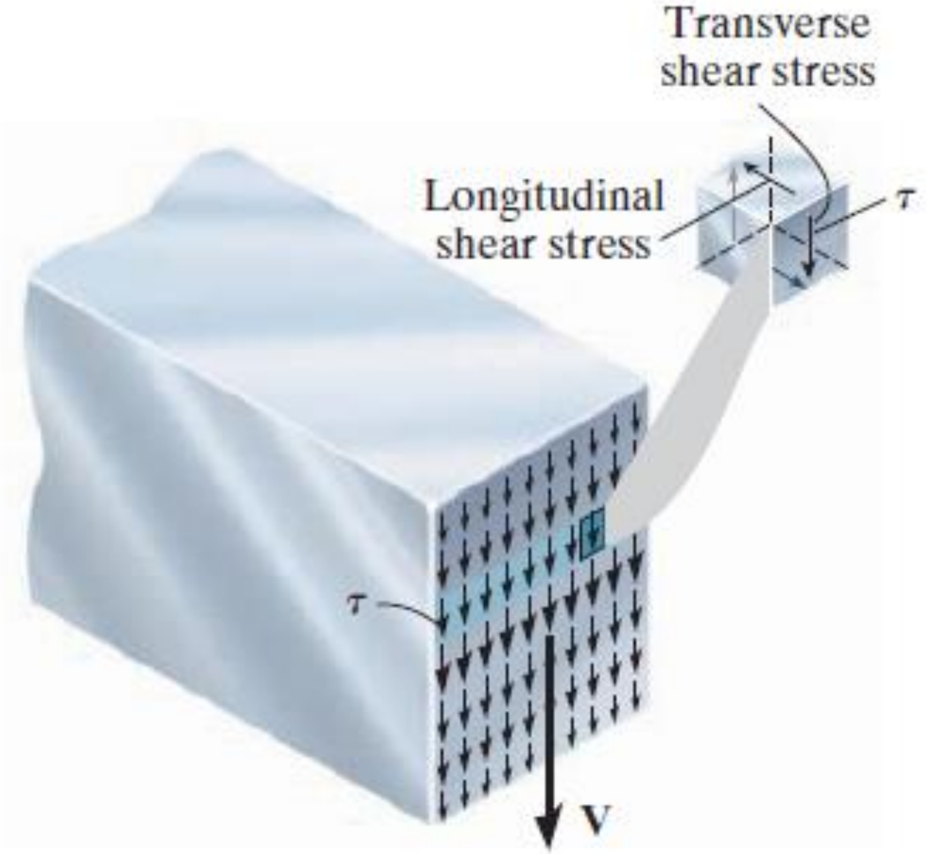
# SHEAR STRESS DISTRIBUTION IN BEAMS and COMBINED LOADINGS

## Objectives

- ✓ Determine shear stress in a prismatic beam
- ✓ Determine the shear flow in a built-up beam
- ✓ Determine the shear flow in thin-walled beam
- ✓ Determine the shear centre of a cross section
- ✓ Determines stresses from combined loadings

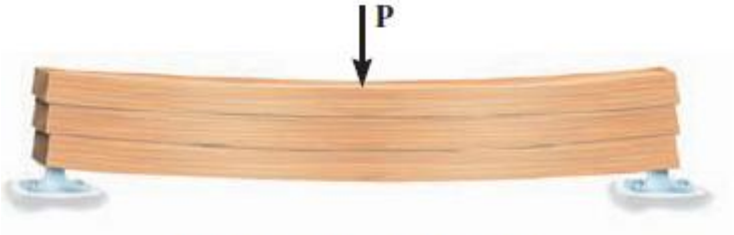
# SHEAR IN A STRAIGHT BEAM

- Transverse shear stress always has its associated longitudinal shear stress acting along longitudinal planes of the beam.

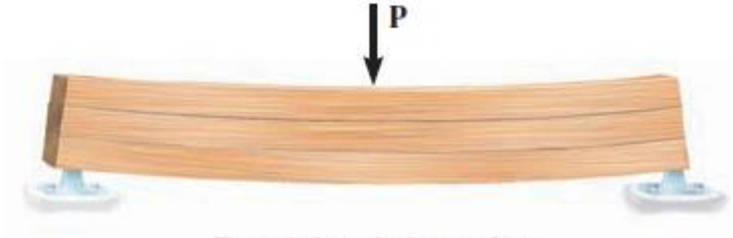


# SHEAR IN A STRAIGHT BEAM (cont)

- Effects of Shear Stresses:

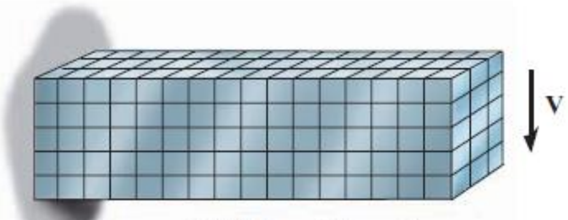


Boards not bonded together
(a)

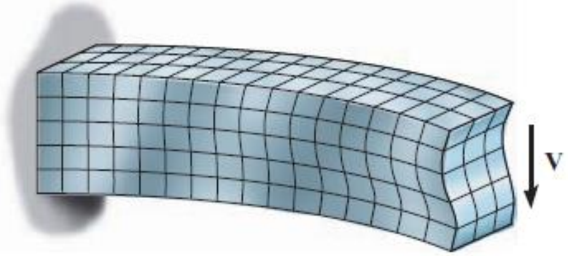


Boards bonded together
(b)

- Nonuniform shear-strain distribution will cause the cross section to warp



(a) Before deformation

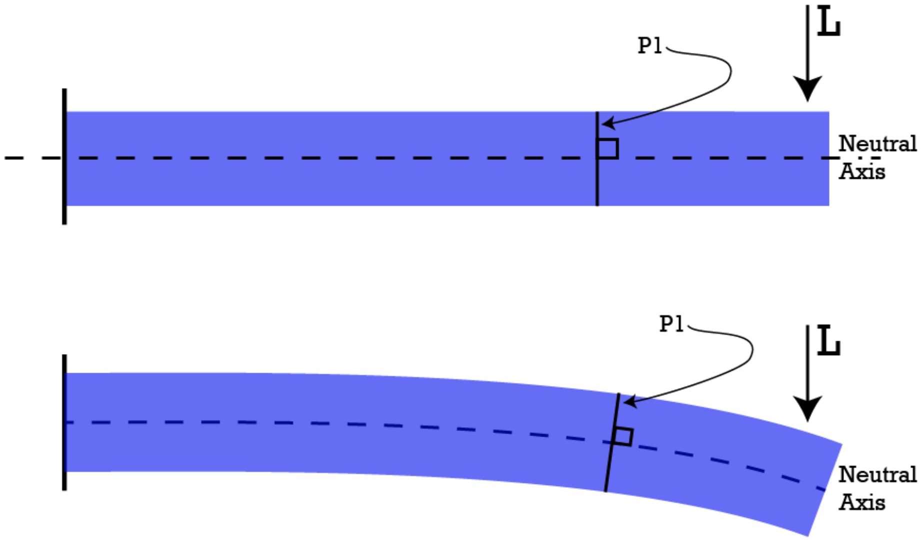


(b) After deformation

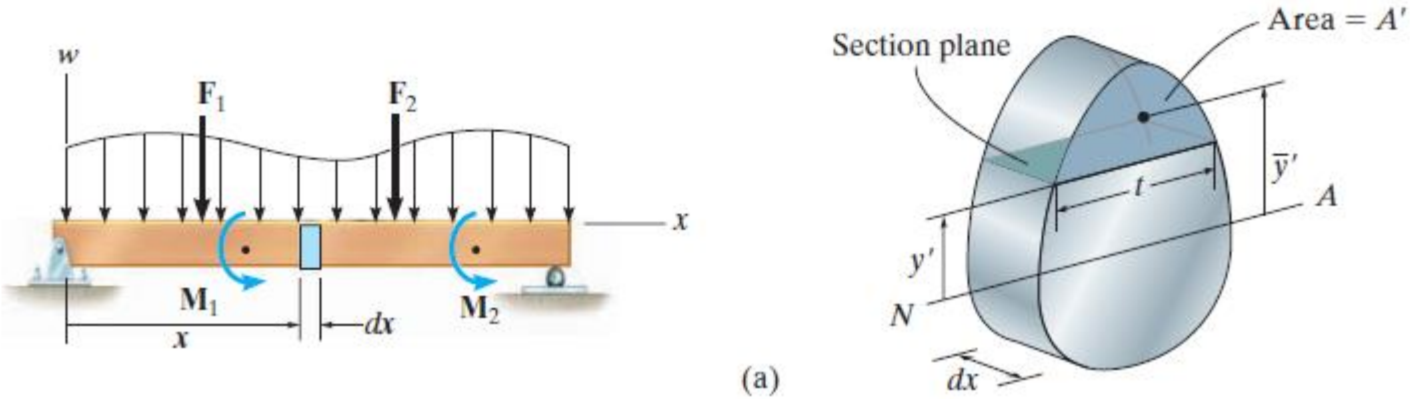
# SHEAR IN A STRAIGHT BEAM (cont)

Note:

1. Warping” violates the assumptions of “plane section remains plane” in flexure and torsion formula
2. However, “Warping” is negligible in “slender beam” (small depth compared with its length)



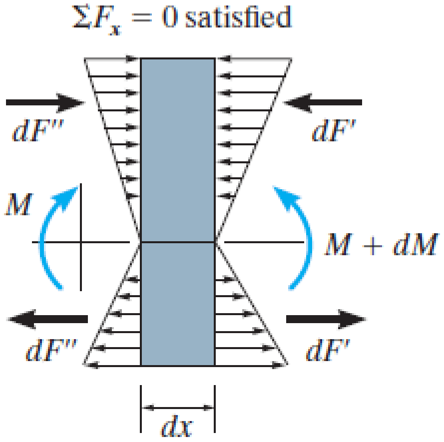
# SHEAR FORMULA



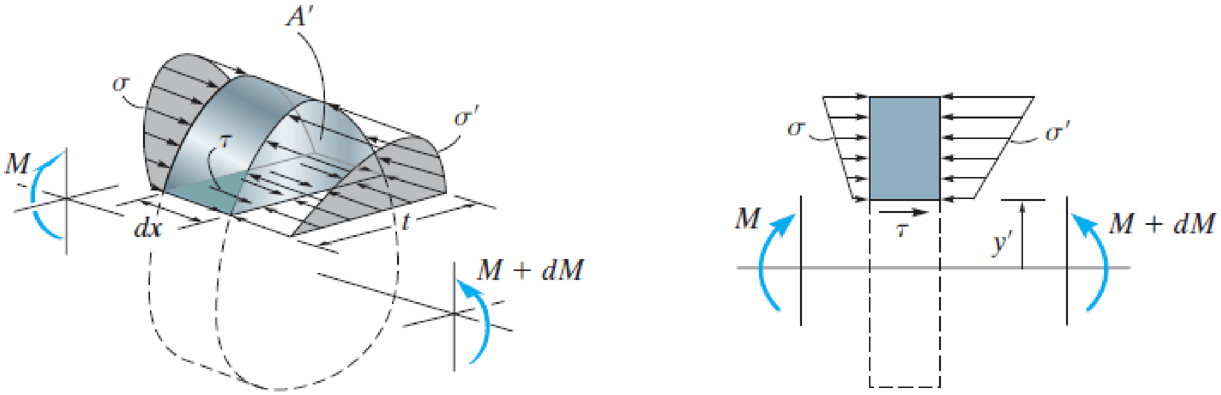
(a)

$$\tau = \frac{VQ}{It}$$

where $Q = \int_{A'} y\,dA = \bar{y}'A'$



(b)



Three-dimensional view

(c)

Profile view

# SHEAR FORMULA

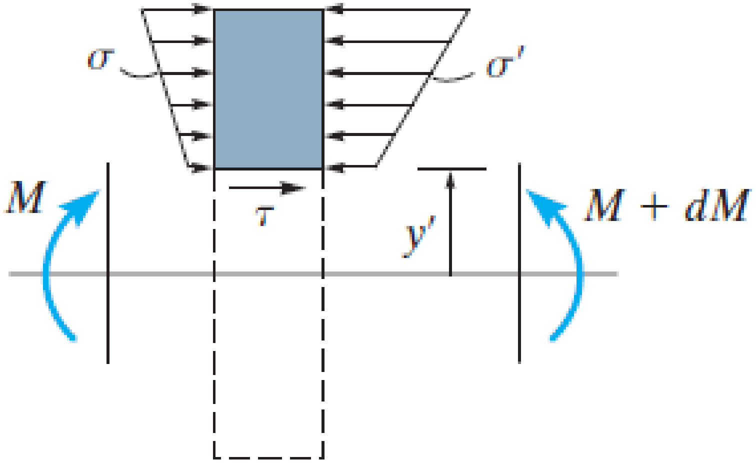
$$\leftarrow + \sum F_X = 0; \int_{A\prime} \sigma' dA' - \int_{A\prime} \sigma dA' - \tau(tdx) = 0$$

$$\int_{A\prime} \left(\frac{M + dM}{I}\right) ydA' - \int_{A\prime} \left(\frac{M}{I}\right) ydA' - \tau(tdx) = 0$$

$$\left(\frac{dM}{I}\right) \int_{A\prime} ydA' = \tau(tdx)$$

*solving for* $\tau$, *we get*

$$\tau = \frac{1}{It}\left(\frac{dM}{dx}\right) \int_{A\prime} ydA'$$



Profile view

# SHEAR FORMULA

$$Note\ V = \frac{dM}{dx}$$

Also, the integral represents the moment of the area, A' about the neutral axis denoted by symbol *Q*.

$$Thus;\quad Q = \int_{A'} y\,dA' = \bar{y}'A'$$

$$Shear\ stress;\quad \tau = \frac{VQ}{It}$$

This equation is referred to as the ***shear formula***

Although in the derivation we considered only the shear stresses acting on the beam's longitudinal plane, the formula applies as well for finding the transverse shear stress on the beam's cross-section since these stresses are complementary and numerically equal

$$Shear\ stress;\ \ \tau = \frac{VQ}{It}$$

- $\tau$ = the shear stress in the member at the point located a distance $y'$ from the neutral axis. This stress is assumed to be constant and therefore *averaged* across the width *t* of the member

- *V* = the shear force, determined from the method of sections and the equations of equilibrium

- *I* = the moment of inertia of the *entire* cross-sectional area calculated about the neutral axis

- *t* = the width of the member's cross section, measured at the point where $\tau$ is to be determined

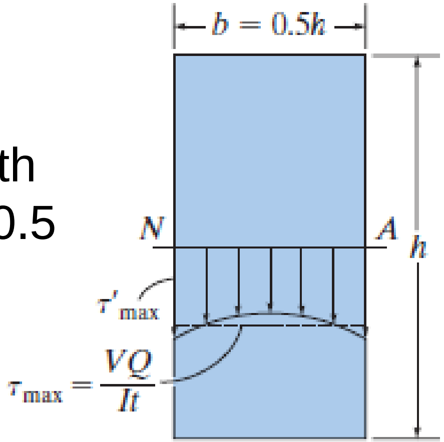
- $Q = \overline{y}'A'$ , where $A'$ is the area of the top (or bottom) portion of the member's cross section, above (or below) the section plane where *t* is measured, and $\overline{y}'$ is the distance from the neutral axis to the centroid of $A'$

# SHEAR IN BEAMS

## Rectangular cross section

- Shear –stress distribution is parabolic

Width/depth
b/h ratio =0.5

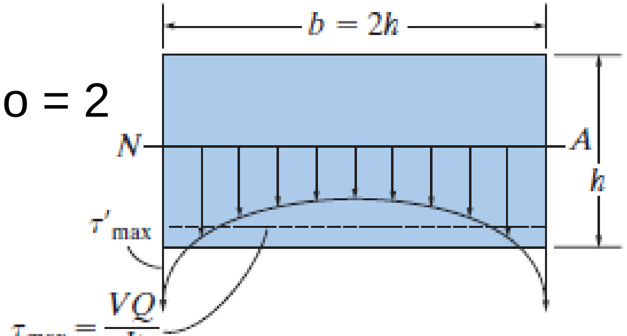


(a)

$$\tau = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right)$$

$$\tau_{\max} = 1.5\frac{V}{A}$$

b/h ratio = 2

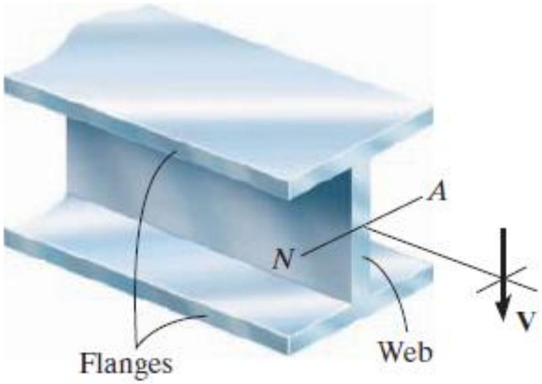


(b)

$b/h = 0.5$, $\tau'_{\max}$ is only about 3% greater than the shear stress calculated from the shear formula

For *flat sections*, e.g $b/h = 2$,
$\tau'_{\max}$ is about 40% greater than $\tau_{\max}$

# SHEAR IN BEAMS (cont)



## Wide-flange beam

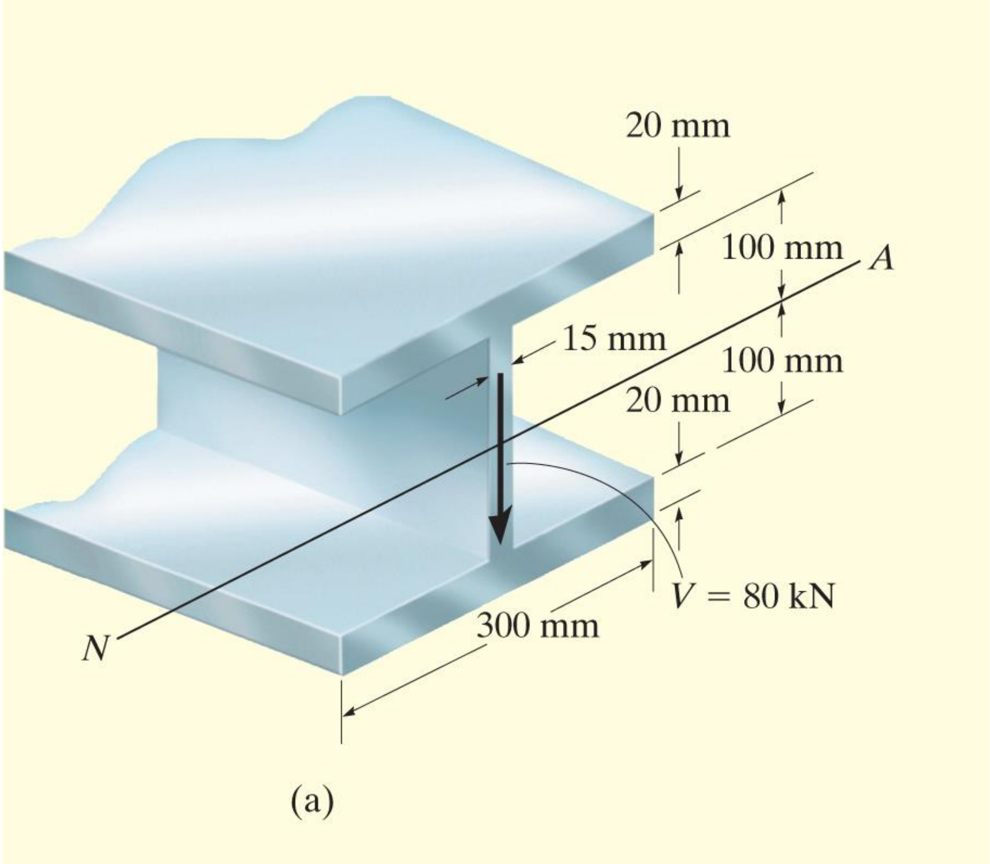
- Shear-stress distribution is parabolic but has a jump at the flange-to-web junctions.

## Limitations on the use of shear formula

- Not on cross sections that are short or flat
- Not at points of sudden cross sectional changes (e.g. flange-to-web junction in wide flange beam)
- Not at a joint on an inclined boundary

# EXAMPLE 1

A steel wide-flange beam has the dimensions shown in Fig. *a*. If it is subjected to a shear of V = 80kN, plot the shear-stress distribution acting over the beam's cross-sectional area



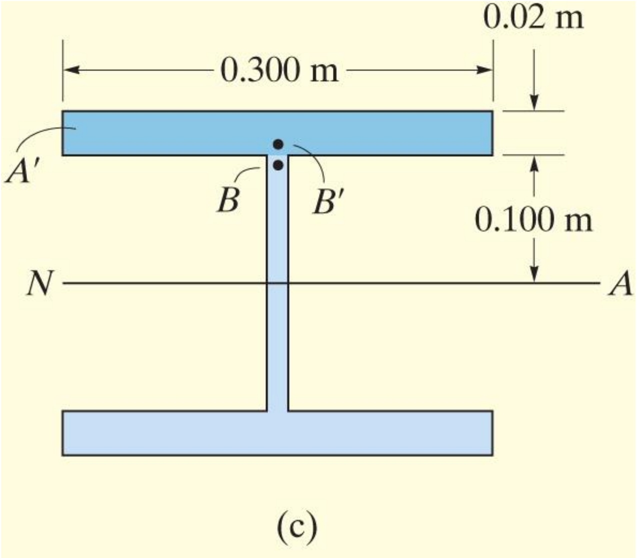
(a)

# EXAMPLE 1 (cont)

## Solutions

- The moment of inertia of the cross-sectional area about the neutral axis is

$$I = \left[\frac{1}{12}(0.015)\left(0.2^3\right)\right] + 2\left[\frac{1}{12}(0.3)\left(0.02^3\right) + (0.3)(0.02)\left(0.11^2\right)\right] = 155.6\left(10^{-6}\right)\text{m}^4$$



(c)

- For point B, $t_{B'}$ = 0.3m, and A' is the dark shaded area shown in Fig. *c*

$$Q_{B'} = \bar{y}' A' = [0.11](0.3)(0.02) = 0.66\left(10^{-3}\right)\text{m}^3$$

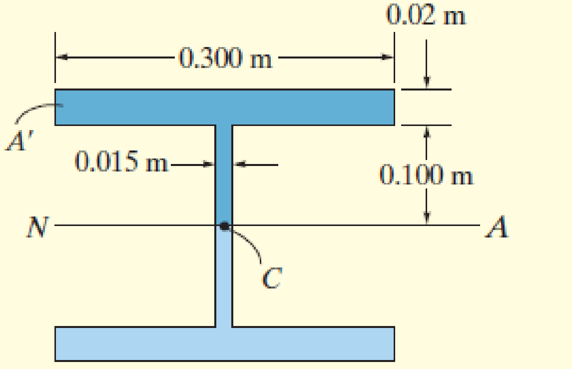
$$\tau_{B'} = \frac{VQ_{B'}}{It_{B'}} = \frac{80\left(10^3\right)0.66\left(10^{-3}\right)}{155.6\left(10^{-6}\right)(0.3)} = 1.13\,\text{MPa}$$

# EXAMPLE 1 (cont)

## Solutions

- For point B, $t_B$ = 0.015m, and $Q_B = Q_{B'}$,

$$\tau_B = \frac{VQ_B}{It_B} = \frac{80\left(10^3\right)0.66\left(10^{-3}\right)}{155.6\left(10^{-6}\right)(0.015)} = 22.6\,\text{MPa}$$



(d)

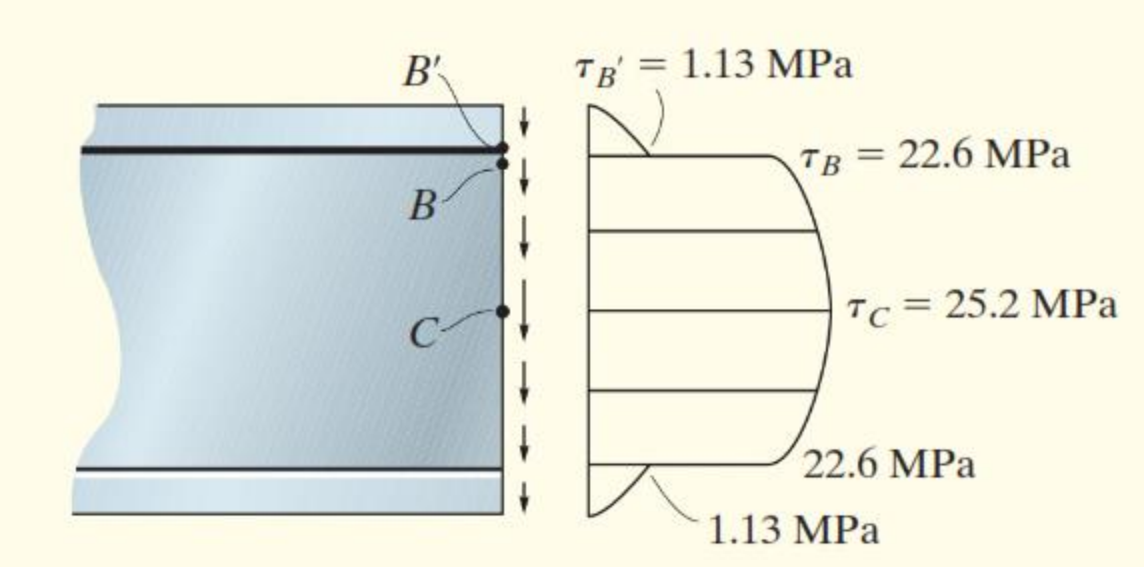
- For point C, $t_C$ = 0.015m, and A’ is the dark shaded area in Fig. *d*.

- Considering this area to be composed of two rectangles,

$$Q_C = \sum \bar{y}' A' = (0.11)(0.3)(0.02) + (0.05)(0.015)(0.1) = 0.735\left(10^{-3}\right)\text{m}^3$$

- Thus, $\tau_C = \tau_{\max} = \dfrac{VQ_C}{It_C} = \dfrac{80\left(10^3\right)(0.735)\left(10^{-3}\right)}{155.6\left(10^{-6}\right)(0.015)} = 25.2\,\text{MPa}$

# EXAMPLE 1 (cont)



- Note that the largest shear stress occurs in the web and is almost uniform throughout its depth, varying from 22.6 MPa to 25.2 Mpa

- It is for this reason that for design, some codes permit the use of calculating the *average shear stress on the cross section of the web, rather* than using the shear formula; that is,

$$\tau_{\text{avg}} = \frac{V}{A_{\text{w}}} = \frac{80(10^3)\,\text{N}}{(0.015\,\text{m})(0.2\,\text{m})} = 26.7\,\text{MPa}$$

# SHEAR FLOW IN BUILT-UP BEAM

- Occasionally in engineering practice, members are “***built up***” from several composite parts in order to achieve a greater resistance to loads.

- An example is shown Below. If the loads cause the members to bend, fasteners such as nails, bolts, welding material, or glue will be needed to keep the component parts from sliding relative to one another

- This loading, when measured as a force per unit length of beam, is referred to as ***shear flow, q***.*

- The magnitude of the shear flow is obtained using a procedure similar to that for finding the shear stress in a beam

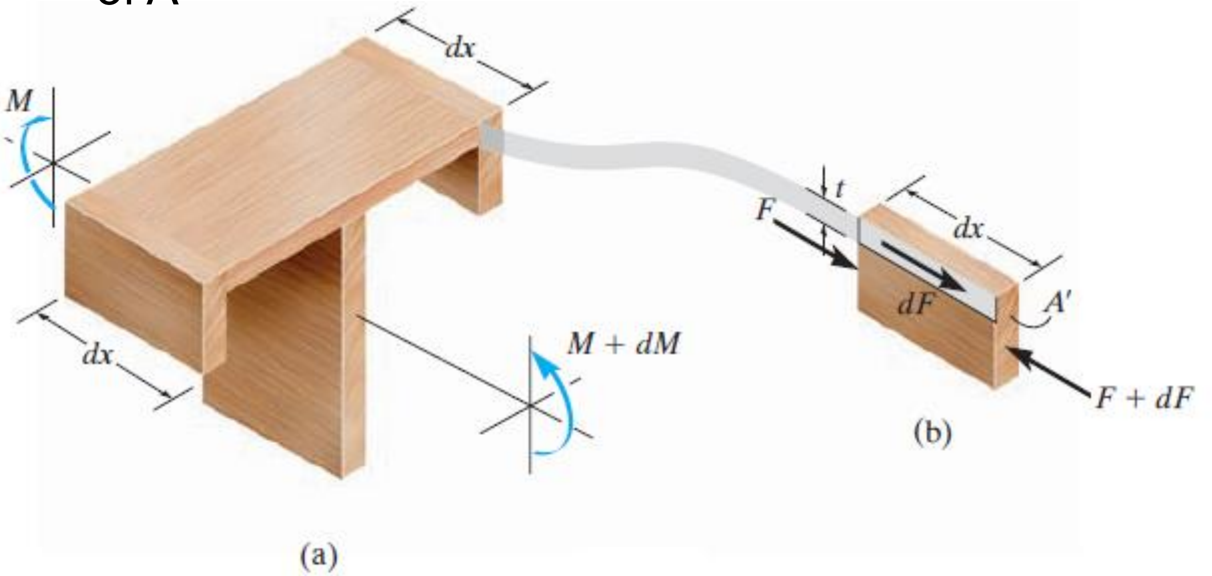
# SHEAR FLOW IN BUILT-UP BEAM

- Shear flow ≡ shear force per unit length along longitudinal axis of a beam.

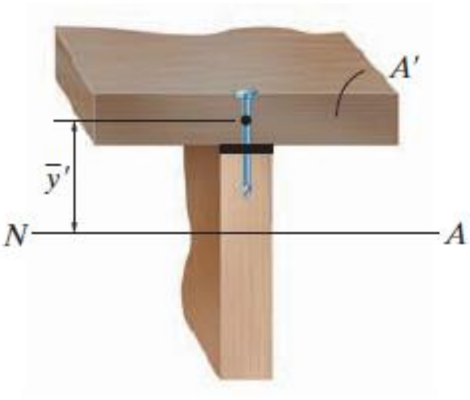
$$q = \frac{VQ}{I}$$

q = shear flow
V = internal resultant shear
I = moment of inertia of the *entire* cross-sectional area

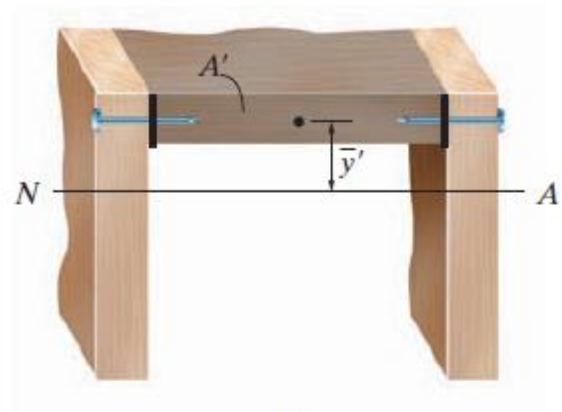
$Q = \bar{y}'A'$, where *A'* is the cross-sectional area of the segment that is *connected to the beam* at the juncture where the shear flow is calculated, and $\bar{y}'$ is the distance from the neutral axis to the centroid of A'



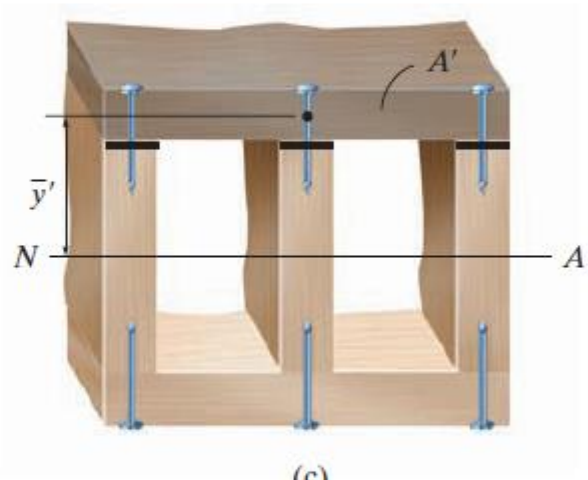
# SHEAR FLOW IN BUILT-UP BEAM (cont)



(a)



(b)

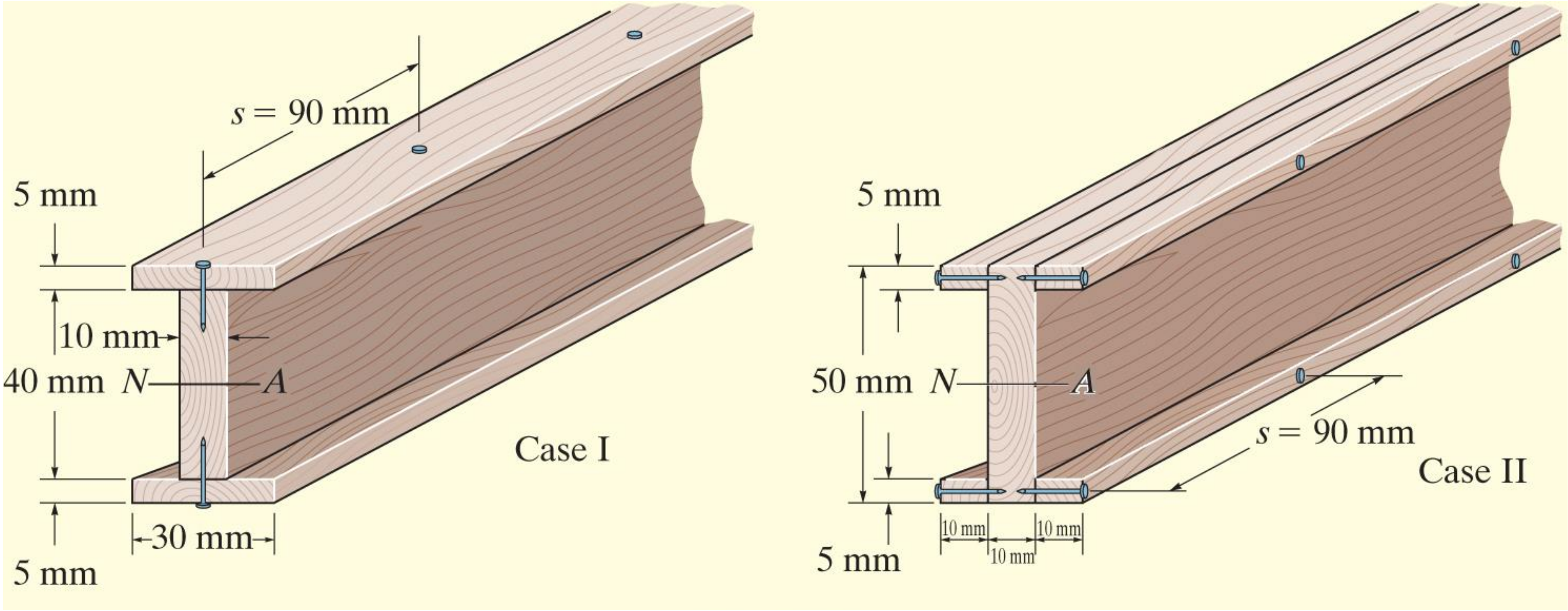


(c)

- The shaded segments connected to built-up beams by fasteners are shown in Fig. above
- The shear flow here must be found at the thick black line, and is determined by using a value of *Q calculated* from *A' and* $\bar{y}'$ *indicated in each figure.*
- *This value of q will be resisted by* a *single fastener in Fig. a, by two fasteners in b, and by three* fasteners in Fig. *c.*

# EXAMPLE 2

Nails having a total shear strength of 40 N are used in a beam that can be constructed either as in Case I or as in Case II, Fig. below. If the nails are spaced at 90 mm, determine the largest vertical shear that can be supported in each case so that the fasteners will not fail.



# EXAMPLE 2 (cont)

## Solutions

- Since the cross section is the same in both cases, the moment of inertia about the neutral axis is

$$I = \frac{1}{12}(30)(50^3) - 2\left[\frac{1}{12}(10)(40^3)\right] = 205833\,\text{mm}^4$$

**Case I**

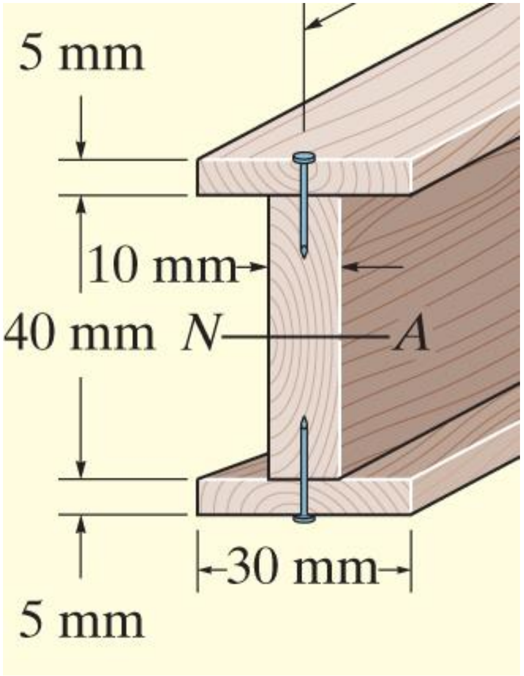
- For this design a single row of nails holds the top or bottom flange onto the web.
- For one of these flanges,

$$Q = \overline{y}'A' = (22.5)(30)(5) = 3375\,\text{mm}^3$$

$$q = \frac{VQ}{I}$$

$$\frac{40}{90} = \frac{V(3375)}{205833}$$

$$V = 27.1\,\text{N} \quad (\text{Ans})$$



# EXAMPLE 2 (cont)

## Solutions

**Case II**

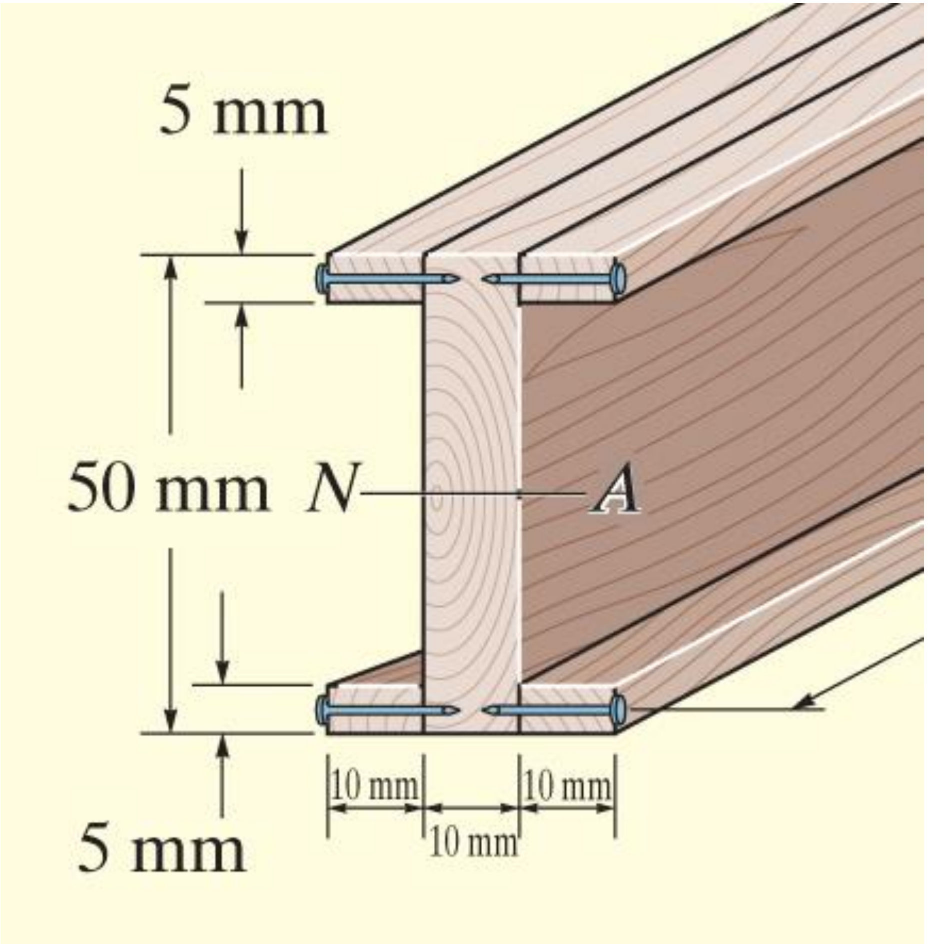
- Here a single row of nails holds one of the side boards onto the web.
- Thus,

$$Q = \bar{y}'A' = (22.5)(10)(5) = 1125\,\mathrm{mm}^3$$

$$q = \frac{VQ}{I}$$

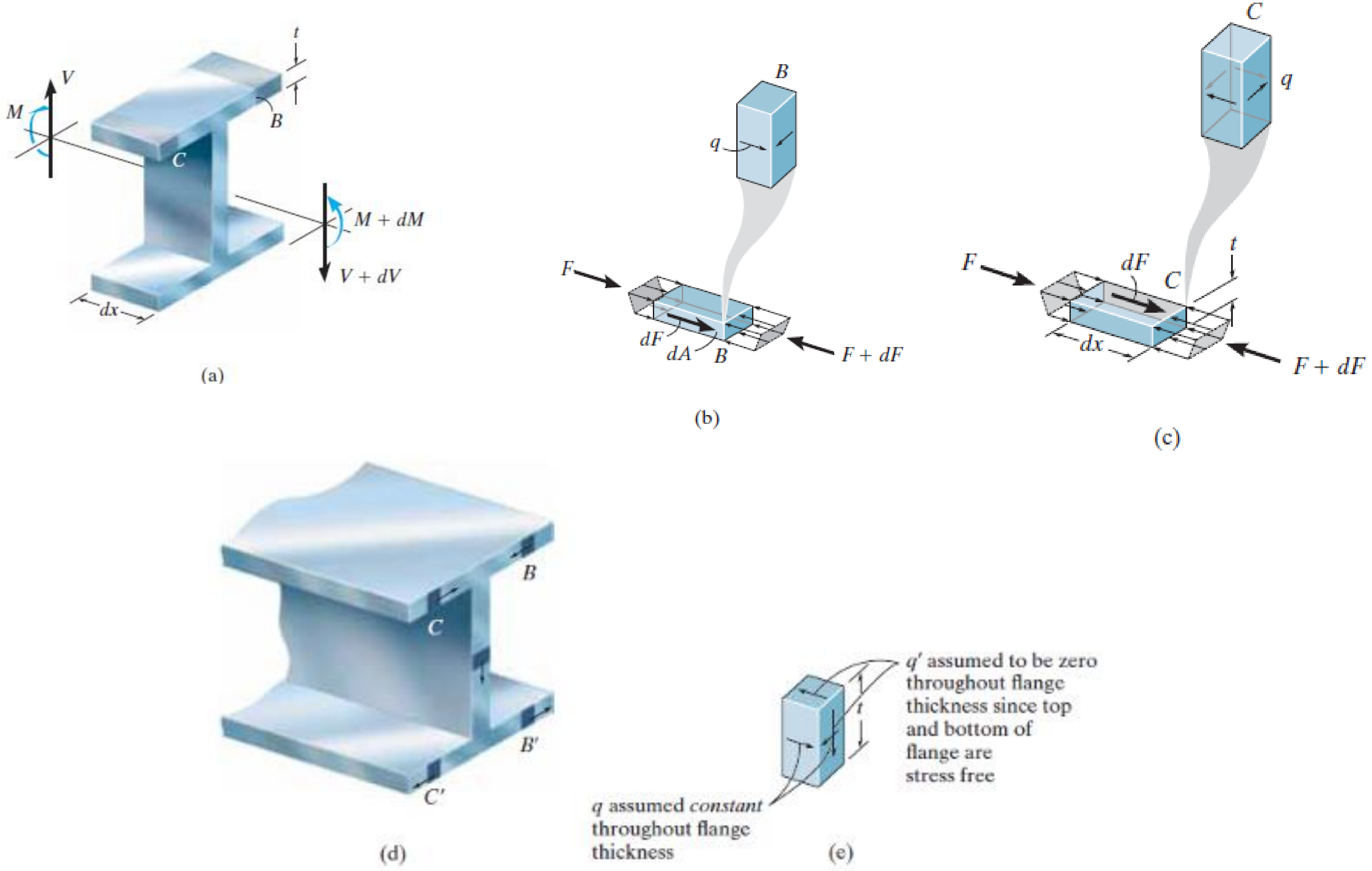
$$\frac{40}{90} = \frac{V(1125)}{205833}$$

$$V = 81.3\,\mathrm{N} \quad (\mathrm{Ans})$$



# SHEAR FLOW IN THIN-WALLED BEAM

- Approximation: only the shear-flow component that acts parallel to the walls of the member will be counted.



(a) (b) (c) (d) (e)

q′ assumed to be zero throughout flange thickness since top and bottom of flange are stress free

q assumed *constant* throughout flange thickness

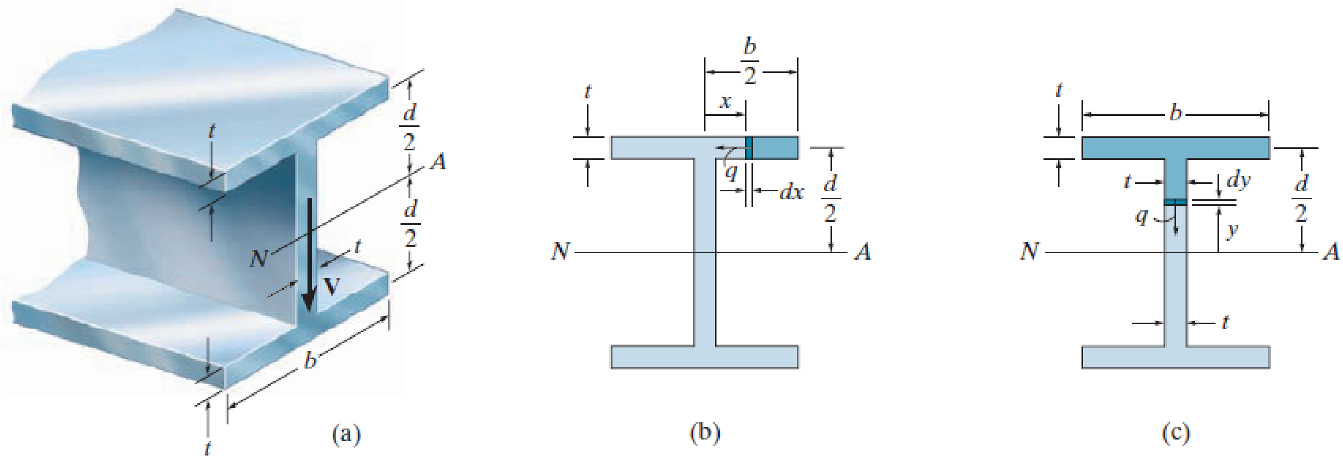
# SHEAR FLOW IN THIN-WALLED BEAM (cont)

- In horizontal flanges, flow varies linearly,

$$q = \frac{VQ}{I} = \frac{V[d/2]((b/2)-x)t}{I} = \frac{Vtd}{2I}\left(\frac{b}{2}-x\right)$$

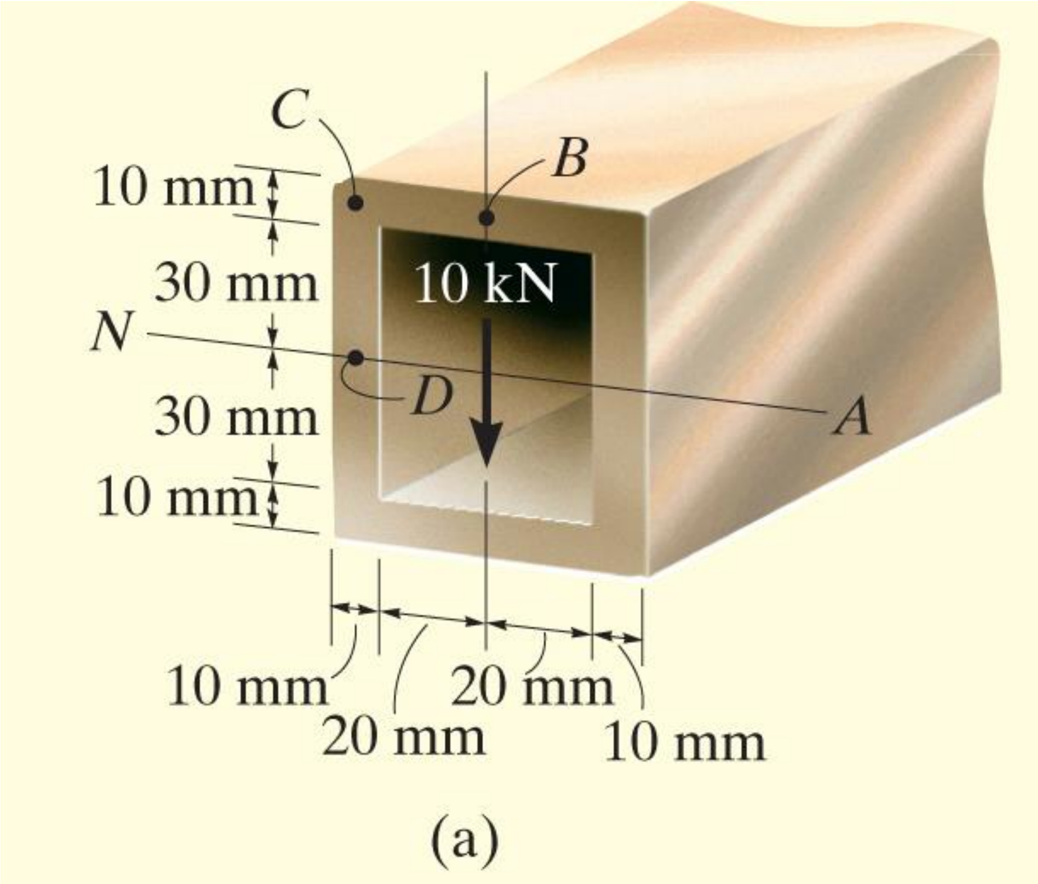
- In vertical web(s), flow varies parabolically,

$$q = \frac{VQ}{I} = \frac{Vt}{I}\left[\frac{db}{2}+\frac{1}{2}\left(\frac{d^2}{4}-y^2\right)\right]$$



# EXAMPLE 3

The thin-walled box beam in Fig. *a* is subjected to a shear of 10 kN. Determine the variation of the shear flow throughout the cross section.



(a)

# EXAMPLE 3 (cont)

## Solutions

- The moment of inertia is $I = \frac{1}{12}(6)(8)^3 - \frac{1}{12}(4)(6)^3 = 184\,\text{mm}^4$
- For point *B*, the area $A' \approx 0$ thus $q'_B = 0$.
- Also,

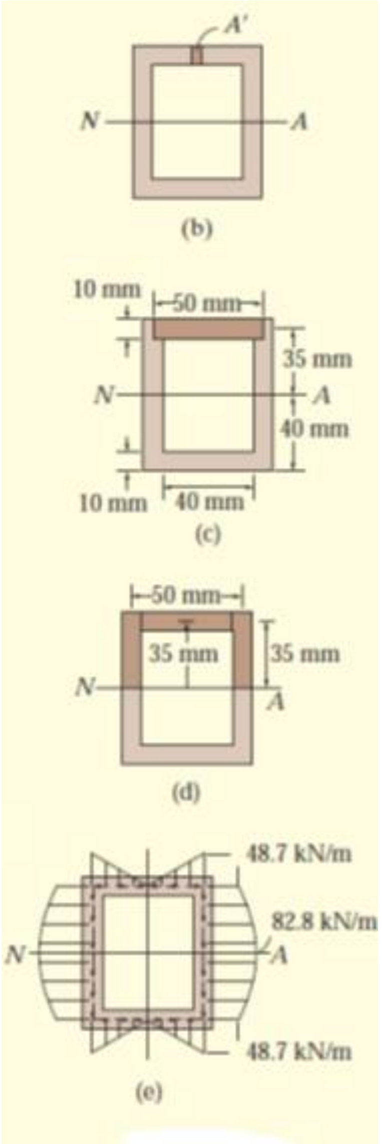
$$Q_C = \bar{y}A' = (3.5)(5)(1) = 17.5\,\text{cm}^3$$
$$Q_D = \sum \bar{y}A' = 2(2)(1)(4) = 30\,\text{cm}^3$$

- For point *C*,

$$q_C = \frac{VQ_C}{I} = \frac{10(17.5/2)}{184} = 0.951\,\text{kN/cm} = 91.5\,\text{N/mm}$$

- The shear flow at *D* is

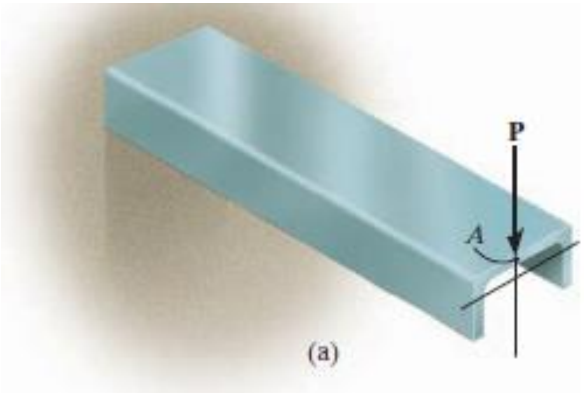
$$q_D = \frac{VQ_D}{I} = \frac{10(30/2)}{184} = 1.63\,\text{kN/cm} = 163\,\text{N/mm}$$



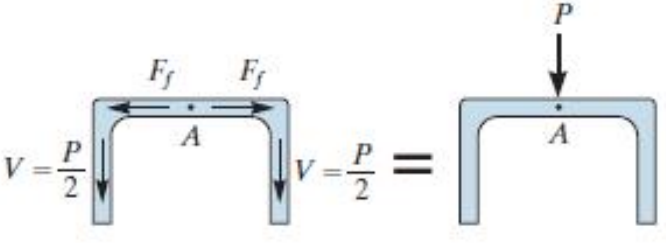
# SHEAR CENTRE

- Shear center is the point through which a force can be applied which will cause a beam to bend and yet not twist.

- The location of the shear center is only a function of geometry of the cross section and does not depend upon the applied load.



(a)



(b)

$$e = \frac{F_f d}{P}$$

# SHEAR CENTRE

- The reason the member twists has to do with the shear-flow distribution along the channel's flanges and web, Fig. *b*.

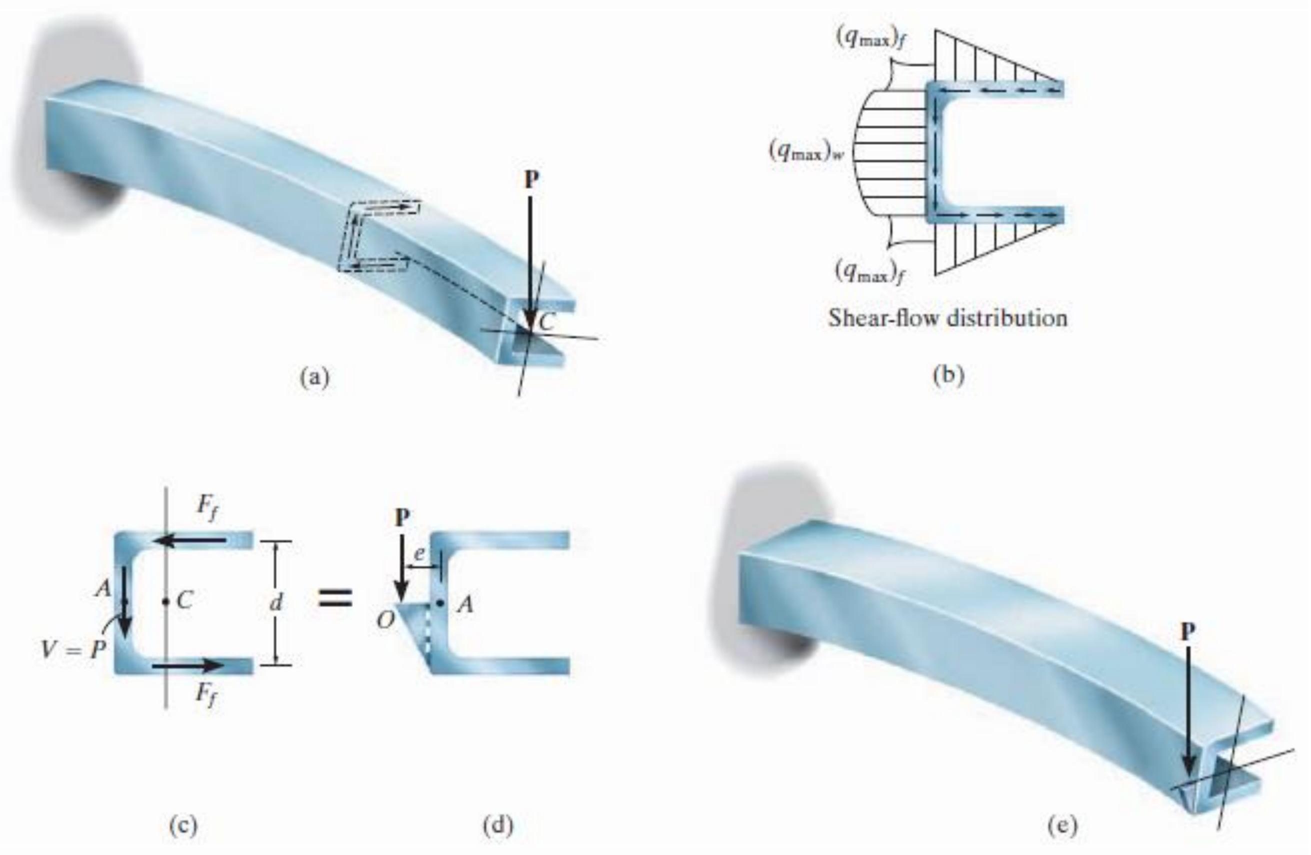
- *When this distribution is* integrated over the flange and web areas, it will give resultant forces of $F_f$ in each flange and a force of $V = P$ *in the web, Fig. c.*

- *If the* moments of these three forces are summed about point *A*, *the unbalanced* couple or torque created by the flange forces is seen to be responsible for twisting the member.

- The actual twist is clockwise when viewed from the front of the beam, as shown in Fig. *a*, *because reactive internal* "equilibrium" forces $F_f$ cause the twisting.

- In order to *prevent this* twisting and therefore cancel the unbalanced moment, it is necessary to apply **P at a point *O located an eccentric distance e from the web, as*** shown in Fig. *d*. Point *O* is called the *SHEAR CENTRE*

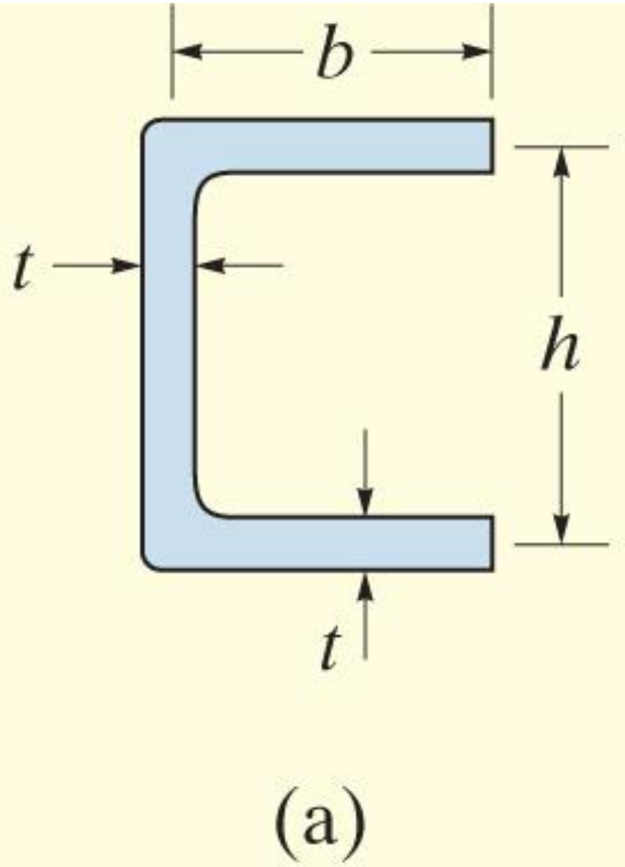
# SHEAR CENTRE (cont)



$$e = \frac{F_f d}{P}$$

# EXAMPLE 4

Determine the location of the shear center for the thin-walled channel section having the dimensions shown in Fig. *a*.



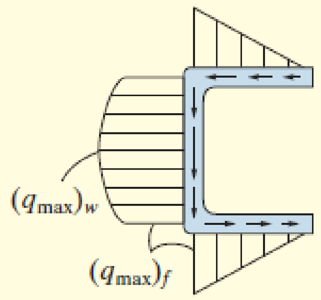
(a)

# EXAMPLE 4 (cont)

## Solutions

- The cross-sectional area can be divided into three component rectangles—a web and two flanges.

$$I = \frac{1}{12} t h^3 + 2\left[ b t \left(\frac{h}{2}\right)^2 \right] = \frac{t h^2}{2}\left(\frac{h}{6} + b\right)$$



Shear flow distribution

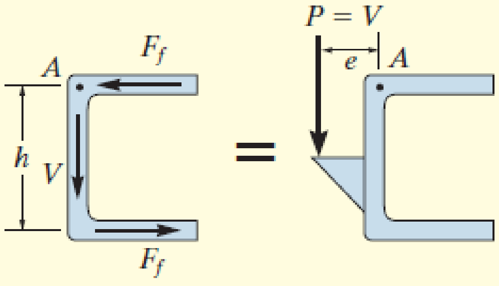
(b)

- *q* at the arbitrary position *x* is

$$q = \frac{VQ}{I} = \frac{V(h/2)[b-x]t}{(th^2/2)[(h/6)+b]} = \frac{V(b-x)}{h[(h/6)+b]}$$

- Hence, the force is

$$F_f = \int_0^b q\,dx = \frac{V}{h[(h/6)+b]}\int_0^b (b-x)\,dx = \frac{Vb^2}{2h[(h/6)+b]}$$



(c)

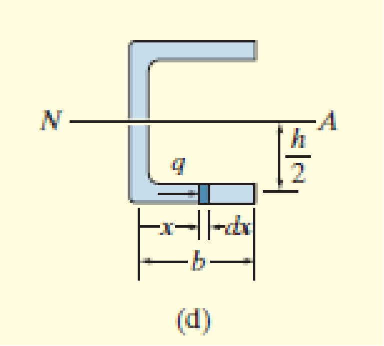
# EXAMPLE 4 (cont)

## Solutions

- Summing moments about point *A*, Fig. *c*, we require

$$Ve = F_f h = \frac{Vb^2 h}{2h\left[(h/6)+b\right]}$$

$$e = \frac{b^2}{\left[(h/3)+2b\right]} \quad \text{(Ans)}$$

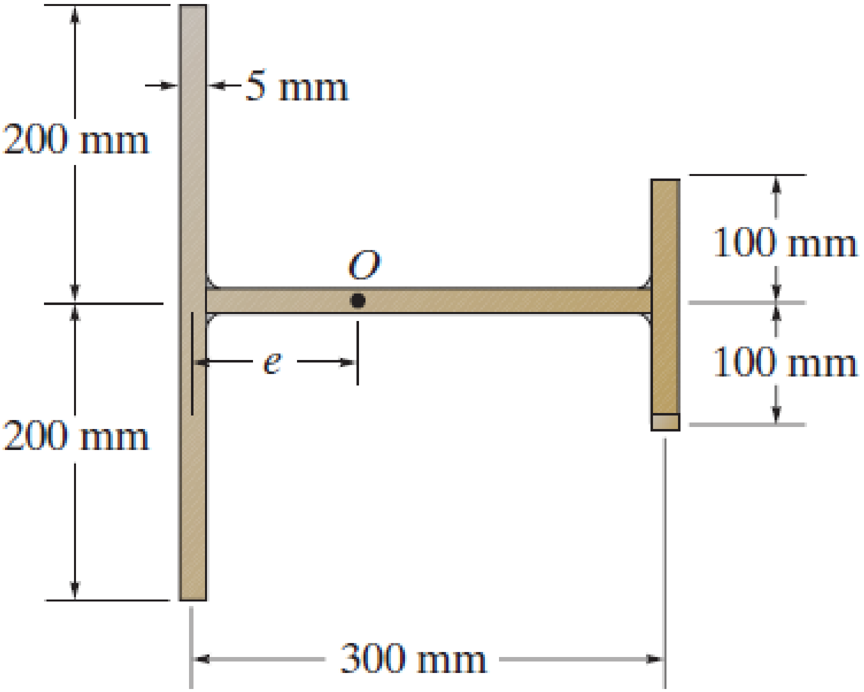


(d)

- As stated previously, *e* depends only on the geometry of the cross section.

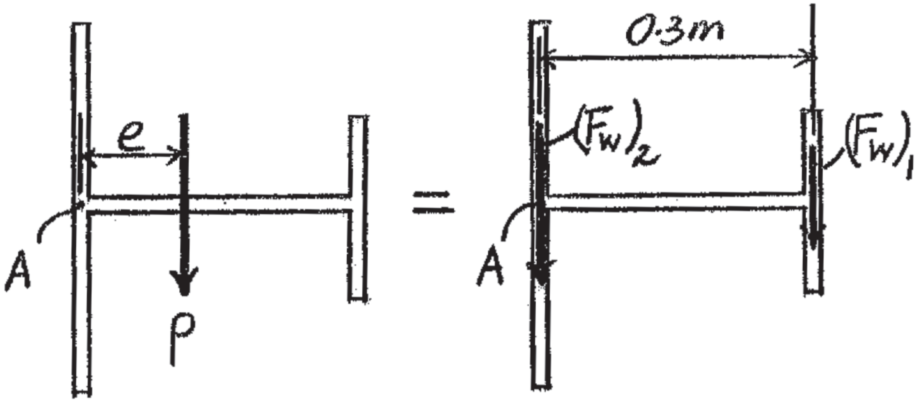
# EXAMPLE 5

The built-up beam is formed by welding together the thin plates of thickness 5 mm. Determine the location of the shear center *O*

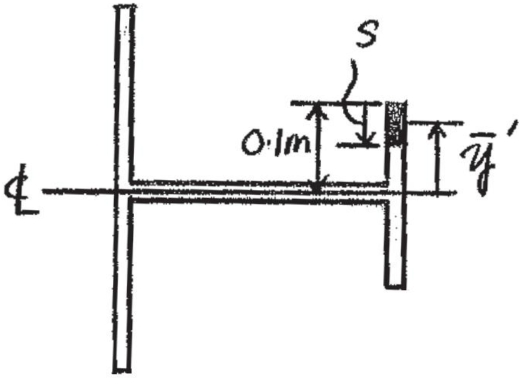


# EXAMPLE 5 Cont'd

**Shear Center:** Referring to Fig. *a* and summing moments about point *A*, we have



(a)



(b)

$$\circlearrowleft + \Sigma (M_R)_A = \Sigma M_A; \qquad -Pe = -(F_w)_1(0.3)$$

$$e = \frac{0.3(F_w)_1}{P} \qquad \text{Eq. (1)}$$

**Section Properties:** The moment of inertia of the cross section about the axis of symmetry is

$$I = \frac{1}{12}(0.005)(0.4^3) + \frac{1}{12}(0.005)(0.2^3) = 30(10^{-6})\ \mathrm{m}^4$$

# EXAMPLE 5 Cont'd

Referring to Fig. $b$, $\bar{y}' = (0.1 - s) + \dfrac{s}{2} = (0.1 - 0.5s)$ m. Thus, Q as a function of $s$ is

$$Q = \bar{y}'A' = (0.1 - 0.5s)(0.005s) = [0.5(10^{-3})s - 2.5(10^{-3})s^2]\ \text{m}^3$$

**Shear Flow:**

$$q = \frac{VQ}{I} = \frac{P[0.5(10^{-3})s - 2.5(10^{-3})s^2]}{30(10^{-6})} = P(16.6667s - 83.3333s^2)$$

**Resultant Shear Force:**

$$(F_w)_1 = 2\int_0^{0.1\text{ m}} q\,ds = 2\int_0^{0.1\text{ m}} P(16.6667s - 83.3333s^2)ds = 0.1111P$$

Substituting this result into Eq. (1),

***e* = 0.03333 m = 33.3 mm**

# STRESS CAUSED BY COMBINED LOADINGS

- So far, we’ve determined the stress in a member subjected to either an internal axial force, a shear force, a bending moment, or a torsional moment.

- Most often, however, the cross-section of a member will be subjected to several of these loadings simultaneously, and when this occurs, then the method of superposition should be used to determine the resultant stress.

- The following procedure for analysis provides a method for doing this

# REVIEW OF STRESS ANALYSES

- **Normal force P** leads to:

$$\textit{uniform normal stress},\ \sigma = \frac{P}{A}$$

- **Shear force V** leads to:

$$\textit{shear} - \textit{stress distribution},\ \tau = \frac{VQ}{It}$$

- **Bending moment M** leads to:

$$\textit{longitudinal stress distribution},\ \sigma = -\frac{My}{I}\ (\text{for straight beam})$$

$$\text{or } \sigma = \frac{My}{\left[Ae\left(R-y\right)\right]}\ (\text{for curved beam})$$

# REVIEW OF STRESS ANALYSES (cont)

- **Torsional moment T** leads to:

$$shear - stress\ distribution, \tau = \frac{T\rho}{J} \quad \text{(for circular shaft)}$$

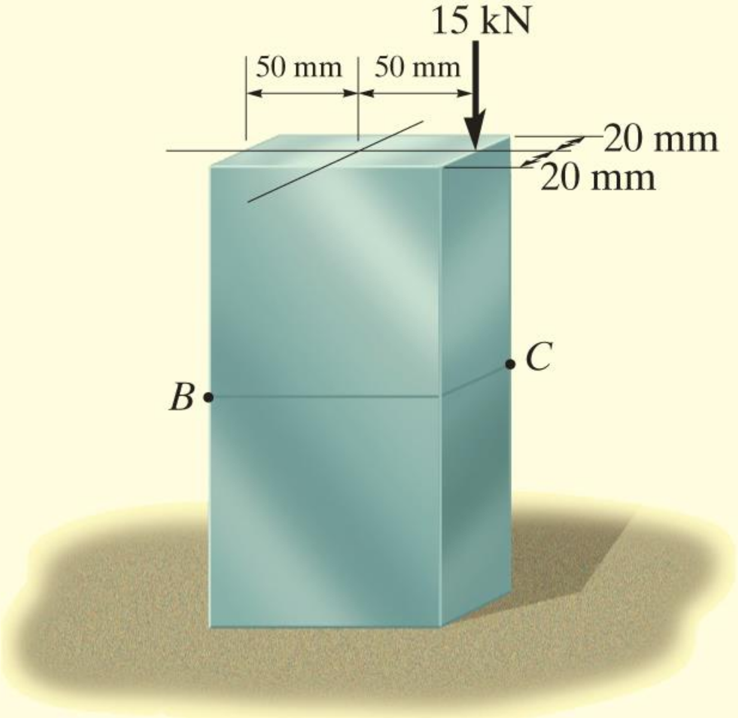
$$\tau = \frac{T}{(2A_m t)} \quad \text{(for closed thin−walled tube)}$$

- **Resultant stresses by superposition:**

  Once the normal and shear stress components for each loading have been calculated, use the principal of superposition to determine the resultant normal and shear stress components.

# EXAMPLE 1

A force of 15 kN is applied to the edge of the member shown below. Neglect the weight of the member and determine the state of stress at points *B* and *C*.



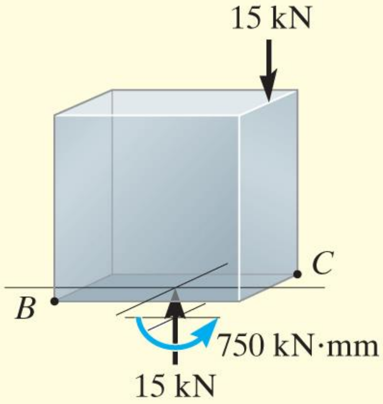
# EXAMPLE 1 (cont)

## Solutions

- For equilibrium at the section there must be an axial force of 15 000 N acting through the centroid and a bending moment of 750 000 N.mm about the centroidal or principal axis.

**Normal force:**

$$\sigma = \frac{P}{A} = \frac{15000}{(100)(40)} = 3.75\,\text{MPa}$$



(b)

**Bending Moment:**

- The maximum stress is

$$\sigma_{\max} = \frac{Mc}{I} = \frac{75000(50)}{\frac{1}{12}(40)(100)^3} = 11.25\,\text{MPa}$$

# EXAMPLE 1 (cont)

**Superposition:**

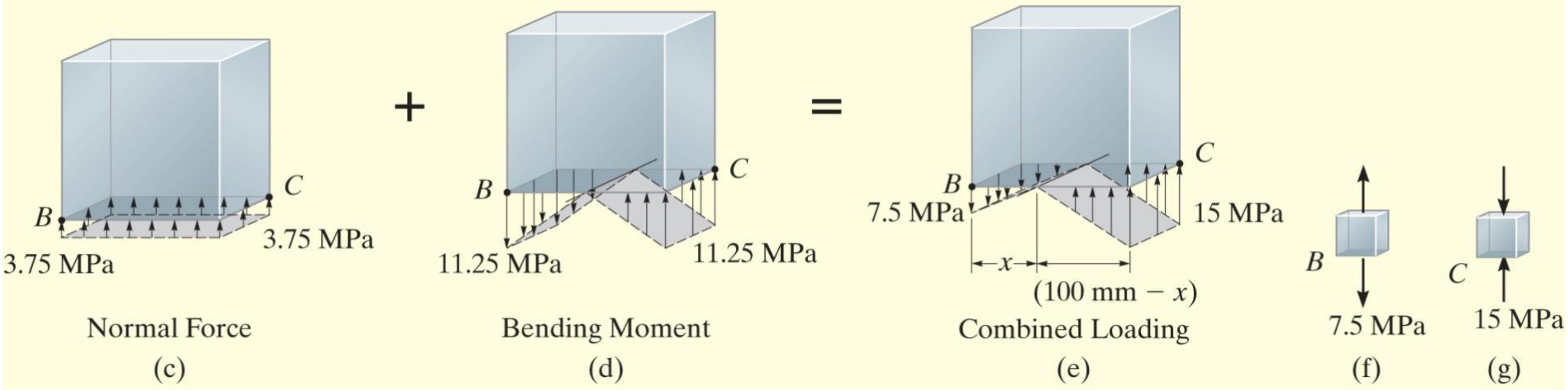
- The location of the line of zero stress can be determined by proportional triangles

$$\frac{7.5}{x} = \frac{15}{(100 - x)}$$

$$x = 33.3 \text{ mm}$$

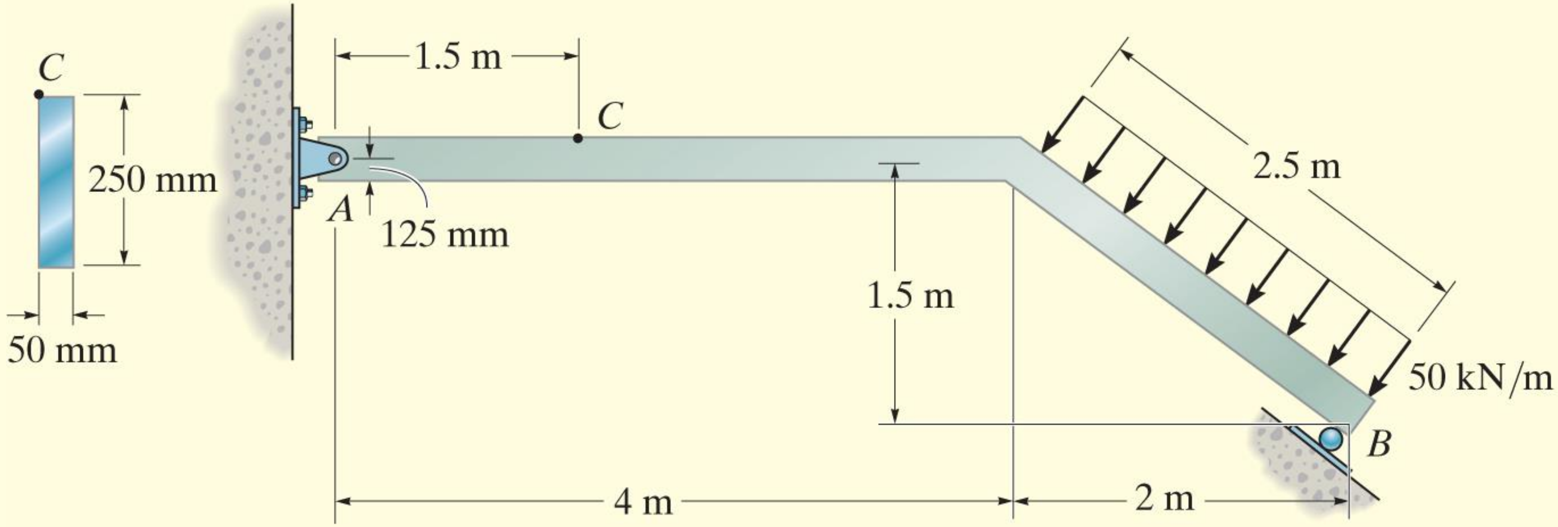
- Elements of material at *B* and *C* are subjected only to normal or *uniaxial stress.*

$$\sigma_B = -3.75 + 11.25 = 7.5 \text{ MPa} \quad \text{(tension)} \quad \text{(Ans)}$$
$$\sigma_C = -3.75 - 11.25 = -15 \text{ MPa} \quad \text{(compression)} \quad \text{(Ans)}$$



# EXAMPLE 2

The member shown below has a rectangular cross section. Determine the state of stress that the loading produces at point *C*.



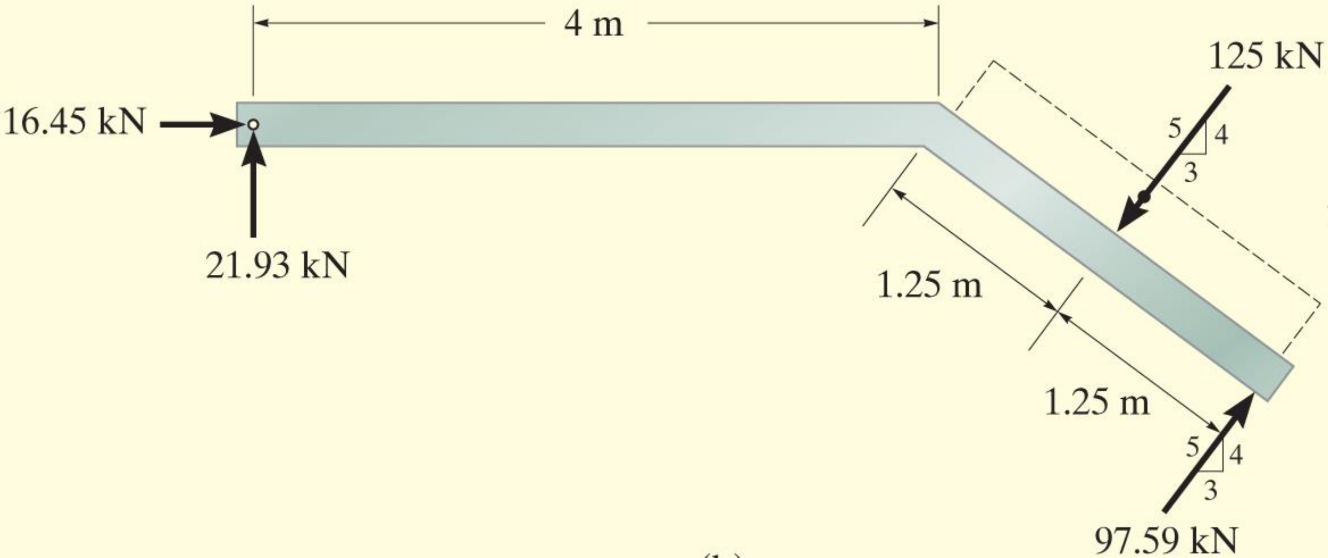
(a)

# EXAMPLE 2 (cont)

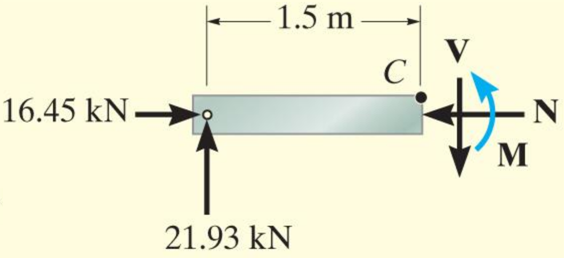
## Solutions

- The resultant internal loadings at the section consist of a normal force, a shear force, and a bending moment.

- Solving, $N = 16.45\,\text{kN},\ V = 21.93\,\text{kN},\ M = 32.89\,\text{kN}$



(b)

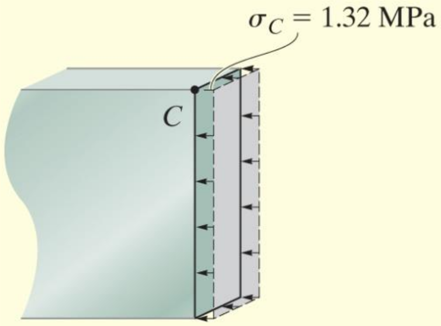


(c)

# EXAMPLE 2 (cont)

## Solutions

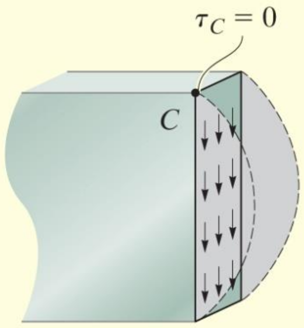
- The uniform normal-stress distribution acting over the cross section is produced by the normal force.
- At Point C, $\sigma_c = \dfrac{P}{A} = \dfrac{16.45\left(10^3\right)}{(0.05)(0.25)} = 1.32\,\text{MPa}$
- In Fig. *e*, the shear stress is zero.



Normal Force

(d)

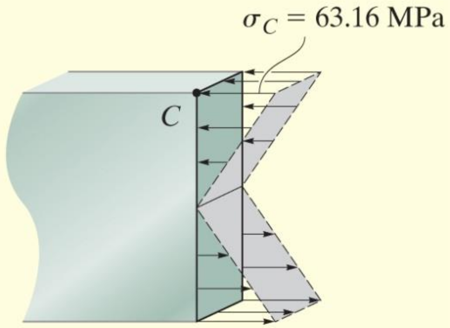
+



Shear Force

(e)

+



Bending Moment

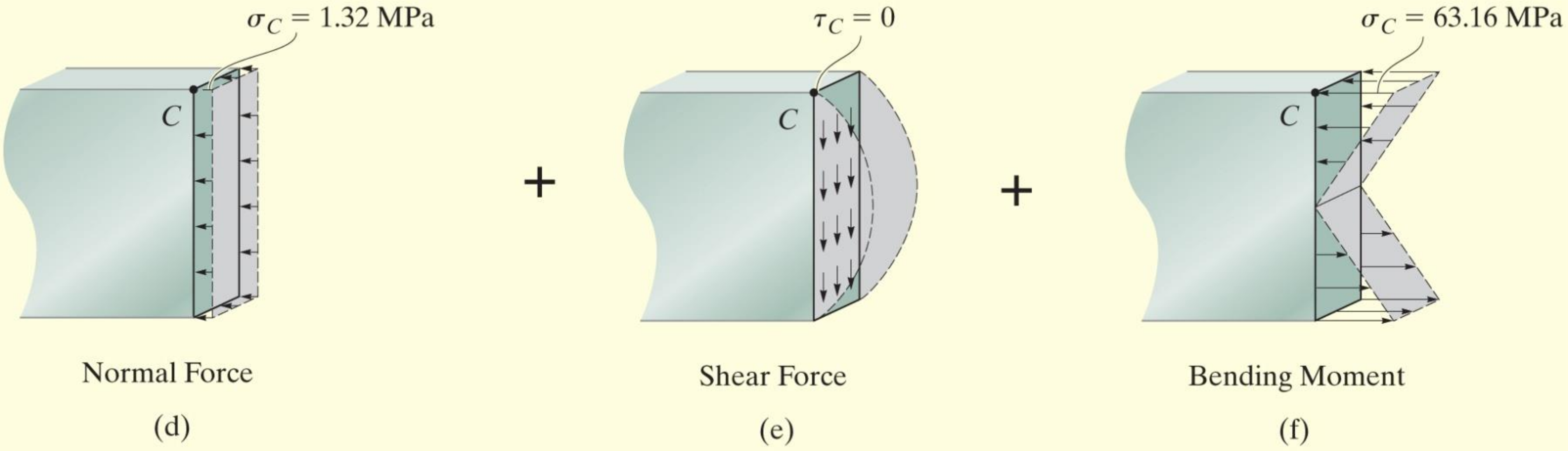
(f)

# EXAMPLE 2 (cont)

## Solutions

- Point *C* is located at y = c = 0.125m from the neutral axis, so the normal stress at *C*, Fig. *f*, is

$$\sigma_c = \frac{Mc}{I} = \frac{\left(32.89\left(10^3\right)\right)\left(0.125\right)}{\left[\frac{1}{2}\left(0.05\right)\left(0.25\right)^3\right]} = 63.16\,\text{MPa}$$



Normal Force
(d)

Shear Force
(e)

Bending Moment
(f)

# EXAMPLE 2 (cont)

## Solutions

- The shear stress is zero.

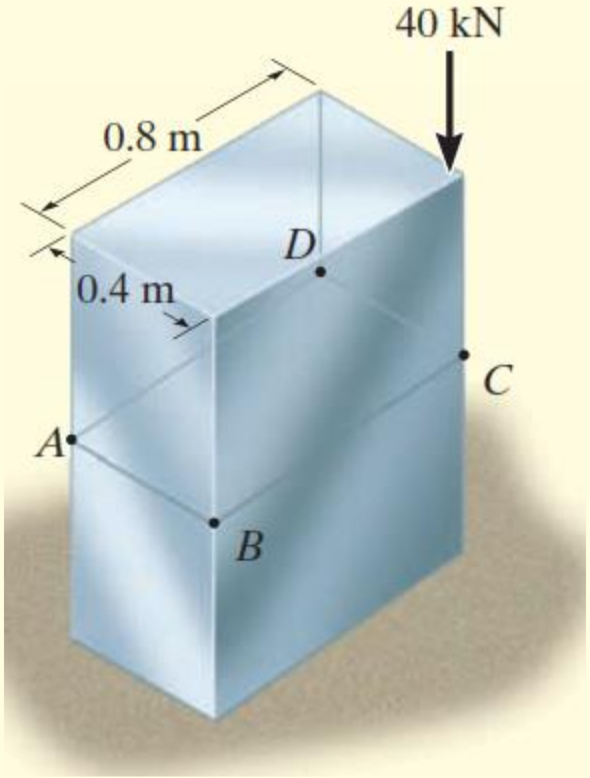
- Adding the normal stresses determined above gives a compressive stress at *C* having a value of

$$\sigma_c = \frac{Mc}{I} = 1.32 + 63.16 = 64.5\,\text{MPa}$$

(g)

# EXAMPLE 3

The rectangular block below of negligible weight is subjected to a vertical force of 40 kN, which is applied to its corner. Determine the largest normal stress acting on a section through *ABCD*.



# EXAMPLE 3 (cont)

## Solutions

- For uniform normal-stress distribution the stress is

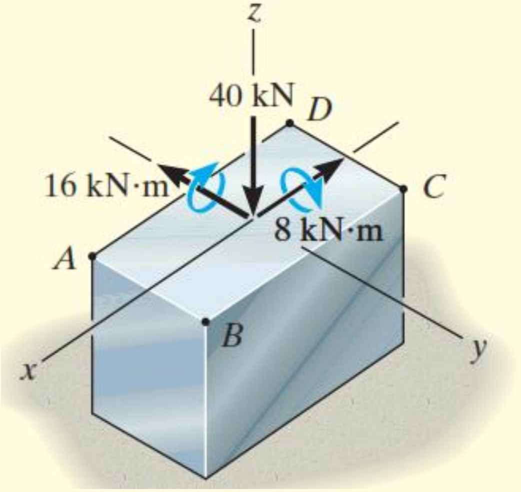
$$\sigma = \frac{P}{A} = \frac{40}{(0.8)(0.4)} = 125\,\text{kPa}$$

- For 8 kN, the maximum stress is

$$\sigma_{\max} = \frac{M_x c_x}{I_x} = \frac{8(0.2)}{\left[\frac{1}{12}(0.8)(0.4)^3\right]} = 375\,\text{kPa}$$

- For 16 kN, the maximum stress is

$$\sigma_{\max} = \frac{M_y c_x}{I_y} = \frac{16(0.4)}{\left[\frac{1}{12}(0.4)(0.8)^3\right]} = 375\,\text{kPa}$$



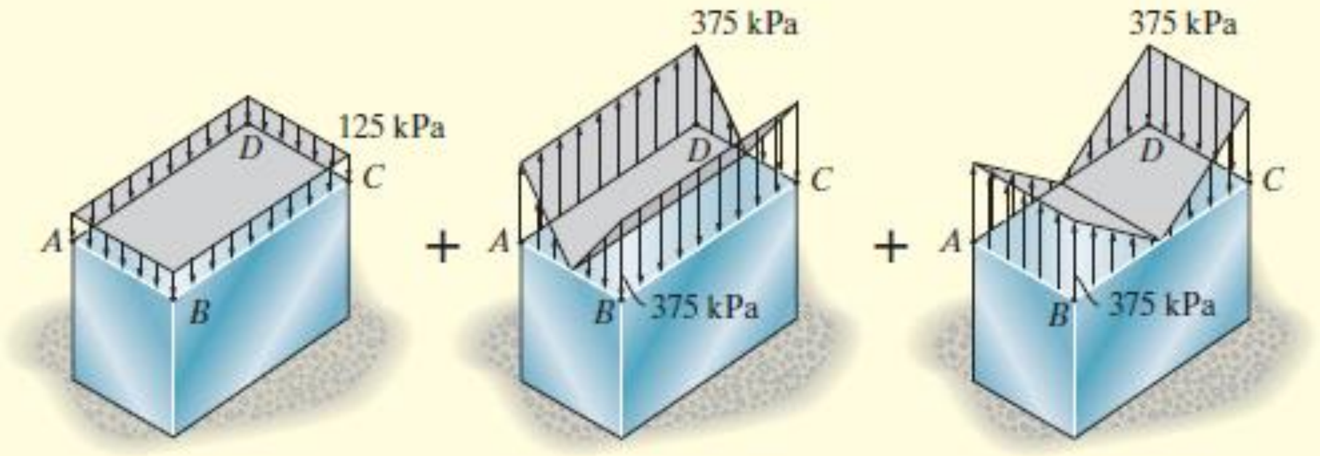
*Equilibrium of the bottom segment*

# EXAMPLE 3 (cont)

## Solutions

- By inspection the normal stress at point *C* is the largest since each loading creates a compressive stress there

$$\sigma_c = -125 - 375 - 375 = -875\,\text{kPa} \quad (\text{Ans})$$



Normal force (40 kN)

(c)

Bending moment (8 kN·m)

(d)

Bending moment (16 kN·m)

(e)