coupled together using the meshed gears. Determine the angle of twist of end *A* of shaft *AB* when the torque 45 Nm is applied. Take G to be 80 GPa. Shaft *AB* is free to rotate within bearings *E* and *F*, whereas shaft *DC* is fixed at *D*. Each shaft has a diameter of 20 mm.

**Figure 9**



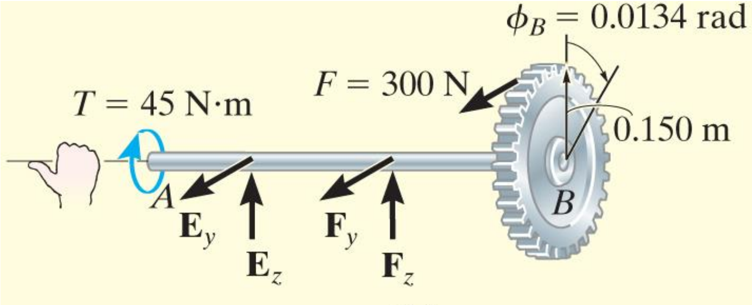
# Example 2 (2 of 6)

**Solutions**

- From free body diagram,

$$F = 45 / 0.15 = 300 \text{ N}$$

$$(T_D)_x = 300(0.075) = 22.5 \text{ Nm}$$



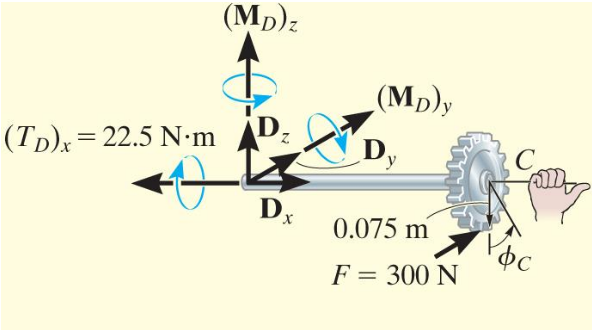
(b)

- Angle of twist at C is

$$\phi_C = \frac{TL_{DC}}{JG} = \frac{(+22.5)(1.5)}{(\pi/2)(0.001)^4\left[80(10)^9\right]} = +0.0269 \text{ rad}$$

- Since the gears at the end of the shaft are in mesh,

$$\phi_B(0.15) = (0.0269)(0.075) \Rightarrow 0.0134 \text{ rad}$$



(c)

# Example 2 (3 of 6)

- Since the angle of twist of end *A* with respect to end *B* of shaft *AB* caused by the torque 45 Nm,

$$\phi_{A/B} = \frac{T_{AB} L_{AB}}{JG} = \frac{(+45)(2)}{(\pi/2)(0.010)^4\left[80(10^9)\right]} = +0.0716 \text{ rad}$$

- The rotation of end *A* is therefore

$$\phi_A = \phi_B + \phi_{A/B} = 0.0134 + 0.0716 = +0.0850 \text{ rad} \quad \text{(Ans)}$$

# Statically Indeterminate Torque-Loaded Members (1 of 2)

- Procedure for analysis:
  - use both equilibrium and compatibility equations

**Equilibrium**

- Draw a free-body diagram of the shaft in order to identify all the torques that act on it. Then write the equations of moment equilibrium about the axis of the shaft.

**Compatibility**

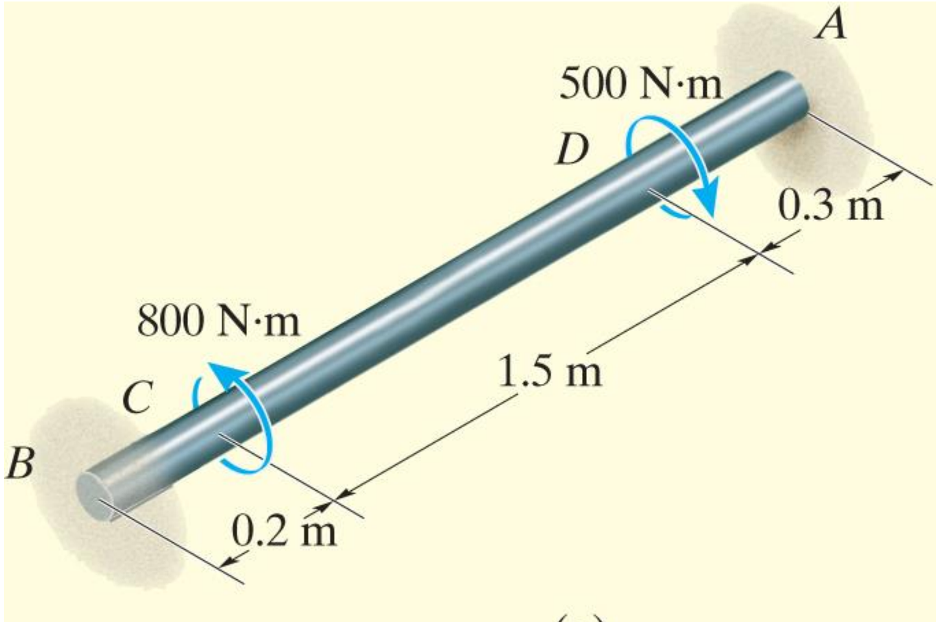
- To write the compatibility equation, investigate the way the shaft will twist when subjected to the external loads, and give consideration as to how the supports constrain the shaft when it is twisted.

# Statically Indeterminate Torque-Loaded Members (2 of 2)

- Express the compatibility condition in terms of the rotational displacements caused by the reactive torques, and then use a torque-displacement relation, such as $\phi = TL/JG$, to relate the unknown torques to the unknown displacements.
- Solve the equilibrium and compatibility equations for the unknown reactive torques. If any of the magnitudes have a negative numerical value, it indicates that this torque acts in the opposite sense of direction to that indicated on the free-body diagram.

# Example 3 (1 of 3)

The solid steel shaft shown below has a diameter of 20 mm. If it is subjected to the two torques, determine the reactions at the fixed supports *A* and *B*.



(a)

# Example 3 (2 of 3)

**Solutions**

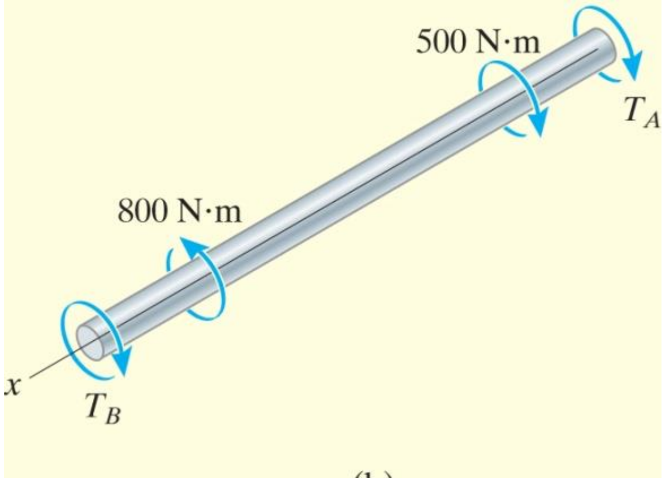
- It is seen that the problem is statically indeterminate since there is only **one** available equation of equilibrium and there are 2 unknowns

$$\sum M_x = 0$$

$$-T_b + 800 - 500 - T_A = 0 \qquad (1)$$

- Since the ends of the shaft are fixed, the angle of twist of one end of the shaft with respect to the other must be zero.

$$\phi_{A/B} = 0$$



(b)

# Example 3 (3 of 3)

- Using the sign convention established,

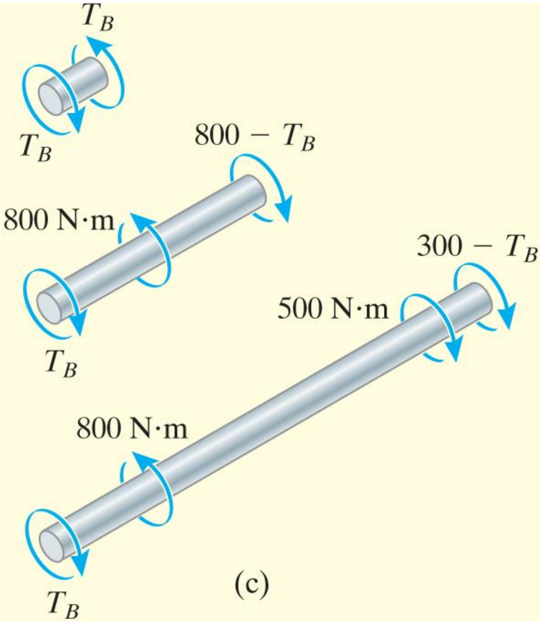
$$\frac{-T_B(0.2)}{JG}+\frac{(800-T_B)(1.5)}{JG}+\frac{(300-T_B)(0.3)}{JG}=0$$

$$T_B = 645\ \text{N}\cdot\text{m} \quad \text{(Ans)}$$

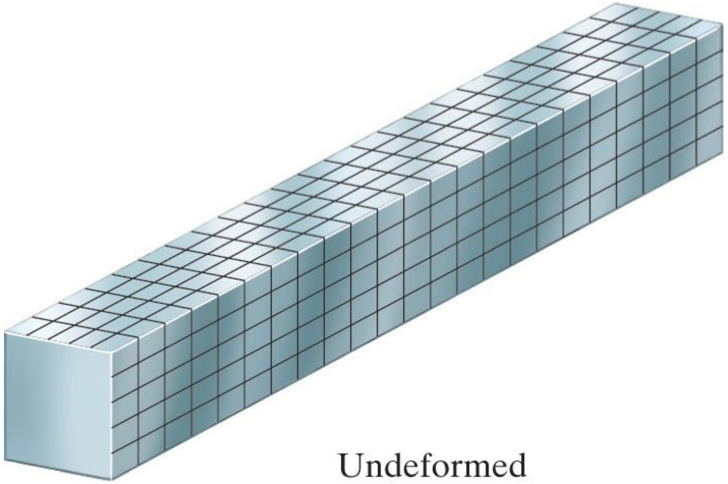
- Using Equation 1,

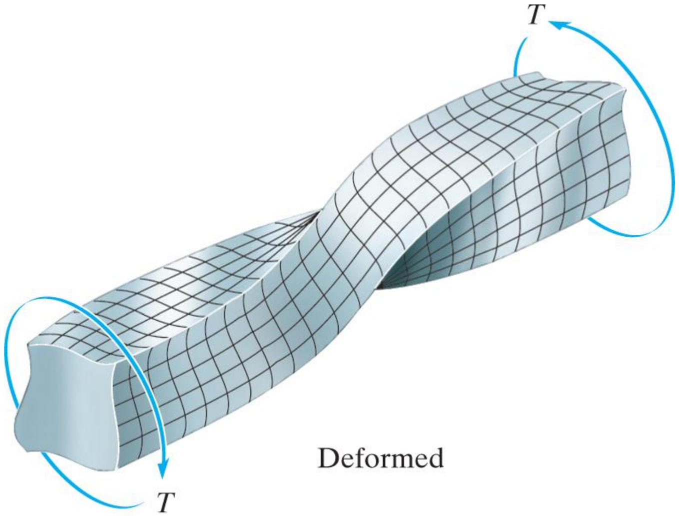
$$T_A = -345\ \text{N}\cdot\text{m}$$

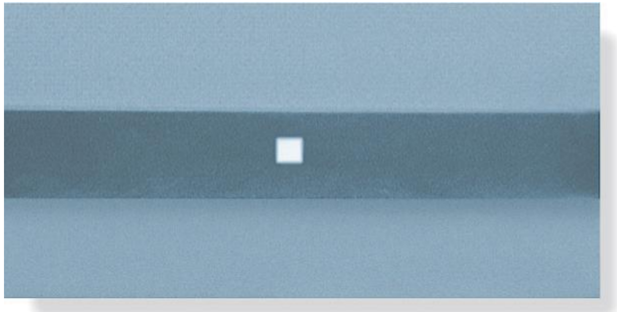
- The negative sign indicates that acts in the opposite direction of that shown in Figure 5–23b.

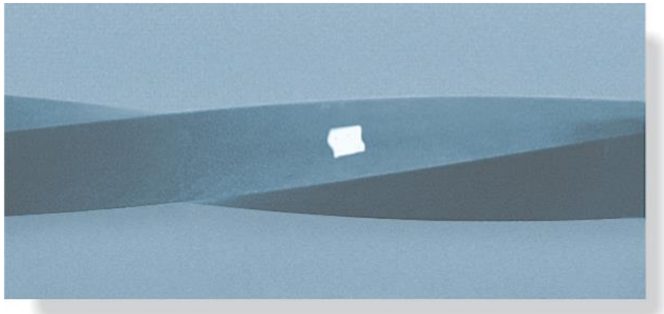


# Solid Non-Circular Shafts (1 of 2)





# Solid Non-Circular Shafts (2 of 2)

**Table-1**

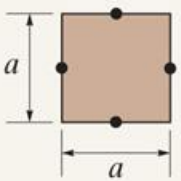
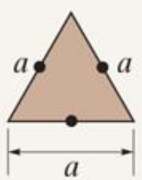
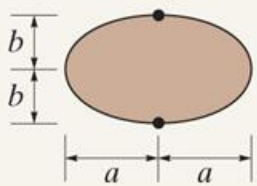
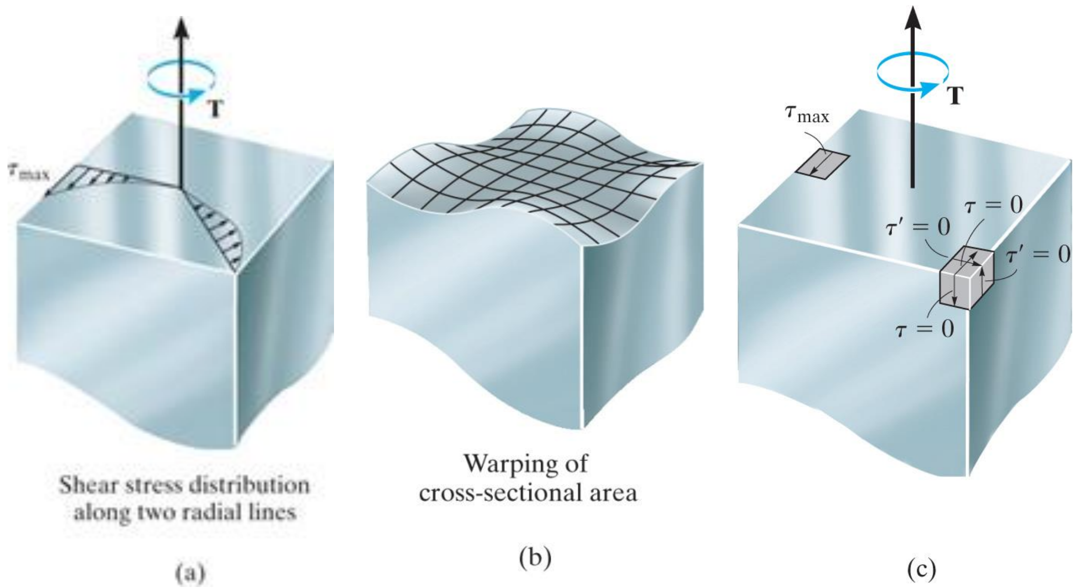
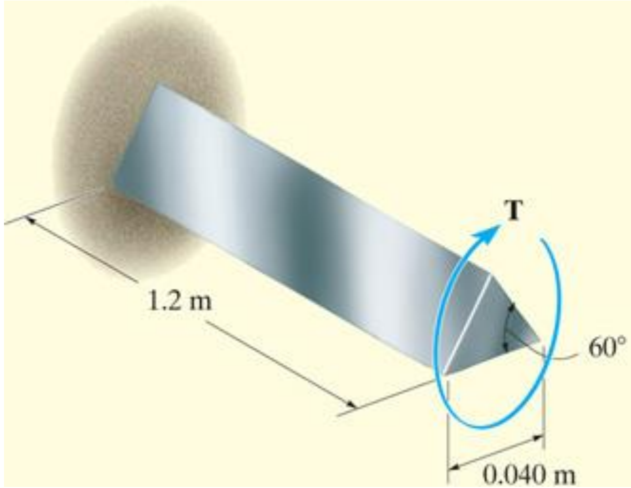
| Shape of cross section | $\tau_{max}$ | $\phi$ |
|---|---|---|
| Square  | $\frac{4.81\,T}{a^3}$ | $\frac{7.10\,TL}{a^4G}$ |
| Equilateral triangle  | $\frac{20\,T}{a^3}$ | $\frac{46\,TL}{a^4G}$ |
| Ellipse  | $\frac{2\,T}{\pi ab^2}$ | $\frac{(a^2+b^2)TL}{\pi a^3b^3G}$ |

**Figure 5-26**



# Example 1 (1 of 3)

The 6061-T6 aluminum shaft shown in Figure below has a cross-sectional area in the shape of an equilateral triangle. Determine the largest torque *T* that can be applied to the end of the shaft if the allowable shear stress is $t_{allow}$ = 56 M P a. and the angle of twist at its end is restricted to $\phi_{allow} = 0.02$ rad. How much torque can be applied to a shaft of circular cross section made from the same amount of material?



# Example 1 (2 of 3)

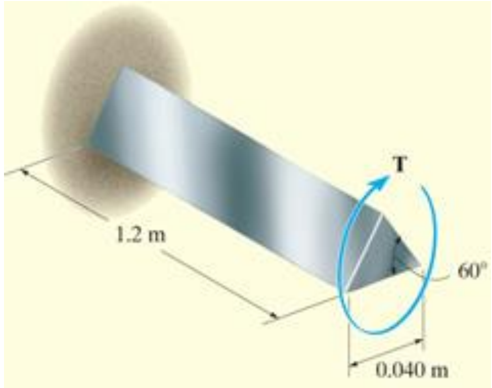
**Solutions**

- By inspection, the resultant internal torque at any cross section along the shaft's axis is also **T**.

$$\tau_{allow} = \frac{20T}{a^3}; \qquad 56 = \frac{20T}{40^3} \Rightarrow T = 1779.2 \text{ Nm}$$

$$\sigma_{allow} = \frac{46T}{a^4 G_{al}}; \qquad 0.02 = \frac{46T(1.2)(10^3)}{40^4\left[26(10^3)\right]} \Rightarrow T = 24.12 \text{ Nm} \quad \text{(Ans)}$$

- By comparison, the torque is limited due to the angle of twist.



# Example 1 (3 of 3)

**Solutions**

- For circular cross section, we have

$$A_{circle} = A_{triangle}; \qquad \pi c^2 = \frac{1}{2}(40)(40\sin 60^\circ) \Rightarrow c = 14.85 \text{ mm}$$

- The limitations of stress and angle of twist then require

$$\tau_{allow} = \frac{Tc}{J}; \qquad 56 = \frac{T(14.85)}{(\pi/2)(14.85)^4} \Rightarrow T = 288.06 \text{ Nm}$$

$$\phi_{allow} = \frac{TL}{JG_{al}}; \qquad 0.02 = \frac{T(1.2)(10^3)}{(\pi/2)(14.85)^4\left[26(10^3)\right]} \Rightarrow T = 33.10 \text{ Nm} \quad \text{(Ans)}$$

- Again, the angle of twist limits the applied torque.

# Thin Wall Tubes Having Closed Sections

- Average shear stress

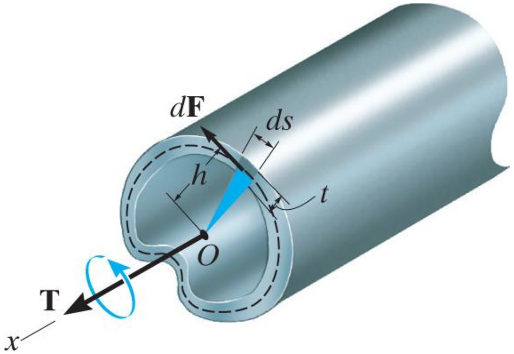
$$\tau_{avg} = \frac{T}{2tA_m}$$

- Shear flow

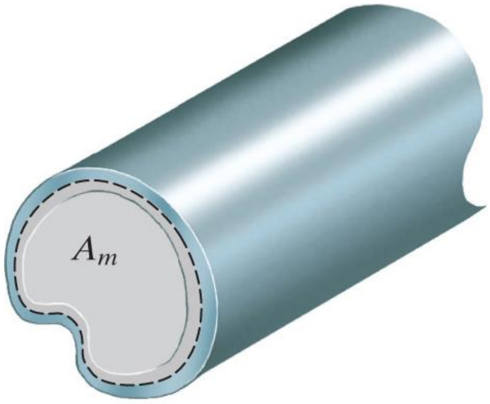
$$q = \frac{T}{2A_m}$$

- Angle of twist

$$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t}$$



(e)

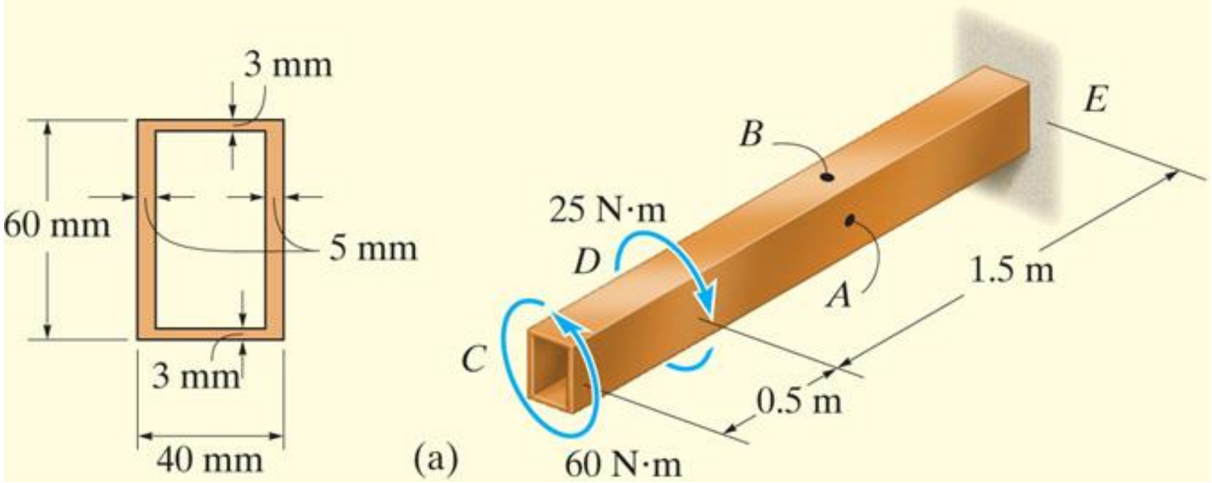


(f)

# Example 2 (4 of 6)

The tube is made of C86100 bronze and has a rectangular cross section as shown in Figure 5–30a. If it is subjected to the two torques, determine the average shear stress in the tube at points *A* and *B*. Also, what is the angle of twist of end *C*? The tube is fixed at *E*.

**Figure 5-30**



# Example 2 (5 of 6)

**Solutions**

- As shown in Figure 5–30d, the mean area is

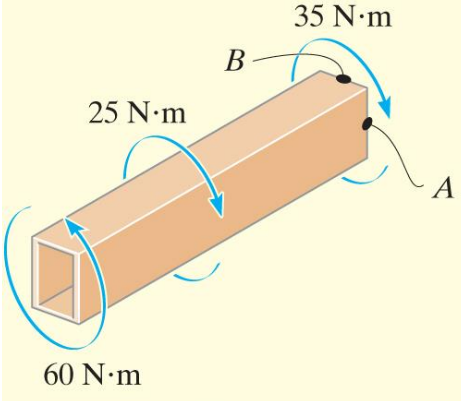
$$A_m = (0.035)(0.057) = 0.002 \text{ m}^2$$

- Applying Equation 5–18 for point *A*,

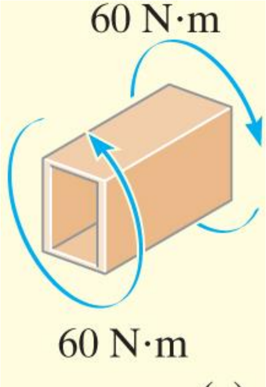
$$\tau_A = \frac{T}{2tA_m} = \frac{35}{2(0.005)(0.002)} = 1.75 \text{ MPa} \quad \text{(Ans)}$$

- And for point *B*,

$$\tau_B = \frac{T}{2tA_m} = \frac{35}{2(0.003)(0.002)} = 2.92 \text{ MPa} \quad \text{(Ans)}$$



(b)



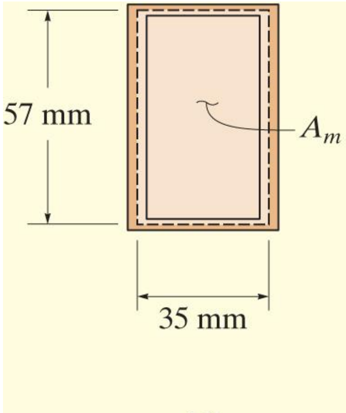
(c)

# Example 2 (6 of 6)

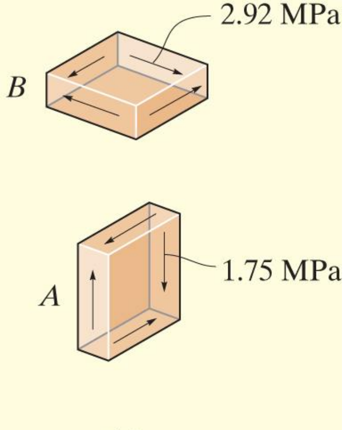
## Solutions

- From the free-body diagrams in Figure 5–30b and 5–30c, the internal torques in regions *DE* and *CD* are 35 Nm and 60 Nm respectively.
- Angle of twist is

$$\phi = \frac{TL}{4A_m^2 G}\oint\frac{ds}{t} = \frac{60(0.5)}{4(0.002)^2[38(10^9)]}\left[2\left(\frac{57}{5}\right)+2\left(\frac{35}{3}\right)\right]$$
$$+\frac{35(1.5)}{4(0.002)^2[38(10^9)]}\left[2\left(\frac{57}{5}\right)+2\left(\frac{35}{3}\right)\right]$$
$$= 6.29(10^{-3})\text{ rad} \quad \text{(Ans)}$$



(d)



(e)