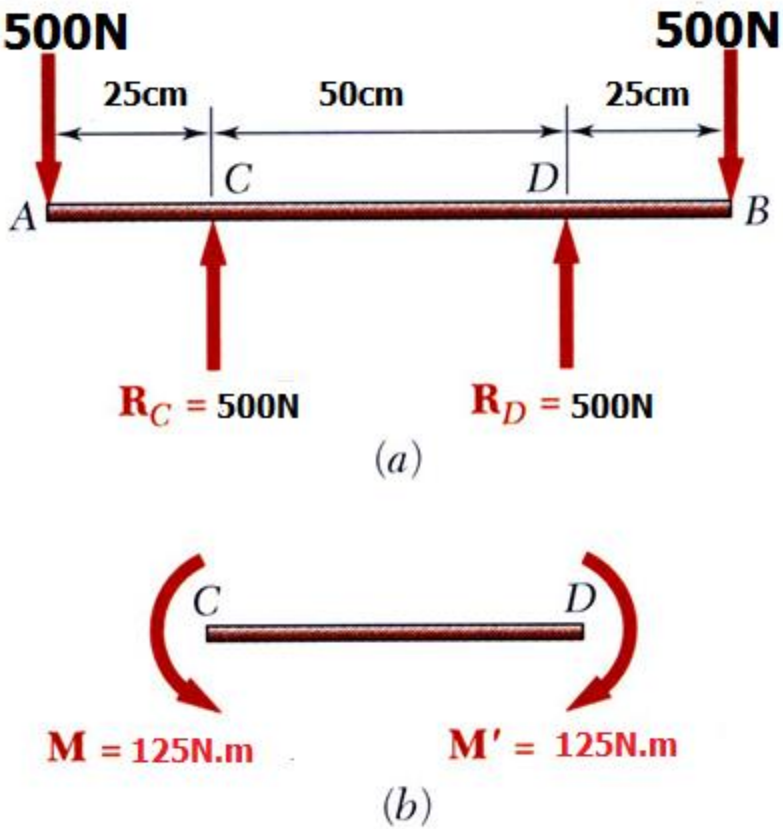
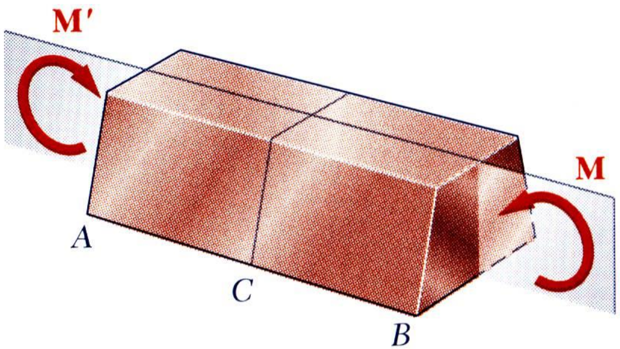
Theory of Bending Stress

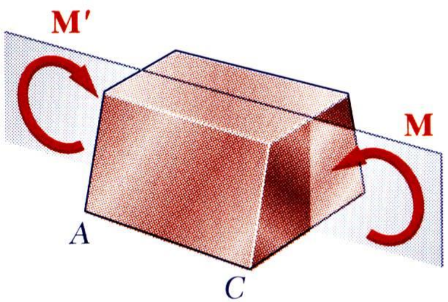
Pure Bending in Beams

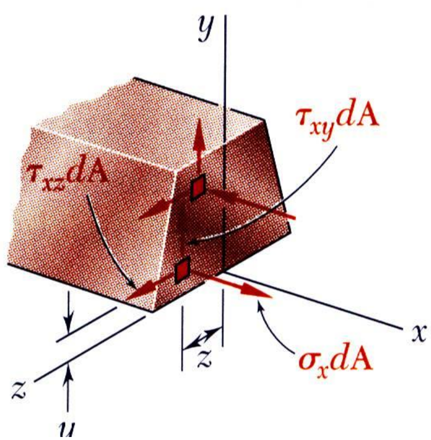


Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

Symmetric Member in Pure Bending







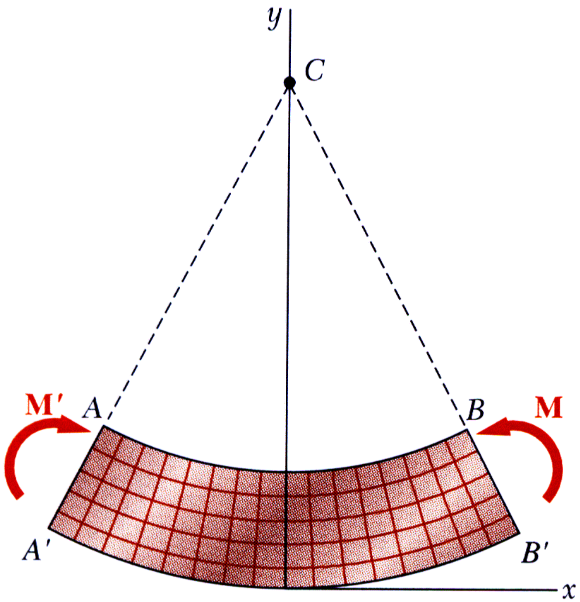
- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.
- From statics, a couple M consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

$$F_x = \int \sigma_x \, dA = 0$$

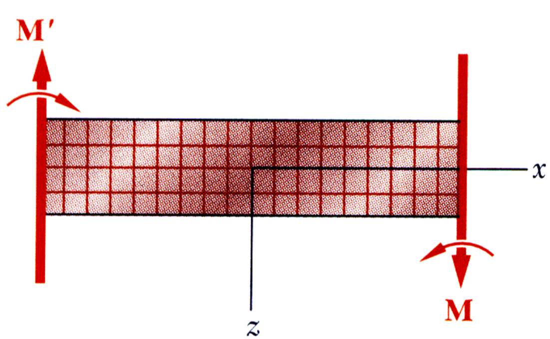
$$M_y = \int z\sigma_x \, dA = 0$$

$$M_z = \int -y\sigma_x \, dA = M$$

Bending Deformations



(*a*) Longitudinal, vertical section (plane of symmetry)

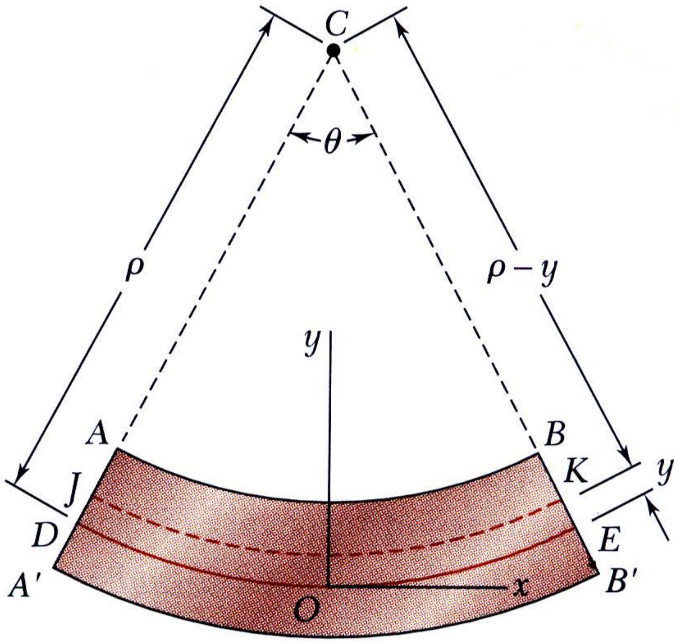


(*b*) Longitudinal, horizontal section

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

Strain Due to Bending



Consider a beam segment of length *L*.

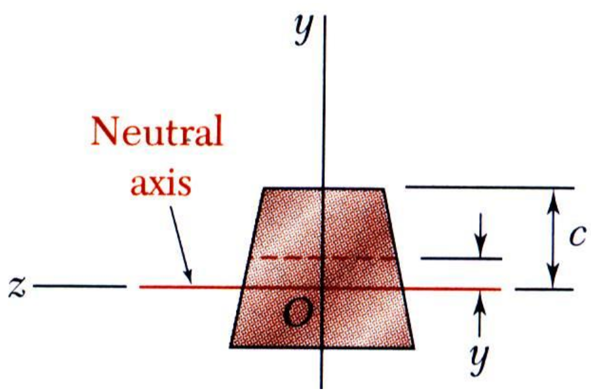
After deformation, the length of the neutral surface remains *L*. At other sections,

observing that the length of *DE* is equal to the length *L* of the undeformed member, we write

$$L = \rho\theta \qquad \text{(i)}$$

Considering now the arc *JK* located at a distance *y* above the neutral surface, we note that its length *L*' is

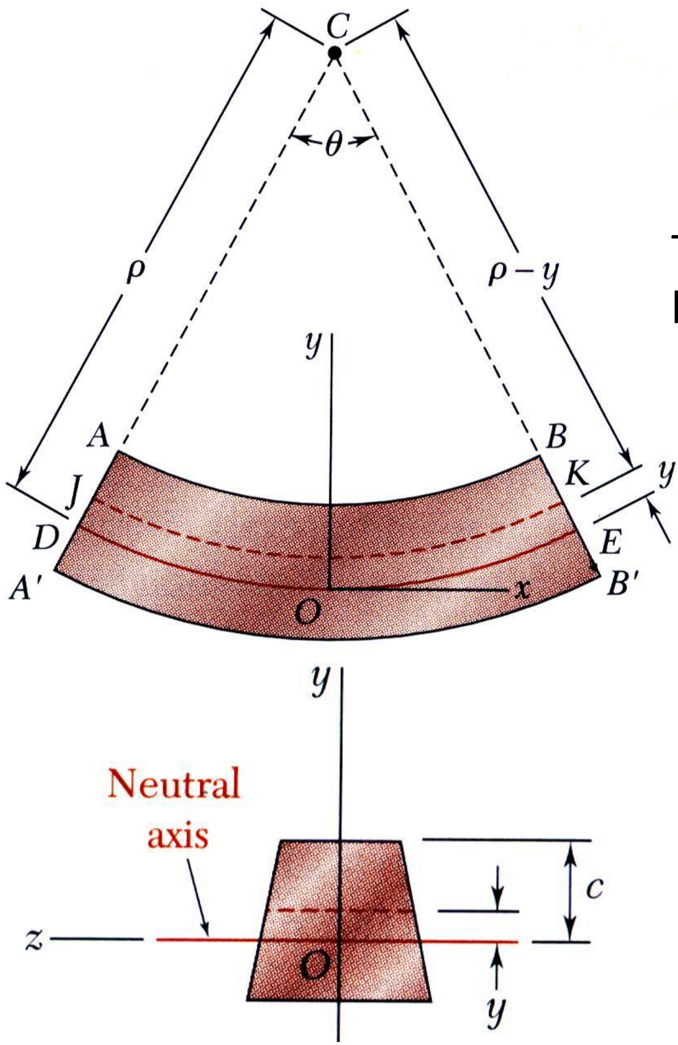
$$L' = (\rho - y)\theta \qquad \text{(ii)}$$



Since the original length of arc *JK* was equal to *L*, the deformation of *JK* is

$$\delta = L - L' = (\rho - y)\theta - \rho\theta = -y\theta \qquad \text{(iii)}$$

Strain Due to Bending



substituting from (i) and (ii) into (iii),

$$\partial = (\rho - y)\theta - \rho\theta = -y\theta$$

The longitudinal strain ε_x in the elements of *JK* is obtained by dividing ∂ by the original length *L* of *JK*.

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta}$$

$$\varepsilon_x = -\frac{y}{\rho} \quad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

where ε_m is the *maximum absolute value*

$$Thus; \quad \varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Maximum strain occurs at the outermost fibre located at distance $y = c$ from the Neutral axis

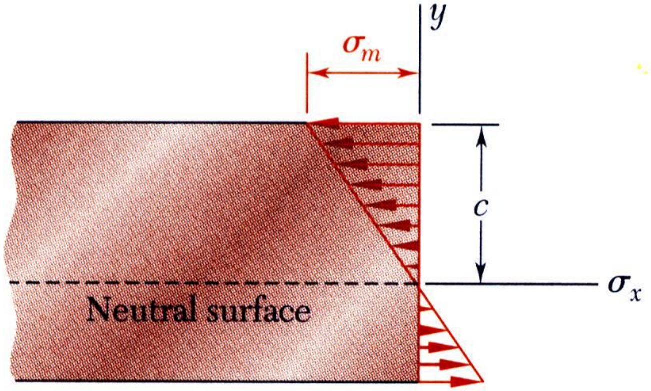
Stress Due to Bending

- For a linearly elastic material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

$$\sigma_x = -\frac{y}{c}\sigma_m \text{ (stress varies linearly)}$$

σ_m denotes the *maximum absolute value*



- For static equilibrium, $dF = \sigma\, dA$

Neutral axis can be located on the x -section by satisfying the condition that the *resultant force* produced by the stress distribution over the x-sectional area must be equal to *zero*

$$F_x = 0 = \int \sigma_x\, dA = \int -\frac{y}{c}\sigma_m\, dA$$

$$0 = -\frac{\sigma_m}{c}\int y\, dA$$

First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.

- For static equilibrium,

$$M = \int -y\sigma_x\, dA = \int -y\left(-\frac{y}{c}\sigma_m\right) dA$$

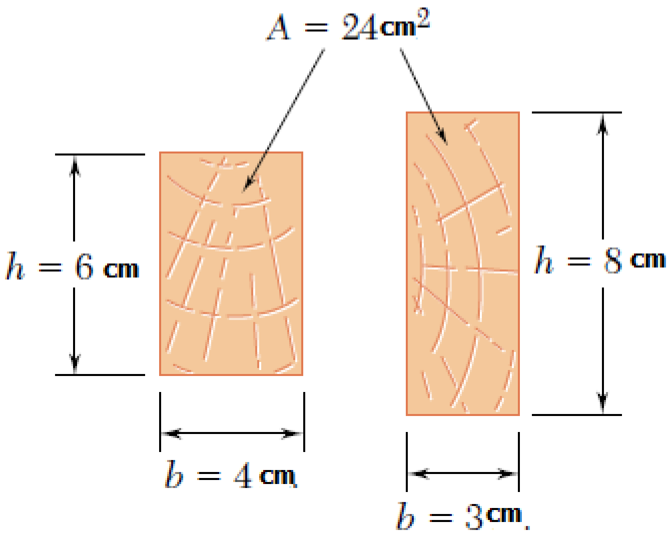
$$M = \frac{\sigma_m}{c}\int y^2\, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

Substituti ng $\sigma_x = -\frac{y}{c}\sigma_m$

$$\sigma_x = -\frac{My}{I}$$ Is called the elastic flexure formulas

Beam Section Properties



$I = 72cm^4$
$S = 24cm^3$

$I = 128cm^4$
$S = 32\ cm^3$

- The maximum normal stress due to bending,

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$I =$ section moment of inertia

The ratio I/c depends only upon the geometry of the cross section.

This ratio is called the *elastic section modulus* and is denoted by S.

$$S = \frac{I}{c} = \text{section modulus}$$

Thus; $\sigma_m = \dfrac{M}{S}$

A beam section with a larger section modulus will have a lower maximum stress

Beam Section Properties

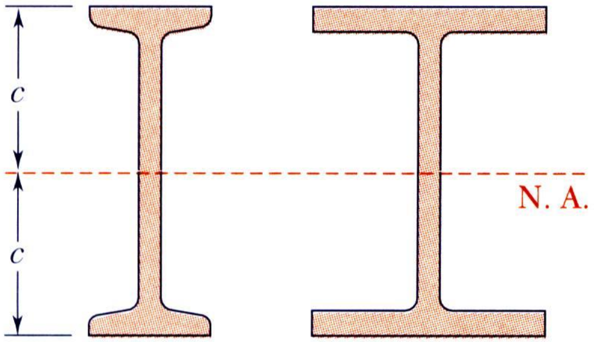
Consider a rectangular beam cross section,

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{\frac{h}{2}} = \frac{1}{6}bh^2 = \frac{1}{6}Ah$$

Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

This shows that, of two beams with the same cross-sectional area *A*, the beam with the larger depth *h* will have the larger section modulus and, thus, will be the more effective in resisting bending.

Beam Section Properties



(a) S-beam (b) W-beam

- Structural steel beams are designed to have a large section modulus.
- In structural steel, standard beams (S-beams) and wide-flange beams (W-beams) are preferred to other shapes because a large portion of their cross section is located far from the neutral axis
- Thus, for a given cross-sectional area and a given depth, their design provides large values of I and, consequently, of S
- Values of the elastic section modulus of commonly manufactured beams can be obtained from tables listing the various geometric properties of such beams

Beam Section Properties

- To determine the maximum stress σ_m in a given section of a standard beam, the engineer needs only to read the value of the elastic section modulus S in a table, and divide the bending moment M in the section by S.

- The deformation of the member caused by the bending moment M is measured by the *curvature* of the neutral surface.

- The curvature is defined as the reciprocal of the radius of curvature ρ, and can be obtained by solving

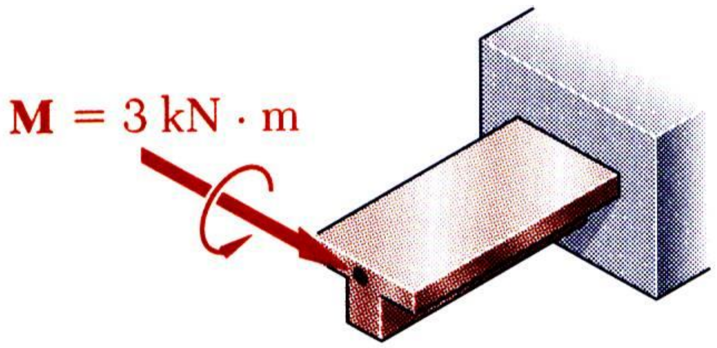
$$\frac{1}{\rho} = \frac{\varepsilon_m}{c}$$

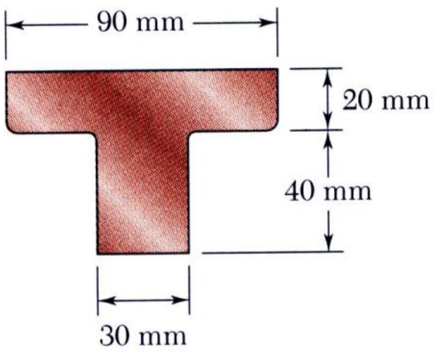
But, in the elastic range, $\varepsilon_m = \sigma_m / E$.

$$\frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$

$$\frac{1}{\rho} = \frac{M}{EI}$$

Example 1





A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing E = 165 GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} \qquad I_{x'} = \sum\left(\bar{I} + Ad^2\right)$$

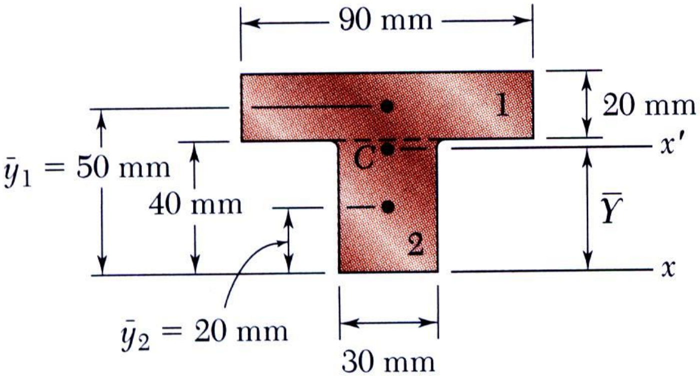
- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

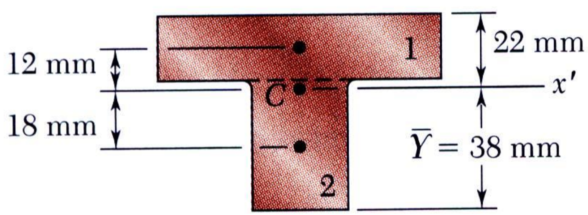
Example 1 – Cont'd



SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$20 \times 90 = 1800$	50	90×10^3
2	$40 \times 30 = 1200$	20	24×10^3
	$\sum A = 3000$		$\sum \bar{y}A = 114 \times 10^3$



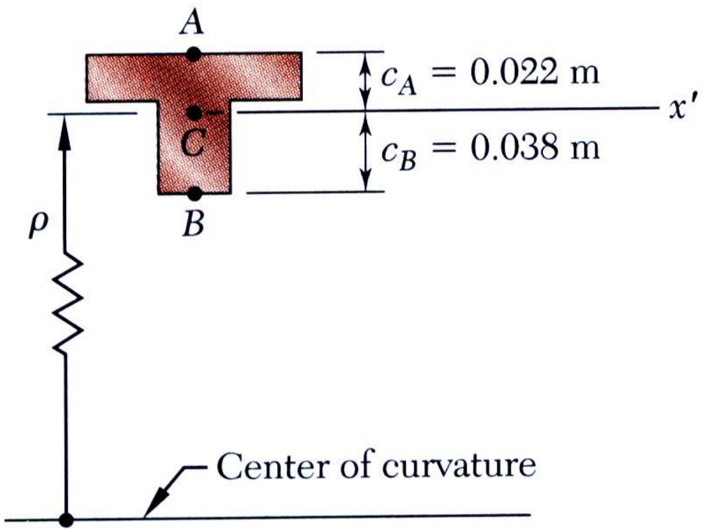
$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{114 \times 10^3}{3000} = 38\,\text{mm}$$

$$I_{x'} = \sum\left(\bar{I} + Ad^2\right) = \sum\left(\tfrac{1}{12}bh^3 + Ad^2\right)$$

$$= \left(\tfrac{1}{12}90 \times 20^3 + 1800 \times 12^2\right) + \left(\tfrac{1}{12}30 \times 40^3 + 1200 \times 18^2\right)$$

$$I = 868 \times 10^3\,\text{mm} = 868 \times 10^{-9}\,\text{m}^4$$

Example 1



- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

$$\sigma_A = \frac{Mc_A}{I} = \frac{3\ \text{kN}\cdot\text{m} \times 0.022\text{m}}{868 \times 10^{-9}\text{m}^4}$$

$$\boxed{\sigma_A = +76.0\ \text{MPa}}$$

$$\sigma_B = -\frac{Mc_B}{I}$$

$$= -\frac{3\ \text{kN}\cdot\text{m} \times 0.038\text{m}}{868 \times 10^{-9}\text{m}^4}$$

$$\boxed{\sigma_B = -131.3\ \text{MPa}}$$

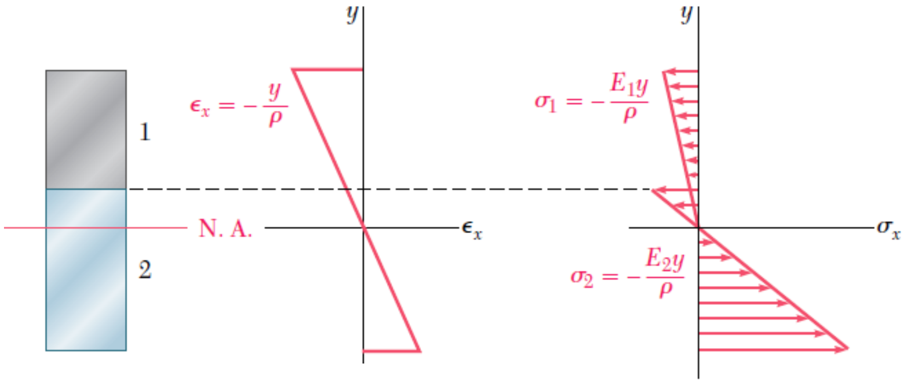
- Calculate the curvature

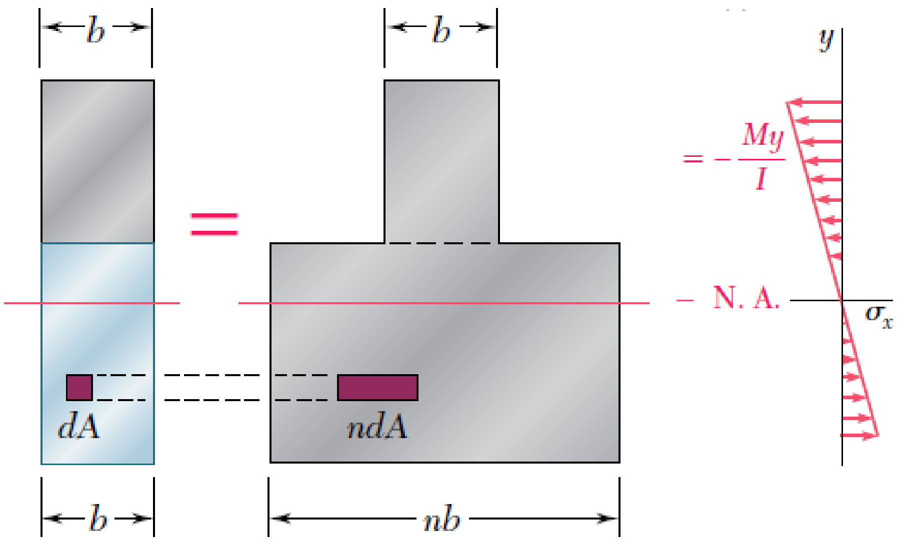
$$\frac{1}{\rho} = \frac{M}{EI}$$

$$= \frac{3\ \text{kN}\cdot\text{m}}{(165\ \text{GPa})(868 \times 10^{-9}\ \text{m}^4)}$$

$$\boxed{\begin{aligned} \frac{1}{\rho} &= 20.95 \times 10^{-3}\ \text{m}^{-1} \\ \rho &= 47.7\ \text{m} \end{aligned}}$$

Bending of Composite Materials





$$\sigma_x = -\frac{My}{I}$$

$$\sigma_1 = \sigma_x \qquad \sigma_2 = n\sigma_x$$

- Consider a composite beam formed from two materials with E_1 and E_2. *with* $E_2 > E_1$
- Normal strain varies linearly.

$$\varepsilon_x = -\frac{y}{\rho}$$

- Piecewise linear normal stress variation.

$$\sigma_1 = E_1 \varepsilon_x = -\frac{E_1 y}{\rho} \qquad \sigma_2 = E_2 \varepsilon_x = -\frac{E_2 y}{\rho}$$

- Neutral axis does not pass through section centroid of composite section.
- Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

- Define a transformed section such that

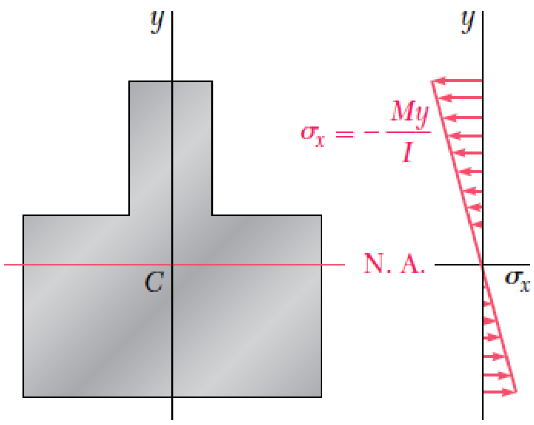
$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho}(n\,dA) \qquad n = \frac{E_2}{E_1}$$

Bending of Composite Materials

- In other words, the resistance to bending of the bar would remain the same if both portions were made of the first material, provided that the width of each element of the lower portion were multiplied by the factor *n*.

- Note that this widening (if $n > 1$), or narrowing (if $n < 1$), must be effected *in a direction parallel to the neutral axis of the section*, since it is essential that the distance *y* of each element from the neutral axis remain the same

- The new cross section obtained in this way is called the *transformed section* of the member



The neutral axis will be drawn *through the centroid of the transformed section* (Fig. 4.23), and the stress σ_x at any point of the corresponding fictitious homogeneous member will be obtained from $\sigma_x = -\frac{My}{I}$

Where: *y* is distance from the neutral surface

I the moment of inertia of the transformed section with respect to its centroidal axis

Bending of Composite Materials

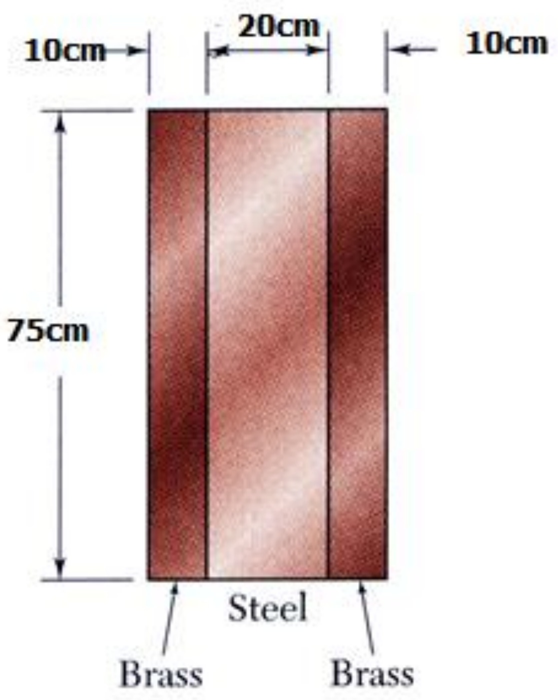
- To obtain the stress σ_1 at a point located in the upper portion of the cross section of the original composite bar, we simply compute the stress σ_x at the corresponding point of the transformed section.

- However, to obtain the stress σ_2 at a point in the lower portion of the cross section, we must *multiply by n* the stress σ_x computed at the corresponding point of the transformed section.

- The same elementary force *dF*2 is applied to an element of area *n dA* of the transformed section and to an element of area *dA* of the original section.

- Thus, the stress σ_2 at a point of the original section must be *n* times larger than the stress at the corresponding point of the transformed section.

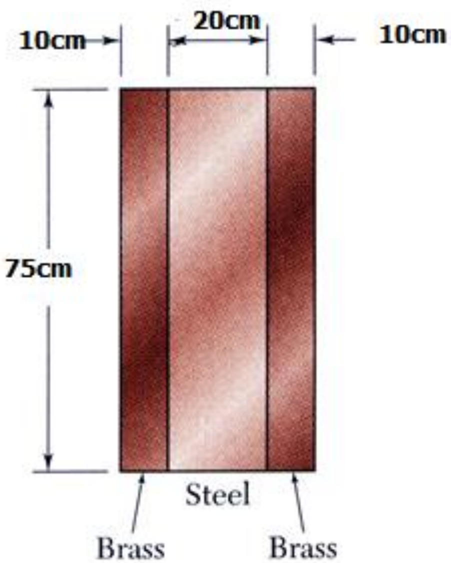
Example 2

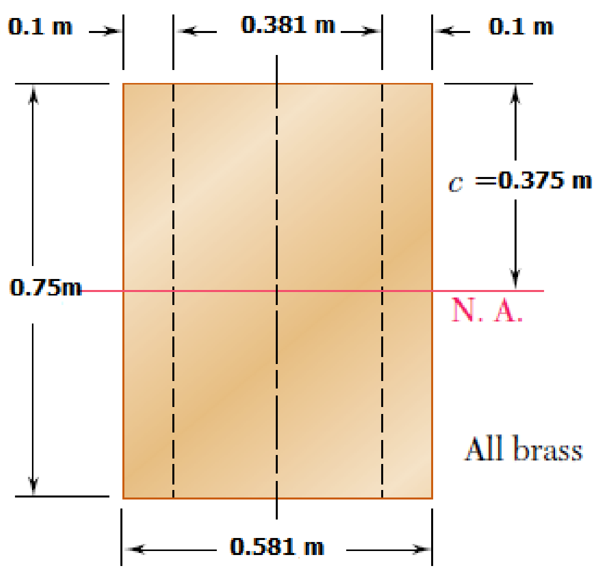


Bar is made from bonded pieces of steel (E_s = 200GPa) and brass (E_b = 105GPa). Determine the maximum stress in the steel and brass when a moment of 40 N.m is applied.

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass
- Evaluate the cross sectional properties of the transformed section
- Calculate the maximum stress in the transformed section. This is the correct maximum stress for the brass pieces of the bar.
- Determine the maximum stress in the steel portion of the bar by multiplying the maximum stress for the transformed section by the ratio of the moduli of elasticity.





Example 2

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.

$$n = \frac{E_s}{E_b} = \frac{200GPa}{105GPa} = 1.905$$

$$b_T = 0.1\text{ m} + 1.905 \times 0.20\text{ m} + 0.1\text{ m} = 0.581\text{ m}$$

Evaluate the transformed cross sectional properties

Central portion of brass = (0.2m)(1.905) = 0.381m

$$I = \tfrac{1}{12} b_T h^3 = \tfrac{1}{12}(0.581\text{ m.})(0.75\text{ m})^3$$

$$= 0.0204\text{ m}^4$$

- Calculate the maximum stresses

$$\sigma_m = \frac{Mc}{I} = \frac{(40\text{ N}\cdot\text{m})(0.375\text{ m})}{0.0204\text{ m}^4} = 735.3\text{Pa}$$

$$(\sigma_b)_{\max} = \sigma_m$$

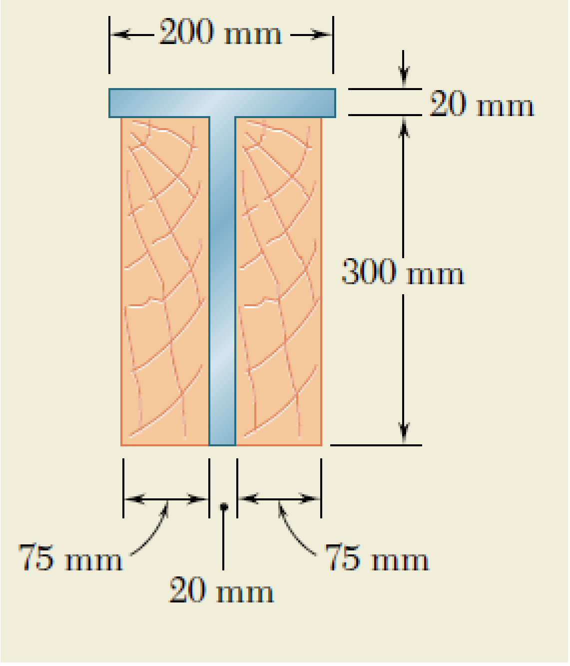
$$(\sigma_s)_{\max} = n\sigma_m = 1.905 \times 735.3\text{ Pa}$$

$$\boxed{\begin{aligned}(\sigma_b)_{\max} &= 735.3\text{Pa}\\ (\sigma_s)_{\max} &= 1400\text{ Pa}\end{aligned}}$$

Example 3

Two steel plates have been welded together to form a beam in the shape of a T that has been strengthened by securely bolting to it the two oak timbers shown. The modulus of elasticity is 12.5 GPa for the wood and 200 GPa for the steel. Knowing that a bending moment M = 50 kN .m is applied to the composite beam, determine (a) the maximum stress in the wood, (b) the stress in the steel along the top edge.

SOLUTION



Transformed Section

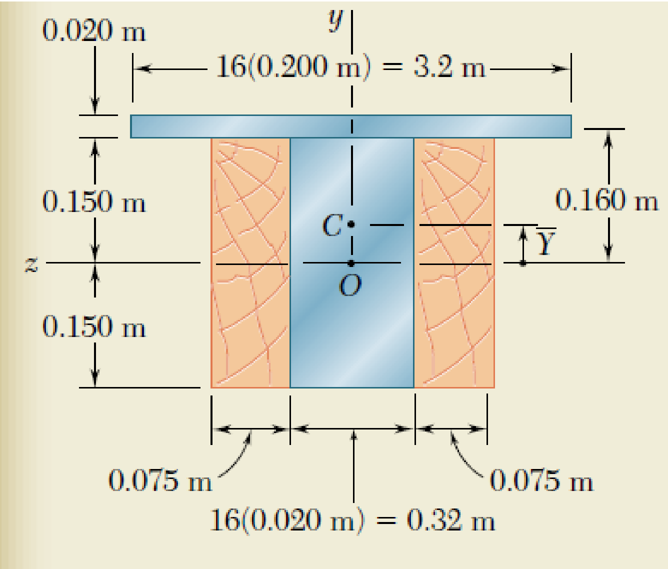
$$n = \frac{E_s}{E_w} = \frac{200GPa}{12.5GPa} = 16$$

Multiply horizontal dimensions of the steel portion of the section by n = 16, we obtain a transformed section made entirely of wood

Example - 3

Neutral Axis. The neutral axis passes through the centroid of the transformed section. Since the section consists of two rectangles, we have

$$b_{t(bottom)} = 0.47\text{m} \; ; \quad b_{t(top)} = 3.2\text{m}$$



$$\bar{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{(0.160\text{m})(3.2\text{m x } 0.020) + 0}{3.2\text{m x } 0.02\text{ m} + 0.470\text{ m x } 0.300\text{ m}}$$

$$= 0.050\text{m}$$

Centroidal Moment of Inertia. Using the parallel-axis theorem:

$$I = \tfrac{1}{12}(0.470)(0.300)^3 + (0.470 \times 0.300)(0.050)^2 + \tfrac{1}{12}(3.2)(0.020)^3 + (3.2 \times 0.020)(0.160 - 0.050)^2$$

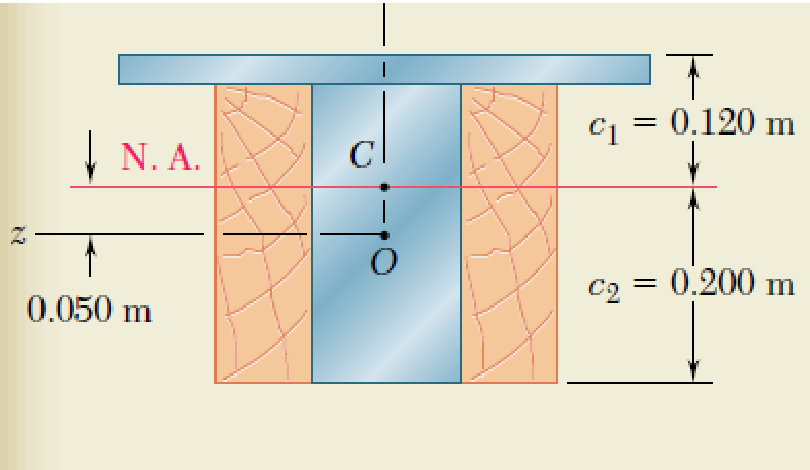
$$I = 2.19 \times 10^{-3}\ \text{m}^4$$

a. Maximum Stress in Wood. The wood farthest from the neutral axis is located along the bottom edge, where $c_2 = 0.200$ m.

$$\sigma_w = \frac{Mc_2}{I} = \frac{(50 \times 10^3\ \text{N} \cdot \text{m})(0.200\ \text{m})}{2.19 \times 10^{-3}\ \text{m}^4}$$

$$\sigma_w = 4.57\ \text{MPa}$$

Example 3



b. Stress in Steel. Along the top edge $c_1 = 0.120$ m. From the transformed section we obtain an equivalent stress in wood, which must be multiplied by n to obtain the stress in steel.

$$\sigma_s = n\frac{Mc_1}{I} = (16)\frac{(50 \times 10^3 \text{ N} \cdot \text{m})(0.120 \text{ m})}{2.19 \times 10^{-3} \text{ m}^4}$$

$$\sigma_s = 43.8 \text{ MPa}$$