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Mechanical Engineering Series

Third Edition

MECHANICS OF MATERIALS

(In SI Units)

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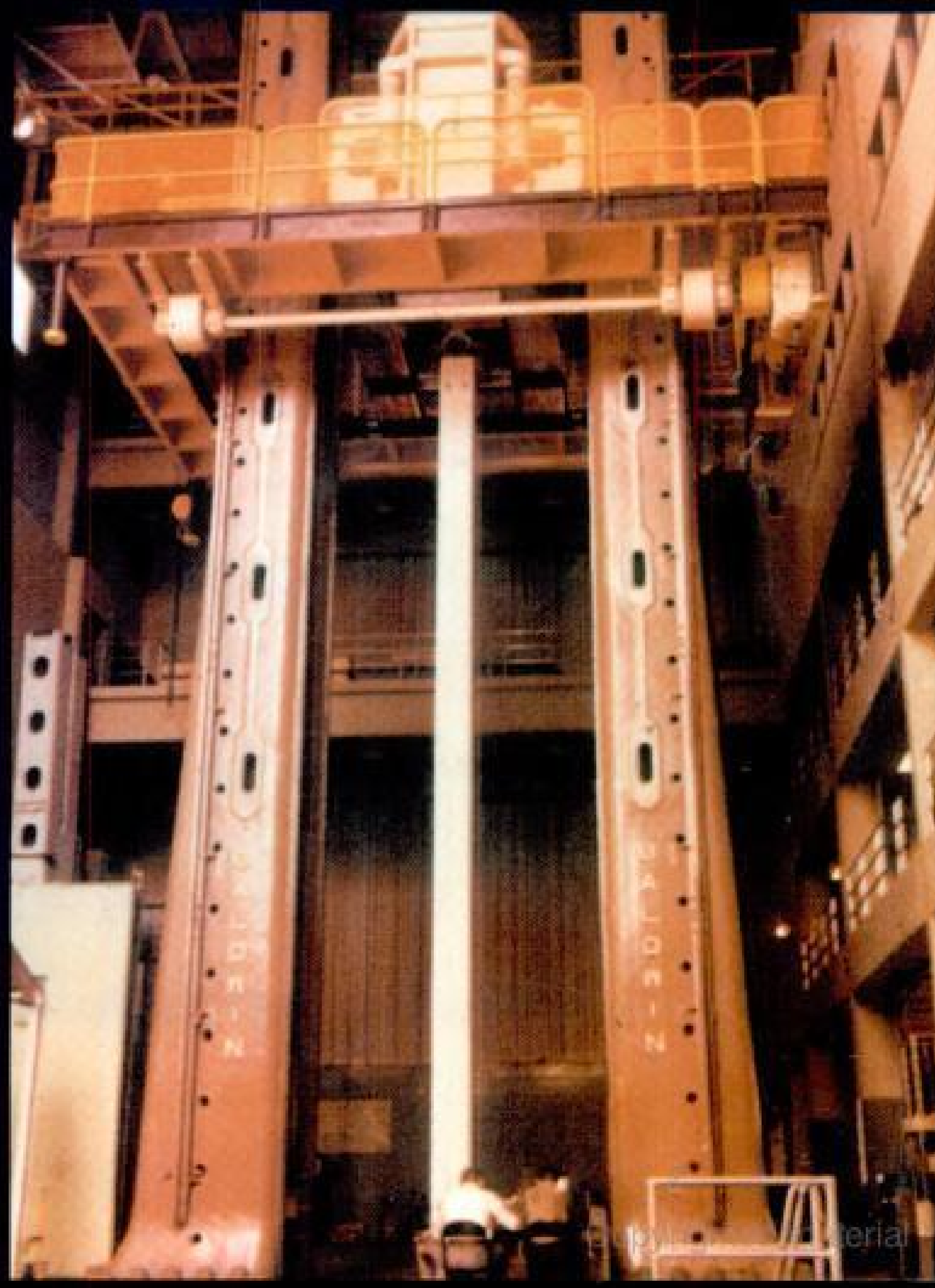

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The photograph on the cover shows a steel wide-flange column being tested in the five-million-pound universal testing machine at Lehigh University, Bethlehem, Pennsylvania. (Courtesy of Fritz Engineering Laboratory.)

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PREFACE

The main objective of a basic mechanics course should be to develop in the engineering student the ability to analyze a given problem in a simple and logical manner and to apply to its solution a few fundamental and well-understood principles. This text is designed for the first course in mechanics of materials—or strength of materials—offered to engineering students in the sophomore or junior year. The authors hope that it will help instructors achieve this goal in that particular course in the same way that their other texts may have helped them in statics and dynamics.

In this text the study of the mechanics of materials is based on the understanding of a few basic concepts and on the use of simplified models. This approach makes it possible to develop all the necessary formulas in a rational and logical manner, and to clearly indicate the conditions under which they can be safely applied to the analysis and design of actual engineering structures and machine components.

Free-body diagrams are used extensively throughout the text to determine external or internal forces. The use of "picture equations" will also help the students understand the superposition of loadings and the resulting stresses and deformations.

It is expected that students using this text will have completed a course in statics. However, Chap. 1 is designed to provide them with an opportunity to review the concepts learned in that course, while shear and bending-moment diagrams are covered in detail in Secs. 5.2 and 5.3. The properties of moments and centroids of areas are described in Appendix A; this material can be used to reinforce the discussion of the determination of normal and shearing stresses in beams (Chaps. 4, 5, and 6).

The first four chapters of the text are devoted to the analysis of the stresses and of the corresponding deformations in various structural members, considering successively axial loading, torsion, and pure bending. Each analysis is based on a few basic concepts, namely, the conditions of equilibrium of the forces exerted on the member, the relations existing between stress and strain in the material, and the conditions imposed by the supports and loading of the member. The study of each type of loading is complemented by a large number of examples, sample problems, and problems to be assigned, all designed to strengthen the students' understanding of the subject.

The concept of stress at a point is introduced in Chap. 1, where it is shown that an axial load can produce shearing stresses as well as normal stresses, depending upon the section considered. The fact that stresses depend upon the orientation of the surface on which they are computed is emphasized again in Chaps. 3 and 4 in the cases of torsion and pure bending. However, the discussion of computational techniques—such as Mohr's circle—used for the transformation of stress at a point is delayed until Chap. 7, after students have had the opportunity to solve problems involving a combination of the basic loadings and have discovered for themselves the need for such techniques.

In this new edition, the discussion in Chap. 2 of the relation between stress and strain in various materials has been expanded to include fiber-reinforced composite materials. Also, the study of beams under transverse loads is now covered in two separate chapters. Chapter 5 is devoted to the determination of the normal stresses in a beam and to the design of beams based on the allowable normal stress in the material used (Sec. 5.4). The chapter begins with a discussion of the shear and bending-moment diagrams (Secs. 5.2 and 5.3) and includes an optional section on the use of singularity functions for the determination of the shear and bending moment in a beam (Sec. 5.5). The chapter ends with an expanded optional section on nonprismatic beams (Sec. 5.6).

Chapter 6 is devoted to the determination of shearing stresses in beams and thin-walled members under transverse loadings. In this new edition, and at the suggestion of our reviewers, the formula for the shear flow, $q = VQ/I$, is derived in the traditional way. More advanced aspects of the design of beams, such as the determination of the principal stresses at the junction of the flange and web of a W-beam, have been moved to Chap. 8, an optional chapter that may be covered after the transformations of stresses have been discussed in Chap. 7. The design of transmission shafts has been moved to that chapter for the same reason, as well as the determination of stresses under combined loadings which, in this new position, can now include the determination of the principal stresses, principal planes, and maximum shearing stress at a given point.

Statically indeterminate problems are first discussed in Chap. 2 and considered throughout the text for the various loading conditions encountered. Thus, students are presented at an early stage with a method of solution which combines the analysis of deformations with the conventional analysis of forces used in statics. In this way, they will have become thoroughly familiar with this fundamental method by the end of the course. In addition, this approach helps the students realize that stresses themselves are statically indeterminate and can be computed only by considering the corresponding distribution of strains.

Design concepts are discussed throughout the text whenever appropriate. A discussion of the application of the factor of safety to design can be found in Chap. 1, where the concepts of both allowable stress design and load and resistance factor design are presented.

The concept of plastic deformation is introduced in Chap. 2, where it is applied to the analysis of members under axial loading. Problems involving the plastic deformation of circular shafts and of prismatic beams are also considered in optional sections of Chaps. 3, 4, and 6. While some of this material can be omitted at the choice of the instructor, its inclusion in the body of the text will help students realize the limitations of the assumption of a linear stress-strain relation and serve to caution them against the inappropriate use of the eleastic torsion and flexure formulas.

The determination of the deflection of beams is discussed in Chap. 9. The first part of the chapter is devoted to the integration method and to the method of superposition, with an optional section (Sec. 9.6) based on the use of singularity functions. (This section should be used only if Sec. 5.5 was covered earlier.) The second part of Chap. 9 is optional. It presents the moment-area method in two lessons instead off three as in our previous edition.

Chapter 10 is devoted to columns and contains new material on the design of wood columns. Chapter 11 covers energy methods, including Castigliano's theorem.

Additional topics such as residual stresses, torsion of noncircular and thin-walled members, bending of curved beams, shearing stresses in nonsymmetrical members, and failure criteria, have been included in optional sections for use in courses of varying emphases. To preserve the integrity of the subject, these topics are presented in the proper sequence, wherever they logically belong. Thus, even when not covered in the course, they are highly visible and can be easily referred to by the students if needed in a later course or in engineering practice. For convenience all optional sections have been indicated by asterisks.

Each chapter begins with an introductory section setting the purpose and goals of the chapter and describing in simple terms the material to be covered and its application to the solution of engineering problems. The body of the text has been divided into units, each consisting of one or several theory sections followed by sample problems and a large number of problems to be assigned. Each unit corresponds to a well-defined topic and generally can be covered in one lesson. Each chapter ends with a review and summary of the material covered in the chapter. Notes in the margin have been included to help the students organize their review work, and cross references provided to help them find the portions of material requiring their special attention. The theory sections include many examples designed to illustrate the material being presented and facilitate its understanding. The sample problems are intended to show some of the applications of the theory to the solution of engineering problems. Since they have been set up in much the same form that students will use in solving the assigned problems, the sample problems serve the double purpose of amplifying the text and demonstrating the type of neat and orderly work that students should cultivate in their own solutions. Most of the problems are of a practical nature and should appeal to engineering students. They are primarily designed, however, to illustrate the material presented in the text and help the students understand the basic principles used in mechanics of materials. The problems have been grouped according to the portions of material they illustrate and have been arranged in order of increasing difficulty. Problems requiring special attention have been indicated by asterisks. Answers to problems are given at the end of the book, except for those with a number set in italics.

The availability of personal computers makes it possible for engineering students to solve a great number of challenging problems. In this new edition of *Mechanics of Materials*, a group of six or more problems designed to be solved with a computer can be found at the end of each chapter. Developing the algorithm required to solve a given problem will benefit the students in two different ways: (1) it will help them gain a better understanding of the mechanics principles involved; (2) it will provide them with an opportunity to apply the skills acquired in their computer programming course to the solution of a meaningful engineering problem.

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John T. DeWolf

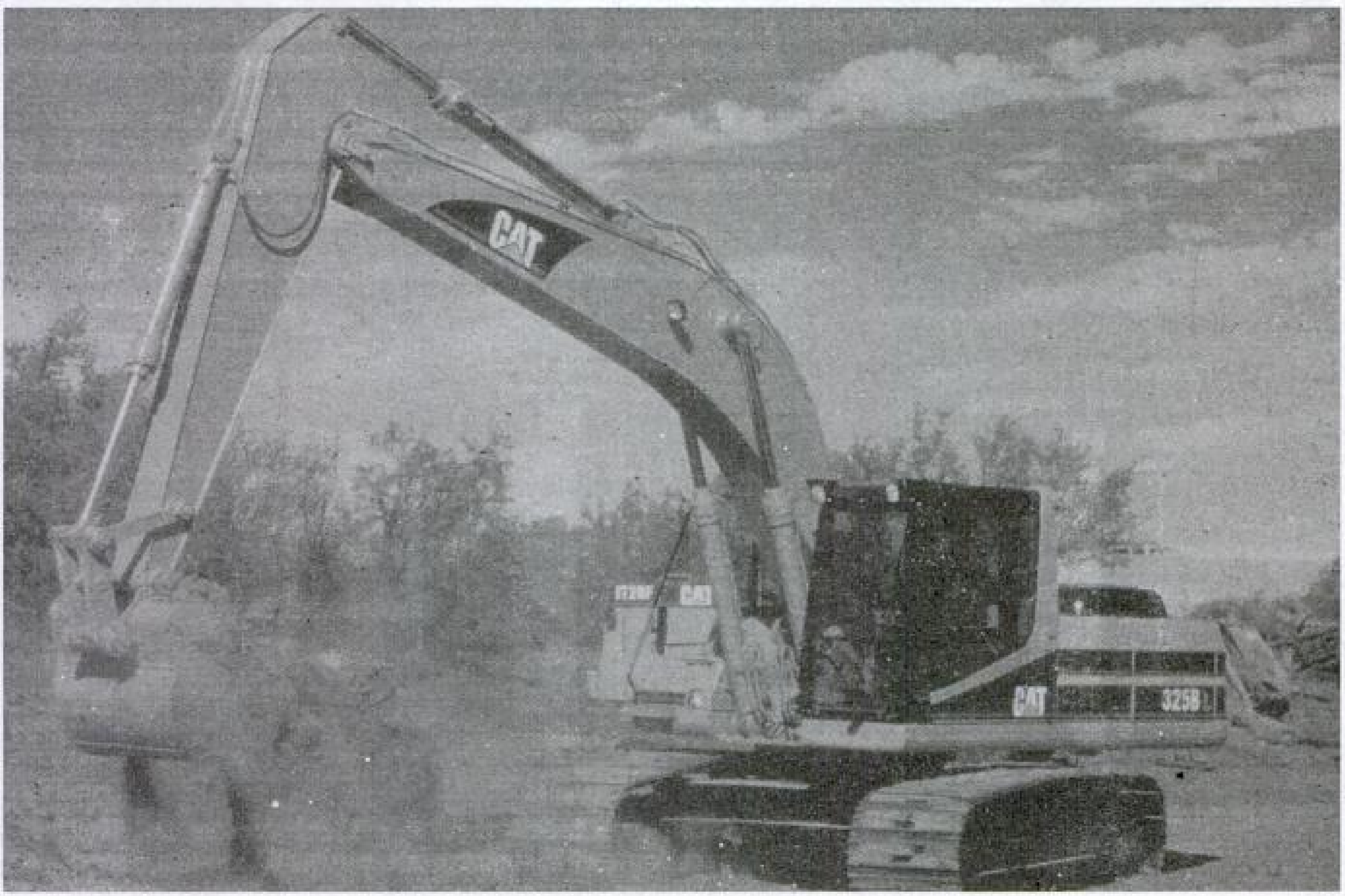
List of Symbols

a	Constant; distance
$\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$	Forces; reactions
$A, B, C, \ldots$	Points
$A, \mathcal{A}$	Area
b	Distance; width
c	Constant; distance; radius
C	Centroid
$C_1, C_2 \ldots$	Constants of integration
C_P	Column stability factor
d	Distance; diameter; depth
D	Diameter
e	Distance; eccentricity; dilatation
E	Modulus of elasticity
f	Frequency; function
$\mathbf{F}$	Force
$F.S.$	Factor of safety
G	Modulus of rigidity; shear modulus
h	Distance; height
$\mathbf{H}$	Force
H, J, K	Points
$I, I_x, \ldots$	Moment of inertia
$I_{xy}, \ldots$	Product of inertia
J	Polar moment of inertia
k	Spring constant; shape factor; bulk modulus; constant
K	Stress concentration factor; torsional spring constant
l	Length; span
L	Length; span
L_e	Effective length
m	Mass
$\mathbf{M}$	Couple
$M, M_x, \ldots$	Bending moment
M_D	Bending moment, dead load (LRFD)
M_L	Bending moment, live load (LRFD)
M_U	Bending moment, ultimate load (LRFD)
n	Number; ratio of moduli of elasticity; normal direction
p	Pressure
$\mathbf{P}$	Force; concentrated load
P_D	Dead load (LRFD)
P_L	Live load (LRFD)
P_U	Ultimate load (LRFD)
q	Shearing force per unit length; shear flow
$\mathbf{Q}$	Force
Q	First moment of area
r	Radius; radius of gyration
$\mathbf{R}$	Force; reaction
R	Radius; modulus of rupture
s	Length
S	Elastic section modulus
t	Thickness; distance; tangential deviation
$\mathbf{T}$	Torque
T	Temperature
u, v	Rectangular coordinates
u	Strain-energy density
U	Strain energy; work
$\mathbf{v}$	Velocity
$\mathbf{V}$	Shearing force
V	Volume; shear
w	Width; distance; load per unit length
$\mathbf{W}, W$	Weight, load
x, y, z	Rectangular coordinates; distance; displacements; deflections
$\bar{x}, \bar{y}, \bar{z}$	Coordinates of centroid
Z	Plastic section modulus
α, β, γ	Angles
α	Coefficient of thermal expansion; influence coefficient
γ	Shearing strain; specific weight
γ_D	Load factor, dead load (LRFD)
γ_L	Load factor, live load (LRFD)
δ	Deformation; displacement
ϵ	Normal strain
θ	Angle; slope
λ	Direction cosine
ν	Poisson's ratio
ρ	Radius of curvature; distance; density
σ	Normal stress
τ	Shearing stress
ϕ	Angle; angle of twist; resistance factor
ω	Angular velocity

C H A P T E R

1

Introduction—Concept of Stress



This chapter is devoted to the study of the stresses occurring in many of the elements contained in this excavator, such as two-force members, axles, bolts, and pins.

1.1. INTRODUCTION

The main objective of the study of the mechanics of materials is to provide the future engineer with the means of analyzing and designing various machines and load-bearing structures.

Both the analysis and the design of a given structure involve the determination of *stresses* and *deformations*. This first chapter is devoted to the concept of *stress*.

Section 1.2 is devoted to a short review of the basic methods of statics and to their application to the determination of the forces in the members of a simple structure consisting of pin-connected members. Section 1.3 will introduce you to the concept of *stress* in a member of a structure, and you will be shown how that stress can be determined from the *force* in the member. After a short discussion of engineering analysis and design (Sec. 1.4), you will consider successively the *normal stresses* in a member under axial loading (Sec. 1.5), the *shearing stresses* caused by the application of equal and opposite transverse forces (Sec. 1.6), and the *bearing stresses* created by bolts and pins in the members they connect (Sec. 1.7). These various concepts will be applied in Sec. 1.8 to the determination of the stresses in the members of the simple structure considered earlier in Sec. 1.2.

The first part of the chapter ends with a description of the method you should use in the solution of an assigned problem (Sec. 1.9) and with a discussion of the numerical accuracy appropriate in engineering calculations (Sec. 1.10).

In Sec. 1.11, where a two-force member under axial loading is considered again, it will be observed that the stresses on an *oblique* plane include both *normal* and *shearing* stresses, while in Sec. 1.12 you will note that *six components* are required to describe the state of stress at a point in a body under the most general loading conditions.

Finally, Sec. 1.13 will be devoted to the determination from test specimens of the *ultimate strength* of a given material and to the use of a *factor of safety* in the computation of the *allowable load* for a structural component made of that material.

1.2. A SHORT REVIEW OF THE METHODS OF STATICS

In this section you will review the basic methods of statics while determining the forces in the members of a simple structure.

Consider the structure shown in Fig. 1.1, which was designed to support a 30-kN load. It consists of a boom *AB* with a 30 × 50-mm rectangular cross section and of a rod *BC* with a 20-mm-diameter circular cross section. The boom and the rod are connected by a pin at *B* and are supported by pins and brackets at *A* and *C* respectively. Our first step should be to draw a *free-body diagram* of the structure by detaching it from its supports at *A* and *C*, and showing the reactions that these supports exert on the structure (Fig. 1.2). Note that the sketch of the structure has been simplified by omitting all unnecessary details. Many of you may have recognized at this point that *AB* and *BC* are *two-force members*. For those of you who have not, we will pursue our analysis, ignoring that fact and assuming that the directions of the reactions at *A* and *C* are unknown. Each of these reactions, therefore, will

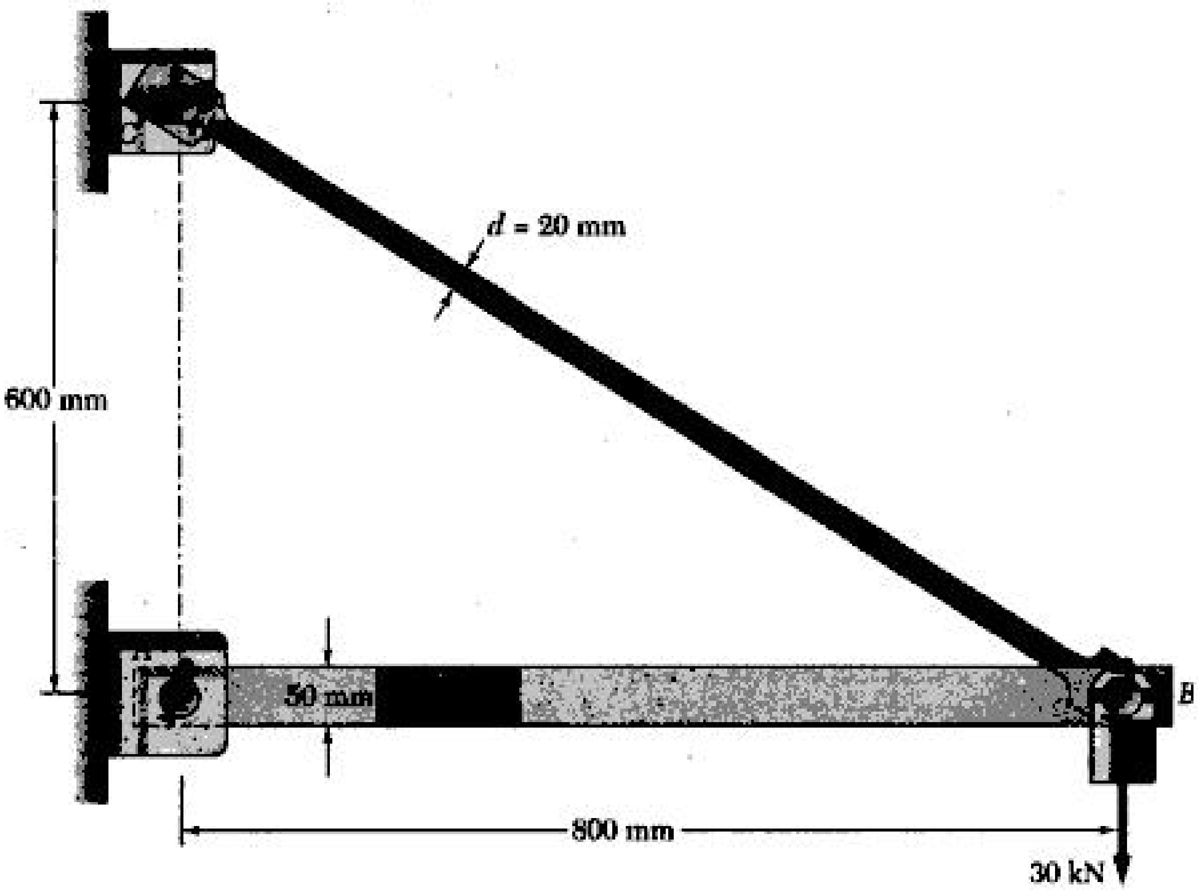


Fig. 1.1

be represented by two components, $\mathbf{A}_x$ and $\mathbf{A}_y$ at A, and $\mathbf{C}_x$ and $\mathbf{C}_y$ at C. We write the following three equilibrium equations:

$$+\circlearrowleft \Sigma M_C = 0: \quad A_x(0.6 \text{ m}) - (30 \text{ kN})(0.8 \text{ m}) = 0$$

$$A_x = +40 \text{ kN} \tag{1.1}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0: \quad A_x + C_x = 0$$

$$C_x = -A_x \qquad C_x = -40 \text{ kN} \tag{1.2}$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + C_y - 30 \text{ kN} = 0$$

$$A_y + C_y = +30 \text{ kN} \tag{1.3}$$

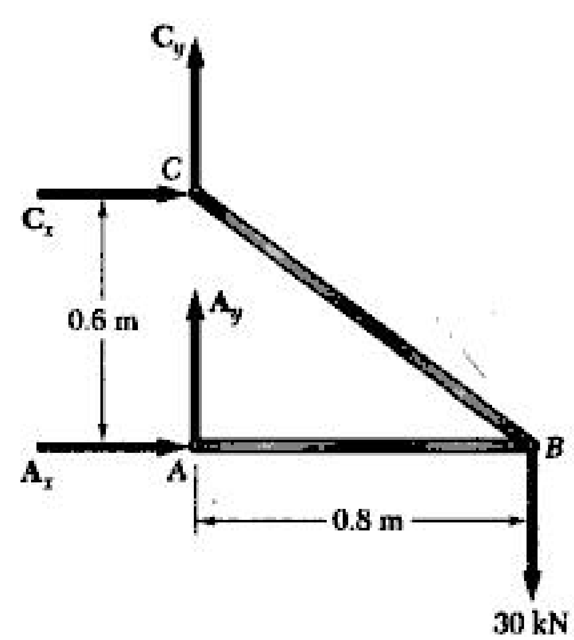


Fig. 1.2

We have found two of the four unknowns, but cannot determine the other two from these equations, and no additional independent equation can be obtained from the free-body diagram of the structure. We must now dismember the structure. Considering the free-body diagram of the boom AB (Fig. 1.3), we write the following equilibrium equation:

$$+\circlearrowleft \Sigma M_B = 0: \quad -A_y(0.8 \text{ m}) = 0 \qquad A_y = 0 \tag{1.4}$$

Substituting for A_y from (1.4) into (1.3), we obtain $C_y = +30$ kN. Expressing the results obtained for the reactions at A and C in vector form, we have

$$\mathbf{A} = 40 \text{ kN} \rightarrow \qquad \mathbf{C}_x = 40 \text{ kN} \leftarrow, \mathbf{C}_y = 30 \text{ kN}\uparrow$$

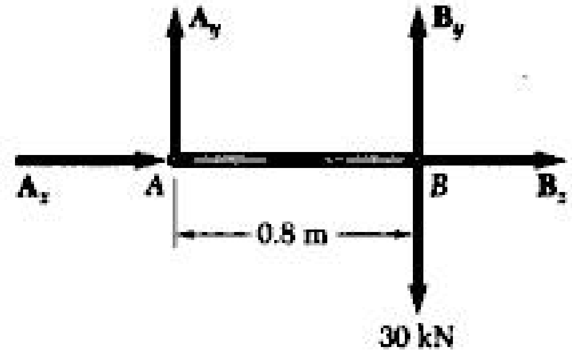


Fig. 1.3

We note that the reaction at A is directed along the axis of the boom AB and causes compression in that member. Observing that the components C_x and C_y of the reaction at C are respectively proportional to the horizontal and vertical components of the distance from B to C, we conclude that the reaction at C is equal to 50 kN, is directed along the axis of the rod BC, and causes tension in that member.

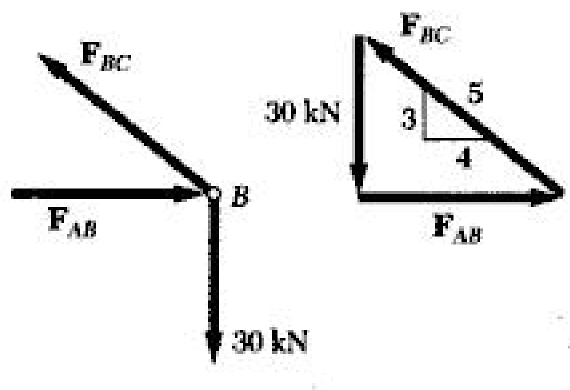


Fig. 1.4

These results could have been anticipated by recognizing that *AB* and *BC* are two-force members, i.e., members that are subjected to forces at only two points, these points being *A* and *B* for member *AB*, and *B* and *C* for member *BC*. Indeed, for a two-force member the lines of action of the resultants of the forces acting at each of the two points are equal and opposite and pass through both points. Using this property, we could have obtained a simpler solution by considering the free-body diagram of pin *B*. The forces on pin *B* are the forces $\mathbf{F}_{AB}$ and $\mathbf{F}_{BC}$ exerted, respectively, by members *AB* and *BC*, and the 30-kN load (Fig. 1.4*a*). We can express that pin *B* is in equilibrium by drawing the corresponding force triangle (Fig. 1.4*b*).

Since the force $\mathbf{F}_{BC}$ is directed along member *BC*, its slope is the same as that of *BC*, namely, 3/4. We can, therefore, write the proportion

$$\frac{F_{AB}}{4} = \frac{F_{BC}}{5} = \frac{30\text{ kN}}{3}$$

from which we obtain

$$F_{AB} = 40\text{ kN} \qquad F_{BC} = 50\text{ kN}$$

The forces $\mathbf{F}'_{AB}$ and $\mathbf{F}'_{BC}$ exerted by pin *B* respectively on boom *AB* and rod *BC* are equal and opposite to $\mathbf{F}_{AB}$ and $\mathbf{F}_{BC}$ (Fig. 1.5).

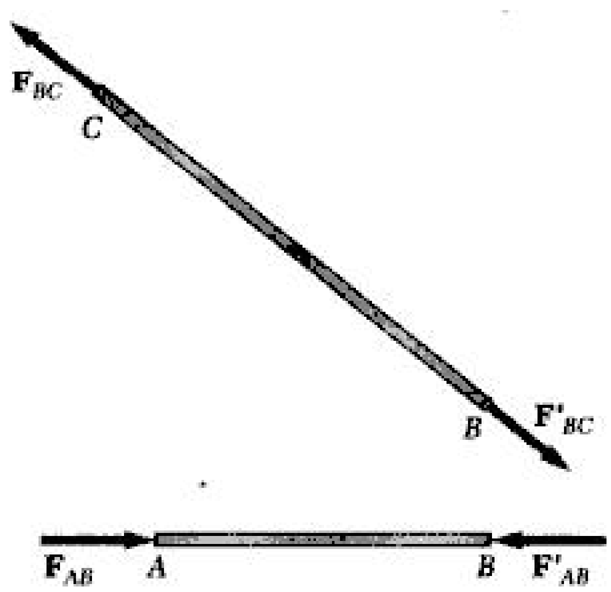


Fig. 1.5

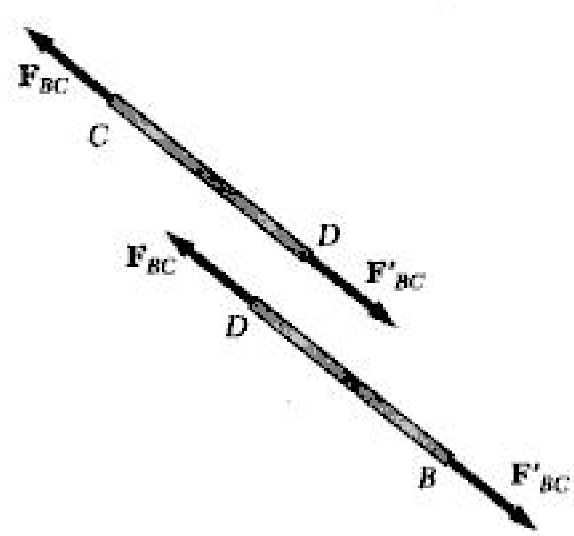


Fig. 1.6

Knowing the forces at the ends of each of the members, we can now determine the internal forces in these members. Passing a section at some arbitrary point *D* of rod *BC*, we obtain two portions *BD* and *CD* (Fig. 1.6). Since 50-kN forces must be applied at *D* to both portions of the rod to keep them in equilibrium, we conclude that an internal force of 50 kN is produced in rod *BC* when a 30-kN load is applied at *B*. We further check from the directions of the forces $\mathbf{F}_{BC}$ and $\mathbf{F}'_{BC}$ in Fig. 1.6 that the rod is in tension. A similar procedure would enable us to determine that the internal force in boom AB is 40 kN and that the boom is in compression.

1.3. STRESSES IN THE MEMBERS OF A STRUCTURE

While the results obtained in the preceding section represent a first and necessary step in the analysis of the given structure, they do not tell us whether the given load can be safely supported. Whether rod BC, for example, will break or not under this loading depends not only upon the value found for the internal force F_{BC}, but also upon the cross-sectional area of the rod and the material of which the rod is made. Indeed, the internal force F_{BC} actually represents the resultant of elementary forces distributed over the entire area A of the cross section (Fig. 1.7) and the average intensity of these distributed forces is equal to the force per unit area, F_{BC}/A, in the section. Whether or not the rod will break under the given loading clearly depends upon the ability of the material to withstand the corresponding value F_{BC}/A of the intensity of the distributed internal forces. It thus depends upon the force F_{BC}, the cross-sectional area A, and the material of the rod.

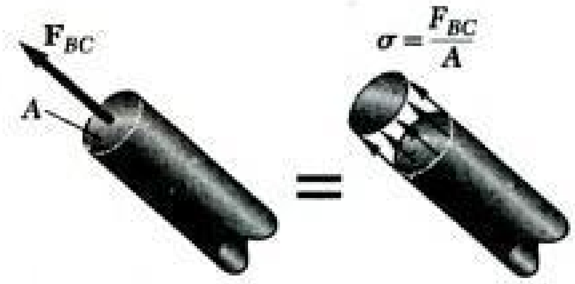


Fig. 1.7

The force per unit area, or intensity of the forces distributed over a given section, is called the *stress* on that section and is denoted by the Greek letter σ (sigma). The stress in a member of cross-sectional area A subjected to an axial load **P** (Fig. 1.8) is therefore obtained by dividing the magnitude P of the load by the area A:

$$\sigma = \frac{P}{A} \tag{1.5}$$

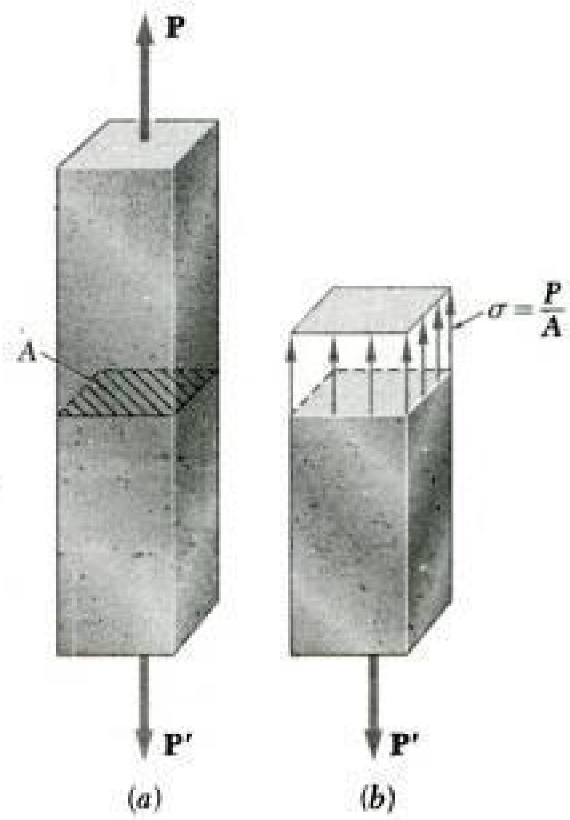


Fig. 1.8

A positive sign will be used to indicate a tensile stress (member in tension) and a negative sign to indicate a compressive stress (member in compression).

Since SI metric units are used in this discussion, with P expressed in newtons (N) and A in square meters (m^2), the stress σ will be expressed in N/m^2. This unit is called a *pascal* (Pa). However, one finds that the pascal is an exceedingly small quantity and that, in practice, multiples of this unit must be used, namely, the kilopascal (kPa), the megapascal (MPa), and the gigapascal (GPa). We have

$$1 \text{ kPa} = 10^3 \text{ Pa} = 10^3 \text{ N/m}^2$$

$$1 \text{ MPa} = 10^6 \text{ Pa} = 10^6 \text{ N/m}^2$$

$$1 \text{ GPa} = 10^9 \text{ Pa} = 10^9 \text{ N/m}^2$$

1.4. ANALYSIS AND DESIGN

Considering again the structure of Fig. 1.1, let us assume that rod *BC* is made of a steel with a maximum allowable stress $\sigma_{all} = 165$ MPa. Can rod *BC* safely support the load to which it will be subjected? The magnitude of the force F_{BC} in the rod was found earlier to be 50 kN. Recalling that the diameter of the rod is 20 mm, we use Eq. (1.5) to determine the stress created in the rod by the given loading. We have

$$P = F_{BC} = +50\text{ kN} = +50 \times 10^3\text{ N}$$

$$A = \pi r^2 = \pi\left(\frac{20\text{ mm}}{2}\right)^2 = \pi(10 \times 10^{-3}\text{ m})^2 = 314 \times 10^{-6}\text{ m}^2$$

$$\sigma = \frac{P}{A} = \frac{+50 \times 10^3\text{ N}}{314 \times 10^{-6}\text{ m}^2} = +159 \times 10^6\text{ Pa} = +159\text{ MPa}$$

Since the value obtained for σ is smaller than the value σ_{all} of the allowable stress in the steel used, we conclude that rod *BC* can safely support the load to which it will be subjected. To be complete, our analysis of the given structure should also include the determination of the compressive stress in boom *AB*, as well as an investigation of the stresses produced in the pins and their bearings. This will be discussed later in this chapter. We should also determine whether the deformations produced by the given loading are acceptable. The study of deformations under axial loads will be the subject of Chap. 2. An additional consideration, required for members in compression involves the *stability* of the member, i.e., its ability to support a given load without experiencing a sudden change in configuration. This will be discussed in Chap. 10.

The engineer's role is not limited to the analysis of existing structures and machines subjected to given loading conditions. Of even greater importance to the engineer is the *design* of new structures and machines, that is, the selection of appropriate components to perform a given task. As an example of design, let us return to the structure of Fig. 1.1, and assume that aluminum with an allowable stress $\sigma_{all} = 100$ MPa is to be used. Since the force in rod *BC* will still be $P = F_{BC} = 50$ kN under the given loading, we must have, from Eq. (1.5),

$$\sigma_{all} = \frac{P}{A} \qquad A = \frac{P}{\sigma_{all}} = \frac{50 \times 10^3\text{ N}}{100 \times 10^6\text{ Pa}} = 500 \times 10^{-6}\text{ m}^2$$

and, since $A = \pi r^2$,

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{500 \times 10^{-6}\text{ m}^2}{\pi}} = 12.62 \times 10^{-3}\text{ m} = 12.62\text{ mm}$$

$$d = 2r = 25.2\text{ mm}$$

We conclude that an aluminum rod 26 mm or more in diameter will be adequate.

1.5. AXIAL LOADING; NORMAL STRESS

As we have already indicated, rod BC of the example considered in the preceding section is a two-force member and, therefore, the forces $\mathbf{F}_{BC}$ and $\mathbf{F}'_{BC}$ acting on its ends B and C (Fig. 1.5) are directed along the axis of the rod. We say that the rod is under *axial loading*. An actual example of structural members under axial loading is provided by the members of the bridge truss shown in Fig. 1.9.

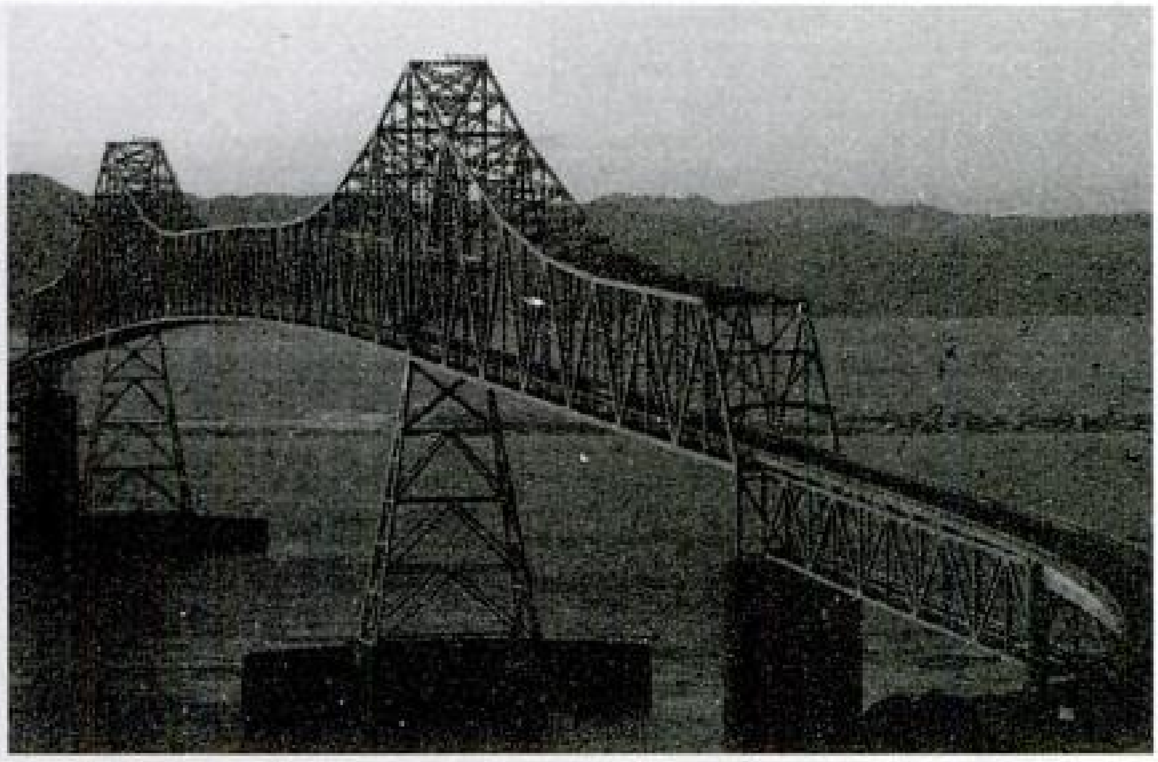


Fig. 1.9 This bridge truss consists of two-force members that may be in tension or in compression.

Returning to rod BC of Fig. 1.5, we recall that the section we passed through the rod to determine the internal force in the rod and the corresponding stress was perpendicular to the axis of the rod; the internal force was therefore normal to the plane of the section (Fig. 1.7) and the corresponding stress is described as a *normal stress*. Thus, formula (1.5) gives us the *normal stress in a member under axial loading*:

$$\sigma = \frac{P}{A} \tag{1.5}$$

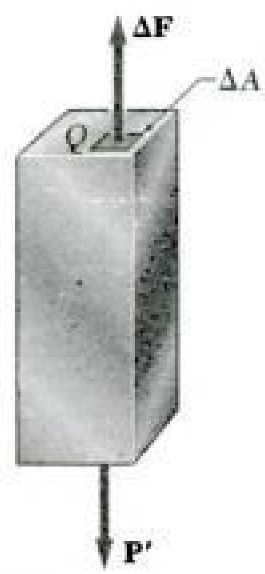


Fig. 1.10

We should also note that, in formula (1.5), σ is obtained by dividing the magnitude P of the resultant of the internal forces distributed over the cross section by the area A of the cross-section; it represents, therefore, the *average value* of the stress over the cross section, rather than the stress at a specific point of the cross section.

To define the stress at a given point Q of the cross section, we should consider a small area ΔA (Fig. 1.10). Dividing the magnitude of $\Delta\mathbf{F}$ by ΔA, we obtain the average value of the stress over ΔA. Letting ΔA approach zero, we obtain the stress at point Q:

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1.6}$$

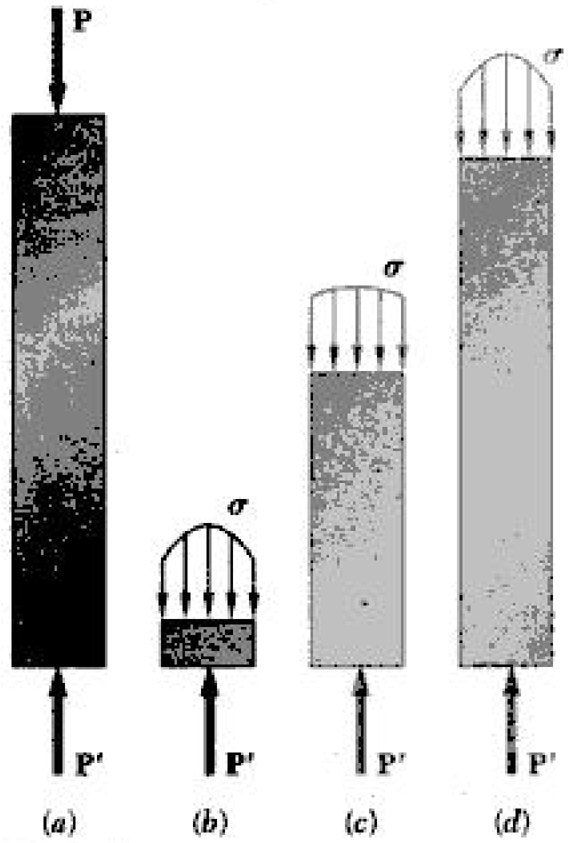


Fig. 1.11

In general, the value obtained for the stress σ at a given point Q of the section is different from the value of the average stress given by formula (1.5), and σ is found to vary across the section. In a slender rod subjected to equal and opposite concentrated loads **P** and **P′** (Fig. 1.11*a*), this variation is small in a section away from the points of application of the concentrated loads (Fig. 1.11*c*), but it is quite noticeable in the neighborhood of these points (Fig. 1.11*b* and *d*).

It follows from Eq. (1.6) that the magnitude of the resultant of the distributed internal forces is

$$\int dF = \int_A \sigma \, dA$$

But the conditions of equilibrium of each of the portions of rod shown in Fig. 1.11 require that this magnitude be equal to the magnitude P of the concentrated loads. We have, therefore,

$$P = \int dF = \int_A \sigma \, dA \tag{1.7}$$

which means that the volume under each of the stress surfaces in Fig. 1.11 must be equal to the magnitude P of the loads. This, however, is the only information that we can derive from our knowledge of statics, regarding the distribution of normal stresses in the various sections of the rod. The actual distribution of stresses in any given section is *statically indeterminate*. To learn more about this distribution, it is necessary to consider the deformations resulting from the particular mode of application of the loads at the ends of the rod. This will be discussed further in Chap. 2.

In practice, it will be assumed that the distribution of normal stresses in an axially loaded member is uniform, except in the immediate vicinity of the points of application of the loads. The value σ of the stress is then equal to σ_{ave} and can be obtained from formula (1.5). However, we should realize that, when we assume a uniform distribution of stresses in the section, i.e., when we assume that the internal forces are uniformly distributed across the section, it follows from elementary statics† that the resultant **P** of the internal forces must be applied at the centroid C of the section (Fig. 1.12). This means that *a uniform distribution of stress is possible only if the line of action of the concentrated loads* **P** *and* **P′** *passes through the centroid of the section considered* (Fig. 1.13). This type of loading is called *centric loading* and will be assumed to take place in all straight two-force members found in trusses and pin-connected structures, such as the one considered in Fig. 1.1. However, if a two-force member

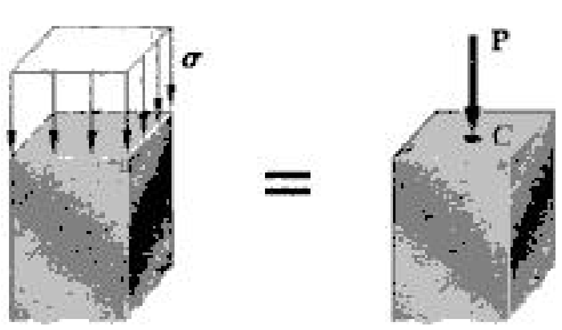


Fig. 1.12

†See Ferdinand P. Beer and E. Russell Johnston, Jr., *Mechanics for Engineers*, 4th ed., McGraw-Hill, New York, 1987, or *Vector Mechanics for Engineers*, 6th ed., McGraw-Hill, New York, 1996, secs. 5.2 and 5.3.

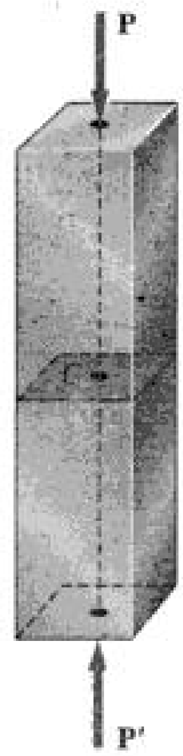


Fig. 1.13

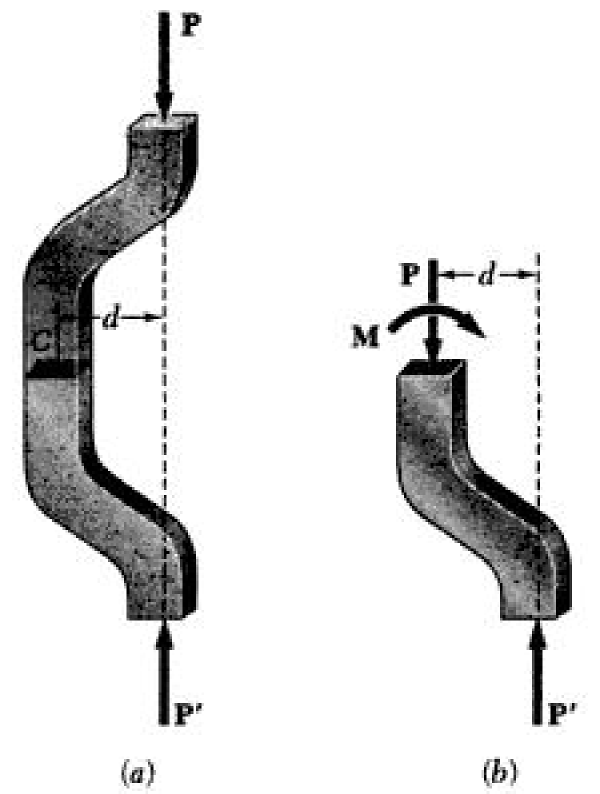


Fig. 1.14

is loaded axially, but *eccentrically* as shown in Fig. 1.14*a*, we find from the conditions of equilibrium of the portion of member shown in Fig. 1.14*b* that the internal forces in a given section must be equivalent to a force **P** applied at the centroid of the section and a couple **M** of moment $M = Pd$. The distribution of forces—and, thus, the corresponding distribution of stresses—*cannot be uniform*. Nor can the distribution of stresses be symmetric as shown in Fig. 1.11. This point will be discussed in detail in Chap. 4.

1.6. SHEARING STRESS

The internal forces and the corresponding stresses discussed in Secs. 1.2 and 1.3 were normal to the section considered. A very different type of stress is obtained when transverse forces **P** and **P′** are applied to a member *AB* (Fig. 1.15). Passing a section at *C* between the points of application of the two forces (Fig. 1.16*a*), we obtain the diagram of portion *AC* shown in Fig. 1.16*b*. We conclude that internal forces must exist in the plane of the section, and that their resultant is equal to **P**. These elementary internal forces are called *shearing forces*, and the magnitude *P* of their resultant is the *shear* in the section. Dividing the shear

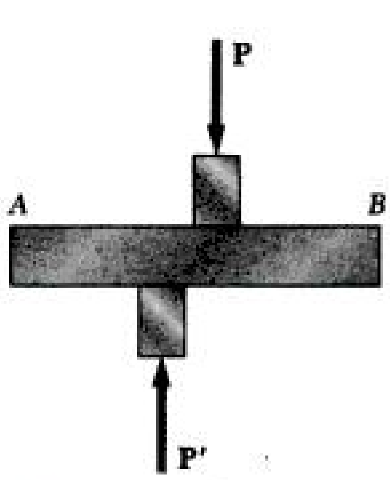


Fig. 1.15

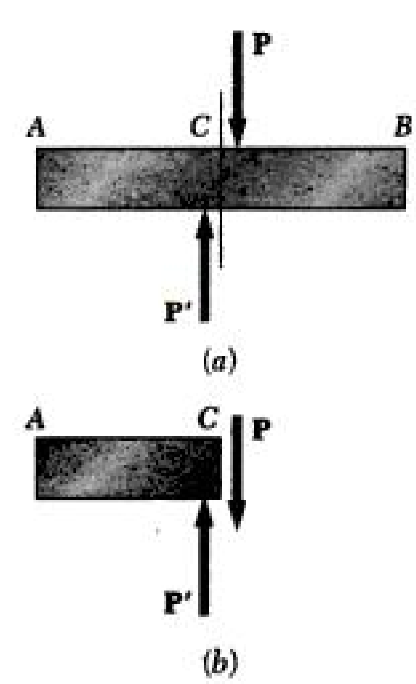


Fig. 1.16

P by the area A of the cross section, we obtain the *average shearing stress* in the section. Denoting the shearing stress by the Greek letter τ (tau), we write

$$\tau_{\text{ave}} = \frac{P}{A} \tag{1.8}$$

It should be emphasized that the value obtained is an average value of the shearing stress over the entire section. Contrary to what we said earlier for normal stresses, the distribution of shearing stresses across the section *cannot* be assumed uniform. As you will see in Chap. 6, the actual value τ of the shearing stress varies from zero at the surface of the member to a maximum value $\tau_{\max}$ that may be much larger than the average value τ_{ave}.

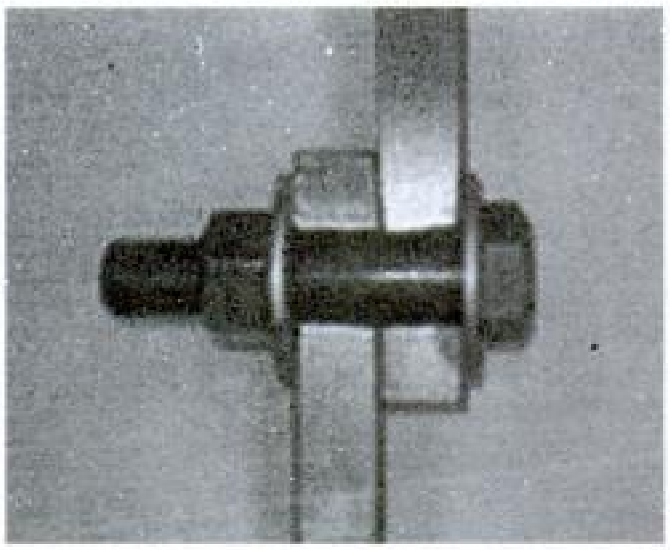


Fig. 1.17 Cutaway view of a connection with a bolt in shear.

Shearing stresses are commonly found in bolts, pins, and rivets used to connect various structural members and machine components (Fig. 1.17). Consider the two plates A and B, which are connected by a bolt CD (Fig. 1.18). If the plates are subjected to tension forces of magnitude F, stresses will develop in the section of bolt corresponding to the plane EE'. Drawing the diagrams of the bolt and of the portion located above the plane EE' (Fig. 1.19), we conclude that the shear P in the section is equal to F. The average shearing stress in the section is obtained, according to formula (1.8), by dividing the shear $P = F$ by the area A of the cross section:

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A} \tag{1.9}$$

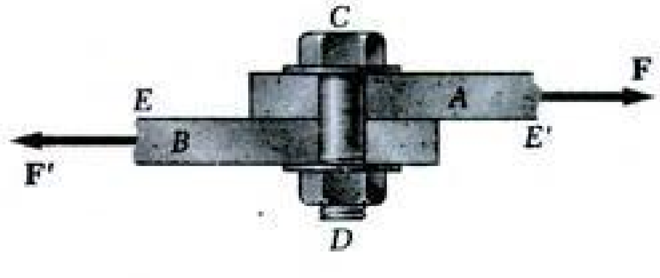


Fig. 1.18

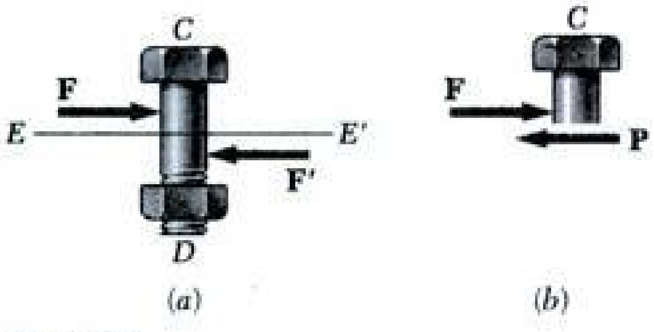


Fig. 1.19

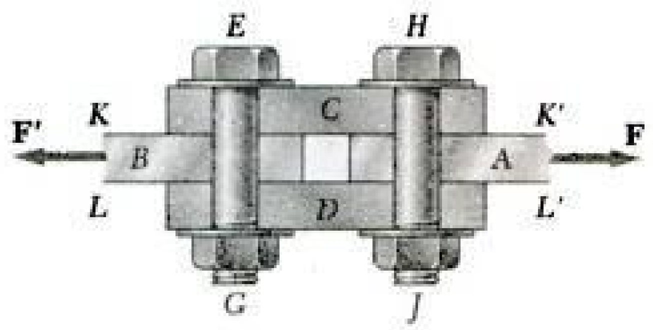


Fig. 1.20

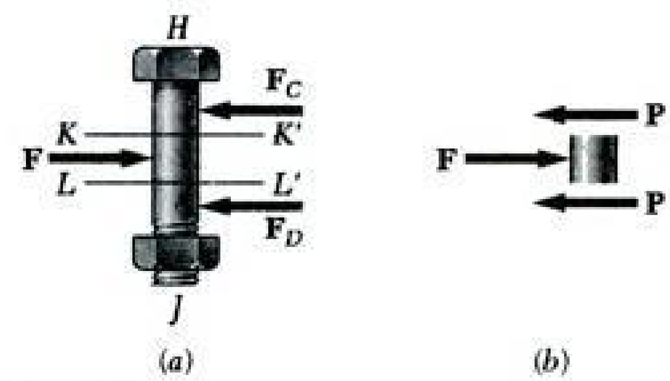


Fig. 1.21

The bolt we have just considered is said to be in *single shear*. Different loading situations may arise, however. For example, if splice plates C and D are used to connect plates A and B (Fig. 1.20), shear will take place in bolt HJ in each of the two planes KK' and LL' (and similarly in bolt EG). The bolts are said to be in *double shear*. To determine the average shearing stress in each plane, we draw free-body diagrams of bolt HJ and of the portion of bolt located between the two planes (Fig. 1.21). Observing that the shear P in each of the sections is $P = F/2$, we conclude that the average shearing stress is

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F/2}{A} = \frac{F}{2A} \tag{1.10}$$

1.7. BEARING STRESS IN CONNECTIONS

Bolts, pins, and rivets create stresses in the members they connect, along the *bearing surface*, or surface of contact. For example, consider again the two plates A and B connected by a bolt CD that we have discussed in the preceding section (Fig. 1.18). The bolt exerts on plate A a force **P** equal and opposite to the force **F** exerted by the plate on the bolt (Fig. 1.22). The force **P** represents the resultant of elementary forces distributed on the inside surface of a half-cylinder of diameter d and of length t equal to the thickness of the plate. Since the distribution of these forces—and of the corresponding stresses—is quite complicated, one uses in practice an average nominal value σ_b of the stress, called the *bearing stress*, obtained by dividing the load P by the area of the rectangle representing the projection of the bolt on the plate section (Fig. 1.23). Since this area is equal to td, where t is the plate thickness and d the diameter of the bolt, we have

$$\sigma_b = \frac{P}{A} = \frac{P}{td} \tag{1.11}$$

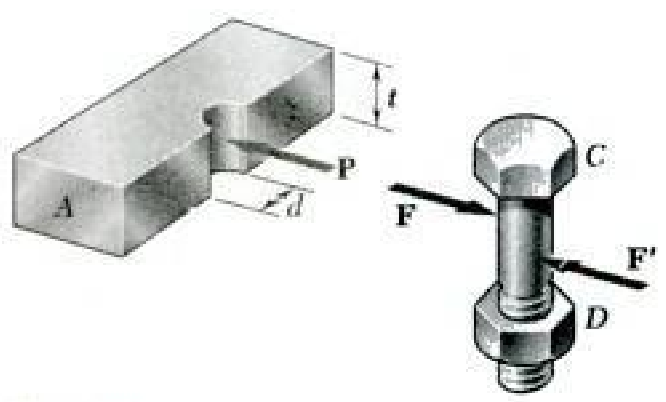


Fig. 1.22

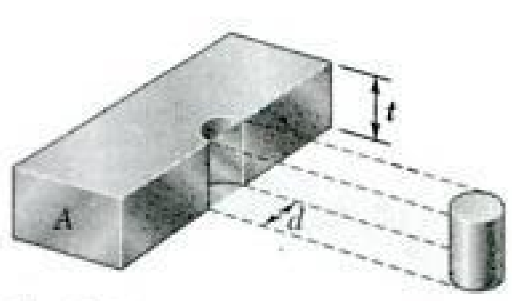


Fig. 1.23

1.8. APPLICATION TO THE ANALYSIS AND DESIGN OF SIMPLE STRUCTURES

We are now in a position to determine the stresses in the members and connections of various simple two-dimensional structures and, thus, to design such structures.

As an example, let us return to the structure of Fig. 1.1 that we have already considered in Sec. 1.2 and let us specify the supports and connections at *A*, *B*, and *C*. As shown in Fig. 1.24, the 20-mm-diameter rod *BC* has flat ends of 20 × 40-mm rectangular cross section, while boom *AB* has a 30 × 50-mm rectangular cross section and is fitted with a clevis at end *B*. Both members are connected at *B* by a pin from which the 30-kN load is suspended by means of a U-shaped bracket. Boom *AB* is supported at *A* by a pin fitted into a double bracket, while rod *BC* is connected at *C* to a single bracket. All pins are 25 mm in diameter.

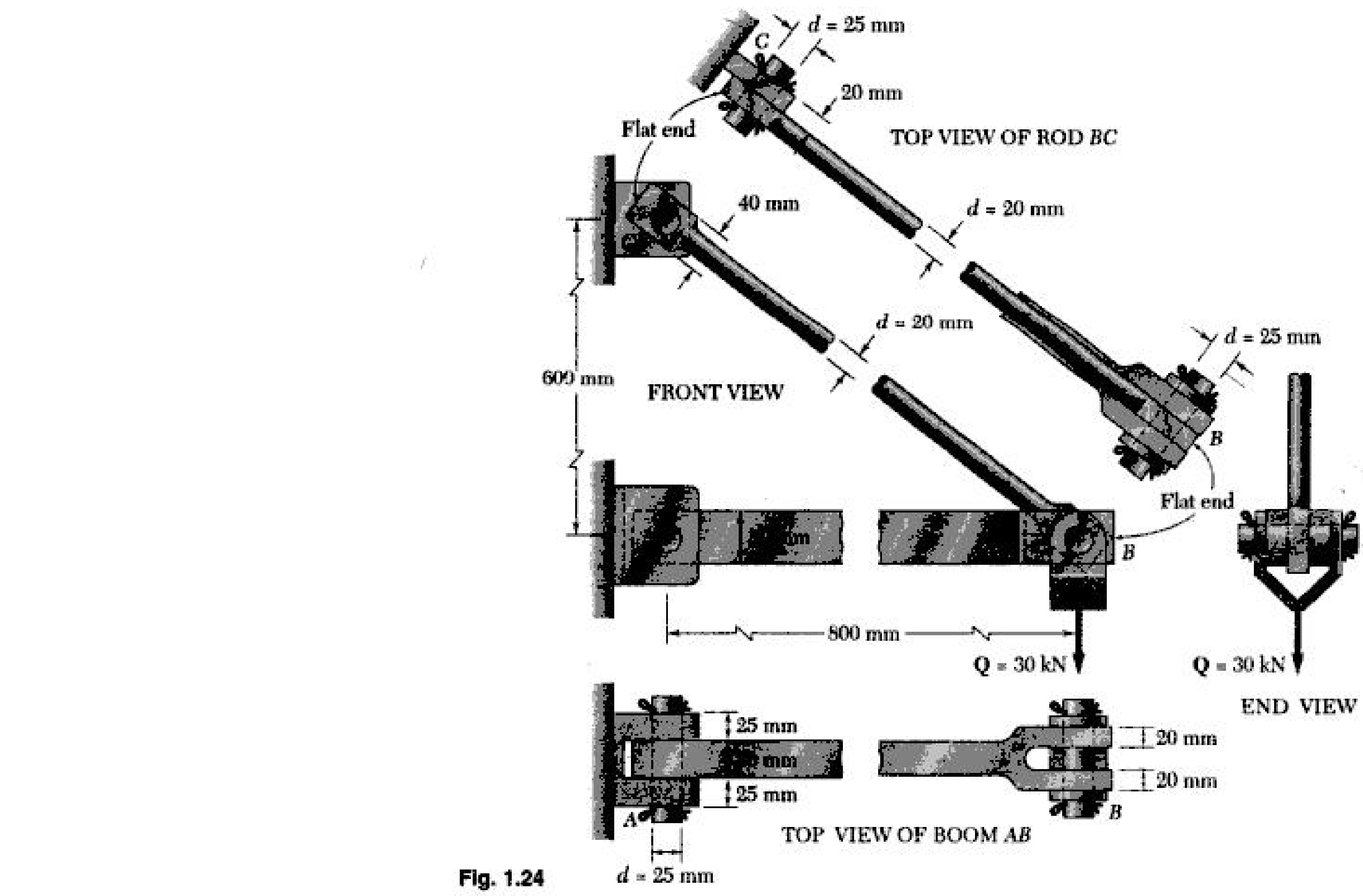


Fig. 1.24

a. Determination of the Normal Stress in Boom *AB* and Rod *BC*. As we found in Secs. 1.2 and 1.4, the force in rod *BC* is $F_{BC} = 50$ kN (tension) and the area of its circular cross section is $A = 314 \times 10^{-6}\,m^2$; the corresponding average normal stress is $\sigma_{BC} = +159$ MPa. However, the flat parts of the rod are also under tension and at the narrowest section, where a hole is located, we have

$$A = (20\text{ mm})(40\text{ mm} - 25\text{ mm}) = 300 \times 10^{-6}\text{ m}^2$$

The corresponding average value of the stress, therefore, is

$$(\sigma_{BC})_{\text{end}} = \frac{P}{A} = \frac{50 \times 10^3\text{ N}}{300 \times 10^{-6}\text{ m}^2} = 167\text{ MPa}$$

Note that this is an *average value*; close to the hole, the stress will actually reach a much larger value, as you will see in Sec. 2.18. It is clear that, under an increasing load, the rod will fail near one of the holes rather than in its cylindrical portion; its design, therefore, could be improved by increasing the width or the thickness of the flat ends of the rod.

Turning now our attention to boom AB, we recall from Sec. 1.2 that the force in the boom is $F_{AB} = 40$ kN (compression). Since the area of the boom's rectangular cross section is $A = 30\text{ mm} \times 50\text{ mm} = 1.5 \times 10^{-3}\text{ m}^2$, the average value of the normal stress in the main part of the rod, between pins A and B, is

$$\sigma_{AB} = -\frac{40 \times 10^3\text{ N}}{1.5 \times 10^{-3}\text{ m}^2} = -26.7 \times 10^6\text{ Pa} = -26.7\text{ MPa}$$

Note that the sections of minimum area at A and B are not under stress, since the boom is in compression, and, therefore, *pushes* on the pins (instead of *pulling* on the pins as rod BC does).

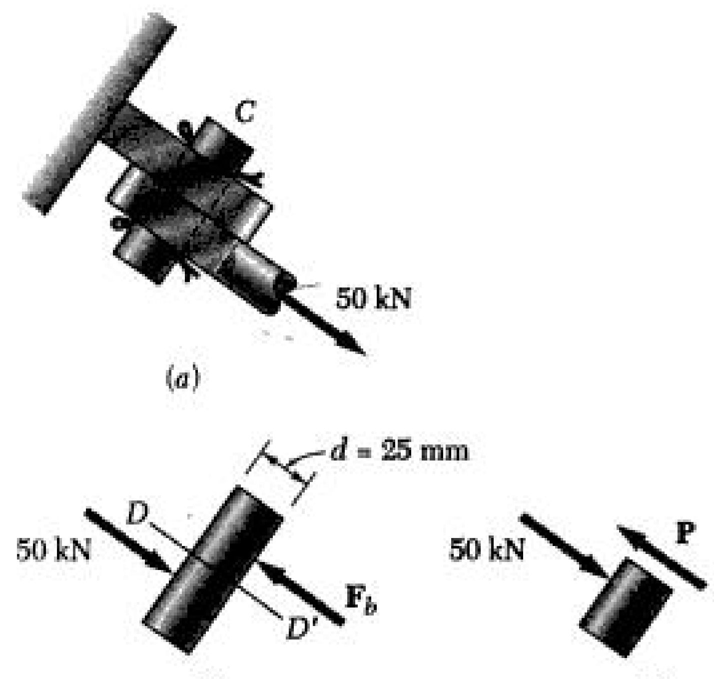


Fig. 1.25

b. Determination of the Shearing Stress in Various Connections. To determine the shearing stress in a connection such as a bolt, pin, or rivet, we first clearly show the forces exerted by the various members it connects. Thus, in the case of pin C of our example (Fig. 1.25*a*), we draw Fig. 1.25*b*, showing the 50-kN force exerted by member BC on the pin, and the equal and opposite force exerted by the bracket. Drawing now the diagram of the portion of the pin located below the plane DD' where shearing stresses occur (Fig. 1.25*c*), we conclude that the shear in that plane is $P = 50$ kN. Since the cross-sectional area of the pin is

$$A = \pi r^2 = \pi\left(\frac{25\text{ mm}}{2}\right)^2 = \pi(12.5 \times 10^{-3}\text{ m})^2 = 491 \times 10^{-6}\text{ m}^2$$

we find that the average value of the shearing stress in the pin at C is

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{50 \times 10^3\text{ N}}{491 \times 10^{-6}\text{ m}^2} = 102\text{ MPa}$$

Considering now the pin at A (Fig. 1.26), we note that it is in double shear. Drawing the free-body diagrams of the pin and of the portion of pin located between the planes DD' and EE' where shearing stresses occur, we conclude that $P = 20$ kN and that

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{20\text{ kN}}{491 \times 10^{-6}\text{ m}^2} = 40.7\text{ MPa}$$

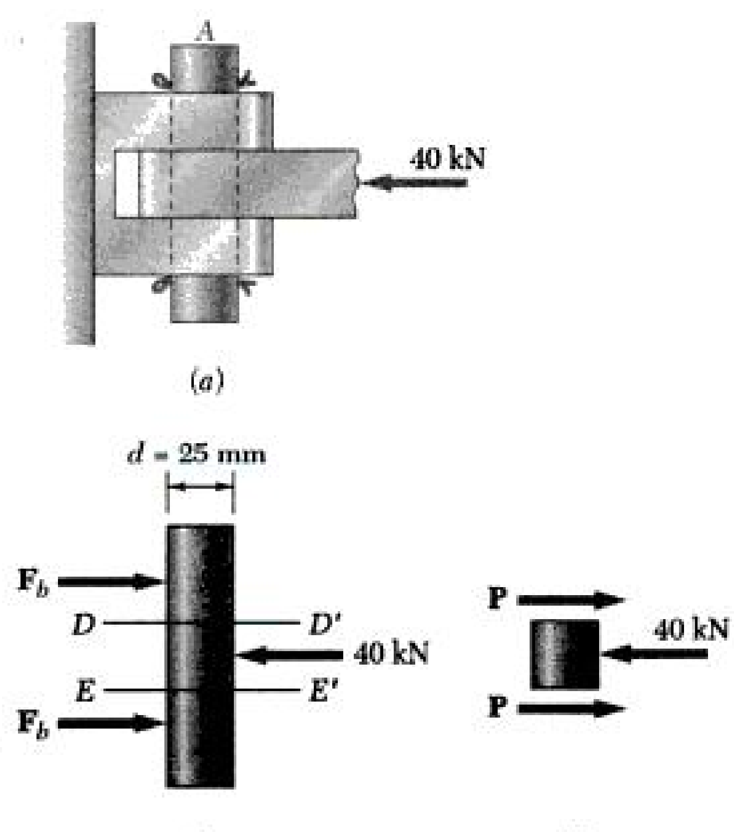


Fig. 1.26

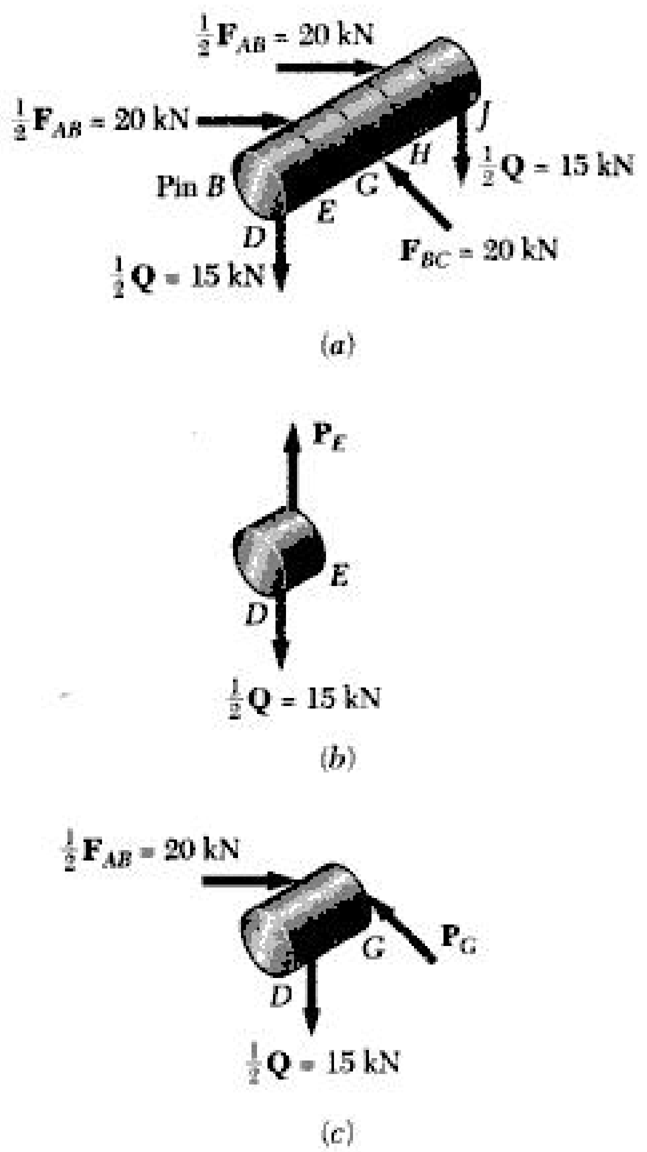


Fig. 1.27

Considering the pin at B (Fig. 1.27a), we note that the pin may be divided into five portions which are acted upon by forces exerted by the boom, rod, and bracket. Considering successively the portions DE (Fig. 1.27b) and DG (Fig. 1.27c), we conclude that the shear in section E is $P_E = 15$ kN, while the shear in section G is $P_G = 25$ kN. Since the loading of the pin is symmetric, we conclude that the maximum value of the shear in pin B is $P_G = 25$ kN, and that the largest shearing stresses occur in sections G and H, where

$$\tau_{\text{ave}} = \frac{P_G}{A} = \frac{25\text{ kN}}{491 \times 10^{-6}\text{ m}^2} = 50.9\text{ MPa}$$

c. Determination of the Bearing Stresses. To determine the nominal bearing stress at A in member AB, we use formula (1.11) of Sec. 1.7. From Fig. 1.24, we have $t = 30$ mm and $d = 25$ mm. Recalling that $P = F_{AB} = 40$ kN, we have

$$\sigma_b = \frac{P}{td} = \frac{40\text{ kN}}{(30\text{ mm})(25\text{ mm})} = 53.3\text{ MPa}$$

To obtain the bearing stress in the bracket at A, we use $t = 2(25\text{ mm}) = 50$ mm and $d = 25$ mm:

$$\sigma_b = \frac{P}{td} = \frac{40\text{ kN}}{(50\text{ mm})(25\text{ mm})} = 32.0\text{ MPa}$$

The bearing stresses at B in member AB, at B and C in member BC, and in the bracket at C are found in a similar way.

1.9. METHOD OF PROBLEM SOLUTION

You should approach a problem in mechanics of materials as you would approach an actual engineering situation. By drawing on your own experience and intuition, you will find it easier to understand and formulate the problem. Once the problem has been clearly stated, however, there is no place in its solution for your particular fancy. Your solution must be based on the fundamental principles of statics and on the principles you will learn in this course. Every step you take must be justified on that basis, leaving no room for your "intuition." After an answer has been obtained, it should be checked. Here again, you may call upon your common sense and personal experience. If not completely satisfied with the result obtained, you should carefully check your formulation of the problem, the validity of the methods used in its solution, and the accuracy of your computations.

The *statement* of the problem should be clear and precise. It should contain the given data and indicate what information is required. A simplified drawing showing all essential quantities involved should be included. The solution of most of the problems you will encounter will necessitate that you first determine the *reactions at supports* and *inter-*

nal forces and couples. This will require the drawing of one or several *free-body diagrams,* as was done in Sec. 1.2, from which you will write *equilibrium equations*. These equations can be solved for the unknown forces, from which the required *stresses* and *deformations* will be computed.

After the answer has been obtained, it should be *carefully checked*. Mistakes in *reasoning* can often be detected by carrying the units through your computations and checking the units obtained for the answer. For example, in the design of the rod discussed in Sec. 1.4, we found, after carrying the units through our computations, that the required diameter of the rod was expressed in millimeters, which is the correct unit for a dimension; if another unit had been found, we would have known that some mistake had been made.

Errors in *computation* will usually be found by substituting the numerical values obtained into an equation which has not yet been used and verifying that the equation is satisfied. The importance of correct computations in engineering cannot be overemphasized.

1.10. NUMERICAL ACCURACY

The accuracy of the solution of a problem depends upon two items: (1) the accuracy of the given data and (2) the accuracy of the computations performed.

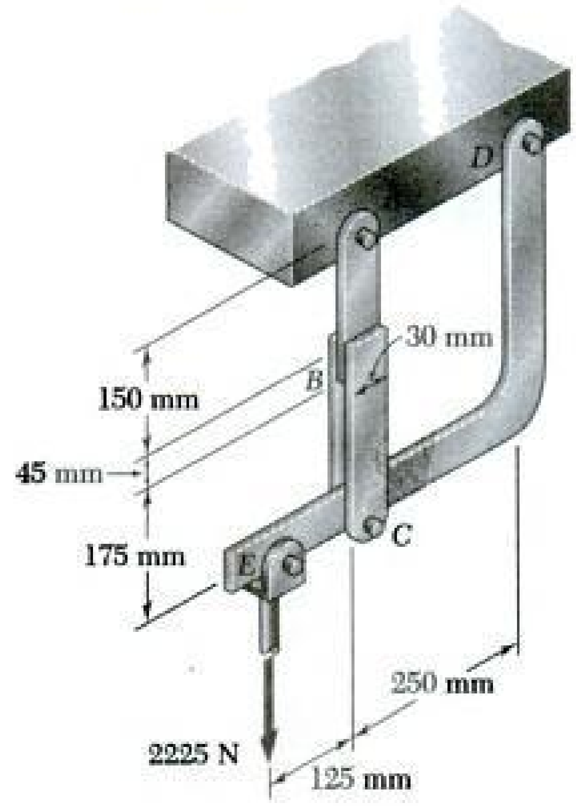
The solution cannot be more accurate than the less accurate of these two items. For example, if the loading of a beam is known to be 300 kN with a possible error of 400 N either way, the relative error which measures the degree of accuracy of the data is

$$\frac{400\ \text{N}}{300\ \text{kN}} = 0.0013 = 0.13\%$$

In computing the reaction at one of the beam supports, it would then be meaningless to record it as 57288 N. The accuracy of the solution cannot be greater than 0.13%, no matter how accurate the computations are, and the possible error in the answer may be as large as $(0.13/100)(57288\ \text{N}) = 74\ \text{N}$. The answer should be properly recorded as $57280 \pm 74\ \text{N}$.

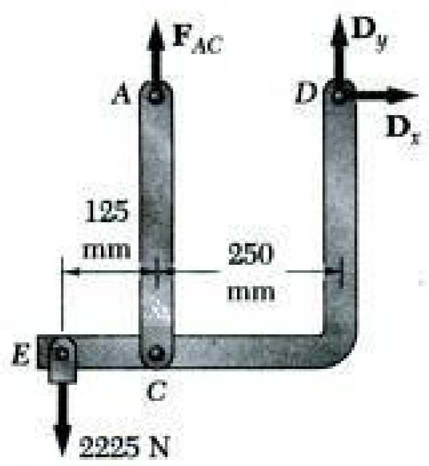
In engineering problems, the data are seldom known with an accuracy greater than 0.2%. It is therefore seldom justified to write the answers to such problems with an accuracy greater than 0.2 percent. A practical rule is to use 4 figures to record numbers beginning with a "1" and 3 figures in all other cases. Unless otherwise indicated, the data given in a problem should be assumed known with a comparable degree of accuracy. A force of 180 N, for example, should be read 180.0 N, and a force of 60 N should be read 60.00 N.

Pocket calculators and computers are widely used by practicing engineers and engineering students. The speed and accuracy of these devices facilitate the numerical computations in the solution of many problems. However, students should not record more significant figures than can be justified merely because they are easily obtained. As noted above, an accuracy greater than 0.2% is seldom necessary or meaningful in the solution of practical engineering problems.



SAMPLE PROBLEM 1.1

In the hanger shown the upper portion of link *ABC* is 10 mm thick and the lower portions are each 6 mm thick. Epoxy resin is used to bond the upper and lower portions together at *B*. The pin at *A* is of 10 mm diameter while a 6.0 mm diameter pin is used at *C*. Determine (*a*) the shearing stress in pin *A*, (*b*) the shearing stress in pin *C*, (*c*) the largest normal stress in link *ABC*, (*d*) the average shearing stress on the bonded surfaces at *B*, (*e*) the bearing stress in the link at *C*.



SOLUTION

Free Body: Entire Hanger. Since the link *ABC* is a two-force member, the reaction at *A* is vertical; the reaction at *D* is represented by its components $\mathbf{D}_x$ and $\mathbf{D}_y$. We write

$$+\circlearrowleft \Sigma M_D = 0: \qquad 2225\text{ N}(375\text{ mm}) - F_{AC}\,(250\text{ mm}) = 0$$

$$F_{AC} = +3337.5\text{ N} \qquad F_{AC} = 3337.5\text{ N} \quad \textit{tension}$$

***a.* Shearing Stress in Pin *A*.** Since this 10 mm diameter pin is in single shear, we write

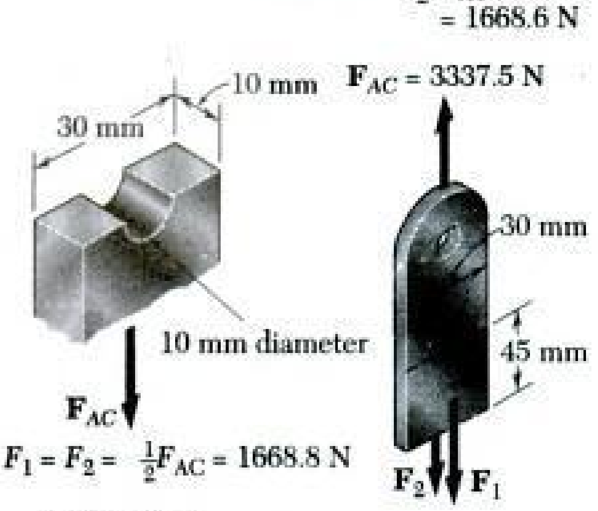
$$\tau_A = \frac{F_{AC}}{A} = \frac{3337.5\text{ N}}{\frac{1}{4}\pi\,(10\text{ mm})^2}$$

***b.* Shearing Stress in Pin *C*.** Since this 6 mm-diameter pin is in double shear, we write

$$\tau_C = \frac{\frac{1}{2}F_{AC}}{A} = \frac{1668.6\text{ N}}{\frac{1}{4}\pi\,(6\text{ mm})^2}$$

***c.* Largest Normal Stress in Link *ABC*.** The largest stress is found where the area is smallest; this occurs at the cross section at *A* where the 10 mm hole is located. We have

$$\sigma_A = \frac{F_{AC}}{A_{net}} = \frac{3337.5\text{ N}}{(10\text{ mm})\,(30\text{ mm} - 10\text{ mm})} = \frac{3337.5\text{ N}}{200\text{ mm}^2}$$

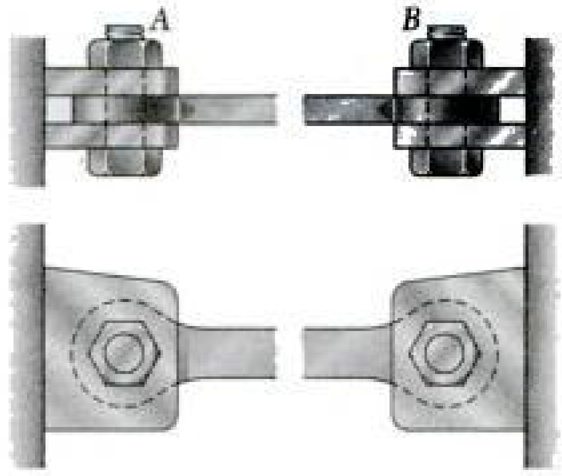


***d.* Average Shearing Stress at *B*.** We note that bonding exists on both sides of the upper portion of the link and that the shear force on each side is $F_1 = (3337.5\text{ N})/2 = 1668.8$ N. The average shearing stress on each surface is thus

$$\tau_B = \frac{F_1}{A} = \frac{1668.8\text{ N}}{(30\text{ mm})(45\text{ mm})}$$

***e.* Bearing Stress in Link at *C*.** For each portion of the link, $F_1 = 1668.8$ N and the nominal bearing area is (6 mm) (6 mm) = 36 mm².

$$\sigma_b = \frac{F_1}{A} = \frac{1668.8\text{ N}}{36\text{ mm}^2}$$



SAMPLE PROBLEM 1.2

The steel tie bar shown is to be designed to carry a tension force of magnitude $P = 120$ kN when bolted between double brackets at A and B. The bar will be fabricated from 20-mm-thick plate stock. For the grade of steel to be used the maximum allowable stresses are: $\sigma = 175$ MPa, $\tau = 100$ MPa, $\sigma_b = 350$ MPa. Design the tie bar by determining the required values of (*a*) the diameter d of the bolt, (*b*) the dimension b at each end of the bar, (*c*) the dimension h of the bar.

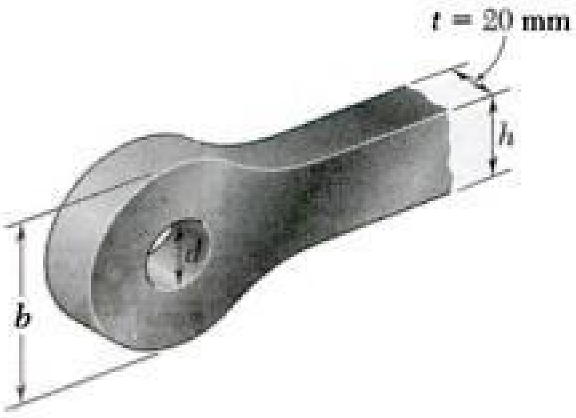
$F_1 = \frac{1}{2}P$

SOLUTION

***a.* Diameter of the Bolt.** Since the bolt is in double shear, $F_1 = \frac{1}{2}P = 60$ kN.

$$\tau = \frac{F_1}{A} = \frac{60 \text{ kN}}{\frac{1}{4}\pi d^2} \qquad 100 \text{ MPa} = \frac{60 \text{ kN}}{\frac{1}{4}\pi d^2} \qquad d = 27.6 \text{ mm}$$

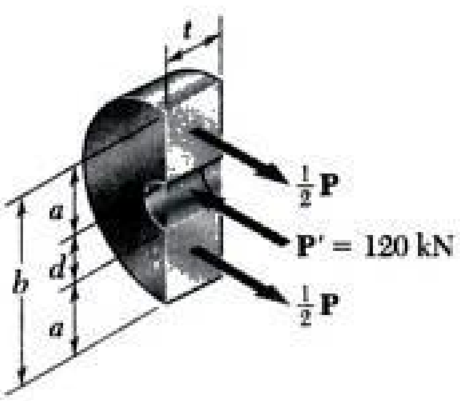
We will use $d = 28$ mm ◀



At this point we check the bearing stress between the 20-mm-thick plate and the 28-mm-diameter bolt.

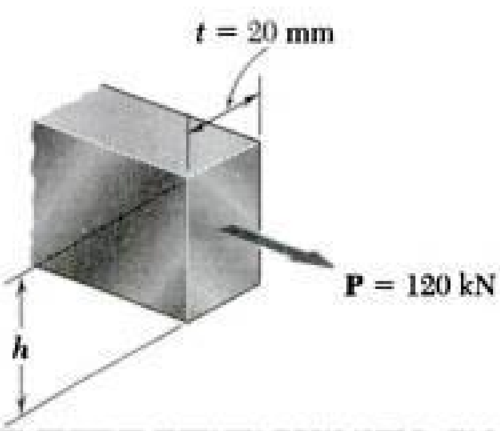
$$\tau_b = \frac{P}{td} = \frac{120 \text{ kN}}{(0.020 \text{ m})(0.028 \text{ m})} = 214 \text{ MPa} < 350 \text{ MPa} \qquad \text{OK}$$

***b.* Dimension *b* at Each End of the Bar.** We consider one of the end portions of the bar. Recalling that the thickness of the steel plate is $t = 20$ mm and that the average tensile stress must not exceed 175 MPa, we write



$$\sigma = \frac{\frac{1}{2}P}{ta} \qquad 175 \text{ MPa} = \frac{60 \text{ kN}}{(0.02 \text{ m})a} \qquad a = 17.14 \text{ mm}$$

$$b = d + 2a = 28 \text{ mm} + 2(17.14 \text{ mm}) \qquad b = 62.3 \text{ mm} \blacktriangleleft$$



***c.* Dimension *h* of the Bar.** Recalling that the thickness of the steel plate is $t = 20$ mm, we have

$$\sigma = \frac{P}{th} \qquad 175 \text{ MPa} = \frac{120 \text{ kN}}{(0.020 \text{ m})h} \qquad h = 34.3 \text{ mm}$$

We will use $h = 35$ mm ◀

PROBLEMS

1.1 Two solid cylindrical rods *AB* and *BC* are welded together at *B* and loaded as shown. Knowing that $d_1 = 30$ mm and $d_2 = 50$ mm, find the average normal stress in the midsection of (*a*) rod *AB*, (*b*) rod *BC*.

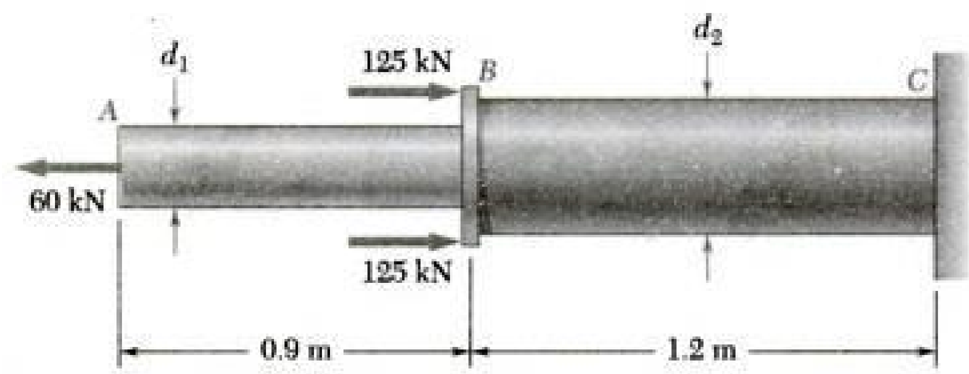


Fig. P1.1 and P1.2

1.2 Two solid cylindrical rods *AB* and *BC* are welded together at *B* and loaded as shown. Knowing that the average normal stress must not exceed 150 MPa in either rod, determine the smallest allowable values of the diameters d_1 and d_2.

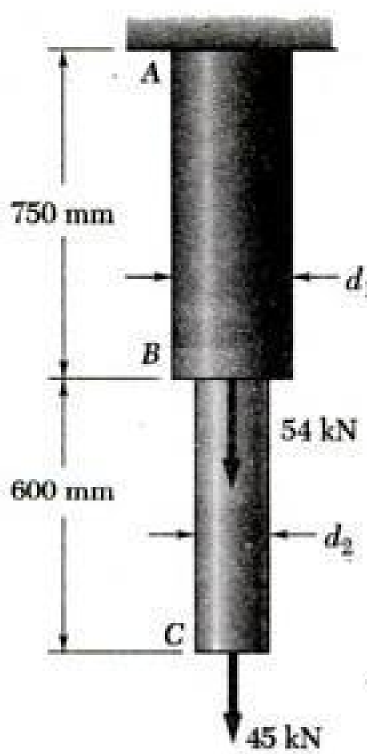


Fig. P1.3 and P1.4

1.3 Two solid cylindrical rods *AB* and *BC* are welded together at *B* and loaded as shown. Knowing that $d_1 = 30$ mm and $d_2 = 20$ mm find the norm:' stress at the midpoint of (*a*) rod *AB*, (*b*) rod *BC*.

1.4 Two solid cylindrical rods *AB* and *BC* are welded together at *B* and loaded as shown. Knowing that the normal stress must not exceed 172 MPa in either rod, determine the smallest allowable values of the diameters d_1 and d_2.

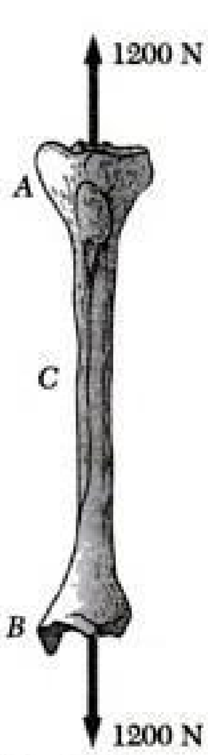


Fig. *P1.5*

1.5 A strain gage located at *C* on the surface of bone *AB* indicates that the average normal stress in the bone is 3.80 MPa when the bone is subjected to two 1200-N forces as shown. Assuming the cross section of the bone at *C* to be annular and knowing that its outer diameter is 25 mm, determine the inner diameter of the bone's cross section at *C*.

1.6 Two steel plates are to be held together by means of 6 mm-diameter high-strength steel bolts fitting snugly inside cylindrical brass spacers. Knowing that the average normal stress must not exceed 200 MPa in the bolts and 120 MPa in the spacers, determine the outer diameter of the spacers that yields the most economical and safe design.

Fig. *P1.6*

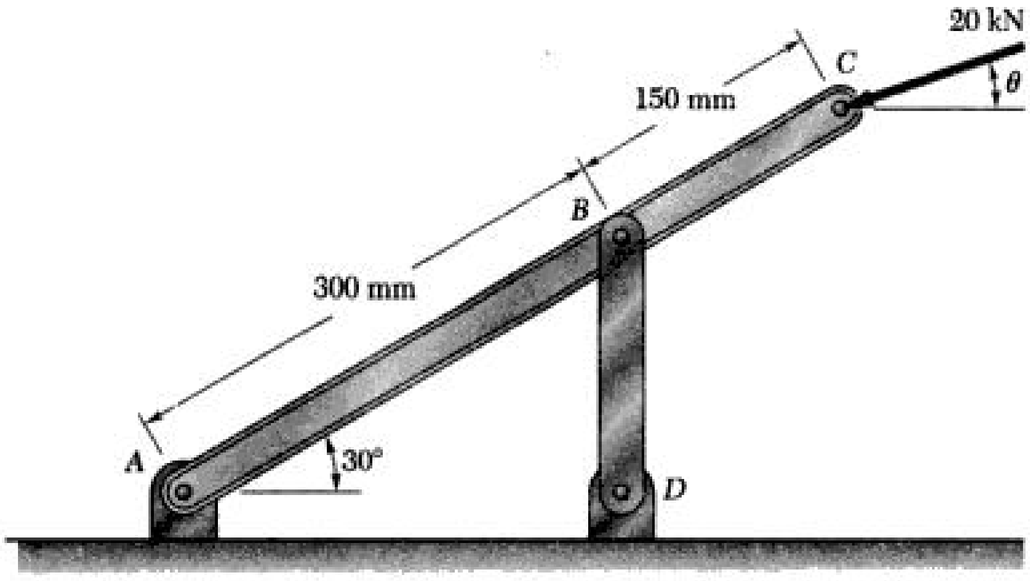


Fig. P1.7

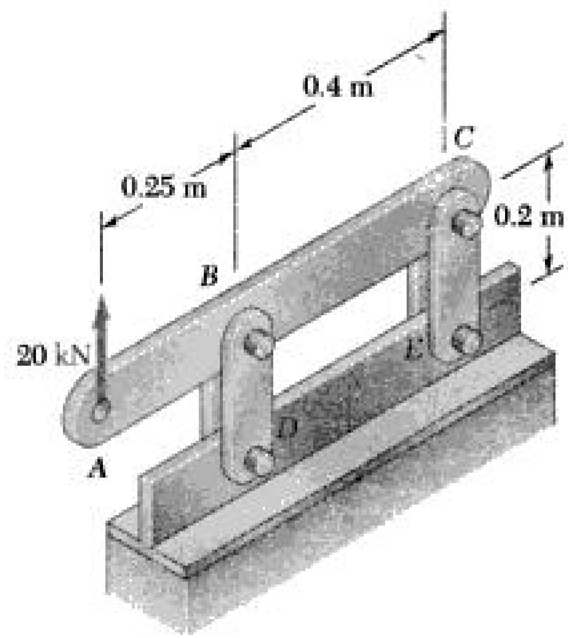


Fig. P1.8

1.7 Link BD consists of a single bar 30 mm wide and 12 mm thick. Knowing that each pin has a 10-mm diameter, determine the maximum value of the average normal stress in link BD if (*a*) $\theta = 0$, (*b*) $\theta = 90°$.

1.8 Each of the four vertical links has an 8×36-mm uniform rectangular cross section and each of the four pins has a 16-mm diameter. Determine the maximum value of the average normal stress in the links connecting (*a*) points B and D, (*b*) points C and E.

1.9 Two horizontal 22 kN forces are applied to pin B of the assembly shown. Knowing that a pin of 20 mm diameter is used at each connection, determine the maximum value of the average normal stress (*a*) in link AB, (*b*) in link BC.

1.10 The frame shown consists of *four* wooden members, ABC, DEF, BE, and CF. Knowing that each member has a 50×100 mm rectangular cross section and that each pin has a 12 mm diameter, determine the maximum value of the average normal stress (*a*) in member BE, (*b*) in member CF.

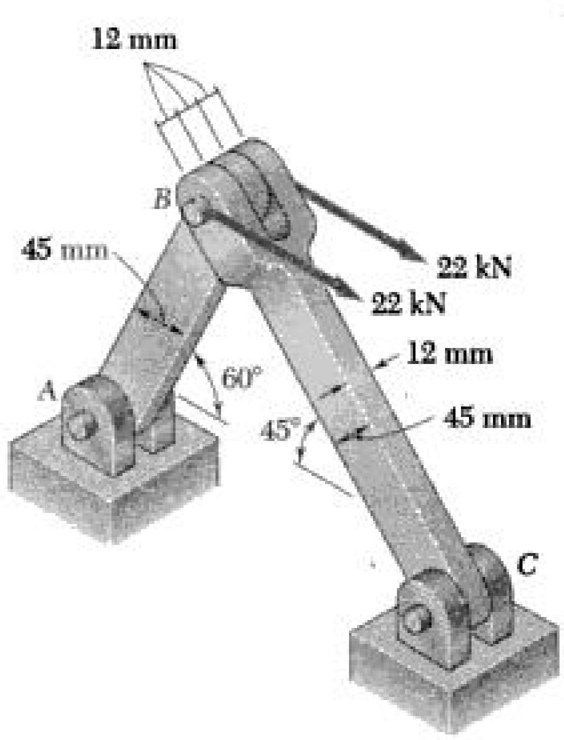


Fig. P1.9

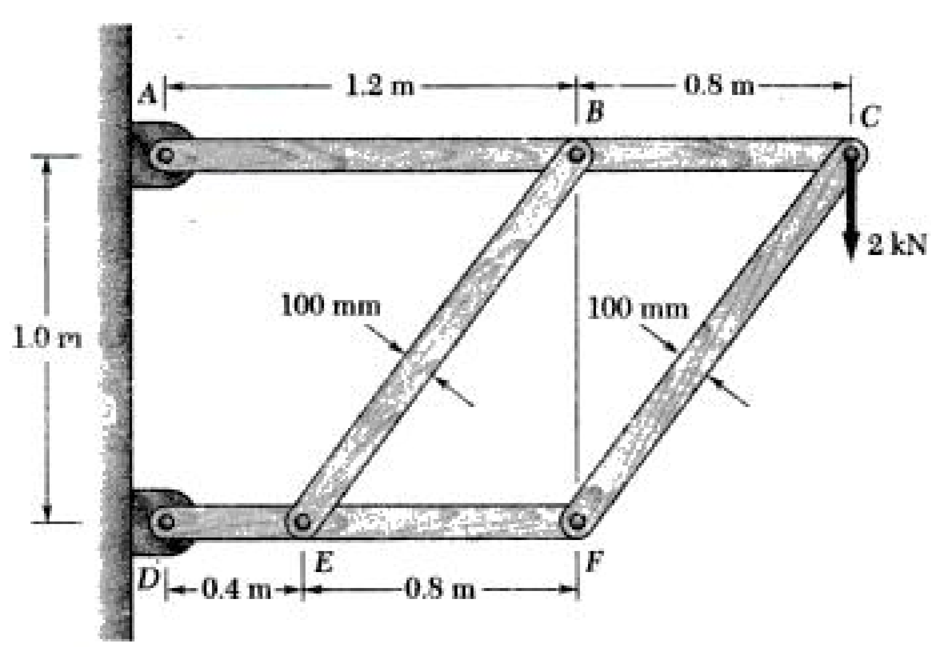


Fig. P1.10

1.11 For the Pratt bridge truss and loading shown, determine the average normal stress in member *BE*, knowing that the cross-sectional area of that member is 3780 mm^2.

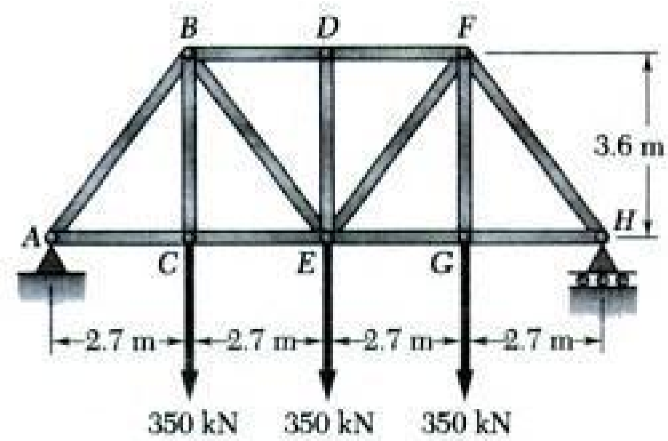


Fig. *P1.11* and *P1.12*

1.12 Knowing that the average normal stress in member *CE* of the Pratt bridge truss shown must not exceed 144 MPa for the given loading, determine the cross-sectional area of that member which will yield the most economical and safe design. Assume that both ends of the member will be adequately reinforced.

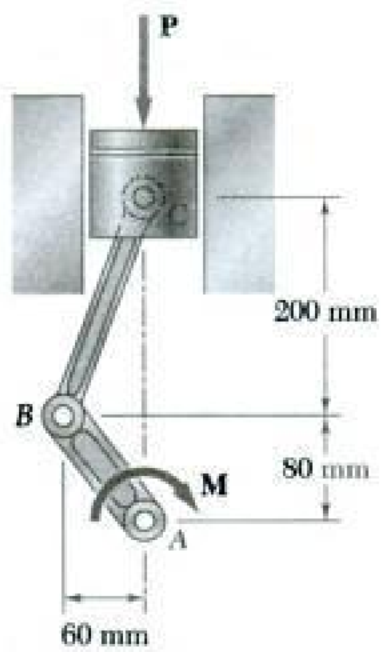


Fig. P1.13

1.13 A couple **M** of magnitude 1500 N · m is applied to the crank of an engine. For the position shown, determine (*a*) the force **P** required to hold the engine system in equilibrium, (*b*) the average normal stress in the connecting rod *BC*, which has a 450-mm^2 uniform cross section.

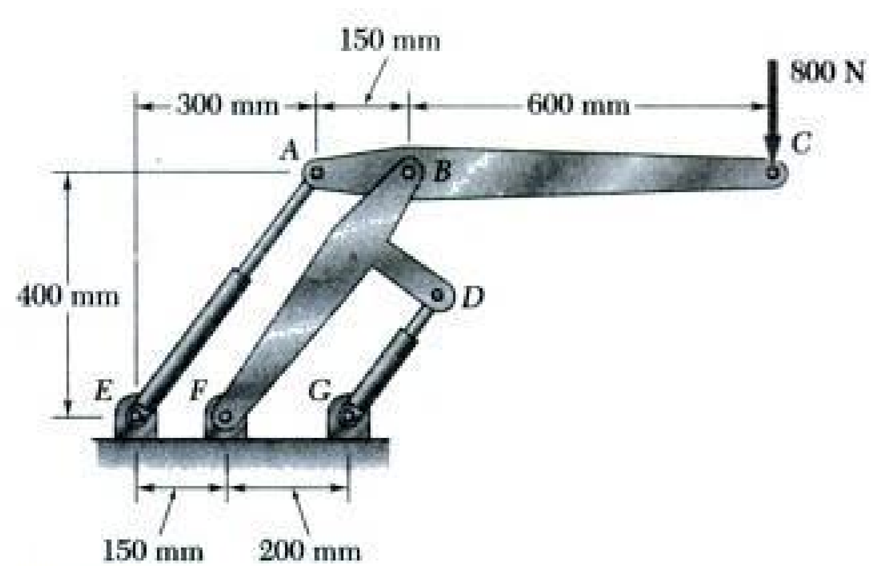


Fig. P1.14

1.14 Two hydraulic cylinders are used to control the position of the robotic arm *ABC*. Knowing that the control rods attached at *A* and *D* each have a 20-mm diameter and happen to be parallel in the position shown, determine the average normal stress in (*a*) member *AE*, (*b*) member *DG*.

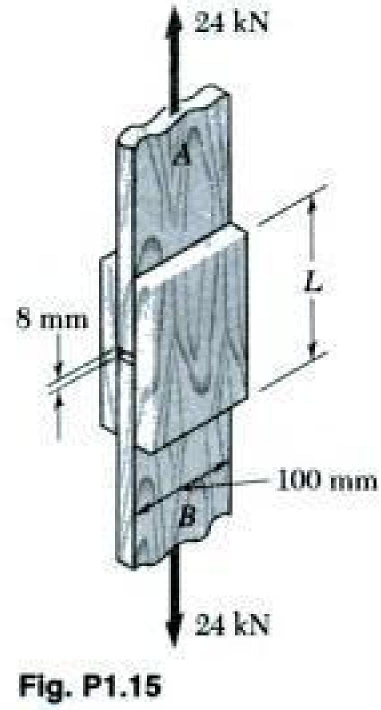


Fig. P1.15

1.15 The wooden members *A* and *B* are to be joined by plywood splice plates which will be fully glued on the surfaces in contact. As part of the design of the joint and knowing that the clearance between the ends of the members is to be 8 mm, determine the smallest allowable length *L* if the average shearing stress in the glue is not to exceed 800 kPa.

1.16 Determine the diameter of the largest circular hole that can be punched into a sheet of polystyrene 6 mm thick, knowing that the force exerted by the punch is 45 kN and that a 55-MPa average shearing stress is required to cause the material to fail.

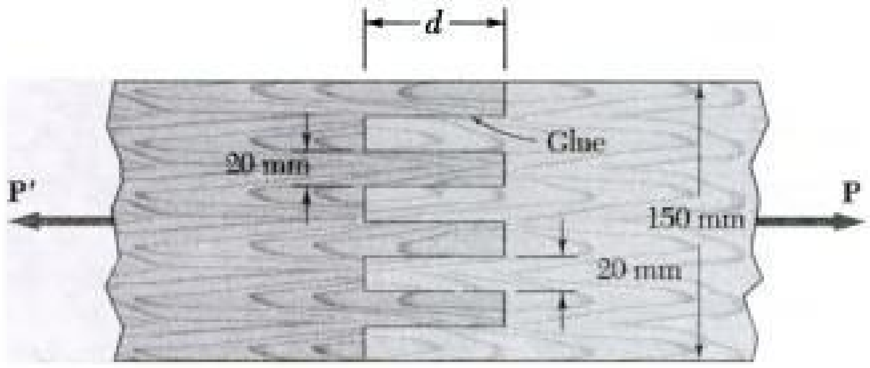


Fig. P1.17

1.17 Two wooden planks, each 22 mm thick and 150 mm wide, are joined by the glued mortise joint shown. Knowing that the joint will fail when the average shearing stress in the glue reaches 0.80 MPa, determine the smallest allowable length d of the cuts if the joint is to withstand an axial load of magnitude $P = 5.4$ kN.

1.18 A load **P** is applied to a steel rod supported as shown by an aluminum plate into which a 15 mm-diameter hole has been drilled. Knowing that the shearing stress must not exceed 120 MPa in the steel rod and 70 MPa in the aluminum plate, determine the largest load **P** that may be applied to the rod.

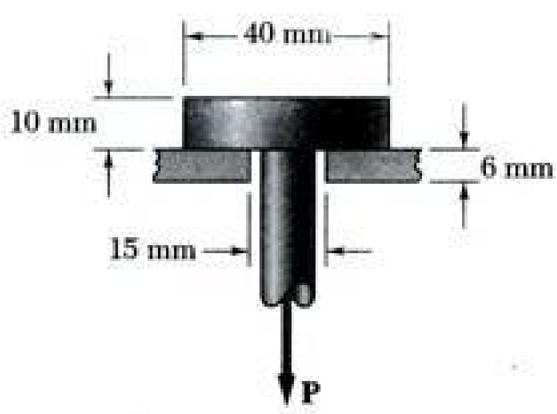


Fig. P1.18

1.19 The axial force in the column supporting the timber beam shown is $P = 75$ kN. Determine the smallest allowable length L of the bearing plate if the bearing stress in the timber is not to exceed 3.0 MPa.

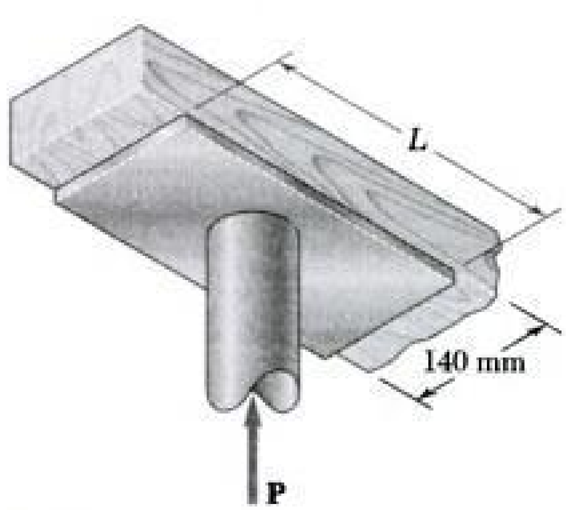


Fig. P1.19

1.20 An axial load **P** is supported by a short W250 × 67 column of cross-sectional area $A = 8580 \text{ mm}^2$ and is distributed to a concrete foundation by a square plate as shown. Knowing that the average normal stress in the column must not exceed 150 MPa and that the bearing stress on the concrete foundation must not exceed 12.5 MPa, determine the side a of the plate that will provide the most economical and safe design.

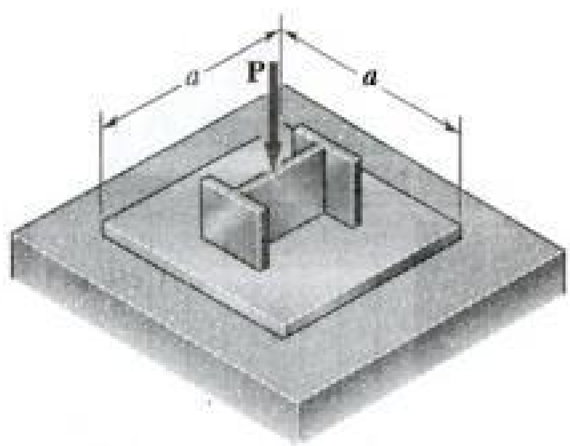


Fig. P1.20

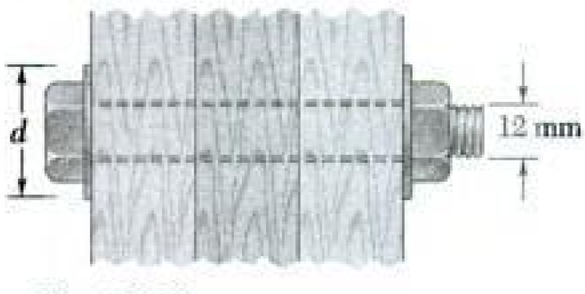


Fig. *P1.21*

1.21 Three wooden planks are fastened together by a series of bolts to form a column. The diameter of each bolt is 12 mm and the inner diameter of each washer is 16 mm, which is slightly larger than the diameter of the holes in the planks. Determine the smallest allowable outer diameter d of the washers, knowing that the average normal stress in the bolts is 35 MPa and that the bearing stress between the washers and the planks must not exceed 8 MPa.

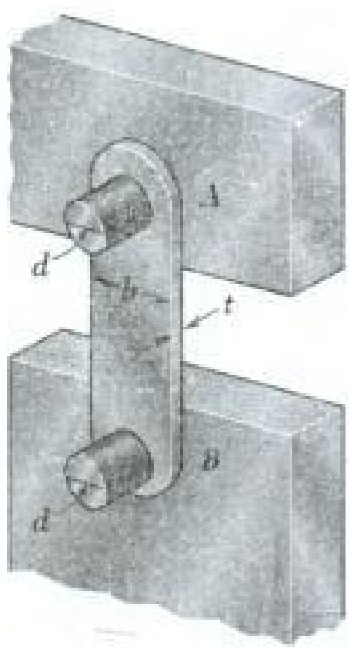


Fig. *P1.22*

1.22 Link AB, of width $b = 50$ mm and thickness $t = 6$ mm, is used to support the end of a horizontal beam. Knowing that the average normal stress in the link is -138 MPa and that the average shearing stress in each of the two pins is 82 MPa, determine (*a*) the diameter d of the pins, (*b*) the average bearing stress in the link.

1.23 For the assembly and loading of Prob. 1.8, determine (*a*) the average shearing stress in the pin at B, (*b*) the average bearing stress at B in link BD, (*c*) the average bearing stress at B in member ABC, knowing that this member has a 10×50-mm uniform rectangular cross section.

1.24 For the assembly and loading of Prob. 1.8, determine (*a*) the average shearing stress in the pin at C, (*b*) the average bearing stress at C in link CE, (*c*) the average bearing stress at C in member ABC, knowing that this member has a 10×50-mm uniform rectangular cross section.

1.25 For the assembly and loading of Prob. 1.9, determine (*a*) the average shearing stress in the pin at A, (*b*) the average bearing stress at A in member AB.

1.26 For the assembly and loading of Prob. 1.9, determine (*a*) the average shearing stress in the pin at C, (*b*) the average bearing stress at C in member BC, (*c*) the average bearing stress at B in member BC.

1.27 Knowing that $\theta = 40°$ and $P = 9$ kN, determine (*a*) the smallest allowable diameter of the pin at B if the average shearing stress in the pin is not to exceed 120 MPa, (*b*) the corresponding average bearing stress in member AB at B, (*c*) the corresponding average bearing stress in each of the support brackets at B.

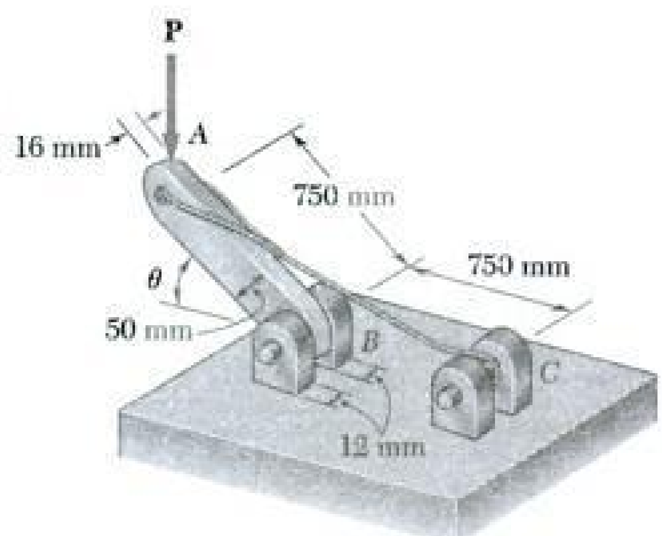


Fig. *P1.27* and *P1.28*

1.28 Determine the largest load **P** that may be applied at A when $\theta = 60°$, knowing that the average shearing stress in the 10-mm-diameter pin at B must not exceed 120 MPa and that the average bearing stress in member AB and in the bracket at B must not exceed 90 MPa.

1.11. STRESS ON AN OBLIQUE PLANE UNDER AXIAL LOADING

In the preceding sections, axial forces exerted on a two-force member (Fig. 1.28*a*) were found to cause normal stresses in that member (Fig. 1.28*b*), while transverse forces exerted on bolts and pins (Fig 1.29*a*) were found to cause shearing stresses in those connections (Fig. 1.29*b*). The reason such a relation was observed between axial forces and normal stresses on one hand, and transverse forces and shearing stresses on the other, was because stresses were being determined only on planes perpendicular to the axis of the member or connection. As you will see in this section, axial forces cause both normal and shearing stresses on planes which are not perpendicular to the axis of the member. Similarly, transverse forces exerted on a bolt or a pin cause both normal and shearing stresses on planes which are not perpendicular to the axis of the bolt or pin.

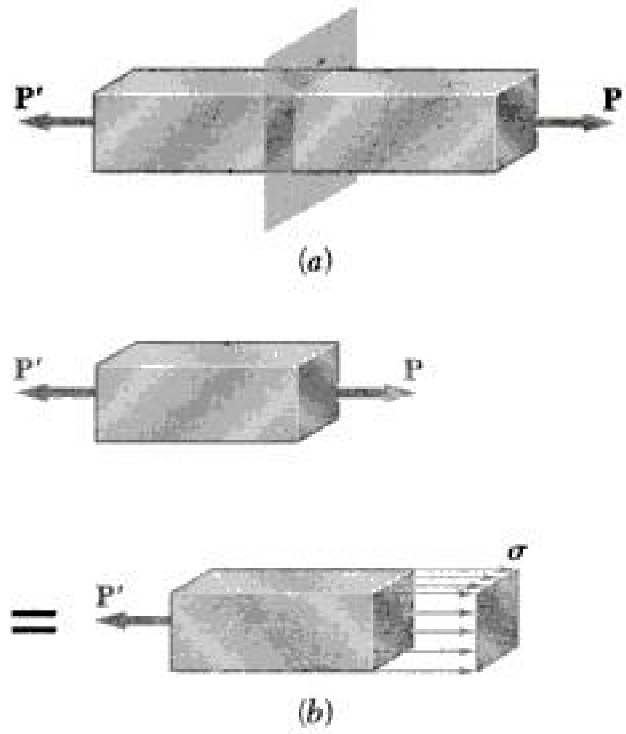


Fig. 1.28

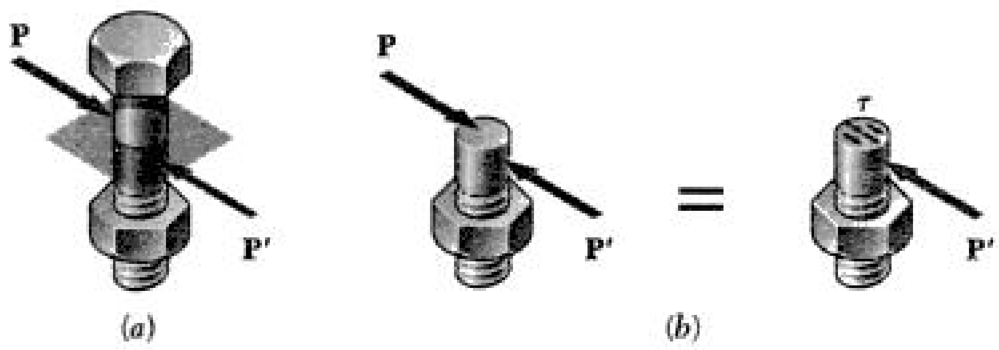


Fig. 1.29

Consider the two-force member of Fig. 1.28, which is subjected to axial forces **P** and **P**′. If we pass a section forming an angle θ with a normal plane (Fig. 1.30*a*) and draw the free-body diagram of the portion of member located to the left of that section (Fig. 1.30*b*), we find from the equilibrium conditions of the free body that the distributed forces acting on the section must be equivalent to the force **P.**

Resolving **P** into components **F** and **V,** respectively normal and tangential to the section (Fig. 1.30*c*), we have

$$F = P\cos\theta \qquad V = P\sin\theta \tag{1.12}$$

The force **F** represents the resultant of normal forces distributed over the section, and the force **V** the resultant of shearing forces (Fig. 1.30*d*). The average values of the corresponding normal and shearing stresses are obtained by dividing, respectively, F and V by the area A_θ of the section:

$$\sigma = \frac{F}{A_\theta} \qquad \tau = \frac{V}{A_\theta} \tag{1.13}$$

Substituting for F and V from (1.12) into (1.13), and observing from Fig. 1.30*c* that $A_0 = A_\theta \cos\theta$, or $A_\theta = A_0/\cos\theta$, where A_0 denotes the

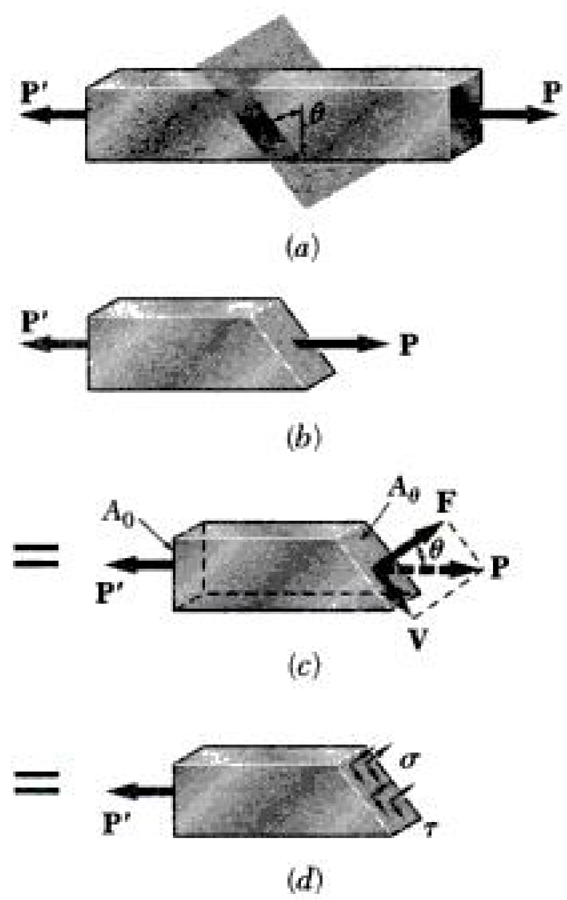


Fig. 1.30

area of a section perpendicular to the axis of the member, we obtain

$$\sigma = \frac{P\cos\theta}{A_0/\cos\theta} \qquad \tau = \frac{P\sin\theta}{A_0/\cos\theta}$$

or

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta \tag{1.14}$$

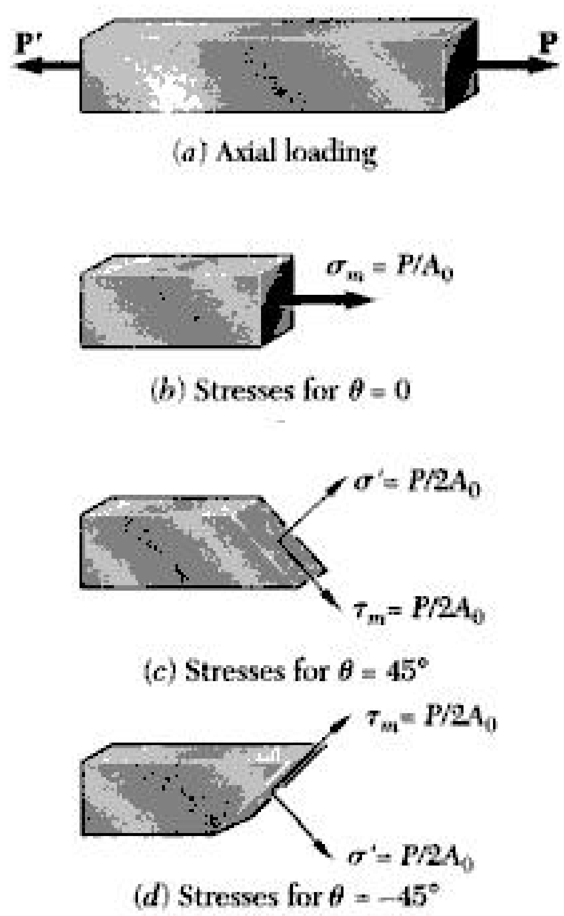


Fig. 1.31

We note from the first of Eqs. (1.14) that the normal stress σ is maximum when $\theta = 0$, i.e., when the plane of the section is perpendicular to the axis of the member, and that it approaches zero as θ approaches 90°. We check that the value of σ when $\theta = 0$ is

$$\sigma_m = \frac{P}{A_0} \tag{1.15}$$

as we found earlier in Sec. 1.3. The second of Eqs. (1.14) shows that the shearing stress τ is zero for $\theta = 0$ and $\theta = 90°$, and that for $\theta = 45°$ it reaches its maximum value

$$\tau_m = \frac{P}{A_0}\sin 45°\cos 45° = \frac{P}{2A_0} \tag{1.16}$$

The first of Eqs. (1.14) indicates that, when $\theta = 45°$, the normal stress σ' is also equal to $P/2A_0$:

$$\sigma' = \frac{P}{A_0}\cos^2 45° = \frac{P}{2A_0} \tag{1.17}$$

The results obtained in Eqs. (1.15), (1.16), and (1.17) are shown graphically in Fig. 1.31. We note that the same loading may produce either a normal stress $\sigma_m = P/A_0$ and no shearing stress (Fig. 1.31*b*), or a normal and a shearing stress of the same magnitude $\sigma' = \tau_m = P/2A_0$ (Fig. 1.31 *c* and *d*), depending upon the orientation of the section.

1.12. STRESS UNDER GENERAL LOADING CONDITIONS; COMPONENTS OF STRESS

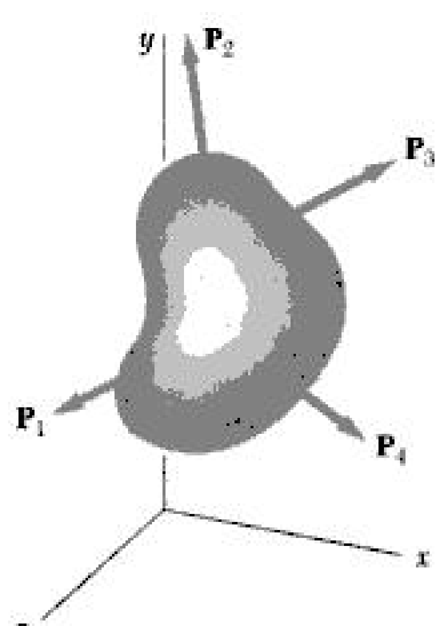


Fig. 1.32

The examples of the previous sections were limited to members under axial loading and connections under transverse loading. Most structural members and machine components are under more involved loading conditions.

Consider a body subjected to several loads $\mathbf{P}_1$, $\mathbf{P}_2$, etc. (Fig. 1.32). To understand the stress condition created by these loads at some point Q within the body, we shall first pass a section through Q, using a plane parallel to the yz plane. The portion of the body to the left of the section is subjected to some of the original loads, and to normal and shearing forces distributed over the section. We shall denote by $\Delta\mathbf{F}^x$ and $\Delta\mathbf{V}^x$, respectively, the normal and the shearing forces acting on a small

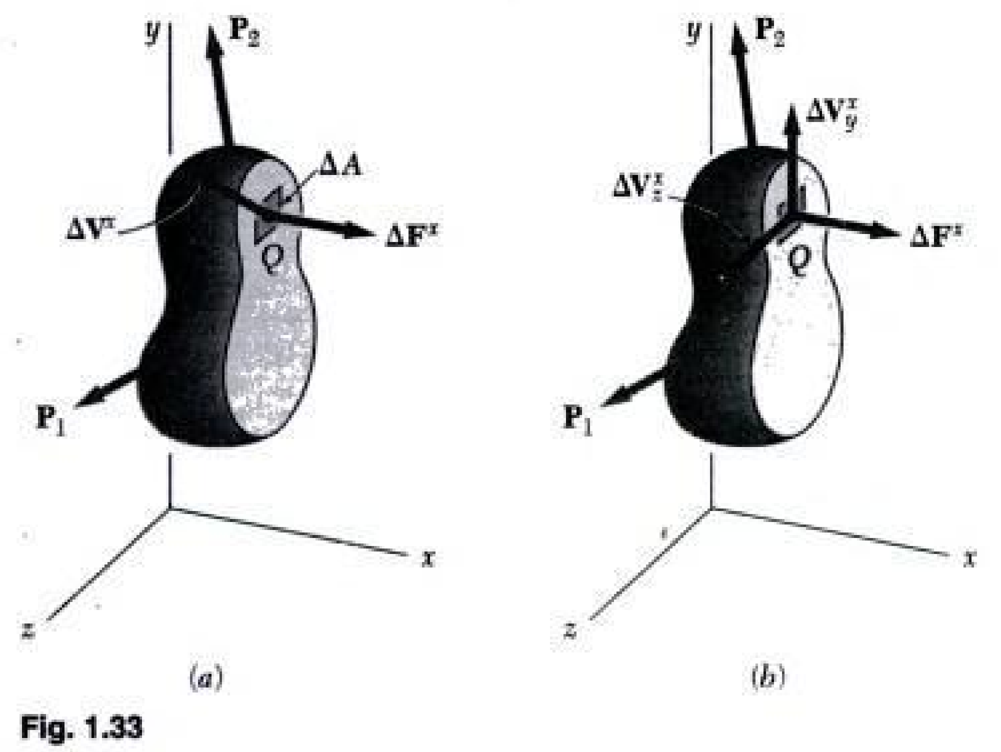


Fig. 1.33

area ΔA surrounding point Q (Fig. 1.33*a*). Note that the superscript x is used to indicate that the forces $\Delta \mathbf{F}^x$ and $\Delta \mathbf{V}^x$ act on a surface perpendicular to the x axis. While the normal force $\Delta \mathbf{F}^x$ has a well-defined direction, the shearing force $\Delta \mathbf{V}^x$ may have any direction in the plane of the section. We therefore resolve $\Delta \mathbf{V}^x$ into two component forces, $\Delta \mathbf{V}_y^x$ and $\Delta \mathbf{V}_z^x$, in directions parallel to the y and z axes, respectively (Fig. 1.33 *b*). Dividing now the magnitude of each force by the area ΔA, and letting ΔA approach zero, we define the three stress components shown in Fig. 1.34:

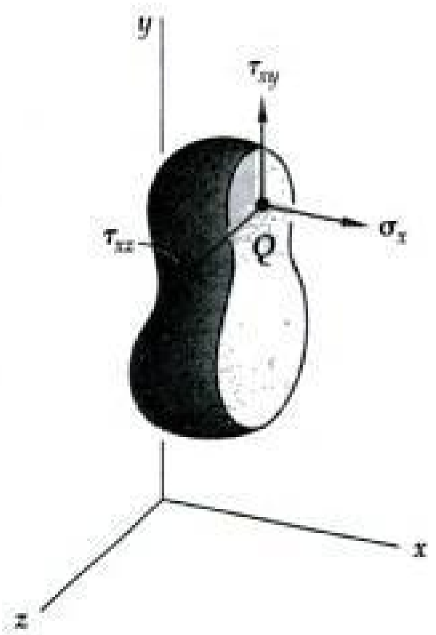


Fig. 1.34

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F^x}{\Delta A}$$
$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_y^x}{\Delta A} \qquad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta V_z^x}{\Delta A} \tag{1.18}$$

We note that the first subscript in σ_x, τ_{xy}, and τ_{xz} is used to indicate that the stresses under consideration are exerted *on a surface perpendicular to the x axis.* The second subscript in τ_{xy} and τ_{xz} identifies *the direction of the component.* The normal stress σ_x is positive if the corresponding arrow points in the positive x direction, i.e., if the body is in tension, and negative otherwise. Similarly, the shearing stress components τ_{xy} and τ_{xz} are positive if the corresponding arrows point, respectively, in the positive y and z directions.

The above analysis may also be carried out by considering the portion of body located to the right of the vertical plane through Q (Fig. 1.35). The same magnitudes, but opposite directions, are obtained for the normal and shearing forces $\Delta \mathbf{F}^x$, $\Delta \mathbf{V}_y^x$, and $\Delta \mathbf{V}_z^x$. Therefore, the same values are also obtained for the corresponding stress components, but since the section in Fig. 1.35 now faces the *negative x axis,* a positive sign for σ_x will indicate that the corresponding arrow points *in the negative x direction.* Similarly, positive signs for τ_{xy} and τ_{xz} will indicate that the corresponding arrows point, respectively, in the negative y and z directions, as shown in Fig. 1.35.

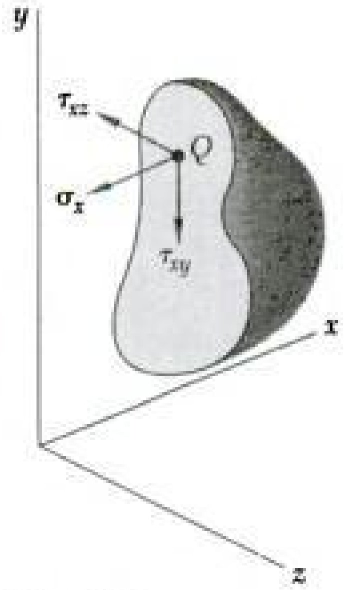


Fig. 1.35

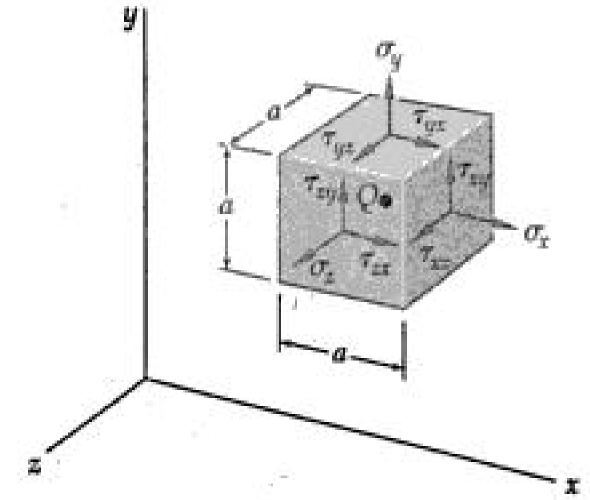


Fig. 1.36

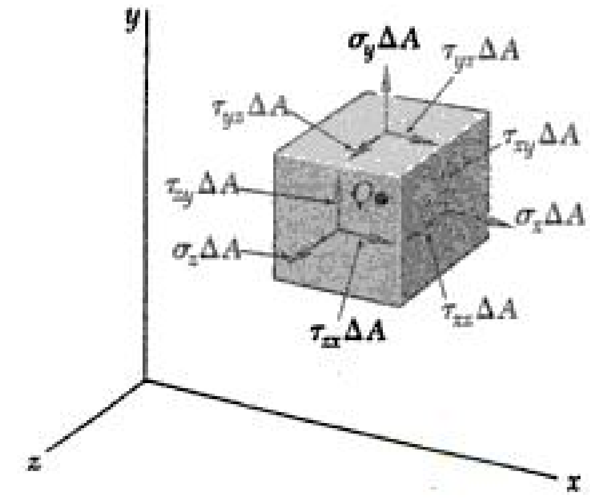


Fig. 1.37

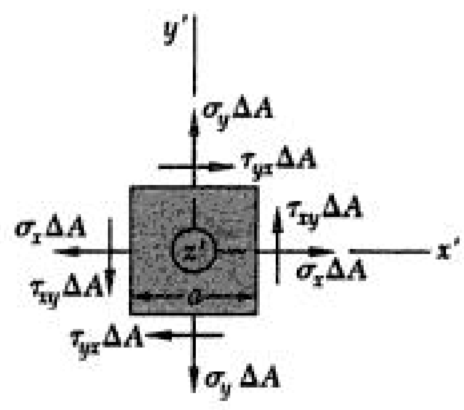


Fig. 1.38

Passing a section through Q parallel to the zx plane, we define in the same manner the stress components, σ_y, τ_{yz}, and τ_{yx}. Finally, a section through Q parallel to the xy plane yields the components σ_z, τ_{zx}, and τ_{zy}.

To facilitate the visualization of the stress condition at point Q, we shall consider a small cube of side a centered at Q and the stresses exerted on each of the six faces of the cube (Fig. 1.36). The stress components shown in the figure are σ_x, σ_y, and σ_z, which represent the normal stress on faces respectively perpendicular to the x, y, and z axes, and the six shearing stress components τ_{xy}, τ_{xz}, etc. We recall that, according to the definition of the shearing stress components, τ_{xy} represents the y component of the shearing stress exerted on the face perpendicular to the x axis, while τ_{yx} represents the x component of the shearing stress exerted on the face perpendicular to the y axis. Note that only three faces of the cube are actually visible in Fig. 1.36, and that equal and opposite stress components act on the hidden faces. While the stresses acting on the faces of the cube differ slightly from the stresses at Q, the error involved is small and vanishes as side a of the cube approaches zero.

Important relations among the shearing stress components will now be derived. Let us consider the free-body diagram of the small cube centered at point Q (Fig. 1.37). The normal and shearing forces acting on the various faces of the cube are obtained by multiplying the corresponding stress components by the area ΔA of each face. We first write the following three equilibrium equations:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{1.19}$$

Since forces equal and opposite to the forces actually shown in Fig. 1.37 are acting on the hidden faces of the cube, it is clear that Eqs. (1.19) are satisfied. Considering now the moments of the forces about axes Qx', Qy', and Qz' drawn from Q in directions respectively parallel to the x, y, and z axes, we write the three additional equations

$$\Sigma M_{x'} = 0 \qquad \Sigma M_{y'} = 0 \qquad \Sigma M_{z'} = 0 \tag{1.20}$$

Using a projection on the $x'y'$ plane (Fig. 1.38), we note that the only forces with moments about the z axis different from zero are the shearing forces. These forces form two couples, one of counterclockwise (positive) moment $(\tau_{xy}\,\Delta A)a$, the other of clockwise (negative) moment $-(\tau_{yx}\,\Delta A)a$. The last of the three Eqs. (1.20) yields, therefore,

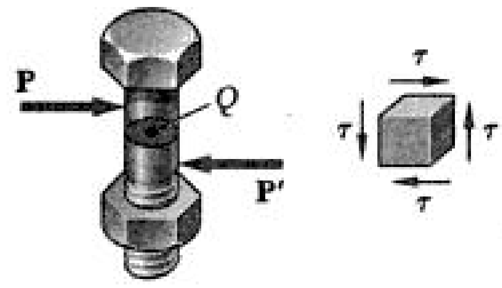
$$+\circlearrowleft \Sigma M_z = 0: \qquad (\tau_{xy}\,\Delta A)a - (\tau_{yx}\,\Delta A)a = 0$$

from which we conclude that

$$\tau_{xy} = \tau_{yx} \tag{1.21}$$

The relation obtained shows that the y component of the shearing stress exerted on a face perpendicular to the x axis is equal to the x component of the shearing stress exerted on a face perpendicular to the y axis. From the remaining two equations (1.20), we derive in a similar manner the relations

$$\tau_{yz} = \tau_{zy} \qquad \tau_{zx} = \tau_{xz} \tag{1.22}$$

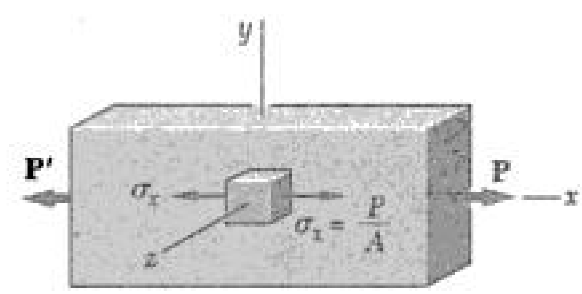


(a) (b)

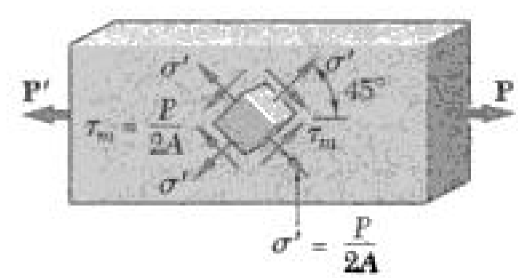
Fig. 1.39

We conclude from Eqs. (1.21) and (1.22) that only six stress components are required to define the condition of stress at a given point Q, instead of nine as originally assumed. These six components are σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}. We also note that, at a given point, *shear cannot take place in one plane only*; an equal shearing stress must be exerted on another plane perpendicular to the first one. For example, considering again the bolt of Fig. 1.29 and a small cube at the center Q of the bolt (Fig. 1.39*a*), we find that shearing stresses of equal magnitude must be exerted on the two horizontal faces of the cube and on the two faces which are perpendicular to the forces **P** and **P′** (Fig. 1.39*b*).

Before concluding our discussion of stress components, let us consider again the case of a member under axial loading. If we consider a small cube with faces respectively parallel to the faces of the member and recall the results obtained in Sec. 1.11, we find that the conditions of stress in the member may be described as shown in Fig. 1.40*a*; the only stresses are normal stresses σ_x exerted on the faces of the cube which are perpendicular to the x axis. However, if the small cube is rotated by 45° about the z axis so that its new orientation matches the orientation of the sections considered in Fig. 1.31*c* and *d*, we conclude that normal and shearing stresses of equal magnitude are exerted on four faces of the cube (Fig. 1.40*b*). We thus observe that the same loading condition may lead to different interpretations of the stress situation at a given point, depending upon the orientation of the element considered. More will be said about this in Chap 7.



(a)



(b)

Fig. 1.40

1.13. DESIGN CONSIDERATIONS

In the preceding sections you learned to determine the stresses in rods, bolts, and pins under simple loading conditions. In later chapters you will learn to determine stresses in more complex situations. In engineering applications, however, the determination of stresses is seldom an end in itself. Rather, the knowledge of stresses is used by engineers to assist in their most important task, namely, the design of structures and machines that will safely and economically perform a specified function.

a. Determination of the Ultimate Strength of a Material. An important element to be considered by a designer is how the material that has been selected will behave under a load. For a given material this is determined by performing specific tests on prepared samples of the material. For example, a test specimen of steel may be prepared and placed in a laboratory testing machine to be subjected to a known centric axial tensile force, as described in Sec. 2.3. As the magnitude of the force is increased, various changes in the specimen are measured, for example, changes in its length and its diameter. Eventually the largest force which may be applied to the specimen is reached, and the specimen either breaks or begins to carry less load. This largest force is called the *ultimate load* for the test specimen and is denoted by P_U. Since the applied load is centric, we may divide the ultimate load by the original cross-sectional area of the rod to obtain the *ultimate normal stress* of the material used. This stress, also known as the *ultimate strength in tension* of the material, is

$$\sigma_U = \frac{P_U}{A} \tag{1.23}$$

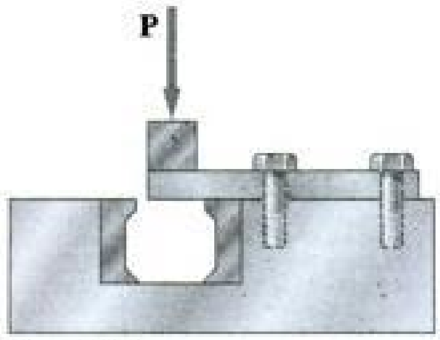


Fig. 1.41

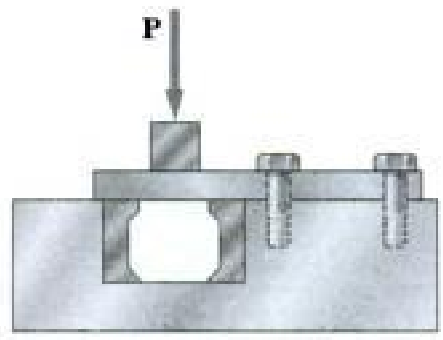


Fig. 1.42

Several test procedures are available to determine the *ultimate shearing stress,* or *ultimate strength in shear,* of a material. The one most commonly used involves the twisting of a circular tube (Sec. 3.5). A more direct, if less accurate, procedure consists in clamping a rectangular or round bar in a shear tool (Fig. 1.41) and applying an increasing load P until the ultimate load P_U for single shear is obtained. If the free end of the specimen rests on both of the hardened dies (Fig. 1.42), the ultimate load for double shear is obtained. In either case, the ultimate shearing stress τ_U is obtained by dividing the ultimate load by the total area over which shear has taken place. We recall that, in the case of single shear, this area is the cross-sectional area A of the specimen, while in double shear it is equal to twice the cross-sectional area.

b. Allowable Load and Allowable Stress; Factor of Safety. The maximum load that a structural member or a machine component will be allowed to carry under normal conditions of utilization is considerably smaller than the ultimate load. This smaller load is referred to as the *allowable load* and, sometimes, as the *working load* or *design load.* Thus, only a fraction of the ultimate-load capacity of the member is utilized when the allowable load is applied. The remaining portion of the load-carrying capacity of the member is kept in reserve to assure its safe performance. The ratio of the ultimate load to the allowable load is used to define the *factor of safety.*† We have

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate load}}{\text{allowable load}} \tag{1.24}$$

An alternative definition of the factor of safety is based on the use of stresses:

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate stress}}{\text{allowable stress}} \tag{1.25}$$

The two expressions given for the factor of safety in Eqs. (1.24) and (1.25) are identical when a linear relationship exists between the load and the stress. In most engineering applications, however, this relationship ceases to be linear as the load approaches its ultimate value, and the factor of safety obtained from Eq. (1.25) does not provide a

†In some fields of engineering, notably aeronautical engineering, the *margin of safety* is used in place of the factor of safety. The margin of safety is defined as the factor of safety minus one; that is, margin of safety = $F.S. - 1.00$.

true assessment of the safety of a given design. Nevertheless, the *allowable-stress method* of design, based on the use of Eq. (1.25), is widely used.

c. Selection of an Appropriate Factor of Safety. The selection of the factor of safety to be used for various applications is one of the most important engineering tasks. On the one hand, if a factor of safety is chosen too small, the possibility of failure becomes unacceptably large; on the other hand, if a factor of safety is chosen unnecessarily large, the result is an uneconomical or nonfunctional design. The choice of the factor of safety that is appropriate for a given design application requires engineering judgment based on many considerations, such as the following:

1. *Variations that may occur in the properties of the member under consideration.* The composition, strength, and dimensions of the member are all subject to small variations during manufacture. In addition, material properties may be altered and residual stresses introduced through heating or deformation that may occur during manufacture, storage, transportation, or construction.
2. *The number of loadings that may be expected during the life of the structure or machine.* For most materials the ultimate stress decreases as the number of load applications is increased. This phenomenon is known as *fatigue* and, if ignored, may result in sudden failure (see Sec. 2.7).
3. *The type of loadings that are planned for in the design, or that may occur in the future.* Very few loadings are known with complete accuracy—most design loadings are engineering estimates. In addition, future alterations or changes in usage may introduce changes in the actual loading. Larger factors of safety are also required for dynamic, cyclic, or impulsive loadings.
4. *The type of failure that may occur.* Brittle materials fail suddenly, usually with no prior indication that collapse is imminent. On the other hand, ductile materials, such as structural steel, normally undergo a substantial deformation called *yielding* before failing, thus providing a warning that overloading exists. However, most buckling or stability failures are sudden, whether the material is brittle or not. When the possibility of sudden failure exists, a larger factor of safety should be used than when failure is preceded by obvious warning signs.
5. *Uncertainty due to methods of analysis.* All design methods are based on certain simplifying assumptions which result in calculated stresses being approximations of actual stresses.
6. *Deterioration that may occur in the future because of poor maintenance or because of unpreventable natural causes.* A larger factor of safety is necessary in locations where conditions such as corrosion and decay are difficult to control or even to discover.
7. *The importance of a given member to the integrity of the whole structure.* Bracing and secondary members may in many cases be designed with a factor of safety lower than that used for primary members.

In addition to the above considerations, there is the additional consideration concerning the risk to life and property that a failure would produce. Where a failure would produce no risk to life and only minimal risk to property, the use of a smaller factor of safety can be considered. Finally, there is the practical consideration that, unless a careful design with a nonexcessive factor of safety is used, a structure or machine might not perform its design function. For example, high factors of safety may have an unacceptable effect on the weight of an aircraft.

For the majority of structural and machine applications, factors of safety are specified by design specifications or building codes written by committees of experienced engineers working with professional societies, with industries, or with federal, state, or city agencies. Examples of such design specifications and building codes are

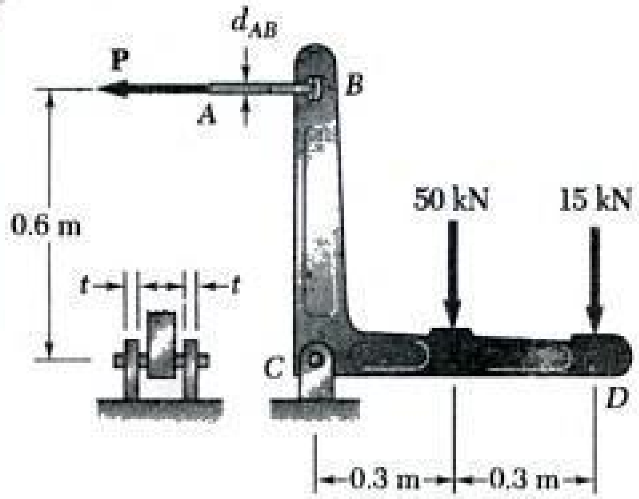
1. *Steel*: American Institute of Steel Construction, Specifications for Structural Steel Buildings
2. *Concrete*: American Concrete Institute, Building Code Requirement for Structural Concrete
3. *Timber*: American Forest and Paper Association, National Design Specification for Wood Construction
4. *Highway bridges*: American Association of State Highway Officials, Standard Specifications for Highway Bridges

***d. Load and Resistance Factor Design.** As we saw above, the allowable-stress method requires that all the uncertainties associated with the design of a structure or machine element be grouped into a single factor of safety. An alternative method of design, which is gaining acceptance chiefly among structural engineers, makes it possible through the use of three different factors to distinguish between the uncertainties associated with the structure itself and those associated with the load it is designed to support. This method, referred to as *Load and Resistance Factor Design (LRFD),* further allows the designer to distinguish between uncertainties associated with the *live load,* P_L, that is, with the load to be supported by the structure, and the *dead load,* P_D, that is, with the weight of the portion of structure contributing to the total load.

When this method of design is used, the *ultimate load,* P_U, of the structure, that is, the load at which the structure ceases to be useful, should first be determined. The proposed design is then acceptable if the following inequality is satisfied:

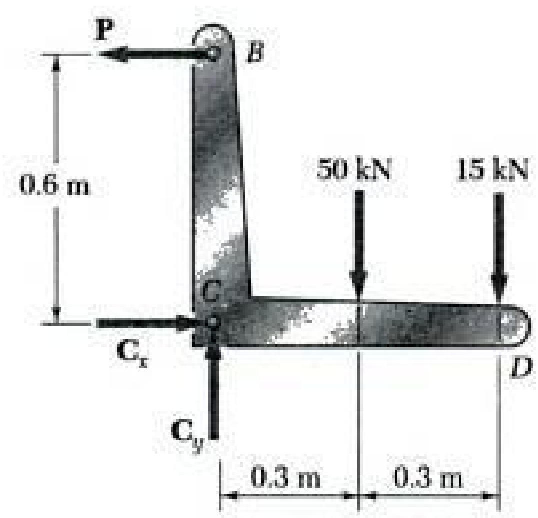
$$\gamma_D P_D + \gamma_L P_L \le \phi P_U \qquad (1.26)$$

The coefficient ϕ is referred to as the *resistance factor;* it accounts for the uncertainties associated with the structure itself and will normally be less than 1. The coefficients γ_D and γ_L are referred to as the *load factors;* they account for the uncertainties associated, respectively, with the dead and live load and will normally be greater than 1, with γ_L generally larger than γ_D. While a few examples or assigned problems using LRFD are included in this chapter and in Chaps. 5 and 10, the allowable-stress method of design will be used in this text.



SAMPLE PROBLEM 1.3

Two forces are applied to the bracket *BCD* as shown. (*a*) Knowing that the control rod *AB* is to be made of a steel having an ultimate normal stress of 600 MPa, determine the diameter of the rod for which the factor of safety with respect to failure will be 3.3. (*b*) The pin at *C* is to be made of a steel having an ultimate shearing stress of 350 MPa. Determine the diameter of the pin *C* for which the factor of safety with respect to shear will also be 3.3. (*c*) Determine the required thickness of the bracket supports at *C* knowing that the allowable bearing stress of the steel used is 300 MPa.



SOLUTION

Free Body: Entire Bracket. The reaction at *C* is represented by its components $\mathbf{C}_x$ and $\mathbf{C}_y$.

$+\circlearrowleft \Sigma M_C = 0$: $P(0.6 \text{ m}) - (50 \text{ kN})(0.3 \text{ m}) - (15 \text{ kN})(0.6 \text{ m}) = 0 \quad P = 40 \text{ kN}$

$\Sigma F_x = 0$: $C_x = 40$ k

$\Sigma F_y = 0$: $C_y = 65$ kN

$$C = \sqrt{C_x^2 + C_y^2} = 76.3 \text{ kN}$$

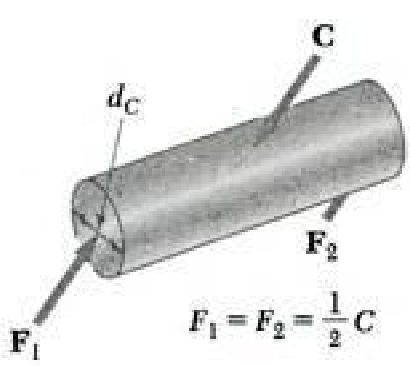
***a.* Control Rod *AB*.** Since the factor of safety is to be 3.3, the allowable stress is

$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{600 \text{ MPa}}{3.3} = 181.8 \text{ MPa}$$

For $P = 40$ kN the cross-sectional area required is

$$A_{\text{req}} = \frac{P}{\sigma_{\text{all}}} = \frac{40 \text{ kN}}{181.8 \text{ MPa}} = 220 \times 10^{-6} \text{ m}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_{AB}^2 = 220 \times 10^{-6} \text{m}^2 \qquad d_{AB} = 16.74 \text{ mm} \blacktriangleleft$$



***b.* Shear in Pin *C*.** For a factor of safety of 3.3, we have

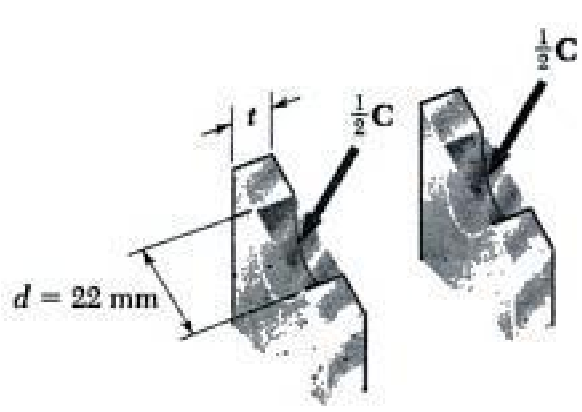
$$\tau_{\text{all}} = \frac{\tau_U}{F.S.} = \frac{350 \text{ MPa}}{3.3} = 106.1 \text{ MPa}$$

Since the pin is in double shear, we write

$$A_{\text{req}} = \frac{C/2}{\tau_{\text{all}}} = \frac{(76.3 \text{ kN})/2}{106.1 \text{ MPa}} = 360 \text{ mm}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_C^2 = 360 \text{ mm}^2 \qquad d_C = 21.4 \text{ mm} \qquad \text{Use: } d_C = 22 \text{ mm} \blacktriangleleft$$

The next larger size pin available is of 22-mm diameter and should be used.

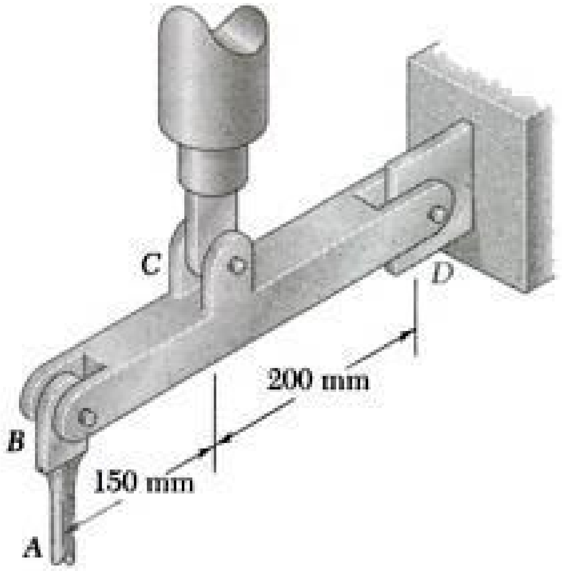


***c.* Bearing at *C*.** Using $d = 22$ mm, the nominal bearing area of each bracket is $22t$. Since the force carried by each bracket is $C/2$ and the allowable bearing stress is 300 MPa, we write

$$A_{\text{req}} = \frac{C/2}{\sigma_{\text{all}}} = \frac{(76.3 \text{ kN})/2}{300 \text{ MPa}} = 127.2 \text{ mm}^2$$

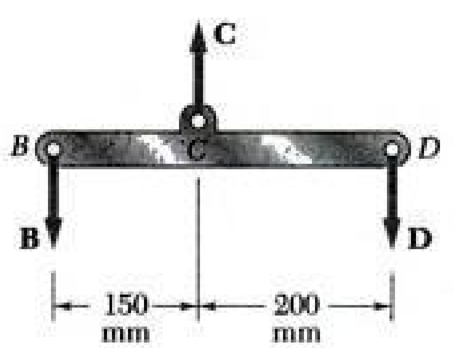
Thus $22t = 127.2 \qquad t = 5.78$ mm $\qquad$ Use: $t = 6$ mm $\blacktriangleleft$

SAMPLE PROBLEM 1.4



The rigid beam BCD is attached by bolts to a control rod at B, to a hydraulic cylinder at C, and to a fixed support at D. The diameters of the bolts used are: $d_B = d_D = 10$ mm, $d_C = 12$ mm. Each bolt acts in double shear and is made from a steel for which the ultimate shearing stress is $\tau_U = 275$ MPa. The control rod AB has a diameter $d_A = 11$ mm and is made of a steel for which the ultimate tensile stress is $\sigma_U = 414$ MPa. If the minimum factor of safety is to be 3.0 for the entire unit, determine the largest upward force which may be applied by the hydraulic cylinder at C.

SOLUTION



The factor of safety with respect to failure must be 3.0 or more in each of the three bolts and in the control rod. These four independent criteria will be considered separately.

Free Body: Beam *BCD*. We first determine the force at C in terms of the force at B and in terms of the force at D.

$+\circlearrowleft \Sigma M_D = 0$: $\quad B(350\text{ mm}) - C(200\text{ mm}) = 0 \qquad C = 1.750B \qquad (1)$

$+\circlearrowleft \Sigma M_B = 0$: $\quad -D(350\text{ mm}) + C(150\text{ mm}) = 0 \qquad C = 2.33D \qquad (2)$

Control Rod. For a factor of safety of 3.0 we have

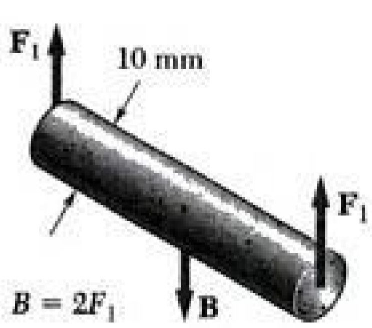
$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{414\text{ MPa}}{3.0} = 138\text{ MPa}$$

The allowable force in the control rod is

$$B = \sigma_{\text{all}}(A) = (138\text{ MPa})\,\tfrac{1}{4}\pi\,(11\times10^{-3}\text{ m})^2 = 13.11$$

Using Eq. (1) we find the largest permitted value of C:

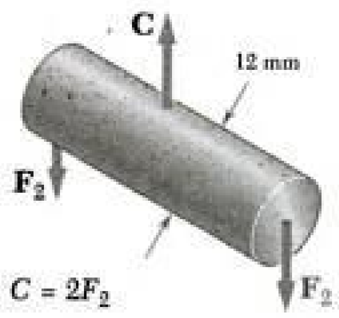
$$C = 1.750\,B = 1.750(13.11\text{ kN})$$



Bolt at *B*. $\tau_{\text{all}} = \tau_U/F.S. = (275\text{ MPa})/3 = 91.67$ MPa. Since the bolt is in double shear, the allowable magnitude of the force **B** exerted on the bolt is

$$B = 2F_1 = 2(\tau_{\text{all}}\,A) = 2(91.67\text{ MPa})\,(\tfrac{1}{4}\pi)(10\times10^{-3}\text{ m})^2 = 14.4\text{ kN}$$

From Eq. (1): $\quad C = 1.750\,B = 1.750(14.4\text{ kN}) \qquad C = 25.2\text{ kN}$ ◀



Bolt at *D*. Since this bolt is the same as bolt B, the allowable force is $D = B = 14.4$ kN. From Eq. (2):

$$C = 2.33\,D = 2.33\,(14.4\text{ kN})$$

Bolt at *C*. We again have $\tau_{\text{all}} = 91.67$ MPa and write

$$C = 2F_2 = 2(\tau_{\text{all}}A) = 2(91.67\text{ MPa})\,(\tfrac{1}{4}\pi)(12\times10^{-3}\text{ m})^2 \qquad C = 20.74\text{ kN} ◀$$

Summary. We have found separately four maximum allowable values of the force C. In order to satisfy all these criteria we must choose the smallest value, namely: $\quad C = 20.74$ kN ◀

PROBLEMS

1.29 The 6-kN load **P** is supported by two wooden members of 75 × 125-mm uniform rectangular cross section that are joined by the simple glued scarf splice shown. Determine the normal and shearing stresses in the glued splice.

1.30 Two wooden members of 75 × 125-mm uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that the maximum allowable tensile stress in the glued splice is 500 kPa, determine (*a*) the largest load **P** which can be safely supported, (*b*) the corresponding shearing stress in the splice.

1.31 Two wooden members of 75 × 150-mm uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that the maximum allowable shearing stress in the glued splice is 0.62 MPa, determine (*a*) the largest load **P** which can be safely applied, (*b*) the corresponding tensile stress in the splice.

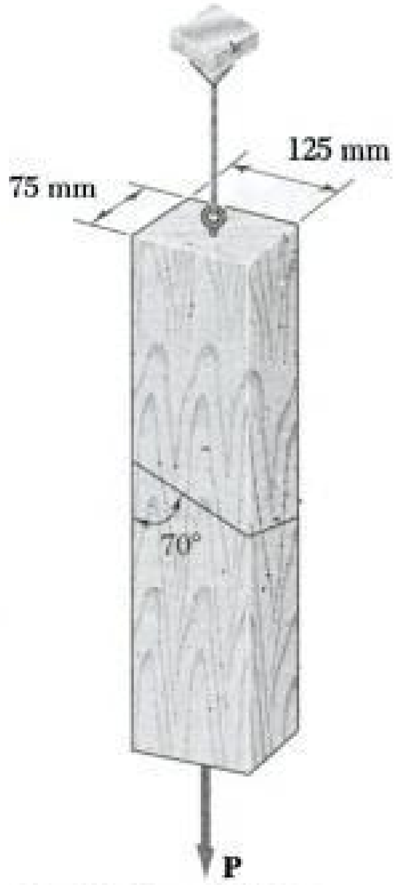


Fig. P1.29 and P1.30

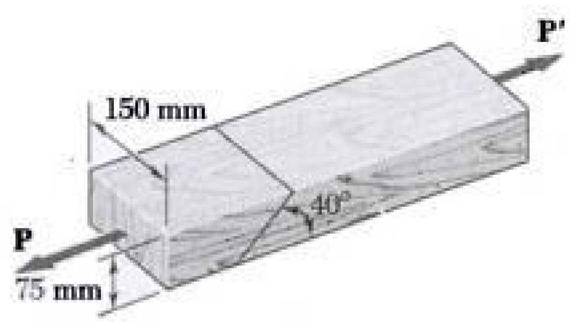


Fig. P1.31 and P1.32

1.32 Two wooden members of 75 × 150 mm uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that $P = 10$ kN determine the normal and shearing stresses in the glued splice.

1.33 A centric load **P** is applied to the granite block shown. Knowing that the resulting maximum value of the shearing stress in the block is 17 MPa, determine (*a*) the magnitude of **P**, (*b*) the orientation of the surface on which the maximum shearing stress occurs, (*c*) the normal stress exerted on that surface, (*d*) the maximum value of the normal stress in the block.

1.34 A 1065 kN load **P** is applied to the granite block shown. Determine the resulting maximum value of (*a*) the normal stress, (*b*) the shearing stress. Specify the orientation of the plane on which each of these maximum values occurs.

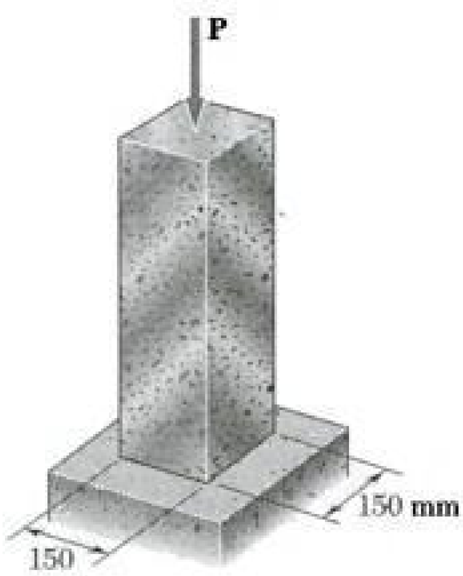


Fig. *P1.33* and *P1.34*

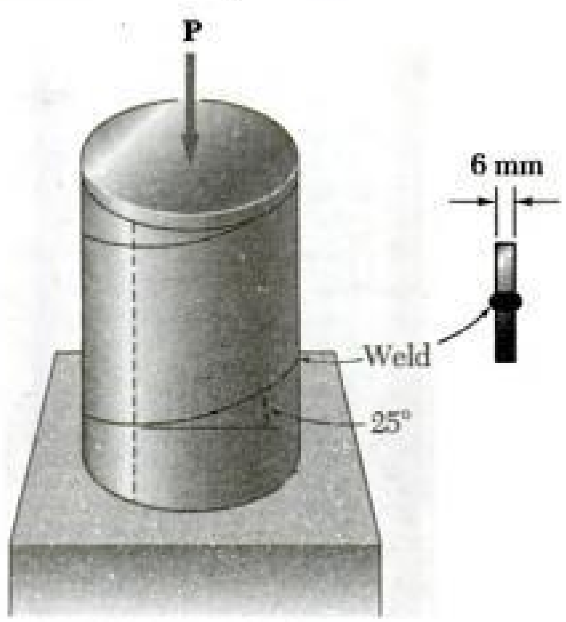


Fig. P1.35 and P1.36

1.35 A steel pipe of 300-mm outer diameter is fabricated from 6-mm-thick plate by welding along a helix which forms an angle of 25° with a plane perpendicular to the axis of the pipe. Knowing that a 250-kN axial force **P** is applied to the pipe, determine the normal and shearing stresses in directions respectively normal and tangential to the weld.

1.36 A steel pipe of 300-mm outer diameter is fabricated from 6-mm-thick plate by welding along a helix which forms an angle of 25° with a plane perpendicular to the axis of the pipe. Knowing that the maximum allowable normal and shearing stresses in directions respectively normal and tangential to the weld are $\sigma = 50$ MPa and $\tau = 30$ MPa, determine the magnitude P of the largest axial force that can be applied to the pipe.

1.37 Link BC is 6 mm thick, has a width $w = 25$ mm, and is made of a steel with a 480-MPa ultimate strength in tension. What was the safety factor used if the structure shown was designed to support a 16-kN load **P**?

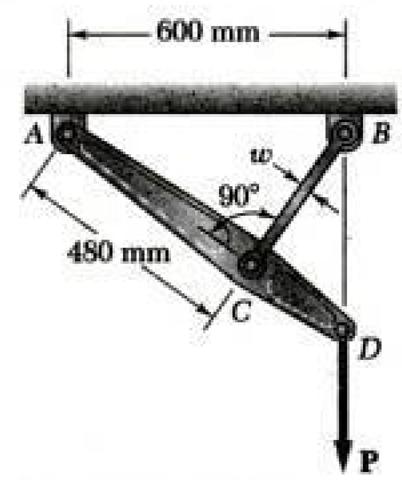


Fig. P1.37 and P1.38

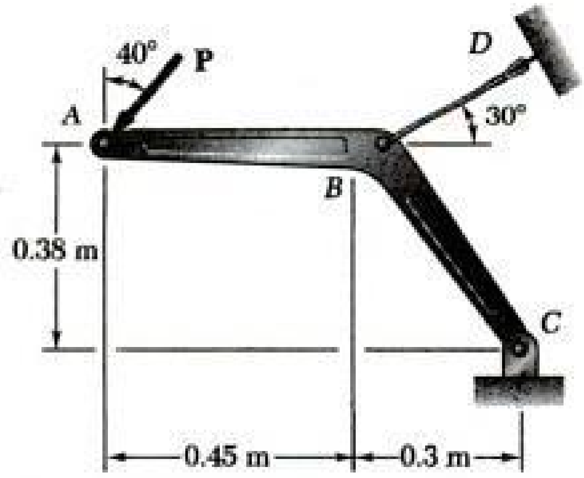


Fig. P1.39 and P1.40

1.38 Link BC is 6 mm thick and is made of a steel with a 450-MPa ultimate strength in tension. What should be its width w if the structure shown is being designed to support a 20-kN load **P** with a factor of safety of 3?

1.39 Member ABC, which is supported by a pin and bracket at C and a cable BD, was designed to support the 18 kN load **P** as shown. Knowing that the ultimate load for cable BD is 110 kN, determine the factor of safety with respect to cable failure.

1.40 Knowing that the ultimate load for cable BD is 110 kN and that a factor of safety of 3.2 with respect to cable failure is required, determine the magnitude of the largest force **P** which can be safely applied as shown to member ABC.

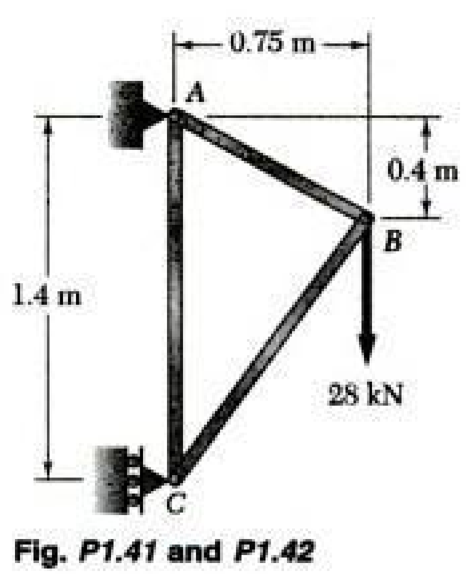


Fig. *P1.41* and *P1.42*

1.41 Members AB and AC of the truss shown consist of bars of square cross section made of the same alloy. It is known that a 20-mm-square bar of the same alloy was tested to failure and that an ultimate load of 120 kN was recorded. If bar AB has a 15-mm-square cross section, determine (*a*) the factor of safety for bar AB, (*b*) the dimensions of the cross section of bar AC if it is to have the same factor of safety as bar AB.

1.42 Members AB and AC of the truss shown consist of bars of square cross section made of the same alloy. It is known that a 20-mm-square bar of the same alloy was tested to failure and that an ultimate load of 120 kN was recorded. If a factor of safety of 3.2 is to be achieved for both bars, determine the required dimensions of the cross section of (*a*) bar AB, (*b*) bar AC.

1.43 The two wooden members shown, which support a 20-kN load, are joined by plywood splices fully glued on the surfaces in contact. The ultimate shearing stress in the glue is 2.8 MPa and the clearance between the members is 8 mm. Determine the factor of safety, knowing that the length of each splice is $L = 200$ mm.

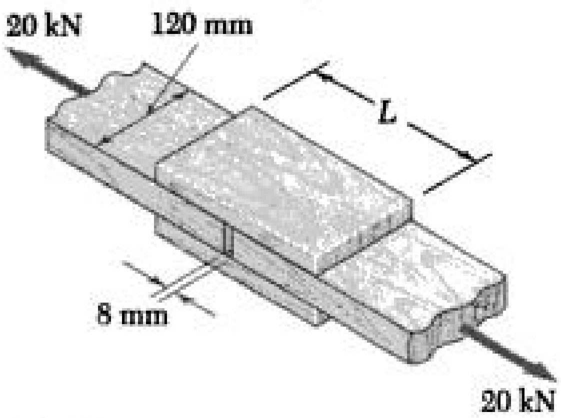


Fig. P1.43

1.44 For the joint and loading of Prob. 1.43, determine the required length L of each splice if a factor of safety of 3.5 is to be achieved.

1.45 Three 20 mm diameter steel bolts are to be used to attach the steel plate shown to a wooden beam. Knowing that the plate will support a 110 kN load and that the ultimate shearing stress for the steel used is 360 MPa, determine the factor of safety for this design.

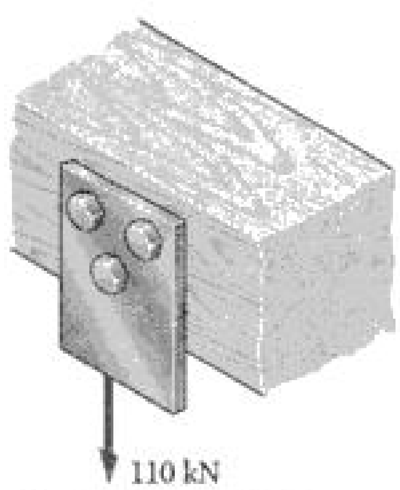


Fig. P1.45 and P1.46

1.46 Three steel bolts are to be used to attach the steel plate shown to a wooden beam. Knowing that the plate will support a 110 kN load, that the ultimate shearing stress for the steel used is 360 MPa, and that a factor of safety of 3.37 is desired, determine the required diameter of the bolts.

1.47 A load **P** is supported as shown by a steel pin that has been inserted in a short wooden member hanging from the ceiling. The ultimate strength of the wood used is 60 MPa in tension and 7.5 MPa in shear, while the ultimate strength of the steel is 150 MPa in shear. Knowing that the diameter of the pin is $d = 16$ mm and that the magnitude of the load is $P = 20$ kN, determine (*a*) the factor of safety for the pin, (*b*) the required values of b and c if the factor of safety for the wooden member is to be the same as that found in part *a* for the pin.

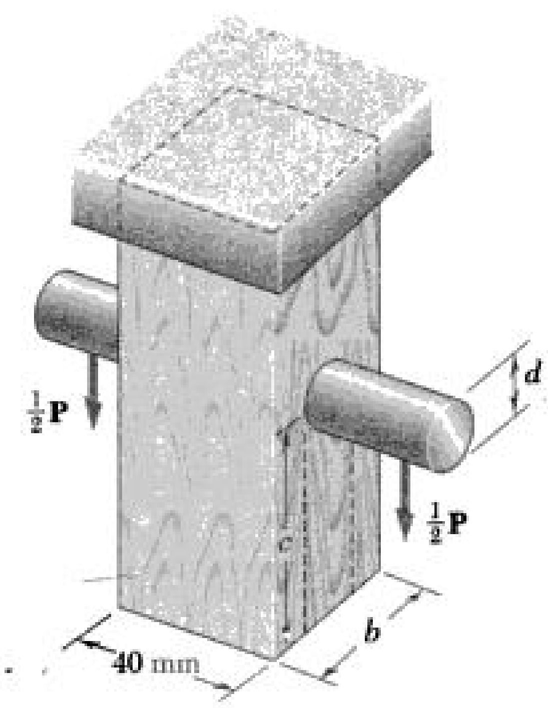


Fig. *P1.47*

1.48 For the support of Prob. 1.47, knowing that $b = 40$ mm, $c = 55$ mm, and $d = 12$ mm, determine the allowable load **P** if an overall factor of safety of 3.2 is desired.

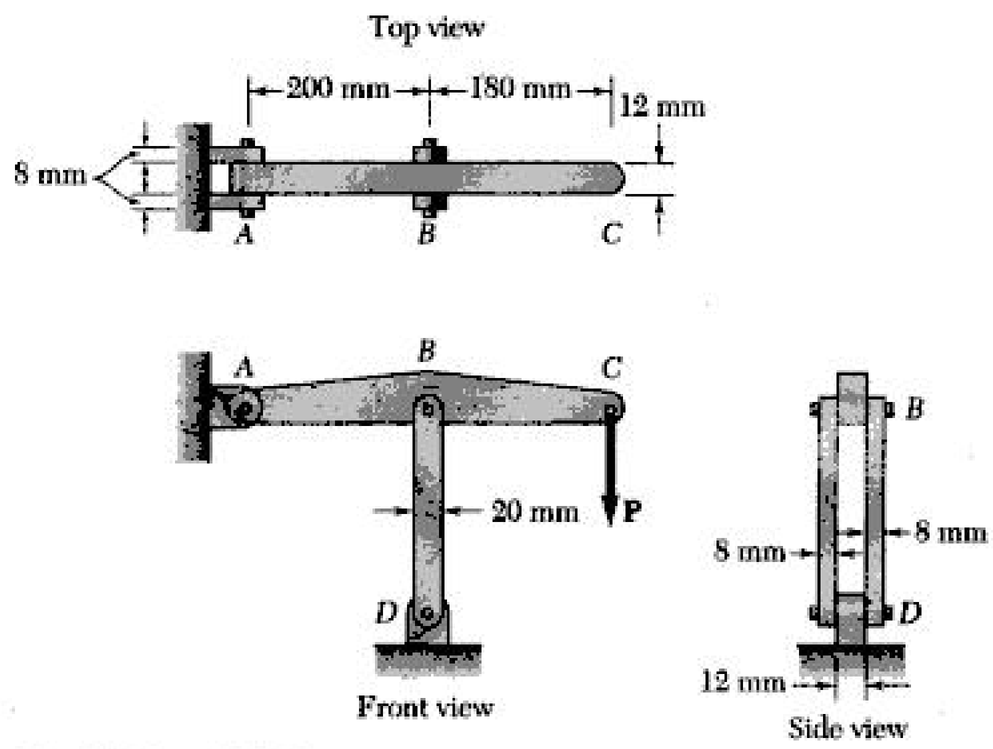


Fig. P1.49 and P1.50

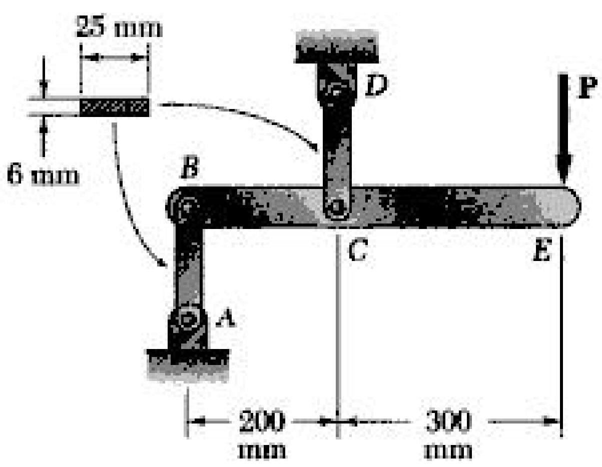


Fig. P1.51 and P1.52

1.49 In the structure shown, an 8-mm-diameter pin is used at A, and 12-mm-diameter pins are used at B and D. Knowing that the ultimate shearing stress is 100 MPa at all connections and that the ultimate normal stress is 250 MPa in each of the two links joining and D, determine the allowable load **P** if an overall factor of safety of 3.0 is desired.

1.50 In an alternative design for the structure of Prob. 1.49, a pin of 10-mm-diameter is to be used at A. Assuming that all other specifications remain unchanged, determine the allowable load **P** if an overall factor of safety of 3.0 is desired.

1.51 Each of the steel links AB and CD is connected to a support and to member BCE by 12 mm diameter steel pins acting in single shear. Knowing that the ultimate shearing stress is 165 MPa for the steel used in the pins and that the ultimate normal stress is 400 MPa for the steel used in the links, determine the allowable load **P** if an overall factor of safety of 3.2 is desired. (Note that the links are not reinforced around the pin holes.)

1.52 An alternative design is being considered to support member BCE of Prob. 1.51, in which link CD will be replaced by two links, each of 3×25 mm cross section, causing the pins at C and D to be in double shear. Assuming that all other specifications remain unchanged, determine the allowable load **P** if an overall factor of safety of 3.2 is desired.

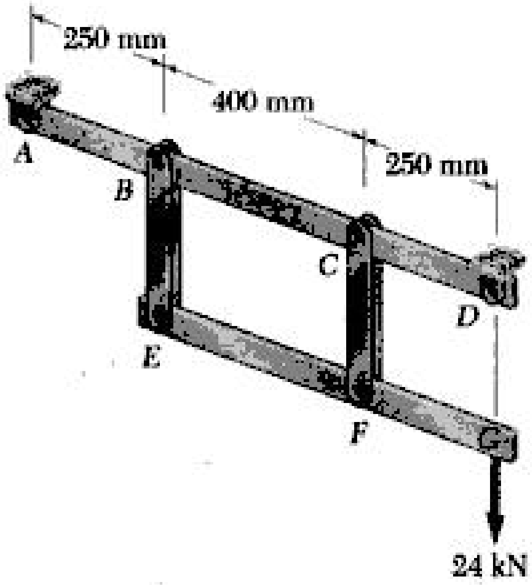


Fig. P1.53

1.53 Each of the two vertical links CF connecting the two horizontal members AD and EG has a 10×40-mm uniform rectangular cross section and is made of a steel with an ultimate strength in tension of 400 MPa, while each of the pins at C and F has a 20-mm diameter and is made of a steel with an ultimate strength in shear of 150 MPa. Determine the overall factor of safety for the links CF and the pins connecting them to the horizontal members.

1.54 Solve Prob. 1.53, assuming that the pins at C and F have been replaced by pins with a 30-mm diameter.

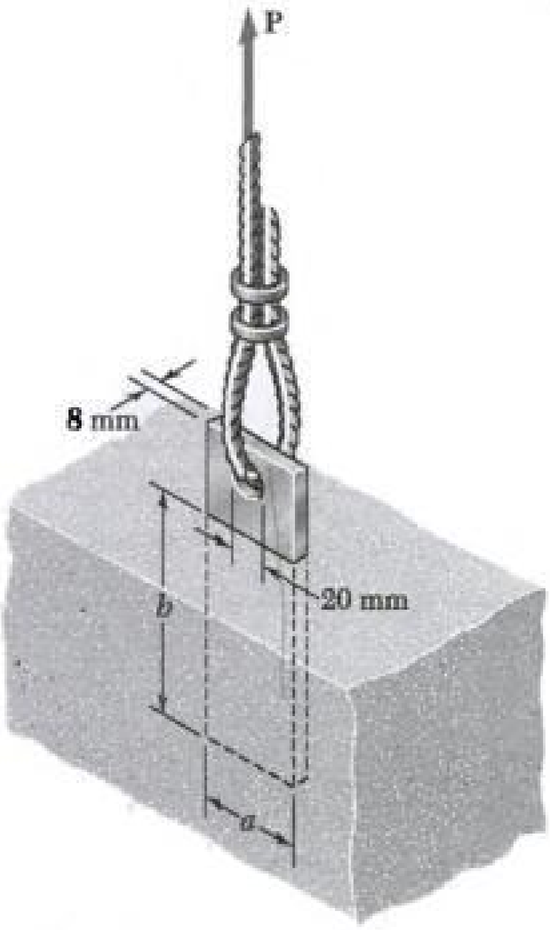


Fig. P1.55

1.55 A steel plate 8 mm thick is embedded in a horizontal concrete slab and is used to anchor a high-strength vertical cable as shown. The diameter of the hole in the plate is 20 mm, the ultimate strength of the steel used is 250 MPa, and the ultimate bonding stress between plate and concrete is 2 MPa. Knowing that a factor of safety of 3.60 is desired when $P = 10$ kN, determine (*a*) the required width a of the plate, (*b*) the minimum depth b to which a plate of that width should be embedded in the concrete slab. (Neglect the normal stresses between the concrete and the lower end of the plate.)

1.56 Determine the factor of safety for the cable anchor of Prob. 1.55 when $P = 14$ kN, knowing that $a = 50$ mm and $b = 190$ mm

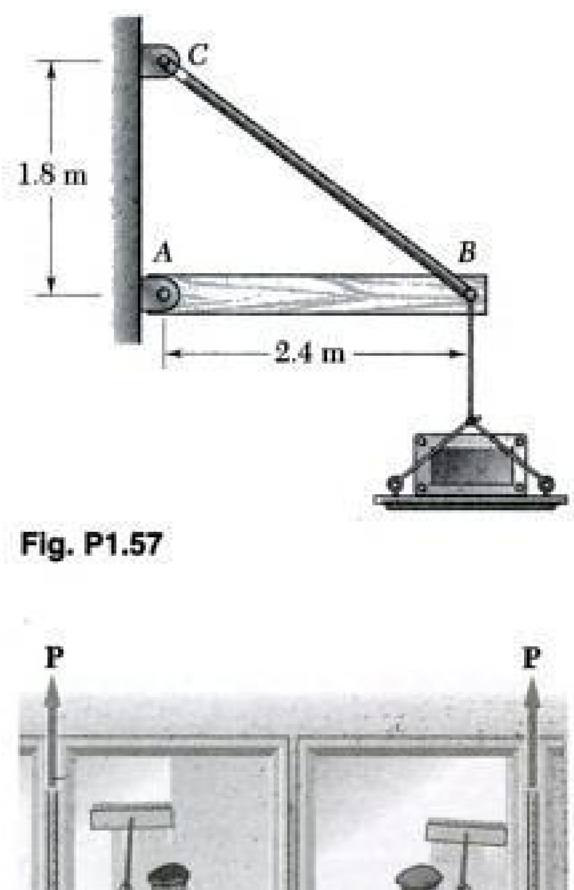


Fig. P1.57

***1.57** A 40-kg platform is attached to the end B of a 50-kg wooden beam AB, which is supported as shown by a pin at A and by a slender steel rod BC with a 12-kN ultimate load. (*a*) Using the Load and Resistance Factor Design method with a resistance factor $\phi = 0.90$ and load factors $\gamma_D = 1.25$ and $\gamma_L = 1.6$, determine the largest load which can be safely placed on the platform. (*b*) What is the corresponding conventional factor of safety for rod BC?

***1.58** The Load and Resistance Factor Design method is to be used to select the two cables which will raise and lower a platform supporting two window washers. The platform weighs 720 N and each of the window washers is assumed to weigh 880 N with equipment. Since these workers are free to move on the platform, 75% of their total weight and of the weight of their equipment will be used as the design live load of each cable. (*a*) Assuming a resistance factor $\phi = 0.90$ and load factors $\gamma_D = 1.2$ and $\gamma_L = 1.5$, determine the required minimum ultimate load of one cable. (*b*) What is the conventional factor of safety for the selected cables?

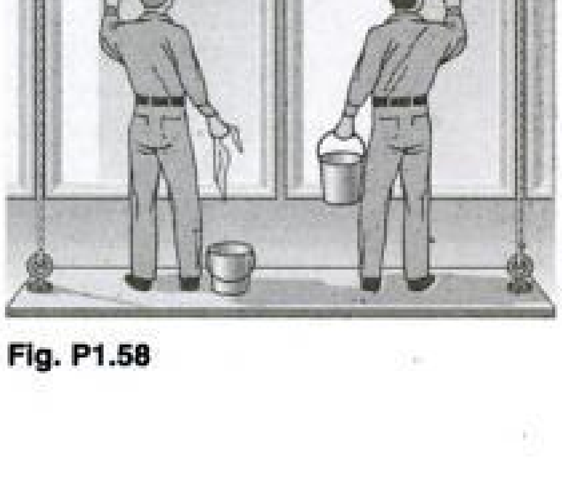


Fig. P1.58

REVIEW AND SUMMARY FOR CHAPTER 1

This chapter was devoted to the concept of stress and to an introduction to the methods used for the analysis and design of machines and load-bearing structures.

Section 1.2 presented a short review of the methods of statics and of their application to the determination of the reactions exerted by its supports on a simple structure consisting of pin-connected members. Emphasis was placed on the use of a *free-body-diagram* to obtain equilibrium equations which were solved for the unknown reactions. Free-body diagrams were also used to find the internal forces in the various members of the structure.

Axial loading. Normal stress

The concept of *stress* was first introduced in Sec. 1.3 by considering a two-force member under an *axial loading*. The *normal stress* in that member was obtained by dividing the magnitude P of the load by the cross-sectional area A of the member (Fig. 1.8*a*). We wrote

Fig. 1.8*a*

$$\sigma = \frac{P}{A} \tag{1.5}$$

Section 1.4 was devoted to a short discussion of the two principal tasks of an engineer, namely, the *analysis* and the *design* of structures and machines.

As noted in Sec. 1.5, the value of σ obtained from Eq. (1.5) represents the *average stress* over the section rather than the stress at a specific point Q of the section. Considering a small area ΔA surrounding Q and the magnitude ΔF of the force exerted on ΔA, we defined the stress at point Q as

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1.6}$$

In general, the value obtained for the stress s at point Q is different from the value of the average stress given by formula (1.5) and is found to vary across the section. However, this variation is small in any section away from the points of application of the loads. In practice, therefore, the distribution of the normal stresses in an axially loaded member is assumed to be *uniform*, except in the immediate vicinity of the points of application of the loads.

However, for the distribution of stresses to be uniform in a given section, it is necessary that the line of action of the loads **P** and **P′** pass through the centroid C of the section. Such a loading is called a *centric* axial loading. In the case of an *eccentric* axial loading, the distribution of stresses is *not* uniform. Stresses in members subjected to an eccentric axial loading will be discussed in Chap 4.

Transverse Forces. Shearing stress

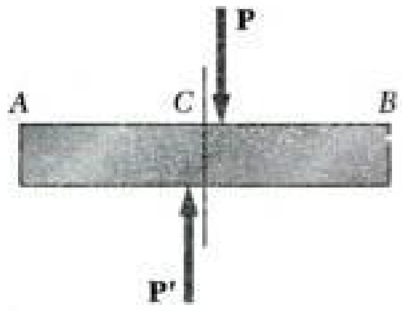


Fig. 1.16*a*

When equal and opposite *transverse forces* **P** and **P**′ of magnitude P are applied to a member AB (Fig. 1.16*a*), *shearing stresses* τ are created over any section located between the points of application of the two forces [Sec 1.6]. These stresses vary greatly across the section and their distribution *cannot* be assumed uniform. However dividing the magnitude P—referred to as the *shear* in the section —by the cross-sectional area A, we defined the *average shearing stress* over the section:

$$\tau_{ave} = \frac{P}{A} \tag{1.8}$$

Single and double shear

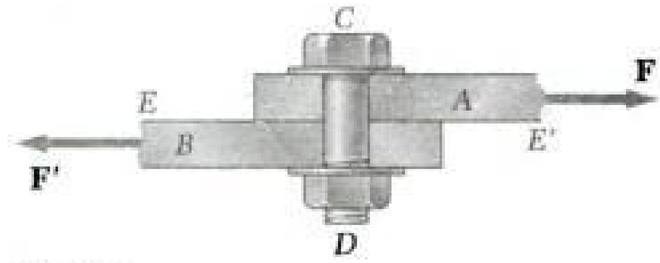


Fig. 1.18

Shearing stresses are found in bolts, pins, or rivets connecting two structural members or machine components. For example, in the case of bolt CD (Fig. 1.18), which is in *single shear,* we wrote

$$\tau_{ave} = \frac{P}{A} = \frac{F}{A} \tag{1.9}$$

while, in the case of bolts EG and HJ (Fig. 1.20), which are both in *double shear,* we had

$$\tau_{ave} = \frac{P}{A} = \frac{F/2}{A} = \frac{F}{2A} \tag{1.10}$$

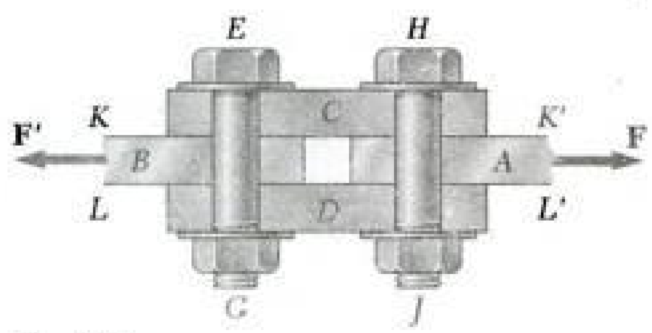


Fig. 1.20

Bearing stress

Bolts, pins, and rivets also create stresses in the members they connect, along the *bearing surface,* or surface of contact [Sec. 1.7]. The bolt CD of Fig. 1.18, for example, creates stresses on the semi-cylindrical surface of plate A with which it is in contact (Fig. 1.22). Since the distribution of these stresses is quite complicated, one uses in practice an average nominal value σ_b of the stress, called *bearing stress,* obtained by dividing the load P by the area of the rectangle representing the projection of the bolt on the plate section. Denoting by t the thickness of the plate and by d the diameter of the bolt, we wrote

$$\sigma_b = \frac{P}{A} = \frac{P}{td} \tag{1.11}$$

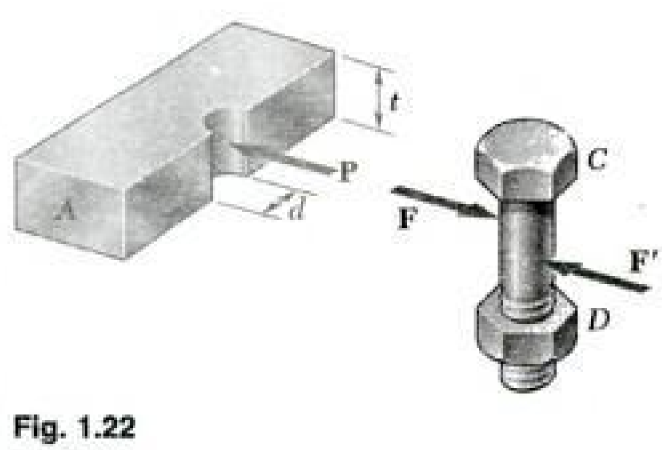


Fig. 1.22

Method of solution

In Sec. 1.8, we applied the concept introduced in the previous sections to the analysis of a simple structure consisting of two pin-connected members supporting a given load. We determined successively the normal stresses in the two members, paying special attention to their narrowest sections, the shearing stresses in the various pins, and the bearing stress at each connection.

The method you should use in solving a problem in mechanics of materials was described in Sec. 1.9. Your solution should begin with a clear and precise *statement* of the problem. You will then draw one or several *free-body diagrams* that you will use to write *equilibrium equations.* These equations will be solved for *unknown forces,* from which the required *stresses* and *deformations* can be computed. Once the answer has been obtained, it should be *carefully checked.*

The first part of the chapter ended with a discussion of *numerical accuracy* in engineering, which stressed the fact that the accuracy of an answer can never be greater than the accuracy of the given data [Sec. 1.10].

Stresses on an oblique section

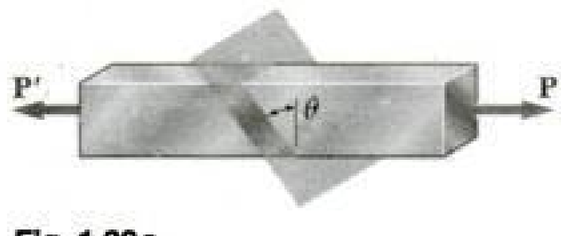


Fig. 1.30a

In Sec. 1.11, we considered the stresses created on an *oblique section* in a two-force member under axial loading. We found that both *normal* and *shearing* stresses occurred in such a situation. Denoting by θ the angle formed by the section with a normal plane (Fig. 1.30*a*) and by A_0 the area of a section perpendicular to the axis of the member, we derived the following expressions for the normal stress σ and the shearing stress τ on the oblique section:

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta \tag{1.14}$$

We observed from these formulas that the normal stress is maximum and equal to $\sigma_m = P/A_0$ for $\theta = 0$, while the shearing stress is maximum and equal to $\tau_m = P/2A_0$ for $\theta = 45°$. We also noted that $\tau = 0$ when $\theta = 0$, while $\sigma = P/2A_0$ when $\theta = 45°$.

Stress under general loading

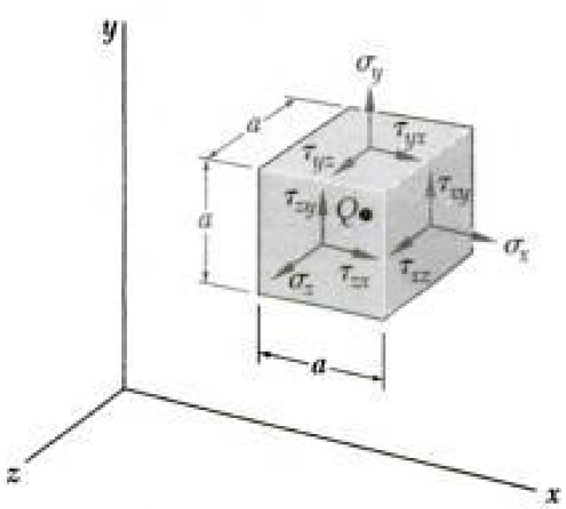


Fig. 1.36

Next, we discussed the state of stress at a point Q in a body under the most general loading condition [Sec. 1.12]. Considering a small cube centered at Q (Fig. 1.36), we denoted by σ_x the normal stress exerted on a face of the cube perpendicular to the x axis, and by τ_{xy} and τ_{xz}, respectively, the y and z components of the shearing stress exerted on the same face of the cube. Repeating this procedure for the other two faces of the cube and observing that $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, and $\tau_{zx} = \tau_{xz}$, we concluded that *six stress components* are required to define the state of stress at a given point Q, namely, σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}.

Section 1.13 was devoted to a discussion of the various concepts used in the design of engineering structures. The *ultimate load* of a given structural member or machine component is the load at which the member or component is expected to fail; it is computed from the *ultimate stress* or *ultimate strength* of the material used, as determined by a laboratory test on a specimen of that material. The ultimate load should be considerably larger than the *allowable load*, i.e., the load that the member or component will be allowed to carry under normal conditions. The ratio of the ultimate load to the allowable load is defined as the *factor of safety*:

Factor of safety

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate load}}{\text{allowable load}} \tag{1.26}$$

The determination of the factor of safety which should be used in the design of a given structure depends upon a number of considerations, some of which were listed in this section.

Load and Resistance Factor Design

Section 1.13 ended with the discussion of an alternative approach to design, known as *Load and Resistance Factor Design*, which allows the engineer to distinguish between the uncertainties associated with the structure and those associated with the load.

REVIEW PROBLEMS

1.59 For the truss and loading shown, determine the average normal stress in member *DF*, knowing that the cross-sectional area of that member is 2500 mm^2.

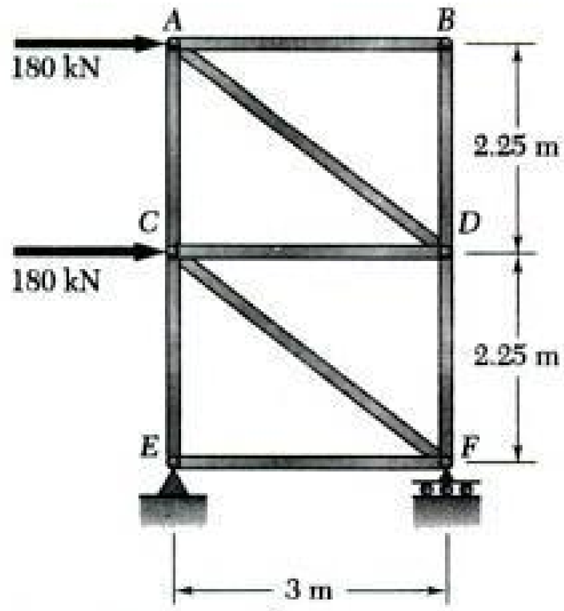


Fig. P1.59

1.60 Link *AC* has a uniform rectangular cross section 3.0 mm thick and 25.0 mm wide. Determine the normal stress in the central portion of that link.

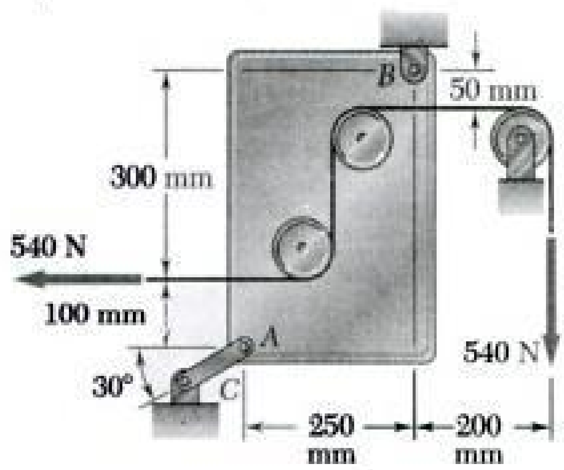


Fig. P1.60

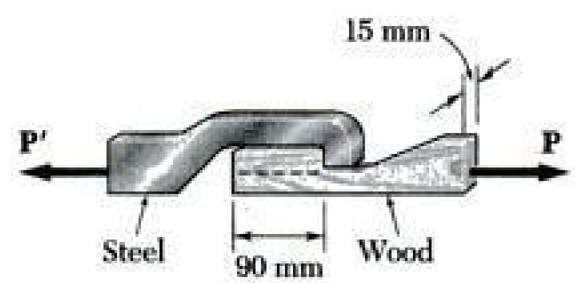


Fig. *P1.61*

1.61 When the force **P** reached 8 kN, the wooden specimen shown failed in shear along the surface indicated by the dashed line. Determine the average shearing stress along that surface at the time of failure.

1.62 Two wooden planks, each 12 mm thick and 225 mm wide, are joined by the dry mortise joint shown. Knowing that the wood used shears off along its grain when the average shearing stress reaches 8 MPa, determine the magnitude *P* of the axial load which will cause the joint to fail.

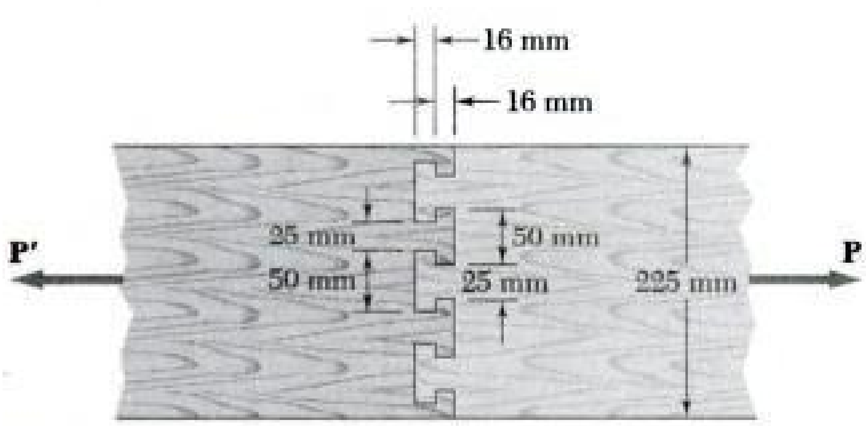


Fig. P1.62

1.63 The hydraulic cylinder *CF*, which partially controls the position of rod *DE*, has been locked in the position shown. Member *BD* is 16 mm thick and is connected to the vertical rod by a 10 mm diameter bolt. Determine (*a*) the average shearing stress in the bolt, (*b*) the bearing stress at *C* in member *BD*.

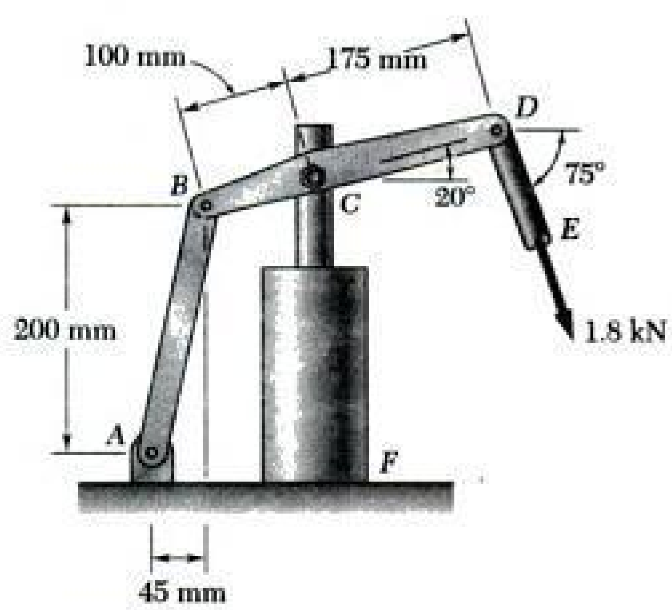


Fig. P1.63

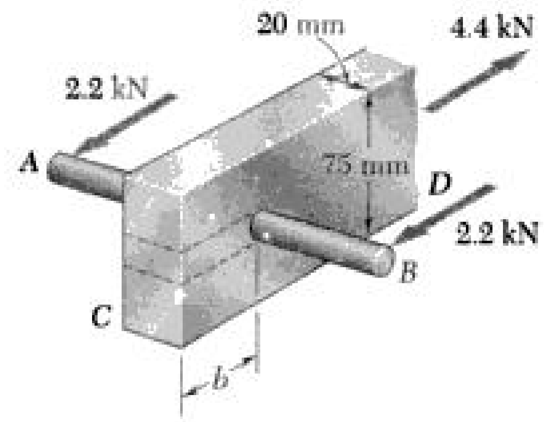


Fig. *P1.64*

1.64 A 12 mm diameter steel rod *AB* is fitted to a round hole near end *C* of the wooden member *CD*. For the loading shown, determine (*a*) the maximum average normal stress in the wood, (*b*) the distance *b* for which the average shearing stress is 0.6 MPa on the surfaces indicated by the dashed lines, (*c*) the average bearing stress on the wood.

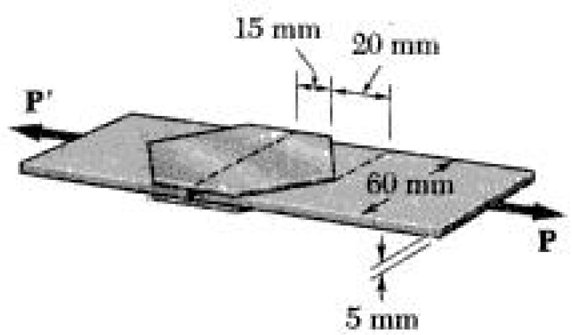


Fig. P1.65

1.65 Two plates, each 3 mm thick, are used to splice a plastic strip as shown. Knowing that the ultimate shearing stress of the bonding between the surfaces is 900 kPa, determine the factor of safety with respect to shear when $P = 1500$ N.

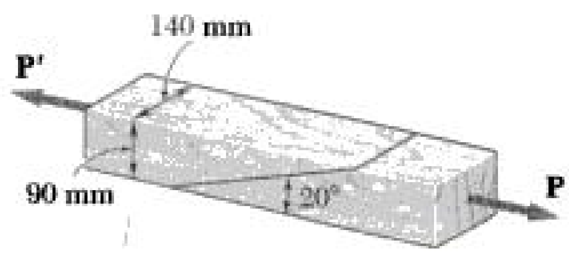


Fig. P1.66

1.66 Two wooden members of 90×140 mm uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that the maximum allowable shearing stress in the glued splice is 0.5 MPa, determine the largest axial load **P** that can be safely applied.

1.67 A steel loop *ABCD* of length 1.2 m and of 10-mm diameter is placed as shown around a 24-mm-diameter aluminum rod *AC*. Cables *BE* and *DF*, each of 12-mm diameter, are used to apply the load **Q**. Knowing that the ultimate strength of the aluminum used for the rod is 260 MPa and that the ultimate strength of the steel used for the loop and the cables is 480 MPa, determine the largest load **Q** that can be applied if an overall factor of safety of 3 is desired.

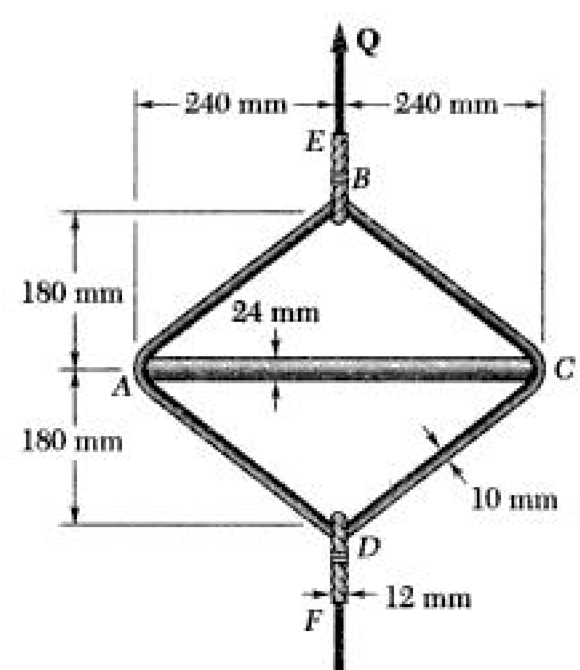


Fig. *P1.67*

1.68 Link AC has a uniform 6×12 mm rectangular cross section and is made of a steel with a 410 MPa ultimate normal stress. It is connected to a support at A and to member BCD at C by 10 mm diameter pins, while member BCD is connected to a support at B by a 8 mm diameter pin. All of the pins are in single shear and are made of a steel with a 170 MPa ultimate shearing stress. Knowing that an overall factor of safety of 3.25 is desired, determine the largest load **P** that can be safely applied at D. Note that link AC is not reinforced around the pin holes.

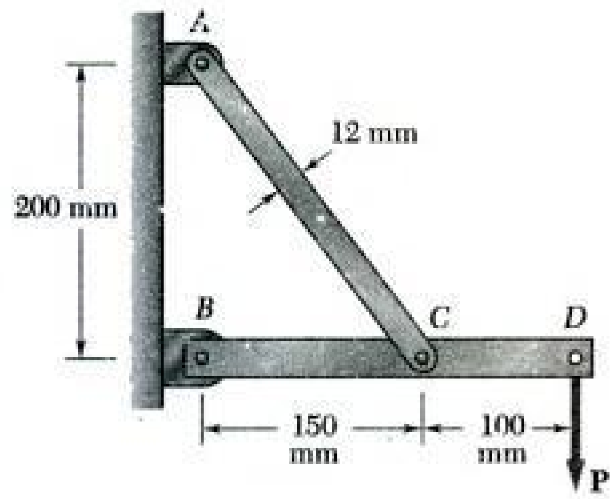


Fig. P1.68

1.69 The two portions of member AB are glued together along a plane forming an angle θ with the horizontal. Knowing that the ultimate stress for the glued joint is 17 MPa in tension and 9 MPa in shear, determine the range of values of θ for which the factor of safety of the member is at least 3.0.

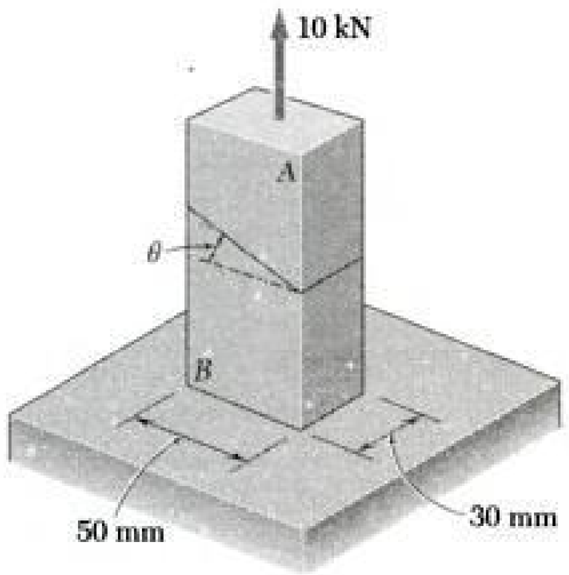


Fig. P1.69 and *P1.70*

1.70 The two portions of member AB are glued together along a plane forming an angle θ with the horizontal. Knowing that the ultimate stress for the glued joint is 17 MPa in tension and 9 MPa in shear, determine (*a*) the value of θ for which the factor of safety of the member is maximum, (*b*) the corresponding value of the factor of safety. (*Hint*: Equate the expressions obtained for the factors of safety with respect to normal stress and shear.)

COMPUTER PROBLEMS

The following problems are designed to be solved with a computer.

1.C1 A solid steel rod consisting of n cylindrical elements welded together is subjected to the loading shown. The diameter of element i is denoted by d_i and the load applied to its lower end by $\mathbf{P}_i$, with the magnitude P_i of this load being assumed positive if $\mathbf{P}_i$ is directed downward as shown and negative otherwise. (*a*) Write a computer program which can be used with SI units to determine the average stress in each element of the rod. (*b*) Use this program to solve Probs. 1.1 and 1.3.

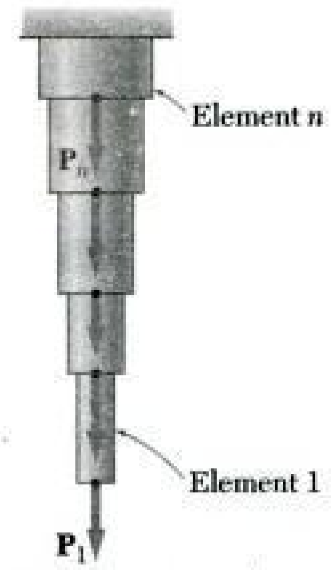


Fig. P1.C1

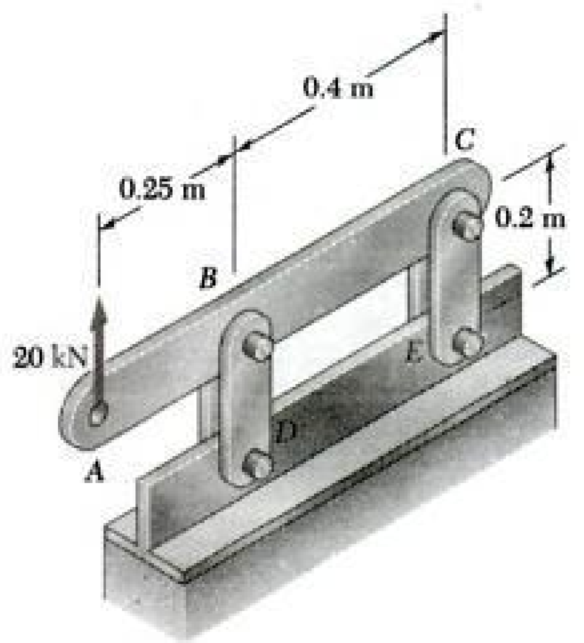


Fig. P1.C2

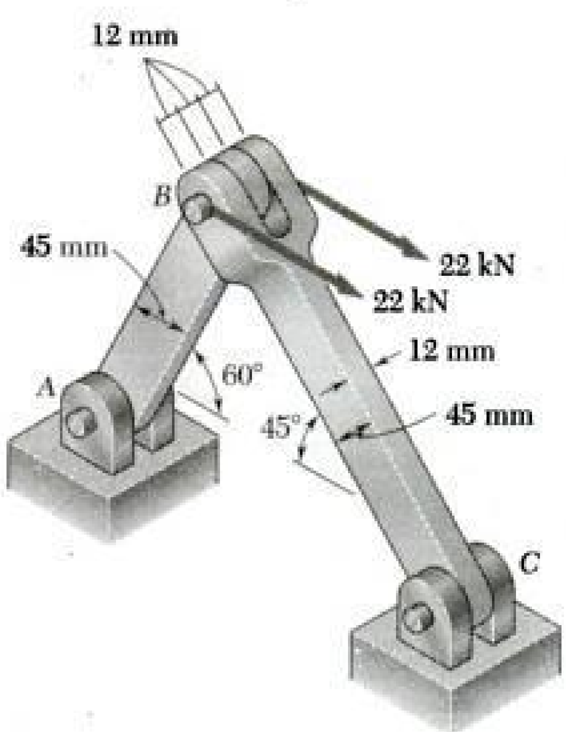


Fig. P1.C3

1.C2 A 20-kN force is applied as shown to the horizontal member *ABC*. Member *ABC* has a 10 × 50-mm uniform rectangular cross section and is supported by four vertical links, each of 8 × 36-mm uniform rectangular cross section. Each of the four pins at *A*, *B*, *C*, and *D* has the same diameter *d* and is in double shear. (*a*) Write a computer program to calculate for values of *d* from 10 to 30 mm, using 1-mm increments, (1) the maximum value of the average normal stress in the links connecting pins *B* and *D*, (2) the average normal stress in the links connecting pins *C* and *E*, (3) the average shearing stress in pin *B*, (4) the average shearing stress in pin *C*, (5) the average bearing stress at *B* in member *ABC*, (6) the average bearing stress at *C* in member *ABC*. (*b*) Check your program by comparing the values obtained for $d = 16$ mm with the answers given for Probs. 1.8, 1.23, and 1.24. (*c*) Use this program to find the permissible values of the diameter *d* of the pins, knowing that the allowable values of the normal, shearing, and bearing stresses for the steel used are, respectively, 150 MPa, 90 MPa, and 230 MPa. (*d*) Solve part *c*, assuming that the thickness of member *ABC* has been reduced from 10 to 8 mm.

1.C3 Two horizontal 22 kN forces are applied to pin *B* of the assembly shown. Each of the three pins at *A*, *B*, and *C* has the same diameter *d* and is in double shear. (*a*) Write a computer program to calculate for values of *d* from 12 to 38 mm, using 1.5 mm increments, (1) the maximum value of the average normal stress in member *AB*, (2) the average normal stress in member *BC*, (3) the average shearing stress in pin *A*, (4) the average shearing stress in pin *C*, (5) the average bearing stress at *A* in member *AB*, (6) the average bearing stress at *C* in member *BC*, (7) the average bearing stress at *B* in member *BC*. (*b*) Check your program by comparing the values obtained for $d = 20$ mm with the answers given for Probs. 1.9, 1.25, and 1.26. (*c*) Use this program to find the permissible values of the diameter *d* of the pins, knowing that the allowable values of the normal, shearing, and bearing stresses for the steel used are, respectively, 150 MPa, 90 MPa, and 250 MPa. (*d*) Solve part *c*, assuming that a new design is being investigated, in which the thickness and width of the two members are changed, respectively, from 12 to 8 mm and from 45 mm to 60 mm.

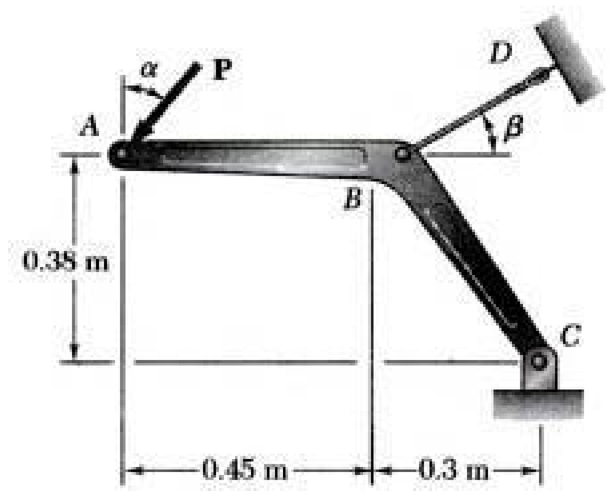


Fig. P1.C4

1.C4 A 18 kN force **P** forming an angle α with the vertical is applied as shown to member *ABC*, which is supported by a pin and bracket at *C* and by a cable *BD* forming an angle β with the horizontal. (*a*) Knowing that the ultimate load of the cable is 110 kN, write a computer program to construct a table of the values of the factor of safety of the cable for values of α and β from 0 to 45°, using increments in α and β corresponding to 0.1 increments in tan α and tan β. (*b*) Check that for any given value of α the maximum value of the factor of safety is obtained for $\beta = 38.66°$ and explain why. (*c*) Determine the smallest possible value of the factor of safety for $\beta = 38.66°$ as well as the corresponding value of α, and explain the result obtained.

1.C5 A load **P** is supported as shown by two wooden members of uniform rectangular cross section which are joined by a simple glued scarf splice. (*a*) Denoting by σ_U and τ_U, respectively, the ultimate strength of the joint in tension and in shear, write a computer program which, for given values of *a*, *b*, *P*, σ_U, and τ_U, expressed in SI units, and for values of α from 5 to 85° at 5° intervals, can be used to calculate (1) the normal stress in the joint, (2) the shearing stress in the joint, (3) the factor of safety relative to failure in tension, (4) the factor of safety relative to failure in shear, (5) the overall factor of safety for the glued joint. (*b*) Apply this program, using the dimensions and loading of the members of Probs. 1.29 and 1.32, knowing that σ_U = 1.26 MPa and τ_U = 1.50 MPa for the glue used in Prob. 1.29, and that σ_U = 1.0 MPa and τ_U = 1.5 MPa for the glue used in Prob. 1.32. (*c*) Verify in each of these two cases that the shearing stress is maximum for α = 45°.

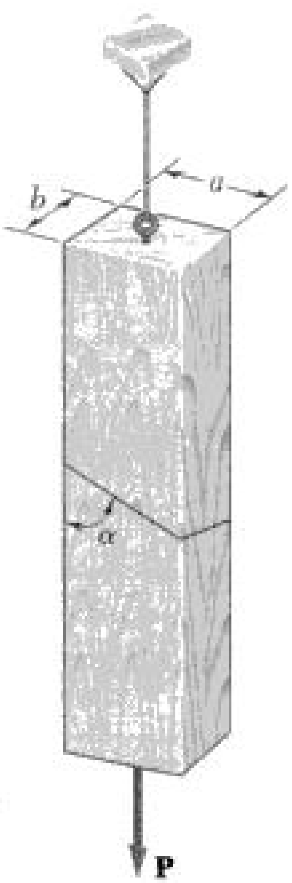


Fig. P1.C5

1.C6 Member *ABC* is supported by a pin and bracket at *A* and by two links, which are pin-connected to the member at *B* and to a fixed support at *D*. (*a*) Write a computer program to calculate the allowable load P_{all} for any given values of (1) the diameter d_1 of the pin at *A*, (2) the common diameter d_2 of the pins at *B* and *D*, (3) the ultimate normal stress σ_U in each of the two links, (4) the ultimate shearing stress τ_U in each of the three pins, (5) the desired overall factor of safety *F.S.* Your program should also indicate which of the following three stresses is critical: the normal stress in the links, the shearing stress in the pin at *A*, or the shearing stress in the pins at *B* and *D*. (*b* and *c*) Check your program by using the data of Probs. 1.49 and 1.50, respectively, and comparing the answers obtained for P_{all} with those given in the text. (*d*) Use your program to determine the allowable load P_{all}, as well as which of the stresses is critical, when $d_1 = d_2$ = 15mm, σ_U = 110 MPa for aluminum links, τ_U = 100 MPa for steel pins, and *F.S.* = 3.2.

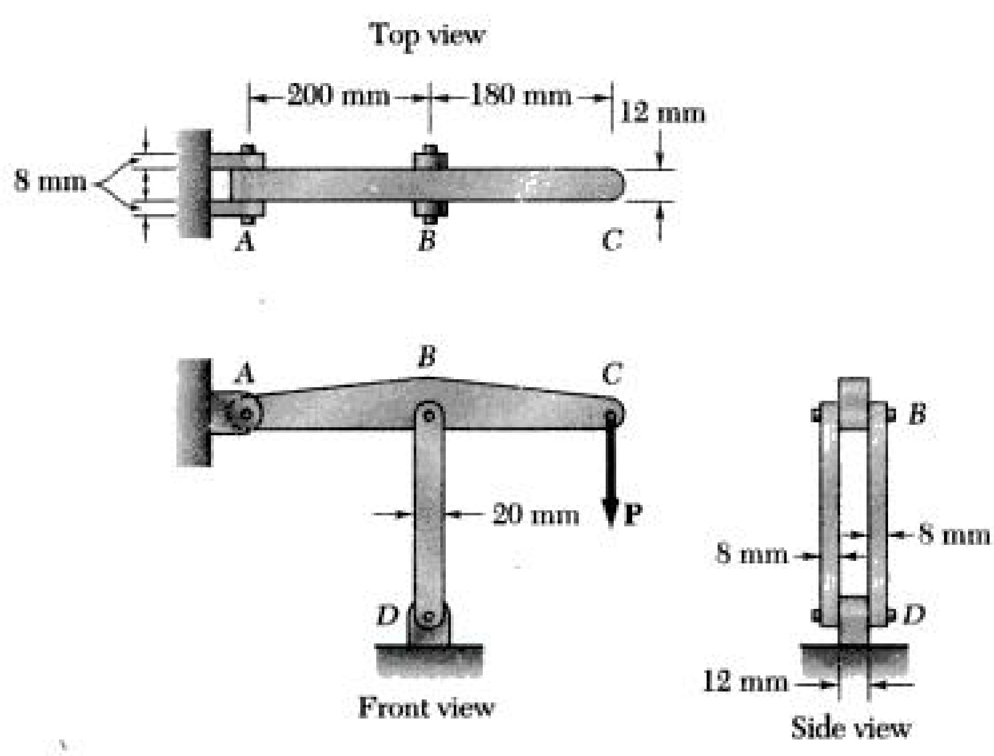
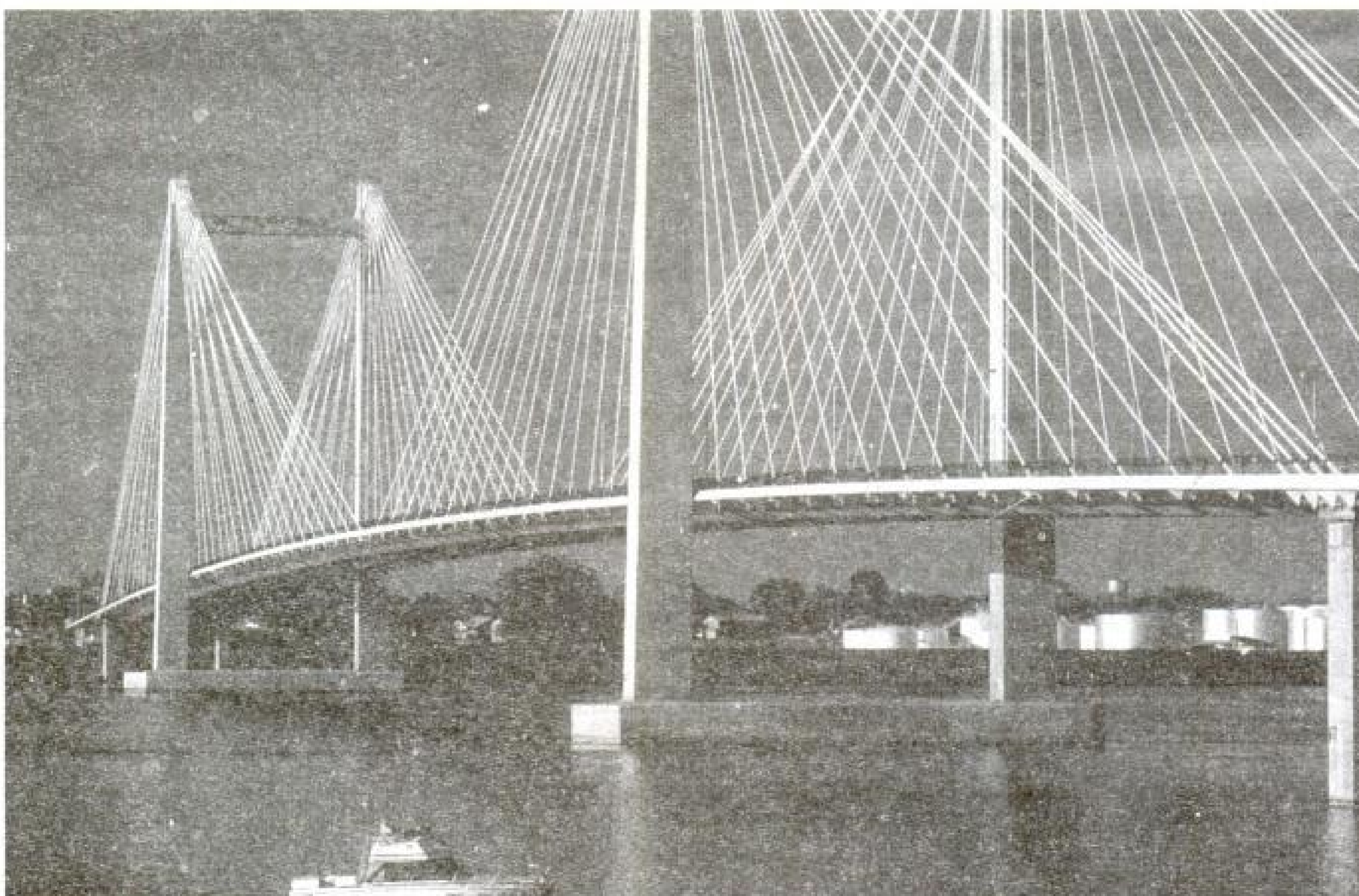


Fig. P1.C6

CHAPTER

2

Stress and Strain—Axial Loading



This chapter is devoted to the study of deformations occurring in structural components subjected to axial loading. The change in length of the diagonal stays was carefully accounted for in the design of this cable-stayed bridge that crosses the Columbia River in Eastern Washington.

lateral and axial strain, the *bulk modulus,* which characterizes the change in volume of a material under hydrostatic pressure, and the *modulus of rigidity*, which relates the components of the shearing stress and shearing strain. Stress-strain relationships for an isotropic material under a multiaxial loading will also be derived.

In Sec. 2.16, stress-strain relationships involving several distinct values of the modulus of elasticity, Poisson's ratio, and the modulus of rigidity, will be developed for fiber-reinforced composite materials under a multiaxial loading. While these materials are not isotropic, they usually display special properties, known as *orthotropic* properties, which facilitate their study.

In the text material described so far, stresses are assumed uniformly distributed in any given cross section; they are also assumed to remain within the elastic range. The validity of the first assumption is discussed in Sec. 2.17, while *stress concentrations* near circular holes and fillets in flat bars are considered in Sec. 2.18. Sections 2.19 and 2.20 are devoted to the discussion of stresses and deformations in members made of a ductile material when the yield point of the material is exceeded. As you will see, permanent *plastic deformations* and *residual stresses* result from such loading conditions.

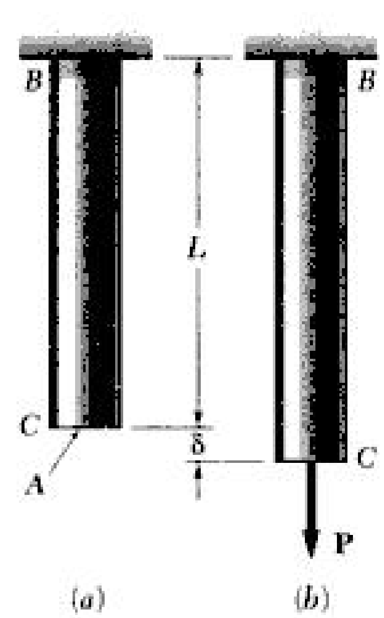


Fig. 2.1

2.2. NORMAL STRAIN UNDER AXIAL LOADING

Let us consider a rod BC, of length L and uniform cross-sectional area A, which is suspended from B (Fig. 2.1*a*). If we apply a load **P** to end C, the rod elongates (Fig. 2.1*b*). Plotting the magnitude P of the load against the deformation δ (Greek letter delta), we obtain a certain load-deformation diagram (Fig. 2.2). While this diagram contains information useful to the analysis of the rod under consideration, it cannot be used directly to predict the deformation of a rod of the same material but of different dimensions. Indeed, we observe that, if a deformation δ is produced in rod BC by a load **P**, a load 2**P** is required to cause the same deformation in a rod $B'C'$ of the same length L, but of cross-sectional area $2A$ (Fig. 2.3). We note that, in both cases, the value of the stress is the same: $\delta = P/A$. On the other hand, a load **P** applied to

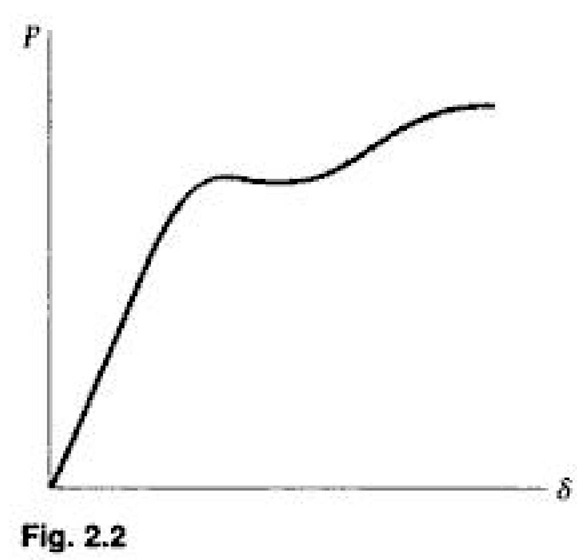


Fig. 2.2

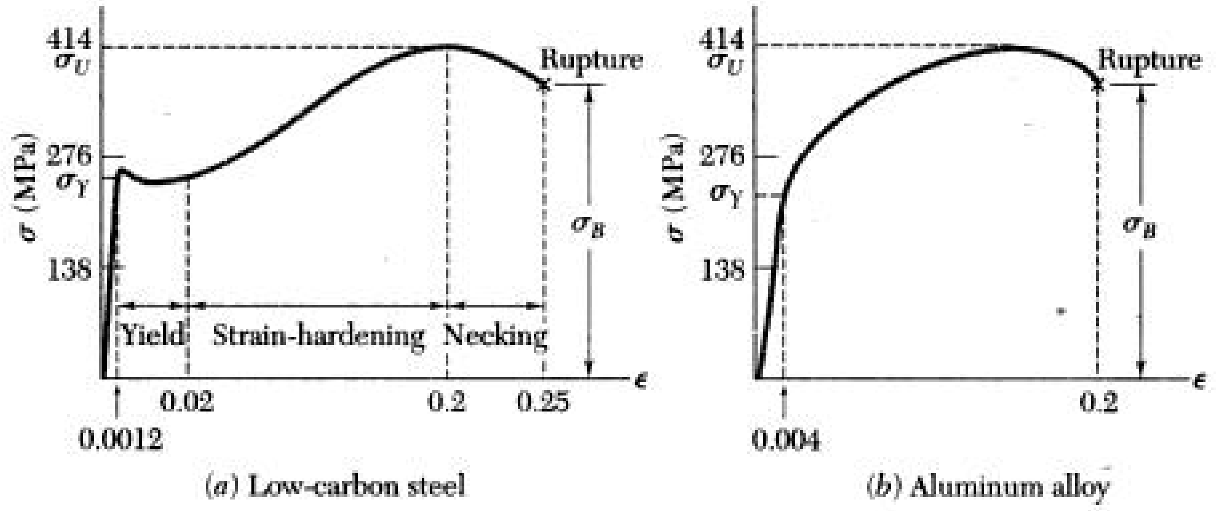


Fig. 2.9 Stress-strain diagrams of two typical ductile materials.

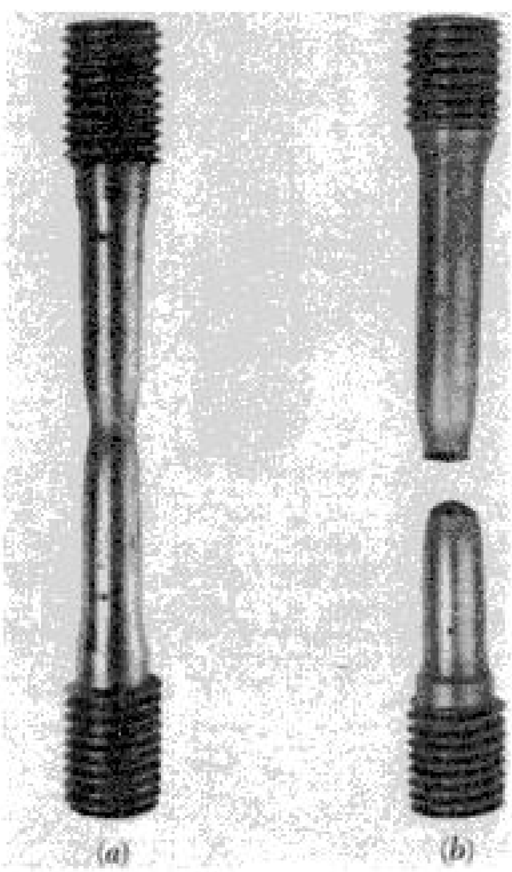


Fig. 2.10 Tested specimen of a ductile material.

the load has been reached, the diameter of a portion of the specimen begins to decrease, because of local instability (Fig. 2.10*a*). This phenomenon is known as *necking*. After necking has begun, somewhat lower loads are sufficient to keep the specimen elongating further, until it finally ruptures (Fig. 2.10*b*). We note that rupture occurs along a cone-shaped surface which forms an angle of approximately 45° with the original surface of the specimen. This indicates that shear is primarily responsible for the failure of ductile materials, and confirms the fact that, under an axial load, shearing stresses are largest on surfaces forming an angle of 45° with the load (cf. Sec. 1.11). The stress σ_Y at which yield is initiated is called the *yield strength* of the material, the stress σ_U corresponding to the maximum load applied to the specimen is known as the *ultimate strength,* and the stress σ_B corresponding to rupture is called the *breaking strength.*

Brittle materials, which comprise cast iron, glass, and stone, are characterized by the fact that rupture occurs without any noticeable prior change in the rate of elongation (Fig. 2.11). Thus, for brittle materials, there is no difference between the ultimate strength and the breaking strength. Also, the strain at the time of rupture is much smaller for brittle than for ductile materials. From Fig. 2.12, we note the absence of any necking of the specimen in the case of a brittle material, and observe that rupture occurs along a surface perpendicular to the load. We conclude from this observation that normal stresses are primarily responsible for the failure of brittle materials.†

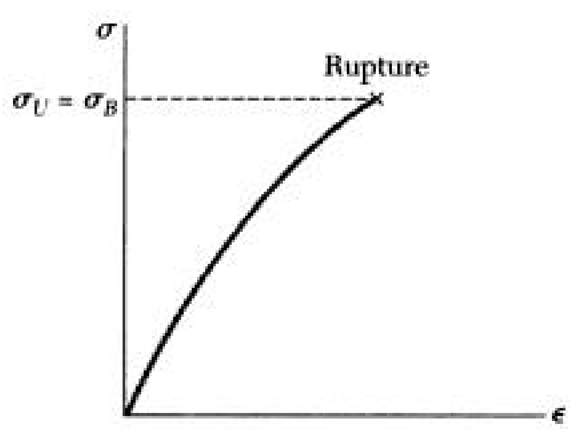


Fig. 2.11 Stress-strain diagram for a typical brittle material.

The stress-strain diagrams of Fig. 2.9 show that structural steel and aluminum, while both ductile, have different yield characteristics. In the case of structural steel (Fig. 2.9*a*), the stress remains constant over a large range of values of the strain after the onset of yield. Later the stress must be increased to keep elongating the specimen, until the max-

†The tensile tests described in this section were assumed to be conducted at normal temperatures. However, a material that is ductile at normal temperatures may display the characteristics of a brittle material at very low temperatures, while a normally brittle material may behave in a ductile fashion at very high temperatures. At temperatures other than normal, therefore, one should refer to *a material in a ductile state* or to *a material in a brittle state,* rather than to a ductile or brittle material.

tests will yield essentially the same plot when true stress and true strain are used. This is not the case for large values of the strain when the engineering stress is plotted versus the engineering strain. However, engineers, whose responsibility is to determine whether a load P will produce an acceptable stress and an acceptable deformation in a given member, will want to use a diagram based on the engineering stress $\sigma = P/A_0$ and the engineering strain $\epsilon = \delta/L_0$ since these expressions involve data that are available to them, namely the cross-sectional area A_0 and the length L_0 of the member in its undeformed state.

2.5. HOOKE'S LAW; MODULUS OF ELASTICITY

Most engineering structures are designed to undergo relatively small deformations, involving only the straight-line portion of the corresponding stress-strain diagram. For that initial portion of the diagram (Fig. 2.9), the stress σ is directly proportional to the strain ϵ, and we can write

$$\sigma = E\epsilon \tag{2.4}$$

This relation is known as *Hooke's law*, after the English mathematician Robert Hooke (1635–1703). The coefficient E is called the *modulus of elasticity* of the material involved, or also *Young's modulus*, after the English scientist Thomas Young (1773–1829). Since the strain ϵ is a dimensionless quantity, the modulus E is expressed in the same units as the stress σ, namely in pascals or one of its multiples.

The largest value of the stress for which Hooke's law can be used for a given material is known as the *proportional limit* of that material. In the case of ductile materials possessing a well-defined yield point, as in Fig. 2.9*a*, the proportional limit almost coincides with the yield point. For other materials, the proportional limit cannot be defined as easily, since it is difficult to determine with accuracy the value of the stress σ for which the relation between σ and ϵ ceases to be linear. But from this very difficulty we can conclude for such materials that using Hooke's law for values of the stress slightly larger than the actual proportional limit will not result in any significant error.

Some of the physical properties of structural metals, such as strength, ductility, and corrosion resistance, can be greatly affected by alloying, heat treatment, and the manufacturing process used. For example, we note from the stress-strain diagrams of pure iron and of three different grades of steel (Fig. 2.16) that large variations in the yield strength, ultimate strength, and final strain (ductility) exist among these four metals. All of them, however, possess the same modulus of elasticity; in other words, their "stiffness," or ability to resist a deformation within the linear range, is the same. Therefore, if a high-strength steel is substituted for a lower-strength steel in a given structure, and if all dimensions are kept the same, the structure will have an increased load-carrying capacity, but its stiffness will remain unchanged.

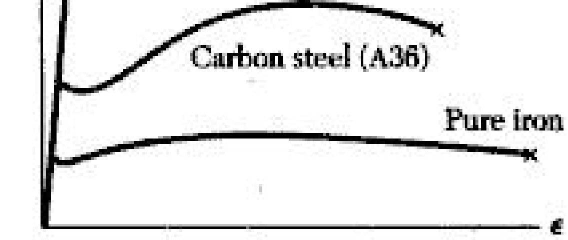


Fig. 2.16 Stress-strain diagrams for iron and different grades of steel.

Fatigue must be considered in the design of all structural and machine components that are subjected to repeated or to fluctuating loads. The number of loading cycles that may be expected during the useful life of a component varies greatly. For example, a beam supporting an industrial crane may be loaded as many as two million times in 25 years (about 300 loadings per working day), an automobile crankshaft will be loaded about half a billion times if the automobile is driven 320,000 km, and an individual turbine blade may be loaded several hundred billion times during its lifetime.

Some loadings are of a fluctuating nature. For example, the passage of traffic over a bridge will cause stress levels that will fluctuate about the stress level due to the weight of the bridge. A more severe condition occurs when a complete reversal of the load occurs during the loading cycle. The stresses in the axle of a railroad car, for example, are completely reversed after each half-revolution of the wheel.

The number of loading cycles required to cause the failure of a specimen through repeated successive loadings and reverse loadings may be determined experimentally for any given maximum stress level. If a series of tests is conducted, using different maximum stress levels, the resulting data may be plotted as a σ-n curve. For each test, the maximum stress σ is plotted as an ordinate and the number of cycles n as an abscissa; because of the large number of cycles required for rupture, the cycles n are plotted on a logarithmic scale.

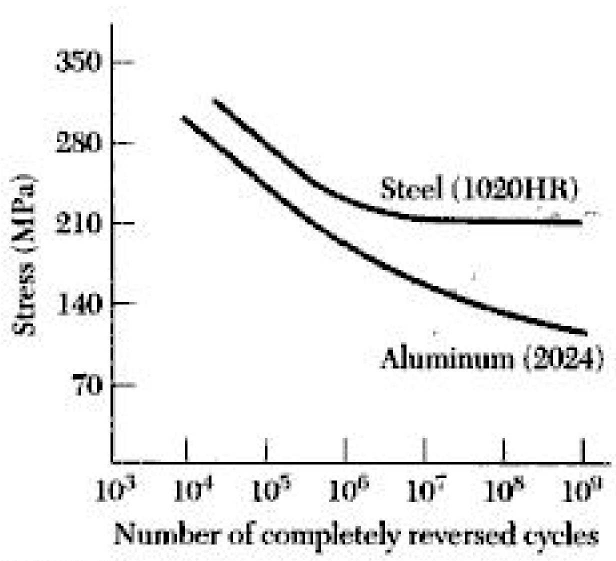
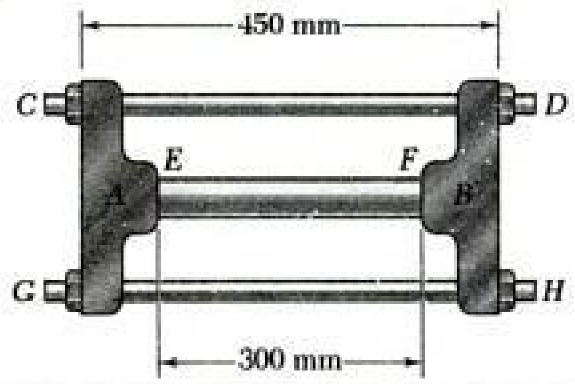


Fig. 2.21

A typical σ-n curve for steel is shown in Fig. 2.21. We note that, if the applied maximum stress is high, relatively few cycles are required to cause rupture. As the magnitude of the maximum stress is reduced, the number of cycles required to cause rupture increases, until a stress, known as the *endurance limit,* is reached. The endurance limit is the stress for which failure does not occur, even for an indefinitely large number of loading cycles. For a low-carbon steel, such as structural steel, the endurance limit is about one-half of the ultimate strength of the steel.

For nonferrous metals, such as aluminum and copper, a typical σ-n curve (Fig. 2.21) shows that the stress at failure continues to decrease as the number of loading cycles is increased. For such metals, one defines the *fatigue limit* as the stress corresponding to failure after a specified number of loading cycles, such as 500 million.

Examination of test specimens, of shafts, of springs, and of other components that have failed in fatigue shows that the failure was initiated at a microscopic crack or at some similar imperfection. At each loading, the crack was very slightly enlarged. During successive loading cycles, the crack propagated through the material until the amount of undamaged material was insufficient to carry the maximum load, and an abrupt, brittle failure occurred. Because fatigue failure may be initiated at any crack or imperfection, the surface condition of a specimen has an important effect on the value of the endurance limit obtained in testing. The endurance limit for machined and polished specimens is higher than for rolled or forged components, or for components that are corroded. In applications in or near seawater, or in other applications where corrosion is expected, a reduction of up to 50% in the endurance limit can be expected.



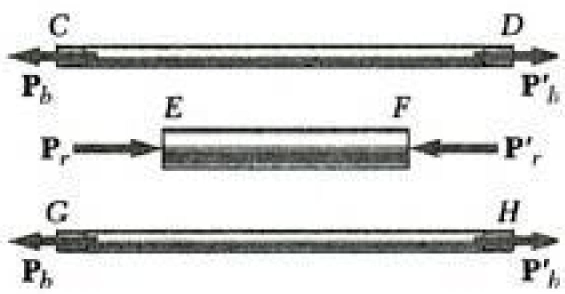
SAMPLE PROBLEM 2.2

The rigid castings *A* and *B* are connected by two 20 mm-diameter steel bolts *CD* and *GH* and are in contact with the ends of a 40 mm-diameter aluminum rod *EF*. Each bolt is single-threaded with a pitch of 2.5 mm, and after being snugly fitted, the nuts at *D* and *H* are both tightened one-quarter of a turn. Knowing that *E* is 200 GPa for steel and 70 GPa for aluminum, determine the normal stress in the rod.

SOLUTION

Deformations

Bolts CD and GH. Tightening the nuts causes tension in the bolts. Because of symmetry, both are subjected to the same internal force P_b and undergo the same deformation δ_b. We have



$$\delta_b = +\frac{P_b L_b}{A_b E_b} = +\frac{P_b\,(450\text{ mm})}{\frac{1}{4}\pi(20\text{ mm})^2(200\times10^3\text{ N/mm}^2)} = +7.162\times10^{-6}\,P_b \quad (1)$$

Rod EF. The rod is in compression. Denoting by P_r the magnitude of the force in the rod and by δ_r the deformation of the rod, we write

$$\delta_r = -\frac{P_r L_r}{A_r E_r} = +\frac{P_r\,(300\text{ mm})}{\frac{1}{4}\pi(40\text{ mm})^2(70\times10^3\text{ N/mm}^2)} = -3.40\times10^{-6}\,P_r \quad (2)$$

Displacement of D Relative to B. Tightening the nuts one-quarter of a turn causes ends *D* and *H* of the bolts to undergo a displacement of $\frac{1}{4}$(2.54 mm) *relative to casting B*. Considering end *D*, we write

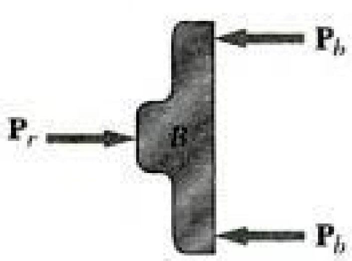
$$\delta_{D/B} = \tfrac{1}{4}(2.54\text{ mm}) = 0.625\text{ mm} \quad (3)$$

But $\delta_{D/B} = \delta_D - \delta_B$, where δ_D and δ_B represent the displacements of *D* and *B*. If we assume that casting *A* is held in a fixed position while the nuts at *D* and *H* are being tightened, these displacements are equal to the deformations of the bolts and of the rod, respectively. We have, therefore,

$$\delta_{D/B} = \delta_b - \delta_r \quad (4)$$

Substituting from (1), (2), and (3) into (4), we obtain

$$0.625\text{ mm} = 7.162\times10^{-6}\,P_b + 3.40\times10^{-6}\,P_r \quad (5)$$



Free Body: Casting *B*

$$\overset{+}{\rightarrow}\Sigma F = 0: \qquad P_r - 2P_b = 0 \qquad P_r = 2P_b \quad (6)$$

Forces in Bolts and Rod

Substituting for P_r from (6) into (5), we have

$$0.625\text{ mm} = 7.162\times10^{-6}\,P_b + 3.40\times10^{-6}(2P_b)$$

$$P_b = 44.76\times10^3\text{ N} = 44.76\text{ kN}$$

$$P_r = 2P_b = 2(44.76\text{ kN}) = 89.52\text{ kN}$$

Stress in Rod

$$\sigma_r = \frac{P_r}{A_r} = \frac{89.52\text{ kN}}{\frac{1}{4}\pi(40)^2} \qquad \sigma_r = 71.2\text{ MPa} \blacktriangleleft$$

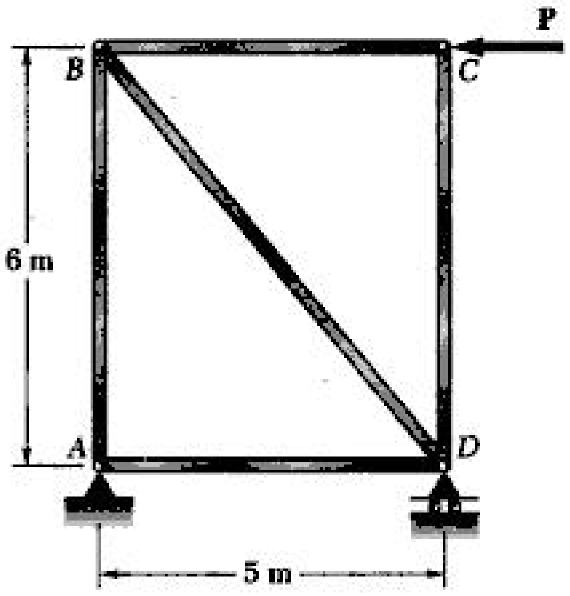


Fig. P2.21

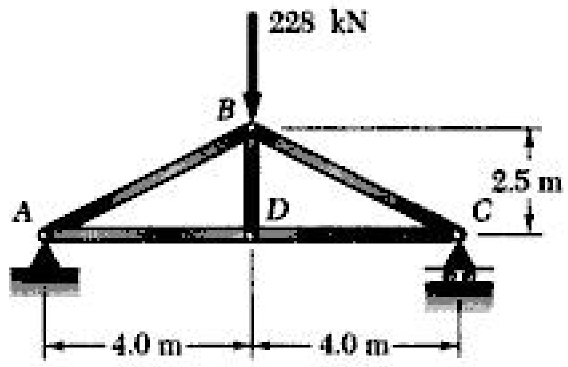


Fig. P2.22

2.21 The steel frame (E = 200 GPa) shown has a diagonal brace BD with an area of 1920 mm^2. Determine the largest allowable load **P** if the change in length of member BD is not to exceed 1.6 mm.

2.22 For the steel truss (E = 200 GPa) and loading shown, determine the deformations of members AB and AD, knowing that their cross-sectional areas are 2400 mm^2 and 1800 mm^2, respectively.

2.23 Members AB and BC are made of steel (E = 200 GPa) with cross-sectional areas of 516 mm^2 and 412 mm^2, respectively. For the loading shown, determine the elongation of (a) member AB, (b) member BC.

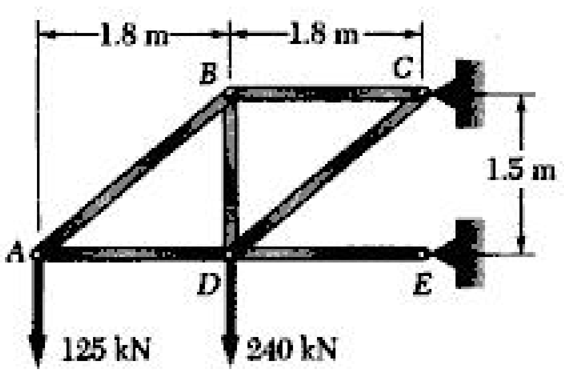


Fig. P2.23

2.24 Members AB and CD are 28 mm-diameter steel rods, and members BC and AD are 22 mm-diameter steel rods. When the turnbuckle is tightened, the diagonal member AC is put in tension. Knowing that E = 200 GPa and h = 1.2 m, determine the largest allowable tension in AC so that the deformations in members AB and CD do not exceed 1.0 mm.

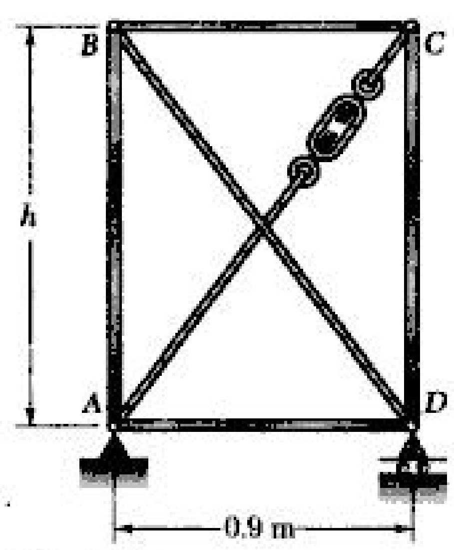


Fig. *P2.24*

2.25 For the structure in Prob. 2.24, determine (a) the distance h so that the deformations in members AB, BC, CD, and AD are all equal to 1.0 mm, (b) the corresponding tension in member AC.

EXAMPLE 2.04

Determine the reactions at A and B for the steel bar and loading shown in Fig. 2.28, assuming a close fit at both supports before the loads are applied.

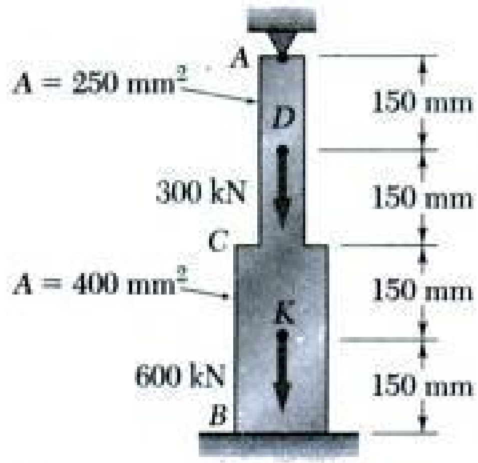


Fig. 2.28

We consider the reaction at B as redundant and release the bar from that support. The reaction $\mathbf{R}_B$ is now considered as an unknown load (Fig. 2.29a) and will be determined from the condition that the deformation δ of the rod must be equal to zero. The solution is carried out by considering separately the deformation δ_L caused by the given loads (Fig. 2.29b) and the deformation δ_R due to the redundant reaction $\mathbf{R}_B$ (Fig. 2.29c).

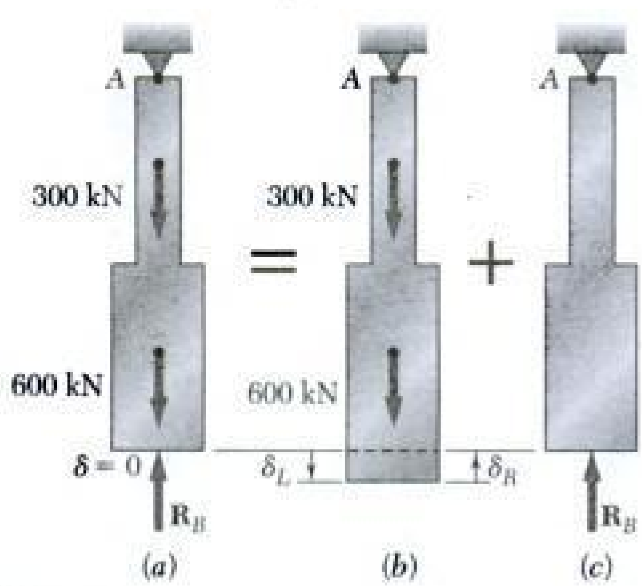


Fig. 2.29

The deformation δ_L is obtained from Eq. (2.8) after the bar has been divided into four portions, as shown in Fig. 2.30.

Following the same procedure as in Example 2.01, we write

$$P_1 = 0 \quad P_2 = P_3 = 600 \times 10^3 \text{ N} \quad P_4 = 900 \times 10^3 \text{ N}$$
$$A_1 = A_2 = 400 \times 10^{-6} \text{ m}^2 \quad A_3 = A_4 = 250 \times 10^{-6} \text{ m}^2$$
$$L_1 = L_2 = L_3 = L_4 = 0.150 \text{ m}$$

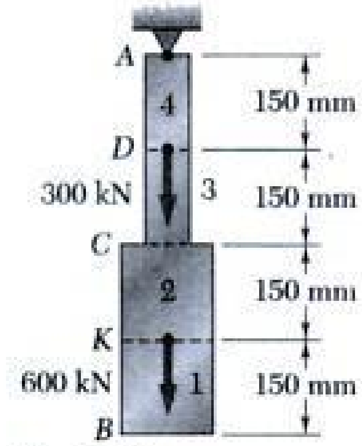


Fig. 2.30

Substituting these values into Eq. (2.8), we obtain

$$\delta_L = \sum_{i=1}^{4} \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3 \text{ N}}{400 \times 10^{-6} \text{ m}^2} + \frac{600 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2} + \frac{900 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2}\right) \frac{0.150 \text{ m}}{E}$$

$$\delta_L = \frac{1.125 \times 10^9}{E} \tag{2.17}$$

Considering now the deformation δ_R due to the redundant reaction R_B, we divide the bar into two portions, as shown in Fig. 2.31, and write

$$P_1 = P_2 = -R_B$$
$$A_1 = 400 \times 10^{-6} \text{ m}^2 \quad A_2 = 250 \times 10^{-6} \text{ m}^2$$
$$L_1 = L_2 = 0.300 \text{ m}$$

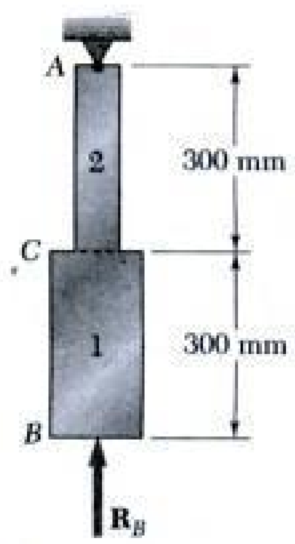


Fig. 2.31

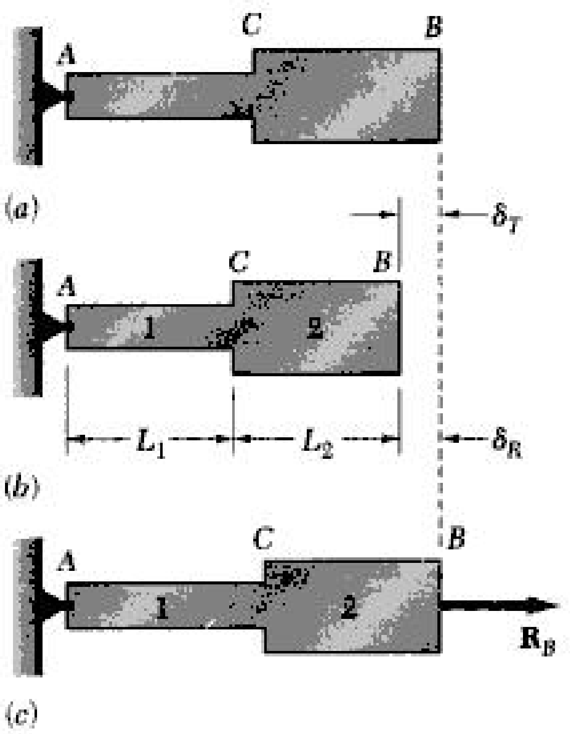


Fig. 2.38

The corresponding deformation (Fig. 2.38*b*) is

$$\delta_T = \alpha(\Delta T)L = (11.7 \times 10^{-6}/°C)(-69°C)(600 \text{ mm})$$
$$= -0.484 \text{ mm}$$

Applying now the unknown force $\mathbf{R}_B$ at end B (Fig. 2.38*c*), we use Eq. (2.8) to express the corresponding deformation δ_R. Substituting

$$L_1 = L_2 = 300 \text{ mm}$$
$$A_1 = 387 \text{ mm}^2 \qquad A_2 = 774 \text{ mm}^2$$
$$P_1 = P_2 = R_B \qquad E = 200 \text{ GPa}$$

into Eq. (2.8), we write

$$\delta_R = \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E}$$
$$= \frac{R_B}{200 \text{ GPa}}\left(\frac{300 \text{ mm}}{387 \text{ mm}^2} + \frac{300 \text{ mm}}{774 \text{ mm}^2}\right)$$
$$= (5.81 \times 10^{-6} \text{mm/N})R_B$$

Expressing that the total deformation of the bar must be zero as a result of the imposed constraints, we write

$$\delta = \delta_T + \delta_R = 0$$
$$= -0.487 \text{ mm} + (5.81 \times 10^{-6} \text{mm/N})R_B = 0$$

from which we obtain

$$R_B = 84 \text{ kN}$$

The reaction at A is equal and opposite.

Noting that the forces in the two portions of the bar are $P_1 = P_2 = 84$ kN, we obtain the following values of the stress in portions AC and CB of the bar:

$$\sigma_1 = \frac{P_1}{A_1} = \frac{84 \text{ kN}}{387 \text{ mm}^2} = +217 \text{ MPa}$$
$$\sigma_2 = \frac{P_2}{A_2} = \frac{84 \text{ kN}}{774 \text{ mm}^2} = +108.5 \text{ MPa}$$

We cannot emphasize too strongly the fact that, while the *total deformation* of the bar must be zero, the deformations of the portions AC and CB *are not zero*. A solution of the problem based on the assumption that these deformations are zero would therefore be wrong. Neither can the values of the strain in AC or CB be assumed equal to zero. To amplify this point, let us determine the strain ϵ_{AC} in portion AC of the bar. The strain ϵ_{AC} can be divided into two component parts; one is the thermal strain ϵ_T produced in the unrestrained bar by the temperature change ΔT (Fig. 2.38*b*). From Eq. (2.22) we write

$$\epsilon_T = \alpha\,\Delta T = (11.7 \times 10^{-6}/°C)(-69°C)$$
$$= 807.3 \times 10^{-6} \text{ mm/mm}$$

The other component of ϵ_{AC} is associated with the stress σ_1 due to the force $\mathbf{R}_B$ applied to the bar (Fig. 2.38*c*). From Hooke's law, we express this component of the strain as

$$\frac{\sigma_1}{E} = \frac{217 \text{ MPa}}{200 \text{ GPa}} = +1085 \times 10^{-6} \text{mm/mm}$$

Adding the two components of the strain in AC, we obtain

$$\epsilon_{AC} = \epsilon_T + \frac{\sigma_1}{E} = -807.3 \times 10^{-6} + 1085 \times 10^{-6}$$
$$= +277 \times 10^{-6} \text{ mm/mm}$$

A similar computation yields the strain in portion CB of the bar:

$$\epsilon_{CB} = \epsilon_T + \frac{\sigma_2}{E} = -807.3 \times 10^{-6} + \frac{108.5 \text{ MPa}}{200 \text{ GPa}}$$
$$= -264.8 \times 10^{-6} \text{ mm/mm}$$

The deformations δ_{AC} and δ_{CB} of the two portions of the bar are expressed respectively as

$$\delta_{AC} = \epsilon_{AC}(AC) = (+277 \times 10^{-6} \text{mm/mm})(300 \text{ mm})$$
$$= 0.0831 \text{ mm}$$
$$\delta_{CB} = \epsilon_{CB}(CB) = (-264.8 \times 10^{-6})(300 \text{ mm})$$
$$= -0.079 \text{ mm}$$

We thus check that, while the sum $\delta = \delta_{AC} + \delta_{CB}$ of the two deformations is zero, neither of the deformations is zero.

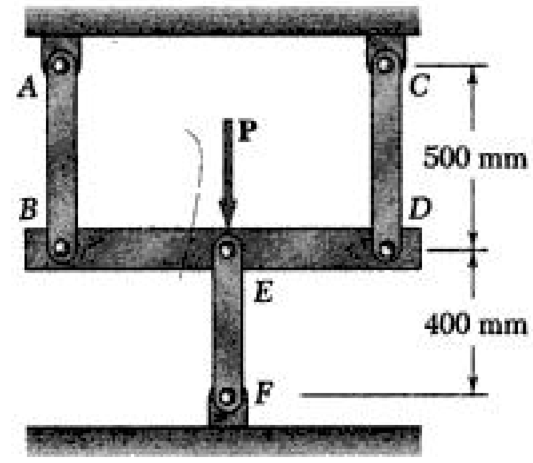


Fig. P2.39

2.39 Three steel rods (E = 200 GPa) support a 36-kN load **P**. Each of the rods AB and CD has a 200-mm^2 cross-sectional area and rod EF has a 625-mm^2 cross-sectional area. Determine the (*a*) the change in length of rod EF, (*b*) the stress in each rod.

2.40 A brass bolt (E_b = 103 GPa) with a 10 mm diameter is fitted inside a steel tube (E_s = 200 GPa) with a 22 mm outer diameter and 3 mm wall thickness. After the nut has been fit snugly, it is tightened one-quarter of a full turn. Knowing that the bolt is single-threaded with a 2.5 mm pitch, determine the normal stress (*a*) in the bolt, (*b*) in the tube.

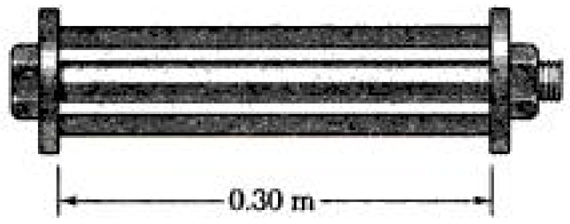


Fig. P2.40

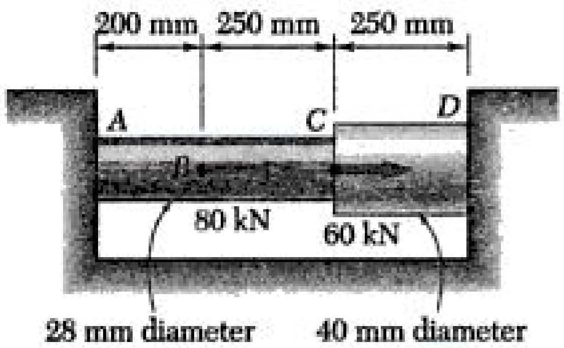


Fig. P2.41

2.41 Two cylindrical rods, CD made of steel (E = 200 GPa) and AC made of aluminum (E = 72 GPa), are joined at C and restrained by rigid supports at A and D. Determine (*a*) the reactions at A and D, (*b*) the deflection of point C.

2.42 A steel tube (E = 200 GPa) with a 32-mm outer diameter and a 4-mm thickness is placed in a vise that is adjusted so that its jaws just touch the ends of the tube without exerting any pressure on them. The two forces shown are then applied to the tube. After these forces are applied, the vise is adjusted to decrease the distance between its jaws by 0.2 mm. Determine (*a*) the forces exerted by the vise on the tube at A and D, (*b*) the change in length of the portion BC of the tube.

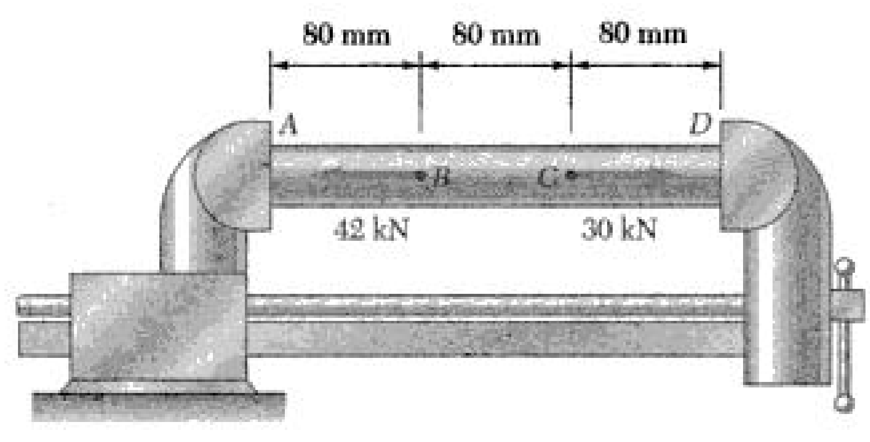


Fig. P2.42

2.43 Solve Prob. 2.42, assuming that after the forces have been applied, the vise is adjusted to decrease the distance between its jaws by 0.1 mm.

2.44 Three wires are used to suspend the plate shown. Aluminum wires are used at A and B with a diameter of 3.0 mm and a steel wire is used at C with a diameter of 2.0 mm. Knowing that the allowable stress for aluminum (E = 72 GPa) is 96 MPa and that the allowable stress for steel (E = 200 GPa) is 124 MPa, determine the maximum load **P** that may be applied.

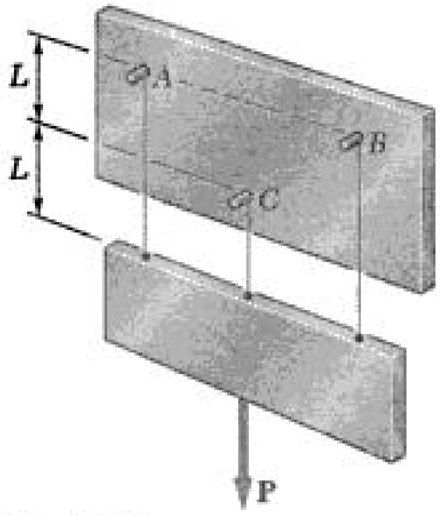


Fig. P2.44

2.11. POISSON'S RATIO

We saw in the earlier part of this chapter that, when a homogeneous slender bar is axially loaded, the resulting stress and strain satisfy Hooke's law, as long as the elastic limit of the material is not exceeded. Assuming that the load **P** is directed along the x axis (Fig. 2.39*a*), we have $\sigma_x = P/A$, where A is the cross-sectional area of the bar, and, from Hooke's law,

$$\epsilon_x = \sigma_x/E \tag{2.24}$$

where E is the modulus of elasticity of the material.

We also note that the normal stresses on faces respectively perpendicular to the y and z axes are zero: $\sigma_y = \sigma_z = 0$ (Fig. 2.39*b*). It would be tempting to conclude that the corresponding strains ϵ_y and ϵ_z are also zero. This, however, *is not the case*. In all engineering materials, the elongation produced by an axial tensile force **P** in the direction of the force is accompanied by a contraction in any transverse direction (Fig. 2.40).† In this section and the following sections (Secs. 2.12 through 2.15), all materials considered will be assumed to be both *homogeneous* and *isotropic*, i.e., their mechanical properties will be assumed independent of both *position* and *direction*. It follows that the strain must have the same value for any transverse direction. Therefore, for the loading shown in Fig. 2.39 we must have $\epsilon_y = \epsilon_z$. This common value is referred to as the *lateral strain*. An important constant for a given material is its *Poisson's ratio*, named after the French mathematician Siméon Denis Poisson (1781–1840) and denoted by the Greek letter ν (nu). It is defined as

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \tag{2.25}$$

or

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x} \tag{2.26}$$

for the loading condition represented in Fig. 2.39. Note the use of a minus sign in the above equations to obtain a positive value for ν, the axial and lateral strains having opposite signs for all engineering materials.‡ Solving Eq. (2.26) for ϵ_y and ϵ_z, and recalling (2.24), we write the following relations, which fully describe the condition of strain under an axial load applied in a direction parallel to the x axis:

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E} \tag{2.27}$$

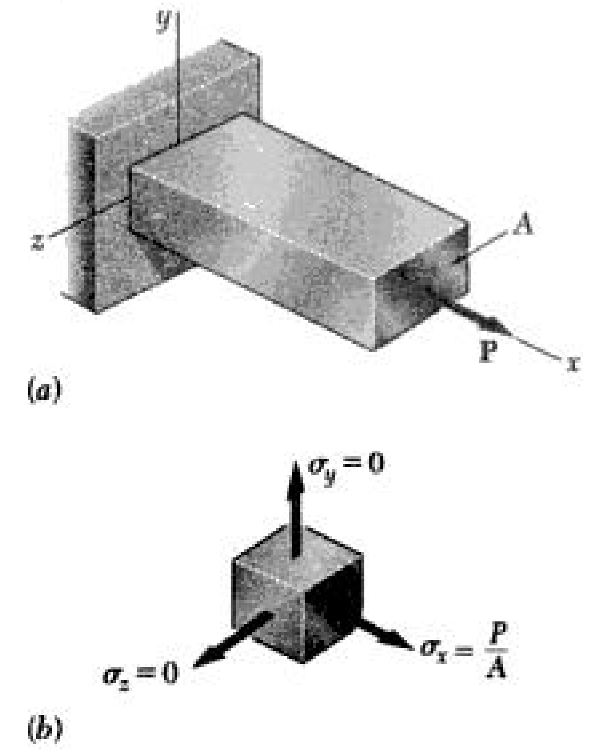


Fig. 2.39

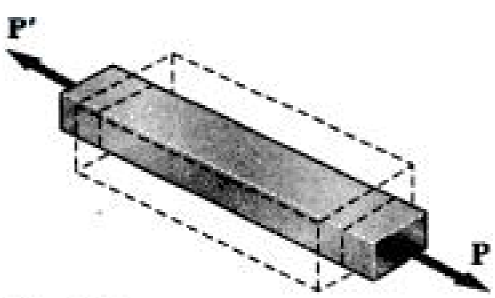


Fig. 2.40

†It would also be tempting, but equally wrong, to assume that the volume of the rod remains unchanged as a result of the combined effect of the axial elongation and transverse contraction (see Sec. 2.13).

‡However, some experimental materials, such as polymer foams, expand laterally when stretched. Since the axial and lateral strains have then the same sign, the Poisson's ratio of these materials is negative. (*See* Roderic Lakes, "Foam Structures with a Negative Poisson's Ratio," *Science*, 27 February 1987, Volume 235, p. 1038–1040.)

Denoting by e the change in volume of our element, we write

$$e = v - 1 = 1 + \epsilon_x + \epsilon_y + \epsilon_z - 1$$

or

$$e = \epsilon_x + \epsilon_y + \epsilon_z \tag{2.30}$$

Since the element had originally a unit volume, the quantity e represents *the change in volume per unit volume;* it is referred to as the *dilatation* of the material. Substituting for ϵ_x, ϵ_y, and ϵ_z from Eqs. (2.28) into (2.30), we write

$$e = \frac{\sigma_x + \sigma_y + \sigma_z}{E} - \frac{2\nu(\sigma_x + \sigma_y + \sigma_z)}{E}$$

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \qquad (2.31)†$$

A case of special interest is that of a body subjected to a uniform hydrostatic pressure p. Each of the stress components is then equal to $-p$ and Eq. (2.31) yields

$$e = -\frac{3(1 - 2\nu)}{E}p \tag{2.32}$$

Introducing the constant

$$k = \frac{E}{3(1 - 2\nu)} \tag{2.33}$$

we write Eq. (2.32) in the form

$$e = -\frac{p}{k} \tag{2.34}$$

The constant k is known as the *bulk modulus* or *modulus of compression* of the material. It is expressed in the same units as the modulus of elasticity E, that is, in pascals or in psi.

Observation and common sense indicate that a stable material subjected to a hydrostatic pressure can only *decrease* in volume; thus the dilatation e in Eq. (2.34) is negative, from which it follows that the bulk modulus k is a positive quantity. Referring to Eq. (2.33), we conclude that $1 - 2\nu > 0$, or $\nu < \frac{1}{2}$. On the other hand, we recall from Sec. 2.11 that ν is positive for all engineering materials. We thus conclude that, for any engineering material,

$$0 < \nu < \tfrac{1}{2} \tag{2.35}$$

We note that an ideal material having a value of ν equal to zero could be stretched in one direction without any lateral contraction. On the other hand, an ideal material for which $\nu = \frac{1}{2}$, and thus $k = \infty$, would

†Since the dilatation e represents a change in volume, it must be independent of the orientation of the element considered. It then follows from Eqs. (2.30) and (2.31) that the quantities $\epsilon_x + \epsilon_y + \epsilon_z$ and $\sigma_x + \sigma_y + \sigma_z$ are also independent of the orientation of the element. This property will be verified in Chap. 7.

EXAMPLE 2.10

A rectangular block of a material with a modulus of rigidity $G = 620$ MPa is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force **P** (Fig. 2.51). Knowing that the upper plate moves through 1.0 mm under the action of the force, determine (*a*) the average shearing strain in the material, (*b*) the force **P** exerted on the upper plate.

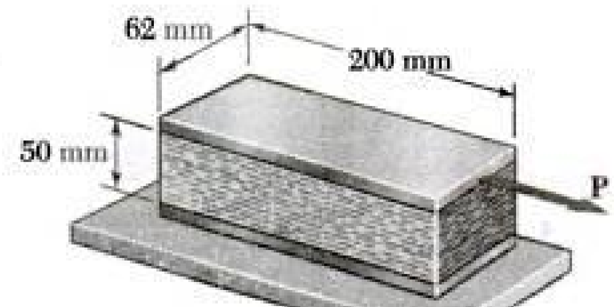


Fig. 2.51

(a) Shearing Strain. We select coordinate axes centered at the midpoint *C* of edge *AB* and directed as shown (Fig. 2.52). According to its definition, the shearing strain γ_{xy} is equal to the angle formed by the vertical and the line *CF* joining the midpoints of edges *AB* and *DE*. Noting that this is a very small angle and recalling that it should be expressed in radians, we write

$$\gamma_{xy} \approx \tan \gamma_{xy} = \frac{1.0 \text{ mm}}{50 \text{ mm}} \qquad \gamma_{xy} = 0.020 \text{ rad}$$

(b) Force Exerted on Upper Plate. We first determine the shearing stress τ_{xy} in the material. Using Hooke's law for shearing stress and strain, we have

$$\tau_{xy} = G\gamma_{xy} (620 \text{ MPa})\ (0.02 \text{ rad}) = 12.4 \text{ MPa}$$

The force exerted on the upper plate is thus

$$P = \tau xy,\ A = (12.4 \text{ MPa})\ (200 \text{ mm})\ (62 \text{ mm}) = 153.76 \text{ kN}$$

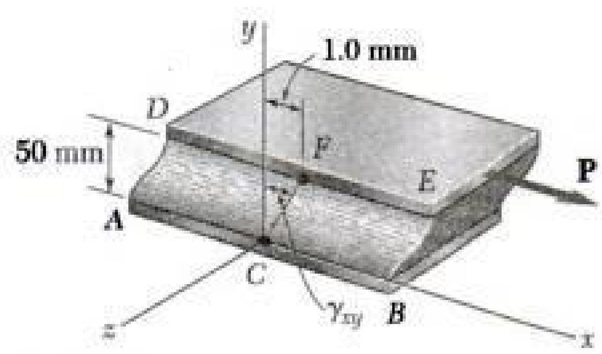


Fig. 2.52

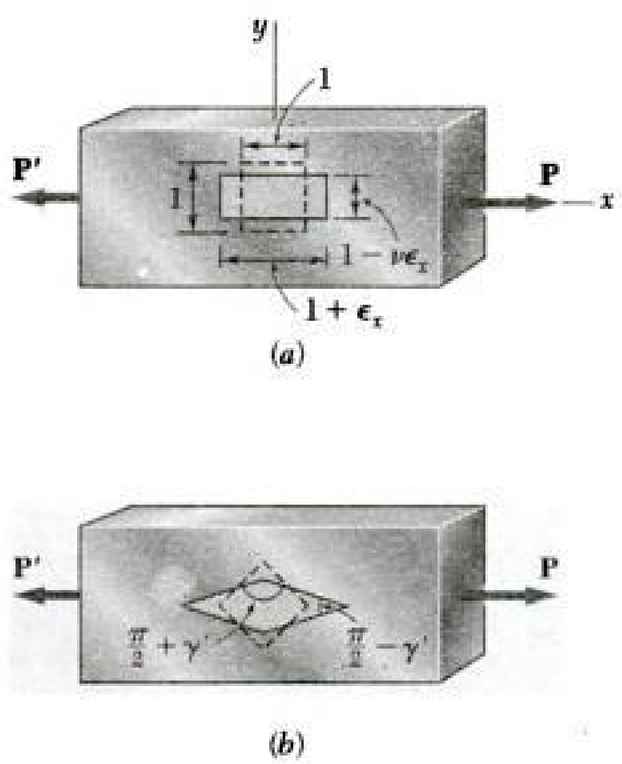


Fig. 2.53

2.15. FURTHER DISCUSSION OF DEFORMATIONS UNDER AXIAL LOADING; RELATION AMONG *E, N* AND *G*

We saw in Sec. 2.11 that a slender bar subjected to an axial tensile load **P** directed along the *x* axis will elongate in the *x* direction and contract in both of the transverse *y* and *z* directions. If ϵ_x denotes the axial strain, the lateral strain is expressed as $\epsilon_y = \epsilon_z = -\nu\epsilon_x$, where ν is Poisson's ratio. Thus, an element in the shape of a cube of side equal to one and oriented as shown in Fig. 2.53*a* will deform into a rectangular parallelepiped of sides $1 + \epsilon_x$, $1 - \nu\epsilon_x$, and $1 - \nu\epsilon_x$. (Note that only one face of the element is shown in the figure.) On the other hand, if the element is oriented at 45° to the axis of the load (Fig. 2.53*b*), the face shown in the figure is observed to deform into a rhombus. We conclude that the axial load **P** causes in this element a shearing strain γ' equal to the amount by which each of the angles shown in Fig. 2.53*b* increases or decreases.†

†Note that the load **P** also produces normal strains in the element shown in Fig. 2.53*b* (see Prob. 2.74).

present case, however, three different values of the modulus of elasticity and six different values of Poisson's ratio will be involved. We write

$$\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E_x} - \frac{\nu_{yx}\sigma_y}{E_y} - \frac{\nu_{zx}\sigma_z}{E_z} \\ \epsilon_y &= -\frac{\nu_{xy}\sigma_x}{E_x} + \frac{\sigma_y}{E_y} - \frac{\nu_{zy}\sigma_z}{E_z} \\ \epsilon_z &= -\frac{\nu_{xz}\sigma_x}{E_x} - \frac{\nu_{yz}\sigma_y}{E_y} + \frac{\sigma_z}{E_z} \end{aligned} \tag{2.45}$$

Equations (2.45) may be considered as defining the transformation of stress into strain for the given layer. It follows from a general property of such transformations that the coefficients of the stress components are symmetric, i.e., that

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y} \qquad \frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z} \qquad \frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x} \tag{2.46}$$

These equations show that, while different, the Poisson's ratios ν_{xy} and ν_{yx} are not independent; either of them can be obtained from the other if the corresponding values of the modulus of elasticity are known. The same is true of ν_{yz} and ν_{zy}, and of ν_{zx} and ν_{xz}.

Consider now the effect of the presence of shearing stresses on the faces of a small element of smeared layer. As pointed out in Sec. 2.14 in the case of isotropic materials, these stresses come in pairs of equal and opposite vectors applied to opposite sides of the given element and have no effect on the normal strains. Thus, Eqs. (2.45) remain valid. The shearing stresses, however, will create shearing strains which are defined by equations similar to the last three of the equations (2.38) of Sec. 2.14, except that three different values of the modulus of rigidity, G_{xy}, G_{yz}, and G_{zx}, must now be used. We have

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G_{yz}} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G_{zx}} \tag{2.47}$$

The fact that the three components of strain ϵ_x, ϵ_y, and ϵ_z can be expressed in terms of the normal stresses only and do not depend upon any shearing stresses characterizes *orthotropic materials* and distinguishes them from other anisotropic materials.

As we saw in Sec. 2.5, a flat *laminate* is obtained by superposing a number of layers or laminas. If the fibers in all layers are given the same orientation to better withstand an axial tensile load, the laminate itself will be orthotropic. If the lateral stability of the laminate is increased by positioning some of its layers so that their fibers are at a right angle to the fibers of the other layers, the resulting laminate will also be orthotropic. On the other hand, if any of the layers of a laminate are positioned so that their fibers are neither parallel nor perpendicular to the fibers of other layers, the lamina; generally, will not be orthotropic.†

†For more information on fiber-reinforced composite materials, see Hyer, M. W., *Stress Analysis of Fiber-Reinforced Composite Materials*, McGraw-Hill, New York, 1998.

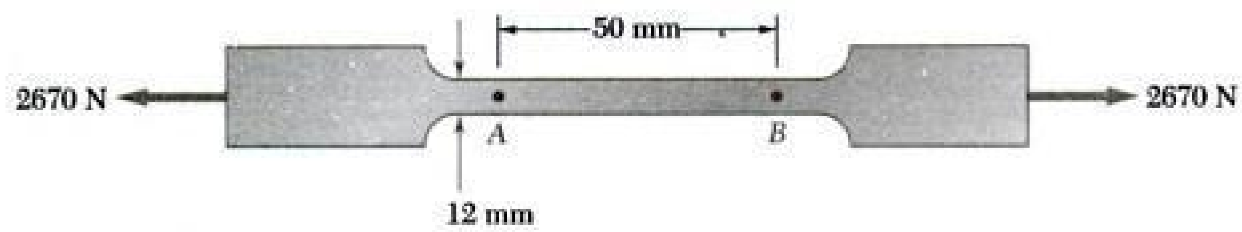


Fig. P2.68

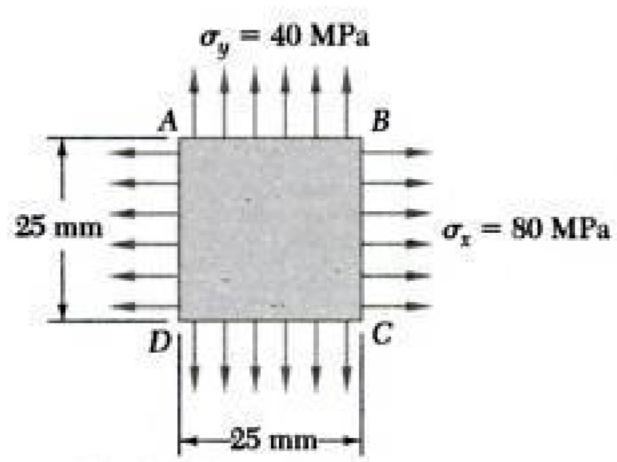


Fig. P2.69

2.68 A 2670 N tensile load is applied to a test coupon made from 1.5 mm flat steel plate (E = 200 GPa, ν = 0.30). Determine the resulting change (*a*) in the 50 mm gage length, (*b*) in the width of portion *AB* of the test coupon, (*c*) in the thickness of portion *AB*, (*d*) in the cross-sectional area of portion *AB*.

2.69 A 25 mm square is scribed on the side of a large steel pressure vessel. After pressurization the biaxial stress condition at the square is as shown. Knowing that E = 200 GPa and ν = 0.30, determine the change in length of (*a*) side *AB*, (*b*) side *BC*, (*c*) diagonal *AC*.

2.70 For the square of Prob. 2.69, determine the percent change in the slope of diagonal *DB* due to the pressurization of the vessel.

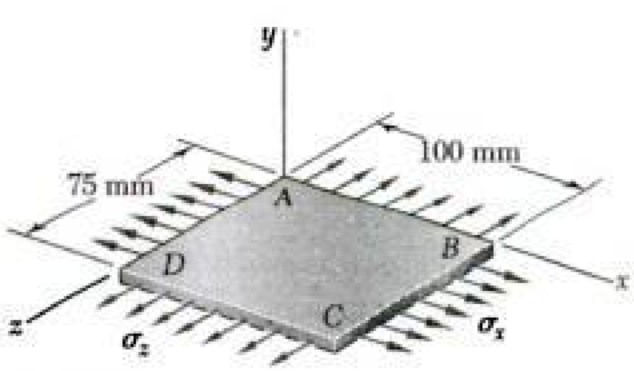


Fig. P2.71

2.71 A fabric used in air-inflated structures is subjected to a biaxial loading that results in normal stresses σ_x = 120 MPa and σ_z = 160 MPa. Knowing that the properties of the fabric can be approximated as E = 87 GPa and ν = 0.34, determine the change in length of (*a*) side *AB*, (*b*) side *BC*, (*c*) diagonal *AC*.

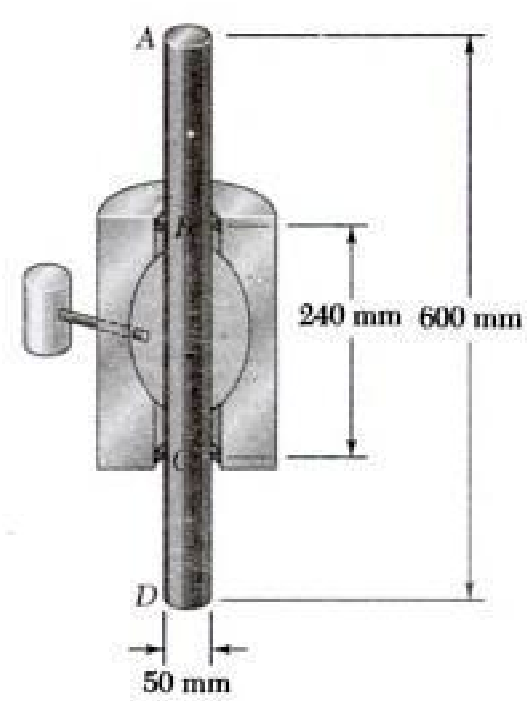


Fig. P2.72

2.72 The brass rod *AD* is fitted with a jacket that is used to apply a hydrostatic pressure of 48 MPa to the 250-mm portion *BC* of the rod. Knowing that E = 105 GPa and ν = 0.33, determine (*a*) the change in the total length *AD*, (*b*) the change in diameter of portion *BC* of the rod.

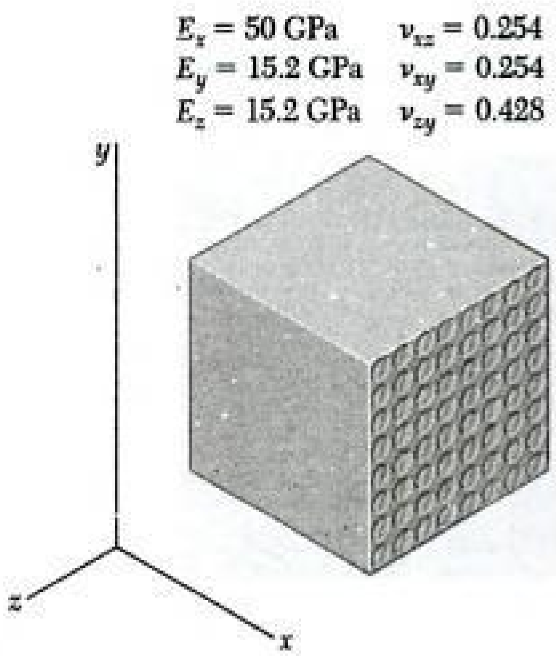


Fig. *P2.89*

***2.89** A composite cube with 40-mm sides and the properties shown is made with glass polymer fibers aligned in the x direction. The cube is constrained against deformations in the y and z directions and is subjected to a tensile load of 65 kN in the x direction. Determine (*a*) the change in the length of the cube in the x direction, (*b*) the stresses σ_x, σ_y, and σ_z.

***2.90** The composite cube of Prob. 2.89 is constrained against deformation in the z direction and elongated in the x direction by 0.035 mm by a tensile load in the x direction. Determine (*a*) the stresses σ_x, σ_y, and σ_z, (*b*) the change in the dimension in the y direction.

***2.91** Show that for any given material, the ratio G/E of the modulus of rigidity over the modulus of elasticity is always less than $\frac{1}{2}$ but more than $\frac{1}{3}$ [*Hint*: Refer to Eq. (2.43) and to Sec. 2.13.]

***2.92** The material constants E, G, k, and ν are related by Eqs. (2.33) and (2.43). Show that any one of these constants may be expressed in terms of any other two constants. For example, show that (*a*) $k = GE/(9G - 3E)$ and (*b*) $\nu = (3k - 2G)/(6k + 2G)$.

2.17. STRESS AND STRAIN DISTRIBUTION UNDER AXIAL LOADING; SAINT-VENANT'S PRINCIPLE

Fig. 2.58

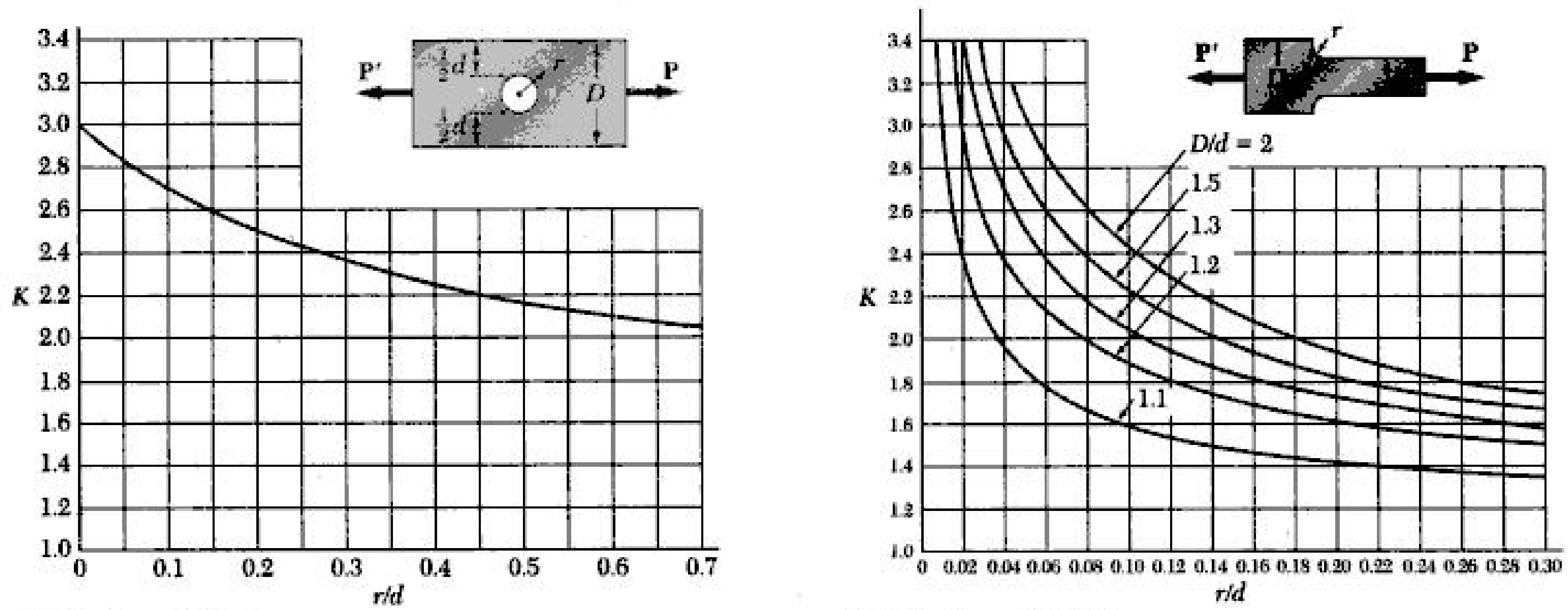
We have assumed so far that, in an axially loaded member, the normal stresses are uniformly distributed in any section perpendicular to the axis of the member. As we saw in Sec. 1.5, such an assumption may be quite in error in the immediate vicinity of the points of application of the loads. However, the determination of the actual stresses in a given section of the member requires the solution of a statically indeterminate problem.

In Sec. 2.9, you saw that statically indeterminate problems involving the determination of *forces* can be solved by considering the *deformations* caused by these forces. It is thus reasonable to conclude that the determination of the *stresses* in a member requires the analysis of the *strains* produced by the stresses in the member. This is essentially the approach found in advanced textbooks, where the mathematical theory of elasticity is used to determine the distribution of stresses corresponding to various modes of application of the loads at the ends of the member. Given the more limited mathematical means at our disposal, our analysis of stresses will be restricted to the particular case when two rigid plates are used to transmit the loads to a member made of a homogeneous isotropic material (Fig. 2.58).

If the loads are applied at the center of each plate,† the plates will move toward each other without rotating, causing the member to get shorter, while increasing in width and thickness. It is reasonable to assume that the member will remain straight, that plane sections will re-

†More precisely, the common line of action of the loads should pass through the centroid of the cross section (cf. Sec. 1.5).

expressed in the form of tables or of graphs, as shown in Fig. 2.64. To determine the maximum stress occurring near a discontinuity in a given member subjected to a given axial load P, the designer needs only to compute the average stress $\sigma_{ave} = P/A$ in the critical section, and multiply the result obtained by the appropriate value of the stress-concentration factor K. You should note, however, that this procedure is valid only as long as σ_{max} does not exceed the proportional limit of the material, since the values of K plotted in Fig. 2.64 were obtained by assuming a linear relation between stress and strain.



(*a*) Flat bars with holes (*b*) Flat bars with fillets

Fig. 2.64 Stress concentration factors for flat bars under axial loading†

Note that the average stress must be computed across the narrowest section: $s_{ave} = P/td$, where t is the thickness of the bar.

EXAMPLE 2.12

Determine the largest axial load **P** that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick, and respectively 40 and 60 mm wide, connected by fillets of radius $r = 8$ mm. Assume an allowable normal stress of 165 MPa.

We first compute the ratios

$$\frac{D}{d} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.50 \qquad \frac{r}{d} = \frac{8 \text{ mm}}{40 \text{ mm}} = 0.20$$

Using the curve in Fig. 2.64*b* corresponding to $D/d = 1.50$, we find that the value of the stress-concentration factor corresponding to $r/d = 0.20$ is

$$K = 1.82$$

Carrying this value into Eq. (2.48) and solving for σ_{ave}, we have

$$\sigma_{ave} = \frac{\sigma_{max}}{1.82}$$

But σ_{max} cannot exceed the allowable stress $\sigma_{all} = 165$ MPa. Substituting this value for σ_{max}, we find that the average stress in the narrower portion ($d = 40$ mm) of the bar should not exceed the value

$$\sigma_{ave} = \frac{165 \text{ MPa}}{1.82} = 90.7 \text{ MPa}$$

Recalling that $\sigma_{ave} = P/A$, we have

$$P = A\sigma_{ave} = (40 \text{ mm})(10 \text{ mm})(90.7 \text{ MPa}) = 36.3 \times 10^3 \text{ N}$$
$$P = 36.3 \text{ kN}$$

†W. D. Pilkey, *Peterson's Stress Concentration Factors*, 2nd ed., John Wiley & Sons, New York, 1997.

We recall that the discussion of stress concentrations of Sec. 2.18 was carried out under the assumption of a linear stress-strain relationship. The stress distributions shown in Figs. 2.62 and 2.63, and the values of the stress-concentration factors plotted in Fig. 2.64 cannot be used, therefore, when plastic deformations take place, i.e., when the value of σ_{max} obtained from these figures exceeds the yield strength σ_Y.

Let us consider again the flat bar with a circular hole of Fig. 2.62, and let us assume that the material is elastoplastic, i.e., that its stress-strain diagram is as shown in Fig. 2.65. As long as no plastic deformation takes place, the distribution of stresses is as indicated in Sec. 2.18 (Fig. 2.69*a*). We observe that the area under the stress-distribution curve represents the integral $\int \sigma\, dA$, which is equal to the load *P*. Thus this area, and the value of σ_{max}, must increase as the load *P* increases. As long as $\sigma_{max} \leq \sigma_Y$, all the successive stress distributions obtained as *P* increases will have the shape shown in Fig. 2.62 and repeated in Fig. 2.69*a*. However, as *P* is increased beyond the value P_Y corresponding to $\sigma_{max} = \sigma_Y$ (Fig. 2.69*b*), the stress-distribution curve must flatten in the vicinity of the hole (Fig. 2.69*c*), since the stress in the material considered cannot exceed the value σ_Y. This indicates that the material is yielding in the vicinity of the hole. As the load *P* is further increased, the plastic zone where yield takes place keeps expanding, until it reaches the edges of the plate (Fig. 2.69*d*). At that point, the distribution of stresses across the plate is uniform, $\sigma = \sigma_Y$, and the corresponding value $P = P_U$ of the load is the largest which may be applied to the bar without causing rupture.

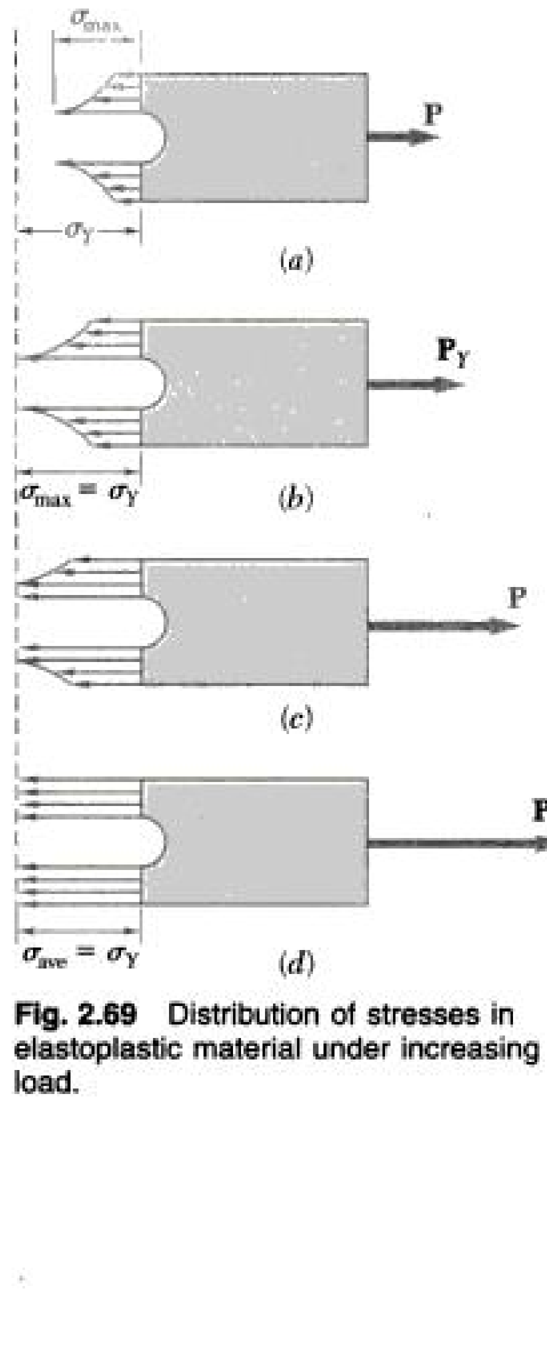


Fig. 2.69 Distribution of stresses in elastoplastic material under increasing load.

It is interesting to compare the maximum value P_Y of the load which can be applied if no permanent deformation is to be produced in the bar, with the value P_U which will cause rupture. Recalling the definition of the average stress, $\sigma_{ave} = P/A$, where *A* is the net cross-sectional area, and the definition of the stress concentration factor, $K = \sigma_{max}/\sigma_{ave}$, we write

$$P = \sigma_{ave}A = \frac{\sigma_{max}A}{K} \tag{2.49}$$

for any value of σ_{max} which does not exceed σ_Y. When $\sigma_{max} = \sigma_Y$ (Fig. 2.69*b*), we have $P = P_Y$, and Eq. (2.49) yields

$$P_Y = \frac{\sigma_Y A}{K} \tag{2.50}$$

On the other hand, when $P = P_U$ (Fig. 2.69*d*), we have $\sigma_{ave} = \sigma_Y$ and

$$P_U = \sigma_Y A \tag{2.51}$$

Comparing Eqs. (2.50) and (2.51), we conclude that

$$P_Y = \frac{P_U}{K} \tag{2.52}$$

PROBLEMS

2.93 Two holes have been drilled through a long steel bar that is subjected to a centric axial load as shown. For $P = 30$ kN, determine the maximum value of the stress (*a*) at *A*, (*b*) at *B*.

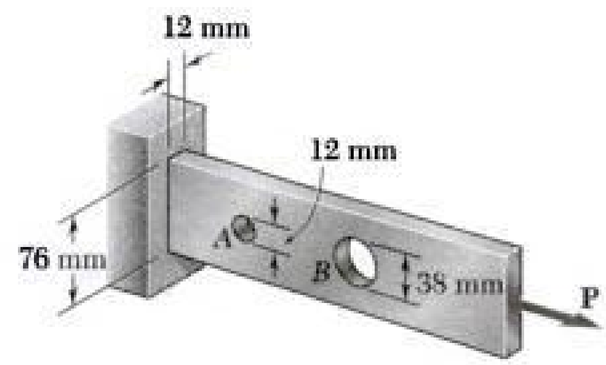


Fig. P2.93 and P2.94

2.94 Knowing that $\sigma_{all} = 110$ MPa, determine the maximum allowable value of the centric axial load **P**.

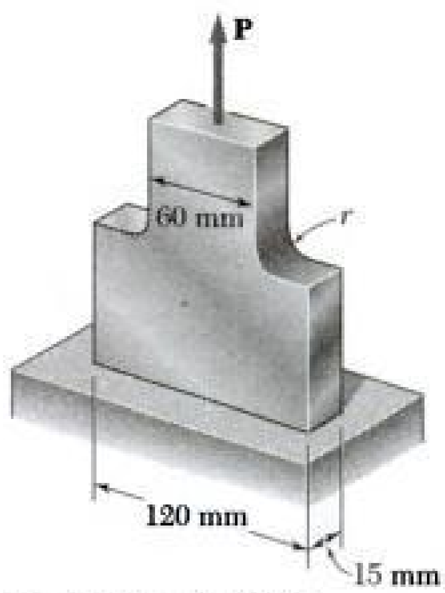


Fig. P2.95 and P2.96

2.95 Knowing that, for the plate shown, the allowable stress is 125 MPa, determine the maximum allowable value of P when (*a*) $r = 12$ mm, (*b*) $r = 18$ mm.

2.96 Knowing that $P = 38$ kN, determine the maximum stress when (*a*) $r = 10$ mm, (*b*) $r = 16$ mm, (*c*) $r = 18$ mm.

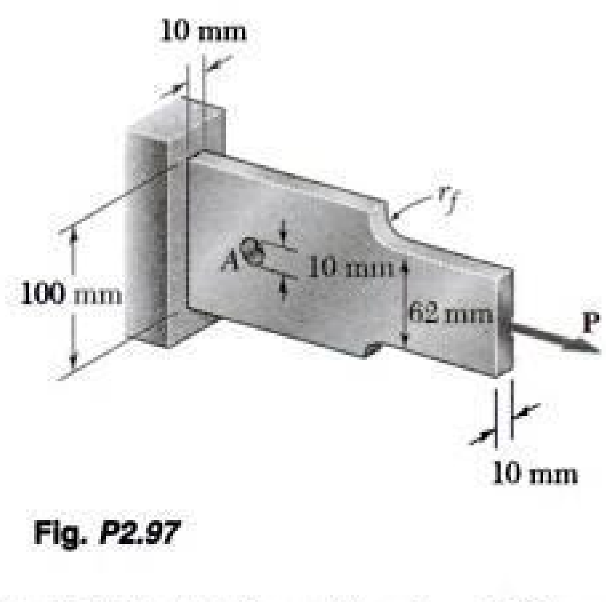


Fig. *P2.97*

2.97 Knowing that the hole has a diameter of 10 mm, determine (*a*) the radius r_f of the fillets for which the same maximum stress occurs at the hole *A* and at the fillets, (*b*) the corresponding maximum allowable load **P** if the allowable stress is 103 MPa.

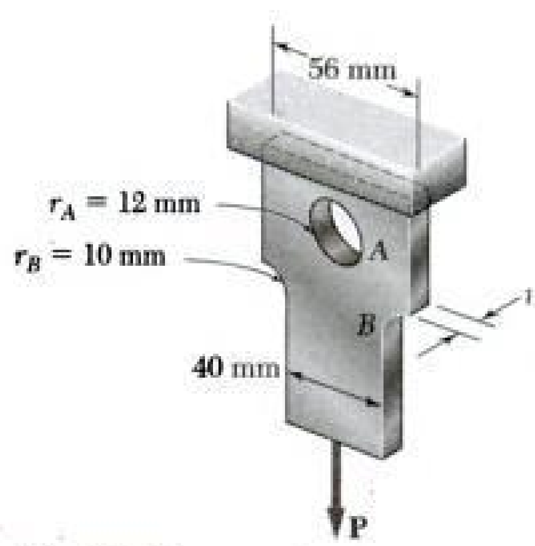


Fig. *P2.98*

2.98 For $P = 38$ kN, determine the minimum plate thickness t required if the allowable stress is 124 MPa.

2.118 A narrow bar of aluminum is bonded to the side of a thick steel plate as shown. Initially, at $T_1 = 20°C$, all stresses are zero. Knowing that the temperature will be slowly raised to T_2 and then reduced to T_1, determine (*a*) the highest temperature T_2 that does *not* result in residual stresses, (*b*) the temperature T_2 that will result in a residual stress in the aluminum equal to 100 MPa. Assume $\alpha_a = 23.6 \times 10^{-6}/°C$ for the aluminum and $\alpha_s = 11.7 \times 10^{-6}/°C$ for the steel. Further assume that the aluminum is elastoplastic, with $E = 70$ GPa and $\sigma_Y = 100$ MPa. (*Hint:* Neglect the small stresses in the plate.)

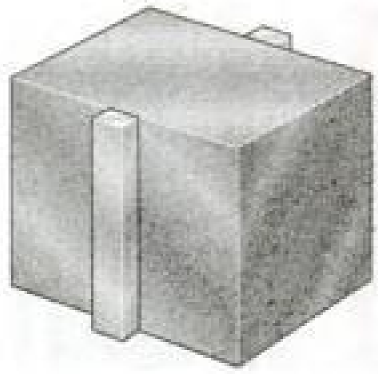


Fig. P2.118

2.119 The steel rod *ABC* is attached to rigid supports and is unstressed at a temperature of 3°C. The steel is assumed elastoplastic, with $\sigma_Y = 248$ MPa and $E = 200$ GPa. The temperature of both portions of the rod is then raised to 120°C. Knowing that $\alpha = 11.7 \times 10^{-6}/°C$, determine (*a*) the stress in portion *AC*, (*b*) the deflection of point *C*.

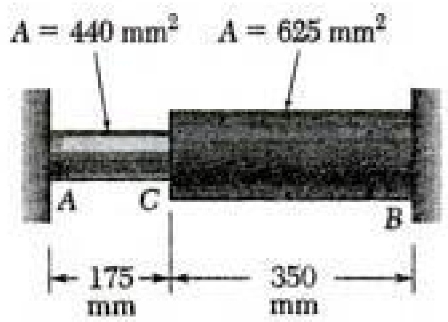


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Fig. P2-119 100% of size
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Fig. *P2.119*

2.120 Solve Prob. 2.119, assuming that the temperature of the rod is raised to 120°C and then returned to 3°C.

2.121 The rigid bar *ABC* is supported by two links, *AD* and *BE*, of uniform 37.5 × 6-mm rectangular cross section and made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. The magnitude of the force **Q** applied at *B* is gradually increased from zero to 260 kN. Knowing that $a = 0.640$ m, determine (*a*) the value of the normal stress in each link, (*b*) the maximum deflection of point *B*.

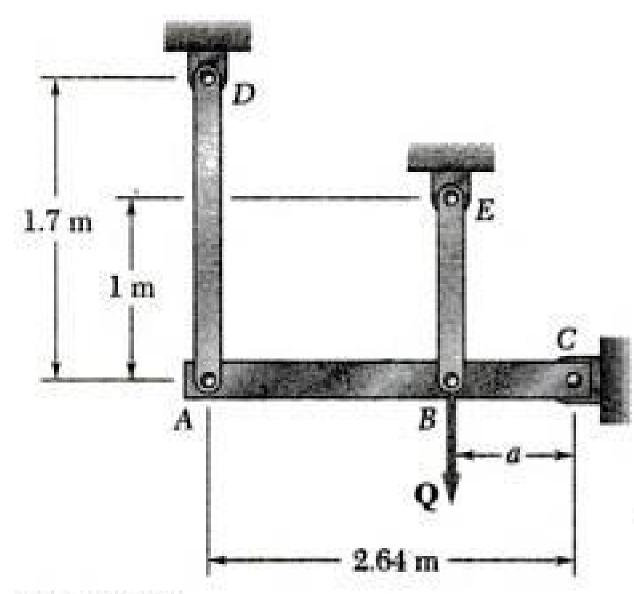


Fig. P2.121

2.122 Solve Prob. 2.121, knowing that $a = 1.76$ m and that the magnitude of the force **Q** applied at *B* is gradually increased from zero to 135 kN.

***2.123** Solve Prob. 2.121, assuming that the magnitude of the force **Q** applied at *B* is gradually increased from zero to 260 kN and then decreased back to zero. Knowing that $a = 0.640$ m, determine (*a*) the residual stress in each link, (*b*) the final deflection of point *B*. Assume that the links are braced so that they can carry compressive forces without buckling.

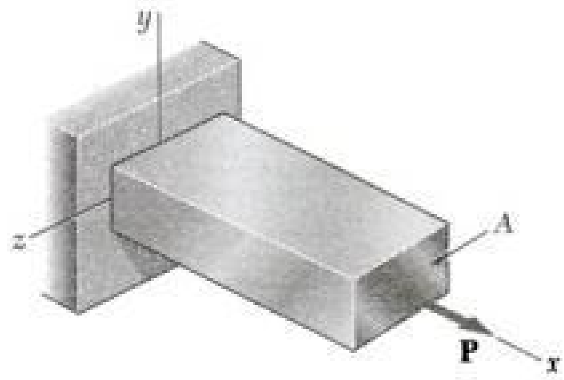


Fig. 2.39a

Lateral strain. Poisson's ratio

When an axial load **P** is applied to a homogeneous, slender bar (Fig. 2.39*a*), it causes a strain, not only along the axis of the bar but in any transverse direction as well [Sec. 2.11]. This strain is referred to as the *lateral strain*, and the ratio of the lateral strain over the axial strain is called *Poisson's ratio* and is denoted by ν (Greek letter nu). We wrote

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \tag{2.25}$$

Recalling that the axial strain in the bar is $\epsilon_x = \sigma_x/E$, we expressed as follows the condition of strain under an axial loading in the x direction:

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E} \tag{2.27}$$

Multiaxial loading

This result was extended in Sec. 2.12 to the case of a *multiaxial loading* causing the state of stress shown in Fig. 2.42. The resulting strain condition was described by the following relations, referred to as the *generalized Hooke's law* for a multiaxial loading.

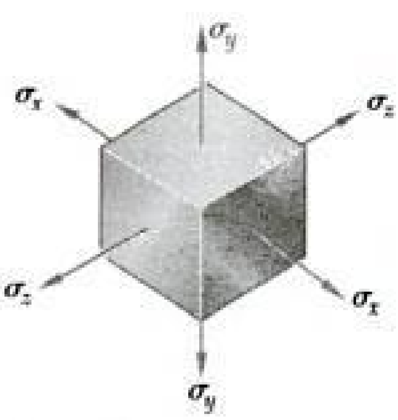


Fig. 2.42

$$\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}
\end{aligned} \tag{2.28}$$

Dilatation

If an element of material is subjected to the stresses $\sigma_x, \sigma_y, \sigma_z$, it will deform and a certain change of volume will result [Sec. 2.13]. The *change in volume per unit volume* is referred to as the *dilatation* of the material and is denoted by e. We showed that

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{2.31}$$

When a material is subjected to a hydrostatic pressure p, we have

$$e = -\frac{p}{k} \tag{2.34}$$

Bulk modulus

where k is known as the *bulk modulus* of the material:

$$k = \frac{E}{3(1 - 2\nu)} \tag{2.33}$$

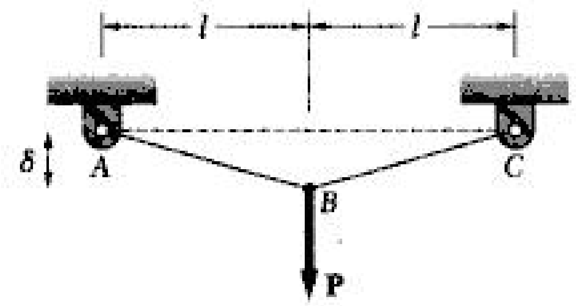


Fig. P2.130

2.130 The uniform wire ABC, of unstretched length $2l$, is attached to the supports shown and a vertical load **P** is applied at the midpoint B. Denoting by A the cross-sectional area of the wire and by E the modulus of elasticity, show that, for $\delta \ll l$, the deflection at the midpoint B is

$$\delta = l\sqrt[3]{\frac{P}{AE}}$$

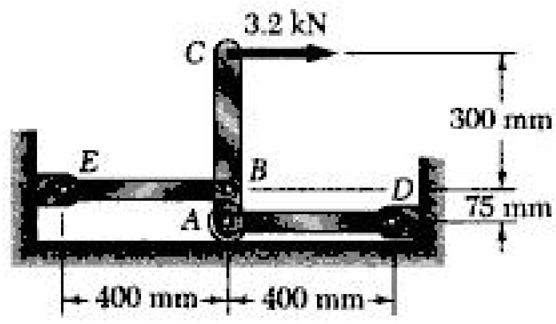


Fig. P2.131

2.131 The steel bars BE and AD each have a 6×18-mm cross section. Knowing that $E = 200$ GPa, determine the deflections of points A, B, and C of the rigid bar ABC.

2.132 In Prob. 2.131, the 3.2-kN force caused point C to deflect to the right. Using $\alpha = 11.7 \times 10^{6}/°C$, determine the (*a*) the overall change in temperature that causes point C to return to its original position, (*b*) the corresponding total deflection of points A and B.

2.133 A hole is to be drilled in the plate at A. The diameters of the bits available to drill the hole range from 9 to 27 mm in 6-mm increments. (*a*) Determine the diameter d of the largest bit that can be used if the allowable load at the hole is not to exceed that at the fillets. (*b*) If the allowable stress in the plate is 145 MPa, what is the corresponding allowable load **P**?

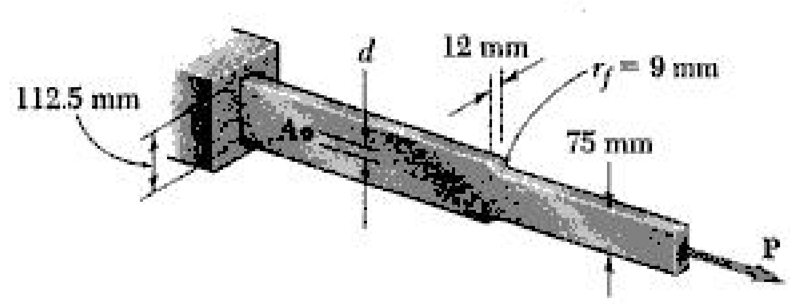


Fig. P2.133 and *P2.134*

2.134 (*a*) For $P = 58$ kN and $d = 12$ mm, determine the maximum stress in the plate shown. (*b*) Solve part *a*, assuming that the hole at A is not drilled.

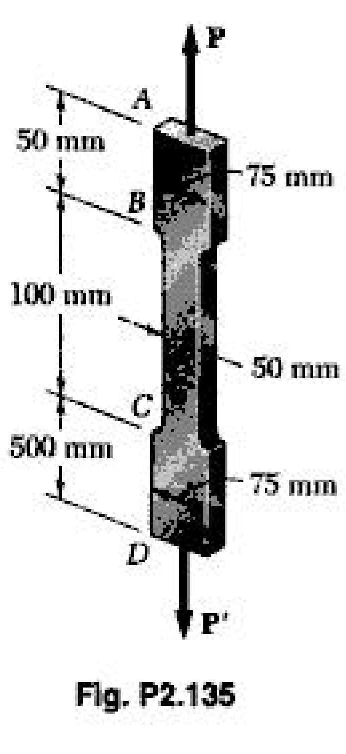


Fig. P2.135

2.135 The steel tensile specimen $ABCD$ ($E = 200$ GPa and $\sigma_y = 345$ MPa) is loaded in tension until the maximum strain is $\epsilon = 0.0025$. (*a*) Neglecting the effect of the fillets on the change in length of the specimen, determine the resulting overall length AD of the specimen after the load is removed. (*b*) Following the removal of the load in part *a*, a compressive load is applied until the maximum compressive strain is $\epsilon = 0.0020$. Determine the resulting overall length AD after the compressive load is removed.

3.1. INTRODUCTION

In the two preceding chapters you studied how to calculate the stresses and strains in structural members subjected to axial loads, that is, to forces directed along the axis of the member. In this chapter structural members and machine parts that are in *torsion* will be considered. More specifically, you will analyze the stresses and strains in members of circular cross section subjected to twisting couples, or *torques*, **T** and **T′** (Fig. 3.1). These couples have a common magnitude T, and opposite senses. They are vector quantities and can be represented either by curved arrows as in Fig. 3.1*a*, or by couple vectors as in Fig. 3.1*b*.

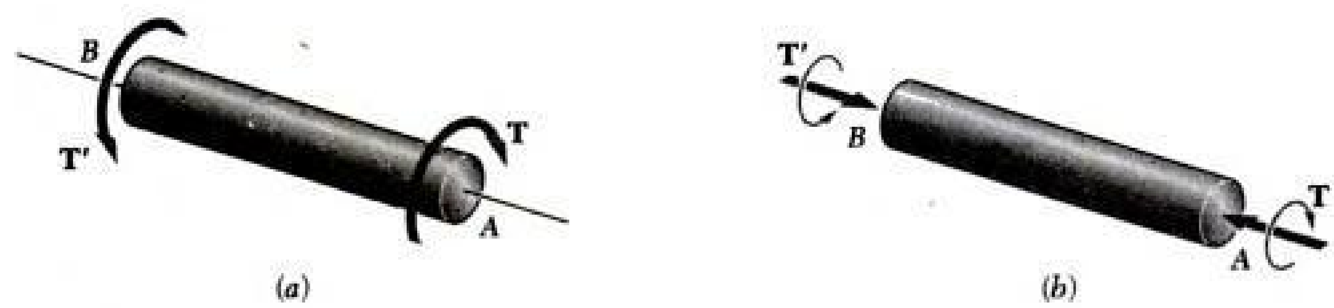


Fig. 3.1

Members in torsion are encountered in many engineering applications. The most common application is provided by *transmission shafts*, which are used to transmit power from one point to another. For example, the shaft shown in Fig. 3.2 is used to transmit power from the engine to the rear wheels of an automobile. These shafts can be either solid, as shown in Fig. 3.1, or hollow.

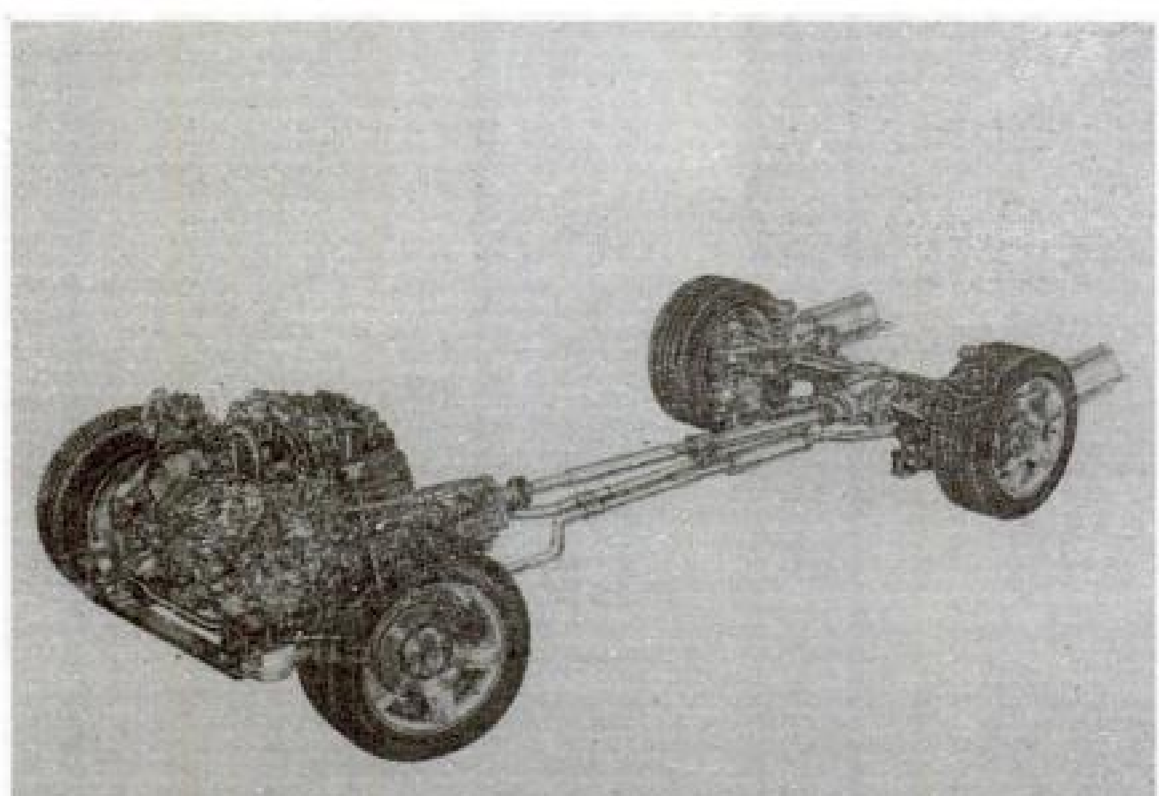
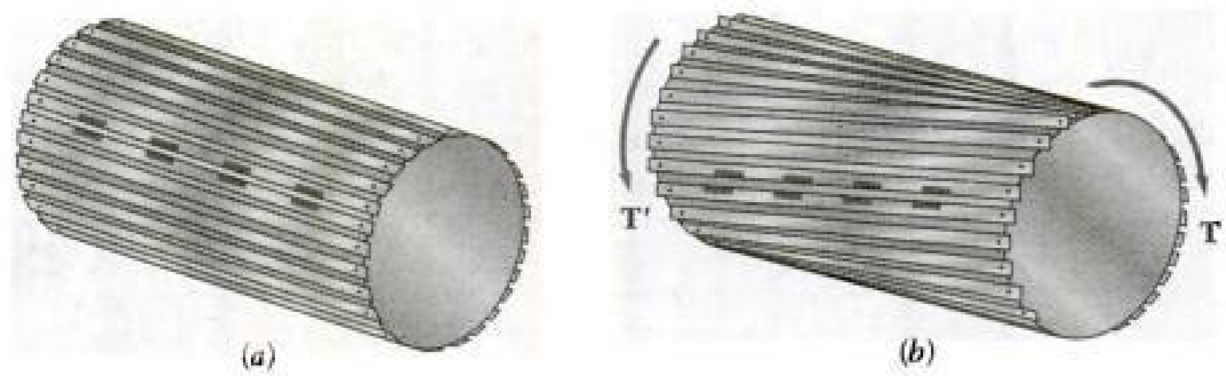


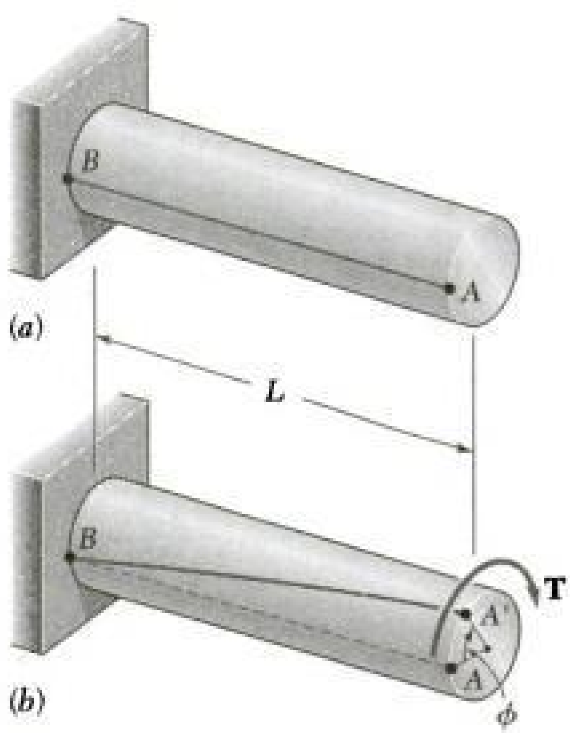
Fig. 3.2 In the automotive power train shown, the shaft transmits power from the engine to the rear wheels.



(a) (b)

Fig. 3.7

by considering a "shaft" made of separate slats pinned at both ends to disks as shown in Fig. 3.7*a*. If markings have been painted on two adjoining slats, it is observed that the slats slide with respect to each other when equal and opposite torques are applied to the ends of the "shaft" (Fig. 3.7*b*). While sliding will not actually take place in a shaft made of a homogeneous and cohesive material, the tendency for sliding will exist, showing that stresses occur on longitudinal planes as well as on planes perpendicular to the axis of the shaft.†



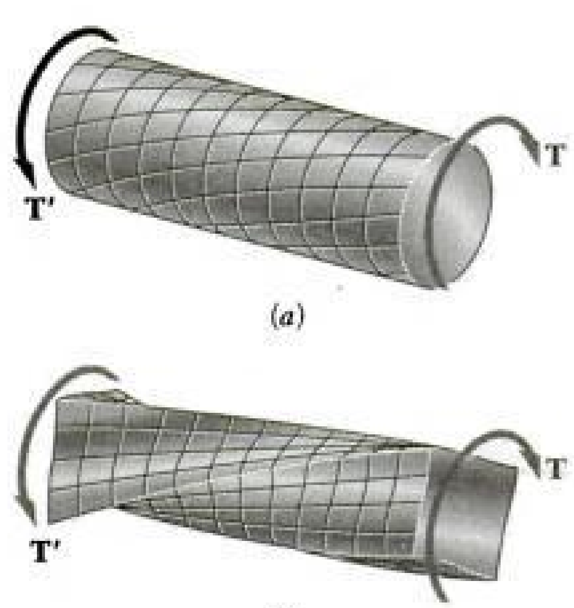
(a) (b)

Fig. 3.8

3.3. DEFORMATIONS IN A CIRCULAR SHAFT

Consider a circular shaft that is attached to a fixed support at one end (Fig. 3.8*a*). If a torque **T** is applied to the other end, the shaft will twist, with its free end rotating through an angle ϕ called *the angle of twist* (Fig. 3.8*b*). Observation shows that, within a certain range of values of T, the angle of twist ϕ is proportional to T. It also shows that ϕ is proportional to the length L of the shaft. In other words, the angle of twist for a shaft of the same material and same cross section, but twice as long, will be twice as large under the same torque **T**. One purpose of our analysis will be to find the specific relation existing among ϕ, L, and T; another purpose will be to determine the distribution of shearing stresses in the shaft, which we were unable to obtain in the preceding section on the basis of statics alone.

At this point, an important property of circular shafts should be noted: When a circular shaft is subjected to torsion, *every cross section remains plane and undistorted*. In other words, while the various cross sections along the shaft rotate through different amounts, each cross section rotates as a solid rigid slab. This is illustrated in Fig. 3.9*a*, which shows the deformations in a rubber model subjected to torsion. The property we are discussing is characteristic of circular shafts, whether solid or hollow; it is not enjoyed by members of noncircular cross section. For example, when a bar of square cross section is subjected to torsion, its various cross sections warp and do not remain plane (Fig. 3.9*b*).



(a) (b)

Fig. 3.9

†The twisting of a cardboard tube that has been slit lengthwise provides another demonstration of the existence of shearing stresses on longitudinal planes.

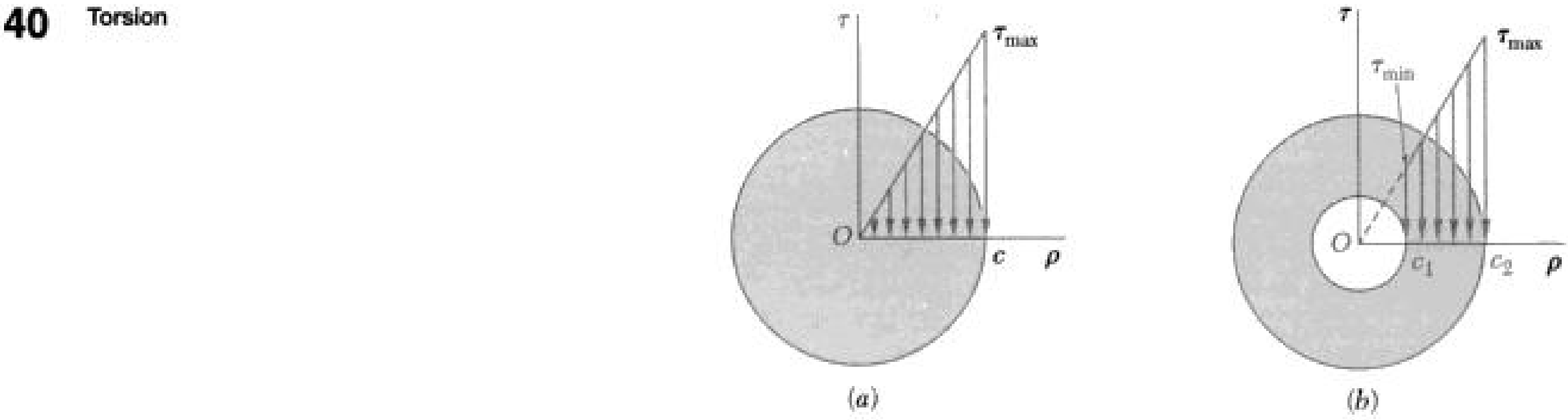


Fig. 3.15 (repeated)

We now recall from Sec. 3.2 that the sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude T of the torque exerted on the shaft:

$$\int \rho(\tau\, dA) = T \tag{3.1}$$

Substituting for τ from (3.6) into (3.1), we write

$$T = \int \rho\tau\, dA = \frac{\tau_{max}}{c}\int \rho^2\, dA$$

But the integral in the last member represents the polar moment of inertia J of the cross section with respect to its center O. We have therefore

$$T = \frac{\tau_{max} J}{c} \tag{3.8}$$

or, solving for τ_{max},

$$\tau_{max} = \frac{Tc}{J} \tag{3.9}$$

Substituting for τ_{max} from (3.9) into (3.6), we express the shearing stress at any distance ρ from the axis of the shaft as

$$\tau = \frac{T\rho}{J} \tag{3.10}$$

Equations (3.9) and (3.10) are known as the *elastic torsion formulas*. We recall from statics that the polar moment of inertia of a circle of radius c is $J = \frac{1}{2}\pi c^4$. In the case of a hollow circular shaft of inner radius c_1 and outer radius c_2, the polar moment of inertia is

$$J = \tfrac{1}{2}\pi c_2^4 - \tfrac{1}{2}\pi c_1^4 = \tfrac{1}{2}\pi(c_2^4 - c_1^4) \tag{3.11}$$

We note that, if SI metric units are used in Eq. (3.9) or (3.10), T will be expressed in N · m, c or ρ in meters, and J in m^4; we check that the resulting shearing stress will be expressed in N/m^2, that is, pascals (Pa). If U.S. customary units are used, T should be expressed in lb · in., c or ρ in inches, and J in in^4, with the resulting shearing stress expressed in psi.

SAMPLE PROBLEM 3.2

The preliminary design of a large shaft connecting a motor to a generator calls for the use of a hollow shaft with inner and outer diameters of 100 mm and 150 mm, respectively. Knowing that the allowable shearing stress is 82 MPa, determine the maximum torque that can be transmitted (*a*) by the shaft as designed, (*b*) by a solid shaft of the same weight, (*c*) by a hollow shaft of the same weight and of 200 mm outer diameter.

SOLUTION

***a*. Hollow Shaft as Designed.** For the hollow shaft we have

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(75 \text{ mm})^4 - (50 \text{ mm})^4] = (39.88)\,10^6 \text{ mm}^4$$

Using Eq. (3.9), we write

$$\tau_{\max} = \frac{Tc_2}{J} \qquad 82 \text{ MPa} = \frac{T\,(75 \times 10^{-3} \text{ m})}{(39.88)10^{-6} \text{ m}^4} \qquad T = 43.6 \text{ kN} \cdot \text{m} \blacktriangleleft$$

***b*. Solid Shaft of Equal Weight.** For the shaft as designed and this solid shaft to have the same weight and length, their cross-sectional areas must be equal.

$$A_{(a)} = A_{(b)}$$

$$\pi[(75 \text{ mm})^2 - (50 \text{ mm})^2] = \pi c_3^2 \qquad c_3 = 55.9 \text{ mm}$$

Since $\tau_{\text{all}} = 82$ MPa, we write

$$\tau_{\max} = \frac{Tc_3}{J} \qquad 82 \text{ MPa} = \frac{T\,(55.9 \times 10^{-3} \text{ m})}{\frac{\pi}{2}(55.9 \times 10^{-3} \text{ m})^4} \qquad T = 22.5 \text{ kN} \cdot \text{m} \blacktriangleleft$$

***c*. Hollow Shaft of 200 mm Diameter.** For equal weight, the cross-sectional areas again must be equal. We determine the inside diameter of the shaft by writing

$$A_{(a)} = A_{(c)}$$

$$\pi[(75 \text{ mm})^2 - (50 \text{ mm})^2] = \pi[(100 \text{ mm})^2 - c_5^2] \qquad c_5 = 82.9 \text{ mm}$$

For $c_5 = 82.9$ mm and $c_4 = 100$ mm,

$$J = \frac{\pi}{2}[(100 \text{ mm})^4 - (82.9 \text{ mm})^4] = (82.89)\,10^6 \text{ mm}^4$$

With $\tau_{\text{all}} = 82$ MPa and $c_4 = 100$ mm,

$$\tau_{\max} = \frac{Tc_4}{J} \qquad 82 \text{ MPa} = \frac{T\,(100 \text{ mm}) \times 10^{-3}}{(82.89)10^{-6} \text{ m}^4} \qquad T = 68.05 \text{ kN} \cdot \text{m} \blacktriangleleft$$

3.21 A torque of magnitude $T = 1000 \text{ N} \cdot \text{m}$ is applied at D as shown. Knowing that the diameter of shaft AB is 56 mm and the diameter of shaft CD is 42 mm, determine the maximum shearing stress in (*a*) shaft AB, (*b*) shaft CD.

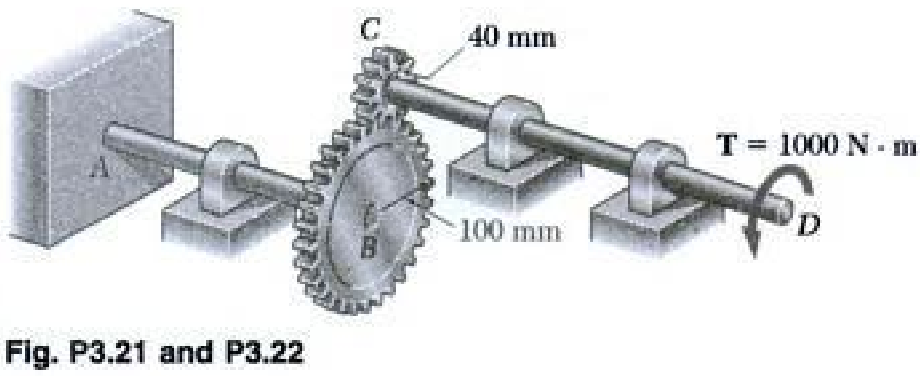


Fig. P3.21 and P3.22

3.22 A torque of magnitude $T = 1000 \text{ N} \cdot \text{m}$ is applied at D as shown. Knowing that the allowable shearing stress is 60 MPa in each shaft, determine the required diameter of (*a*) shaft AB, (*b*) shaft CD.

3.23 and *3.24* Under normal operating conditions a motor exerts a torque of magnitude $T_F = 136 \text{ N} \cdot \text{m}$ at F. Knowing that the allowable shearing stress is 72 MPa in each shaft, determine for the given data the required diameter of (*a*) shaft CDE, (*b*) shaft FGH.

3.23 $r_D = 200$ mm, $r_G = 75$ mm
3.24 $r_D = 75$ mm, $r_G = 200$ mm

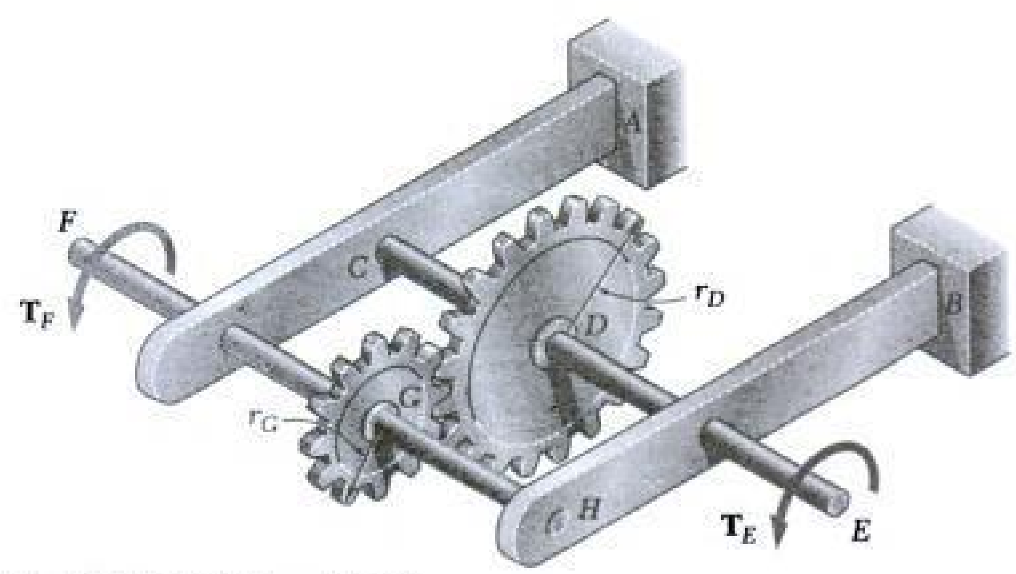


Fig. P3.23, *P3.24*, and P3.25

3.25 Under normal operating conditions a motor exerts a torque of magnitude T_F at F. The shafts are made of a steel for which the allowable shearing stress is 82 MPa and have diameters $d_{CDE} = 24$ mm and $d_{FGH} = 20$ mm. Knowing that $r_D = 165$ mm and $r_G = 114$ mm, determine the largest allowable value of T_F.

the portion of shaft located on one side of the section. This procedure, which has already been explained in Sec. 3.4 and illustrated in Fig. 3.17, is applied in Sample Prob. 3.3.

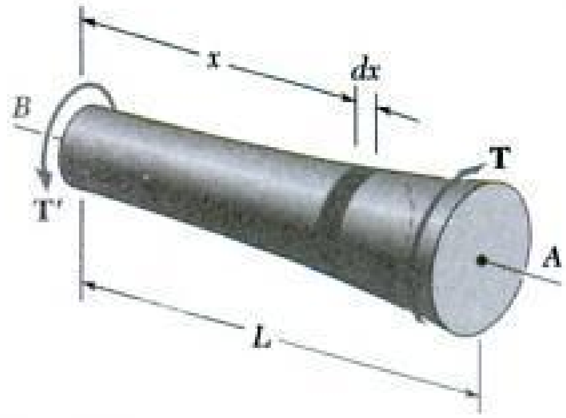


Fig. 3.25

In the case of a shaft with a variable circular cross section, as shown in Fig. 3.25, formula (3.16) may be applied to a disk of thickness dx. The angle by which one face of the disk rotates with respect to the other is thus

$$d\phi = \frac{T\,dx}{JG}$$

where J is a function of x which may be determined. Integrating in x from 0 to L, we obtain the total angle of twist of the shaft:

$$\phi = \int_0^L \frac{T\,dx}{JG} \tag{3.18}$$

The shaft shown in Fig. 3.22, which was used to derive formula (3.16), and the shaft of Fig. 3.16, which was discussed in Examples 3.02 and 3.03, both had one end attached to a fixed support. In each case, therefore, the angle of twist ϕ of the shaft was equal to the angle of rotation of its free end. When both ends of a shaft rotate, however, the angle of twist of the shaft is equal to the angle through which one end of the shaft rotates *with respect to the other*. Consider, for instance, the assembly shown in Fig. 3.26*a*, consisting of two elastic shafts AD and BE, each of length L, radius c, and modulus of rigidity G, which are attached to gears meshed at C. If a torque **T** is applied at E (Fig. 3.26*b*), both shafts will be twisted. Since the end D of shaft AD is fixed, the angle of twist of AD is measured by the angle of rotation ϕ_A of end A. On the other hand, since both ends of shaft BE rotate, the angle of twist of BE is equal to the difference between the angles of rotation ϕ_B and ϕ_E, i.e., the angle of twist is equal to the angle through which end E rotates with respect to end B. Denoting this relative angle of rotation by $\phi_{E/B}$, we write

$$\phi_{E/B} = \phi_E - \phi_B = \frac{TL}{JG}$$

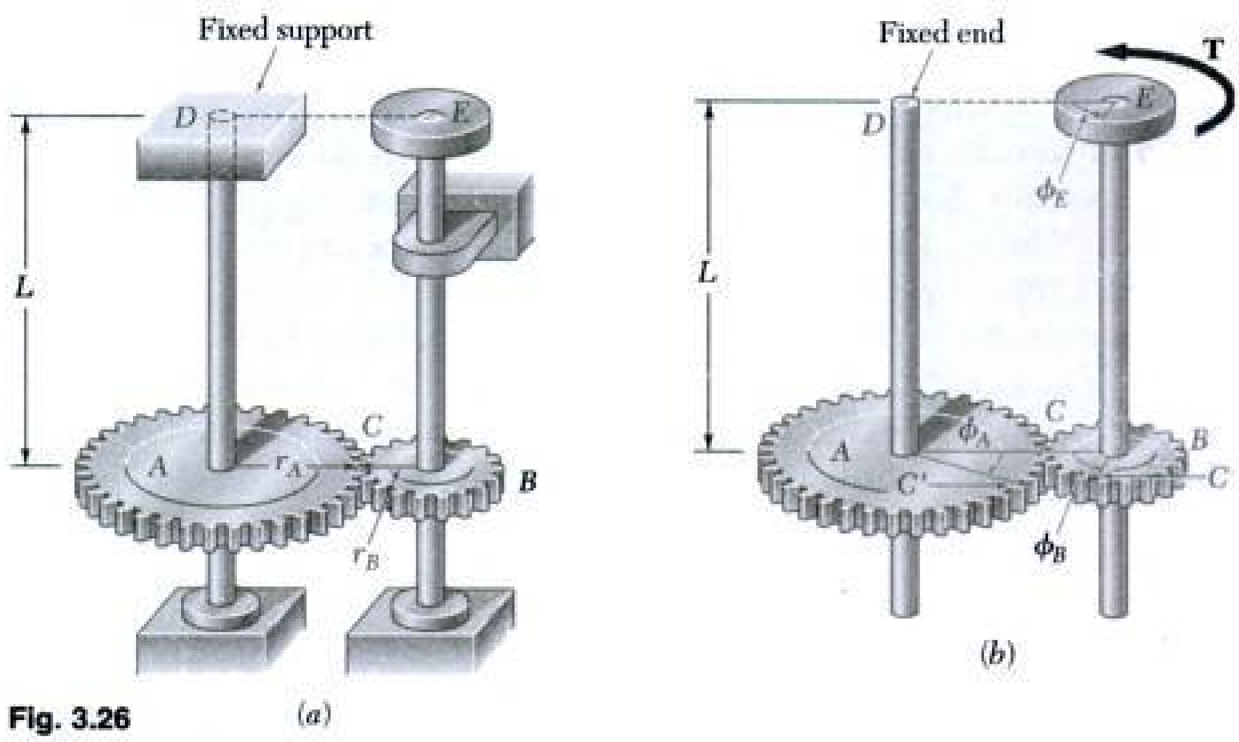
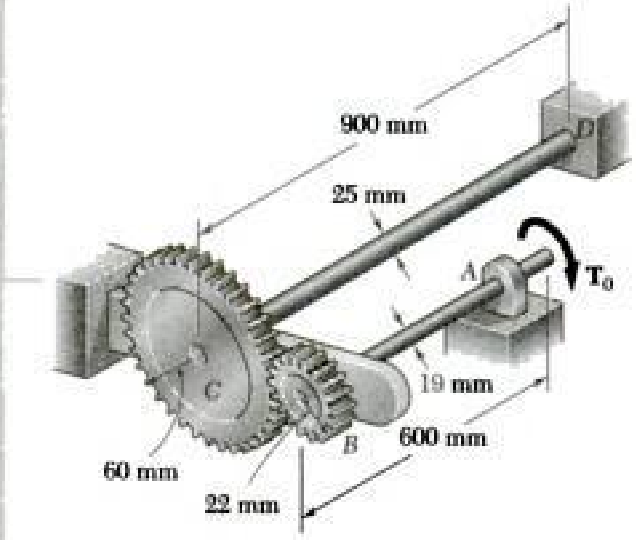
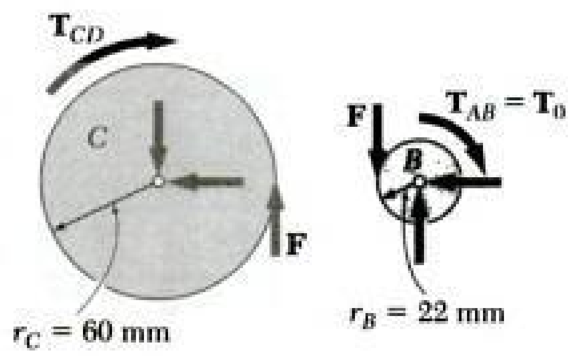


Fig. 3.26

SAMPLE PROBLEM 3.4



Two solid steel shafts are connected by the gears shown. Knowing that for each shaft $G = 77$ GPa and that the allowable shearing stress is 55 MPa, determine (*a*) the largest torque $\mathbf{T}_0$ that may be applied to end *A* of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.



SOLUTION

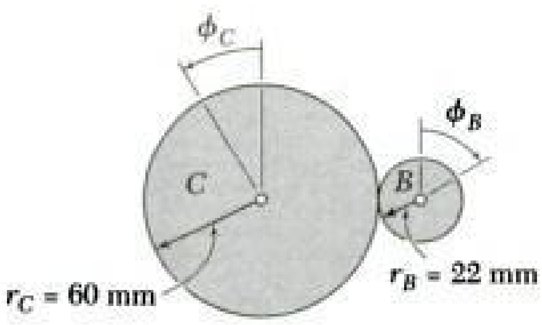
Statics. Denoting by F the magnitude of the tangential force between gear teeth, we have

Gear B. $\Sigma M_B = 0$: $\quad F(22 \text{ mm}) - T_0 = 0$

Gear C. $\Sigma M_C = 0$: $\quad F(60 \text{ mm}) - T_{CD} = 0$

$$T_{CD} = 2.72\, T_0 \qquad (1)$$

Kinematics. Noting that the peripheral motions of the gears are equal, we write



$$r_B\phi_B = r_C\phi_C \qquad \phi_B = \phi_C\frac{r_C}{r_B} = \phi_C\frac{60 \text{ mm}}{22 \text{ mm}} = 2.72\,\phi_C \qquad (2)$$

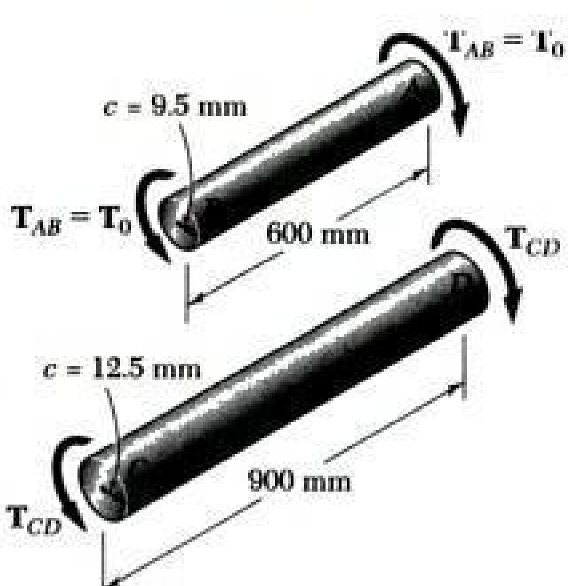
***a.* Torque T_0**

Shaft AB. With $T_{AB} = T_0$ and $c = 9.5$ mm, together with a maximum permissible shearing stress of 55 MPa, we write

$$\tau = \frac{T_{AB}c}{J} \qquad 55 \text{ MPa} = \frac{T_0(9.5\times10^{-3}\text{ m})}{\frac{1}{2}\pi(9.5\times10^{-3}\text{ m})^4} \qquad T_0 = 74.07 \text{ N}\cdot\text{m} \quad \triangleleft$$

Shaft CD. From (1) we have $T_{CD} = 2.72T_0$. With $c = 12.5$ mm and $\tau_{all} = 55$ MPa, we write

$$\tau = \frac{T_{CD}c}{J} \qquad 55 \text{ MPa} = \frac{2.72\,T_0(12.5\times10^{-3}\text{ m})}{\frac{1}{2}\pi(12.5\times10^{-3}\text{ m})^4} \qquad T_0 = 62 \text{ N}\cdot\text{m} \quad \triangleleft$$



Maximum Permissible Torque. We choose the smaller value obtained for T_0

$$T_0 = 62 \text{ N}\cdot\text{m} \blacktriangleleft$$

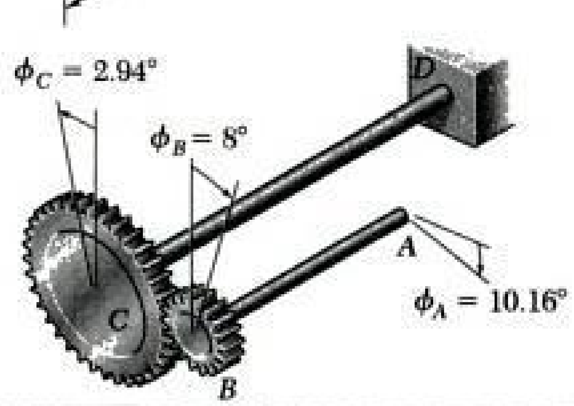
***b.* Angle of Rotation at End *A*.** We first compute the angle of twist for each shaft.

Shaft AB. For $T_{AB} = T_0 = 62$ N · m, we have

$$\phi_{A/B} = \frac{T_{AB}L}{JG} = \frac{(62 \text{ N}\cdot\text{m})(0.6 \text{ m})}{\frac{1}{2}\pi\,(0.0095 \text{ m})^4\,(77\times10^9 \text{ Pa})} = 0.0377 \text{ rad} = 2.16°$$

Shaft CD. $\quad T_{CD} = 2.72\,T_0 = 2.72\,(62 \text{ N}\cdot\text{m}) = 168.64 \text{ N}\cdot\text{m}$

$$\phi_{C/D} = \frac{T_{CD}L}{JG} = \frac{(168.64 \text{ N}\cdot\text{m})\,(0.9 \text{ m})}{\frac{1}{2}\pi\,(0.0125 \text{ m})^4\,(77\times10^9 \text{ Pa})} = 0.0513 \text{ rad} = 2.94°$$



Since end *D* of shaft *CD* is fixed, we have $\phi_C = \phi_{C/D} = 2.94°$. Using (2), we find the angle of rotation of gear *B* to be

$$\phi_B = 2.72\,\phi_C = 2.72\,(2.94°) = 8°$$

For end *A* of shaft *AB*, we have

$$\phi_A = \phi_B + \phi_{A/B} = 8° + 2.16° \qquad \phi_A = 10.16° \blacktriangleleft$$

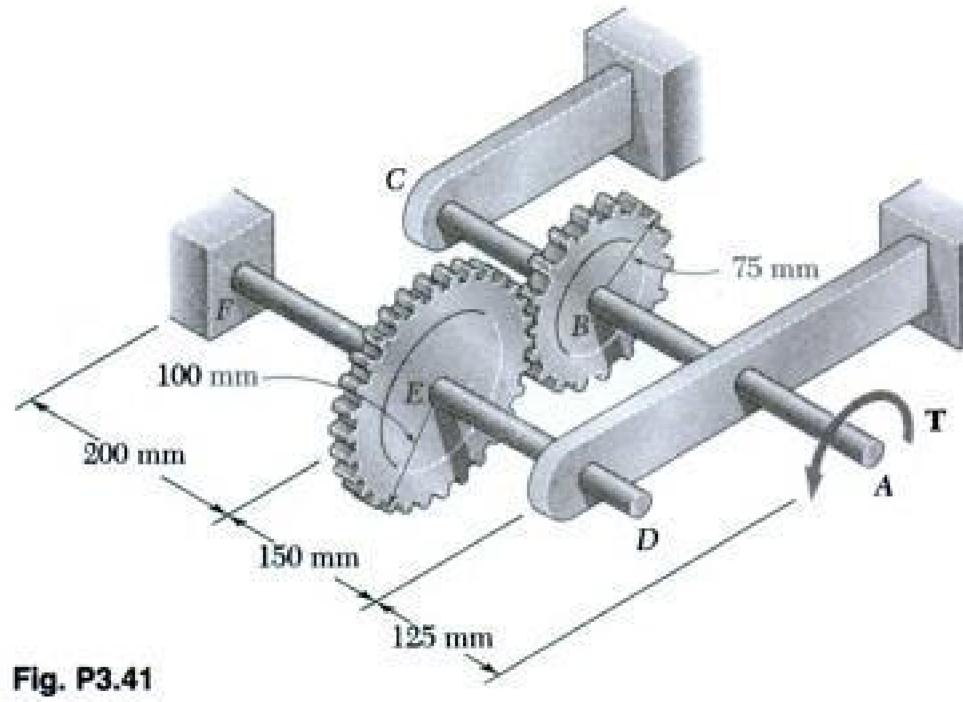


Fig. P3.41

3.41 Two shafts, each of 20 mm diameter, are connected by the gears shown. Knowing that $G = 77$ GPa and that the shaft at F is fixed, determine the angle through which end A rotates when a 85 N · m torque is applied at A.

3.42 Solve Prob. 3.41, assuming that after a design change the radius of gear B is 100 mm and the radius of gear E is 75 mm.

3.43 A coder F, used to record in digital form the rotation of shaft A, is connected to the shaft by means of the gear train shown, which consists of four gears and three solid steel shafts each of diameter d. Two of the gears have a radius r and the other two a radius nr. If the rotation of the coder F is prevented, determine in terms of T, l, G, J, and n the angle through which end A rotates.

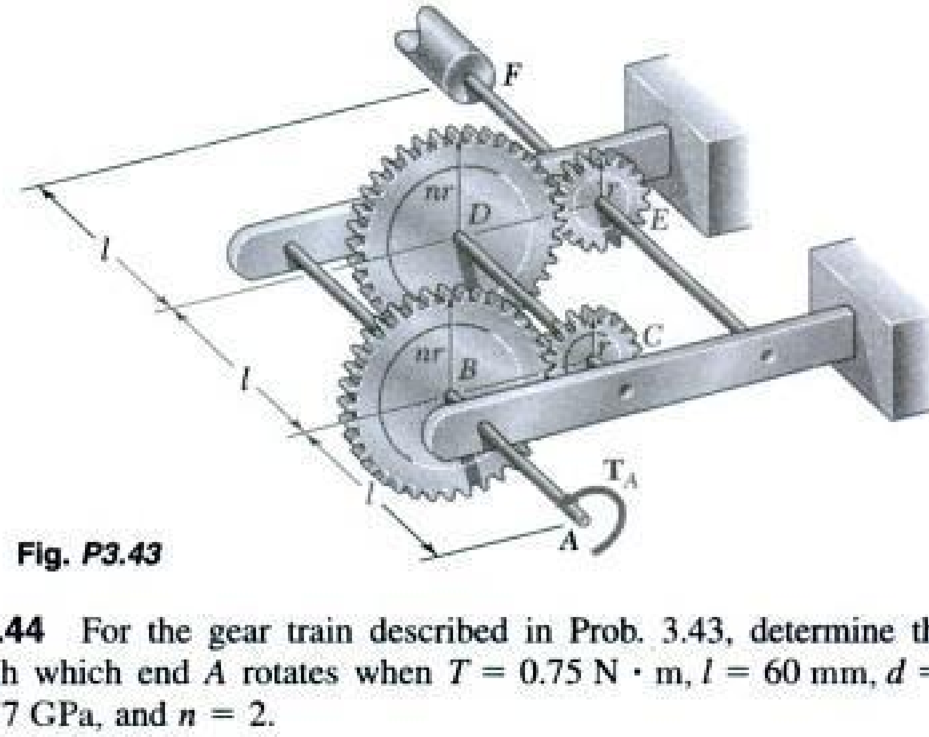


Fig. *P3.43*

3.44 For the gear train described in Prob. 3.43, determine the angle through which end A rotates when $T = 0.75$ N · m, $l = 60$ mm, $d = 4$ mm, $G = 77$ GPa, and $n = 2$.

3.45 The design specifications of a 2-m-long solid circular transmission shaft require that the angle of twist of the shaft not exceed 3° when a torque of 9 kN · m is applied. Determine the required diameter of the shaft, knowing that the shaft is made of (*a*) a steel with an allowable shearing stress of 90 MPa and a modulus of rigidity of 77 GPa, (*b*) a bronze with an allowable shearing stress of 35 MPa and a modulus of rigidity of 42 GPa.

3.62 The mass moment of inertia of a gear is to be determined experimentally by using a torsional pendulum consisting of a 2 m steel wire. Knowing that $G = 77$ GPa, determine the diameter of the wire for which the torsional spring constant will be 5.8 N · m/rad.

Fig. P3.62

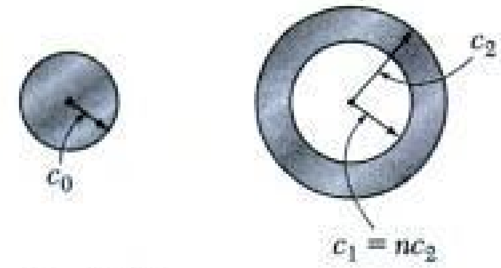


Fig. P3.63

3.63 A solid shaft and a hollow shaft are made of the same material and are of the same weight and length. Denoting by n the ratio c_1/c_2, show that the ratio T_s/T_h of the torque T_s in the solid shaft to the torque T_h in the hollow shaft is (*a*) $\sqrt{1 - n^2}/(1 + n^2)$ if the maximum shearing stress is the same in each shaft, (*b*) $(1 - n^2)/(1 + n^2)$ if the angle of twist is the same for each shaft.

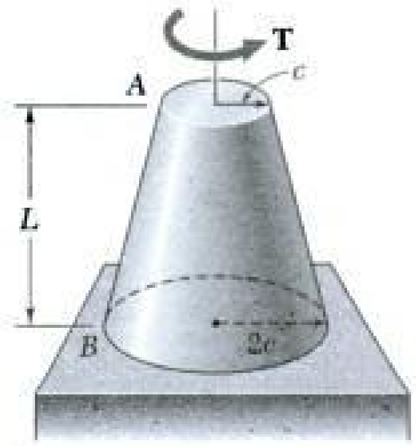


Fig. P3.64

3.64 A torque **T** is applied as shown to a solid tapered shaft *AB*. Show by integration that the angle of twist at *A* is

$$\phi = \frac{7TL}{12\pi Gc^4}$$

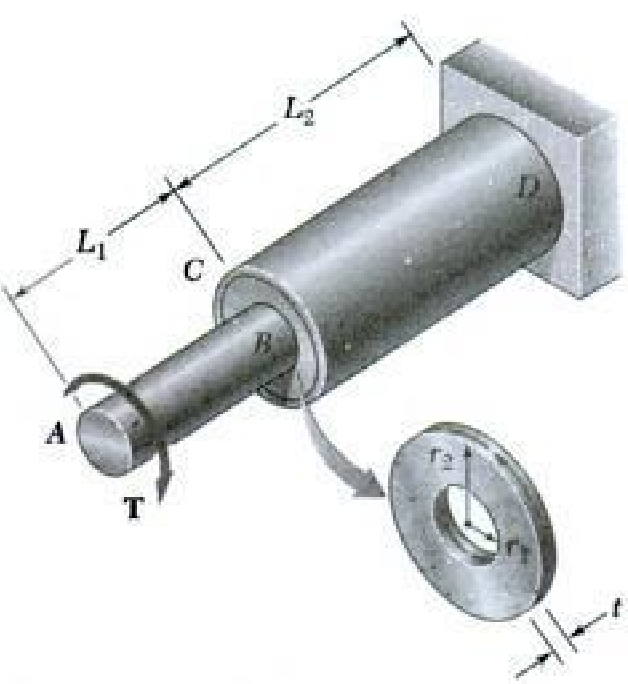


Fig. P3.65 and P3.66

3.65 An annular plate of thickness t and modulus of rigidity G is used to connect shaft *AB* of radius r_1 to tube *CD* of inner radius r_2. Knowing that a torque **T** is applied to end *A* of shaft *AB* and that end *D* of tube *CD* is fixed, (*a*) determine the magnitude and location of the maximum shearing stress in the annular plate, (*b*) show that the angle through which end *B* of the shaft rotates with respect to end *C* of the tube is

$$\phi_{B/C} = \frac{T}{4\pi Gt}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

3.66 An annular aluminum plate ($G = 27$ GPa), of thickness $t = 6$ mm, is used to connect the aluminum shaft *AB*, of length $L_1 = 90$ mm and radius $r_1 = 30$ mm, to the aluminum tube *CD*, of length $L_2 = 150$ mm, inner radius $r_2 = 75$ mm and 4 mm thickness. Knowing that a torque of magnitude $T = 2500$ N · m is applied to end *A* of shaft *AB* and that end *D* of tube *CD* is fixed, determine (*a*) the maximum shearing stress in the shaft-plate-tube sys-

SAMPLE PROBLEM 3.6

The stepped shaft shown is to rotate at 900 rpm as it transmits power from a turbine to a generator. The grade of steel specified in the design has an allowable shearing stress of 55 MPa. (*a*) For the preliminary design shown, determine the maximum power that can be transmitted. (*b*) If in the final design the radius of the fillet is increased so that $r = 24$ mm, what will be the percent change, relative to the preliminary design, in the power which may be transmitted?

SOLUTION

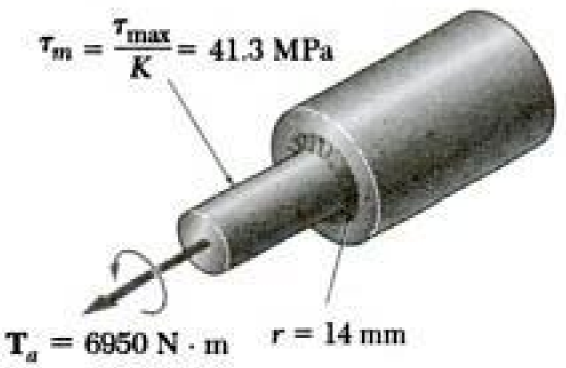
***a.* Preliminary Design.** Using the notation of Fig. 3.32, we have: $D = 190$ mm, $d = 95$ mm, $r = 14$ mm

$$\frac{D}{d} = \frac{190}{95} = 2 \qquad \frac{r}{d} = \frac{14\text{ mm}}{95\text{ mm}} = 0.15$$

A stress concentration factor $K = 1.33$ is found from Fig. 3.32.

Torque. Recalling Eq. (3.25), we write

$$\tau_{max} = K\frac{Tc}{J} \qquad T = \frac{J}{c}\frac{\tau_{max}}{K} \tag{1}$$



where J/c refers to the smaller-diameter shaft:

$$J/c = \tfrac{1}{2}\pi c^3 = \tfrac{1}{2}\pi\,(47.5)^3 = 168.3\,(10^3)\text{ mm}^3$$

and where
$$\frac{\tau_{max}}{K} = \frac{55\text{ MPa}}{1.33} = 41.3\text{ MPa}$$

Substituting into Eq. (1), we find $T = 168.3\,(10^3)\text{ mm}^3\,(41.3\text{ MPa}) = 6950\text{ N}\cdot\text{m}$

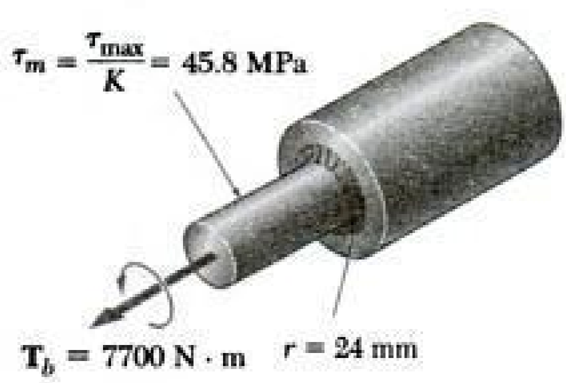
Power. Since $f = (900\text{ rpm})\dfrac{1\text{ Hz}}{60\text{ rpm}} = 15\text{ Hz} = 15\text{ s}^{-1}$, we write

$$P_a = 2\pi f T = 2\pi\,(15\text{ s}^{-1})(6950\text{ N}\cdot\text{m}) = 655\,(10^3)\text{ N}\cdot\text{m/s}$$

$$P_a = 655\text{ kW} ◀$$

***b.* Final Design.** For $r = 24$ mm

$$\frac{D}{d} = 2 \qquad \frac{r}{d} = \frac{24\text{ mm}}{95\text{ mm}} = 0.250 \qquad K = 1.20$$



Following the procedure used above, we write

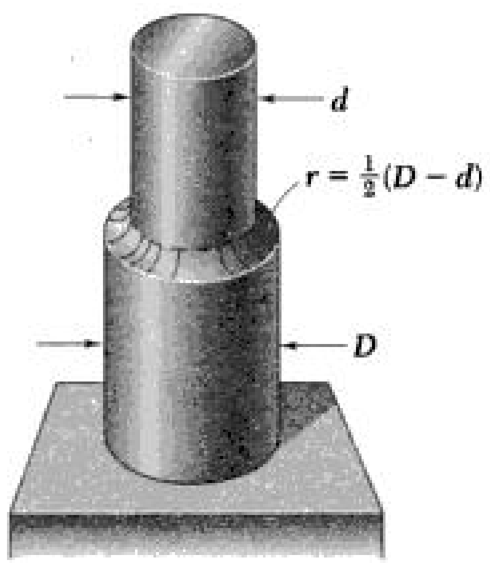
$$\frac{\tau_{max}}{K} = \frac{55\text{ MPa}}{1.20} = 45.8\text{ MPa}$$

$$T = \frac{J}{c}\frac{\tau_{max}}{K} = 168.3(10^3)\text{ mm}^3(45.8\text{ MPa}) = 7700\text{ N}\cdot\text{m}$$

$$P_b = 2\pi f T = 2\pi(15\text{ s}^{-1})(7700\text{ N}\cdot\text{m}) = 725\text{ kW}$$

Percent Change in Power

$$\text{Percent change} = 100\,\frac{P_b - P_a}{P_a} = 100\,\frac{725 - 655}{655} = +10.68\% ◀$$



Full quarter-circular fillet extends to edge of larger shaft

Fig. P3.91, *P3.92*, and *P3.93*

3.91 A torque of magnitude $T = 25$ N · m is applied to the stepped shaft shown, which has a full quarter-circular fillet. Knowing that $D = 24$ mm, determine the maximum shearing stress in the shaft when (*a*) $d = 20$ mm, (*b*) $d = 21.6$ mm.

3.92 In the stepped shaft shown, which has a full quarter-circular fillet, $D = 38$ mm and $d = 30$ mm. Knowing that the frequency of the shaft is 30 Hz and that the allowable shearing stress is 55 MPa, determine the maximum power that may be transmitted by the shaft.

3.93 In the stepped shaft shown, which has a full quarter-circular fillet, the allowable shearing stress is 82 MPa. Knowing that $D = 32$ mm, determine the largest allowable torque that may be applied to the shaft if (*a*) $d = 28$ mm, (*b*) $d = 25$ mm.

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*3.9. PLASTIC DEFORMATIONS IN CIRCULAR SHAFTS

When we derived Eqs. (3.10) and (3.16), which define respectively the stress distribution and the angle of twist for a circular shaft subjected to a torque **T**, we assumed that Hooke's law applied throughout the shaft. If the yield strength is exceeded in some portion of the shaft, or if the material involved is a brittle material with a nonlinear shearing-stress-strain diagram, these relations cease to be valid. The purpose of this section is to develop a more general method—which may be used when Hooke's law does not apply—for determining the distribution of stresses in a solid circular shaft, and for computing the torque required to produce a given angle of twist.

We first recall that no specific stress-strain relationship was assumed in Sec. 3.3, when we proved that the shearing strain γ varies lin-

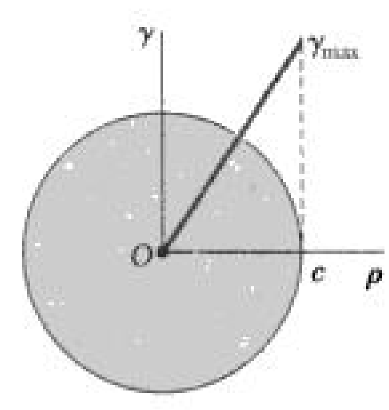


Fig. 3.33

early with the distance ρ from the axis of the shaft (Fig. 3.33). Thus, we may still use this property in our present analysis and write

$$\gamma = \frac{\rho}{c}\gamma_{max} \tag{3.4}$$

where c is the radius of the shaft.

elastic core is obtained by making γ equal to the yield strain γ_Y in Eq. (3.2) and solving for the corresponding value ρ_Y of the distance ρ. We have

$$\rho_Y = \frac{L\gamma_Y}{\phi} \tag{3.34}$$

Let us denote by ϕ_Y the angle of twist at the onset of yield, i.e., when $\rho_Y = c$. Making $\phi = \phi_Y$ and $\rho_Y = c$ in Eq. (3.34), we have

$$c = \frac{L\gamma_Y}{\phi_Y} \tag{3.35}$$

Dividing (3.34) by (3.35), member by member, we obtain the following relation:†

$$\frac{\rho_Y}{c} = \frac{\phi_Y}{\phi} \tag{3.36}$$

If we carry into Eq. (3.32) the expression obtained for ρ_Y/c, we express the torque T as a function of the angle of twist ϕ,

$$T = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\phi_Y^3}{\phi^3}\right) \tag{3.37}$$

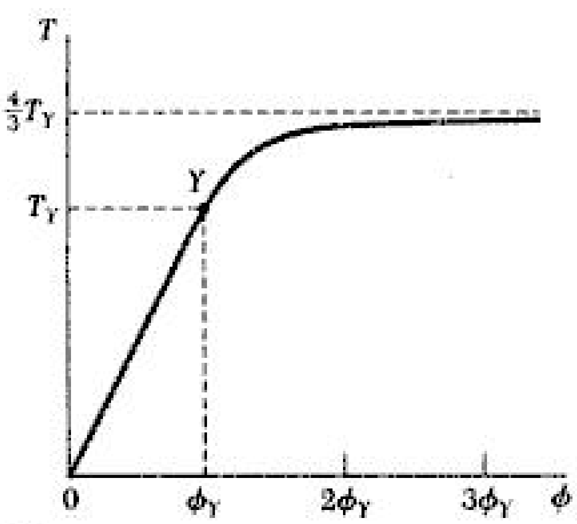


Fig. 3.39

where T_Y and ϕ_Y represent, respectively, the torque and the angle of twist at the onset of yield. Note that Eq. (3.37) may be used only for values of ϕ larger than ϕ_Y. For $\phi < \phi_Y$, the relation between T and ϕ is linear and given by Eq. (3.16). Combining both equations, we obtain the plot of T against ϕ represented in Fig. 3.39. We check that, as ϕ increases indefinitely, T approaches the limiting value $T_p = \frac{4}{3}T_Y$ corresponding to the case of a fully developed plastic zone (Fig. 3.38*d*). While the value T_p cannot actually be reached, we note from Eq. (3.37) that it is rapidly approached as ϕ increases. For $\phi = 2\phi_Y$, T is within about 3% of T_p, and for $\phi = 3\phi_Y$ within about 1 percent.

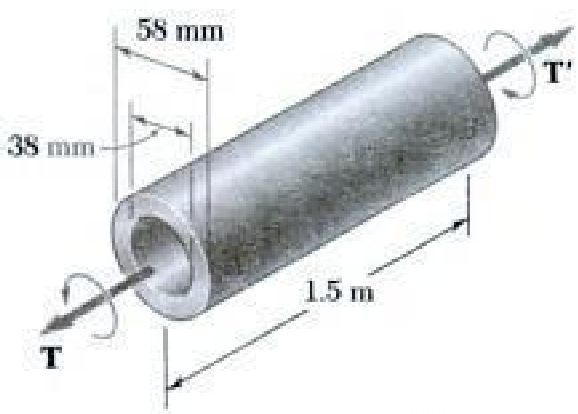
Since the plot of T against ϕ that we have obtained for an idealized elastoplastic material (Fig. 3.39) differs greatly from the shearing-stress-strain diagram of that material (Fig. 3.37), it is clear that the shearing-stress-strain diagram of an actual material cannot be obtained directly from a torsion test carried out on a solid circular rod made of that material. However, a fairly accurate diagram may be obtained from a torsion test if the specimen used incorporates a portion consisting of a thin circular tube.‡ Indeed, we may assume that the shearing stress will have a constant value τ in that portion. Equation (3.1) thus reduces to

$$T = \rho A\tau$$

where ρ denotes the average radius of the tube and A its cross-sectional area. The shearing stress is thus proportional to the torque, and successive values of τ can be easily computed from the corresponding values of T. On the other hand, the values of the shearing strain γ may be obtained from Eq. (3.2) and from the values of ϕ and L measured on the tubular portion of the specimen.

†Equation (3.36) applies to any ductile material with a well-defined yield point, since its derivation is independent of the shape of the stress-strain diagram beyond the yield point.

‡In order to minimize the possibility of failure by buckling, the specimen should be made so that the length of the tubular portion is no longer than its diameter.



SAMPLE PROBLEM 3.7

Shaft *AB* is made of a mild steel which is assumed to be elastoplastic with $G = 77$ GPa and $\tau_Y = 145$ MPa. A torque **T** is applied and gradually increased in magnitude. Determine the magnitude of **T** and the corresponding angle of twist (*a*) when yield first occurs, (*b*) when the deformation has become fully plastic.

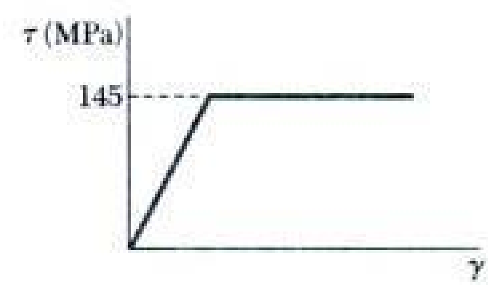
SOLUTION

Geometric Properties

The geometric properties of the cross section are

$$c_1 = \tfrac{1}{2}(38 \text{ mm}) = 19 \text{ mm} \qquad c_2 = \tfrac{1}{2}(58 \text{ mm}) = 29 \text{ mm}$$

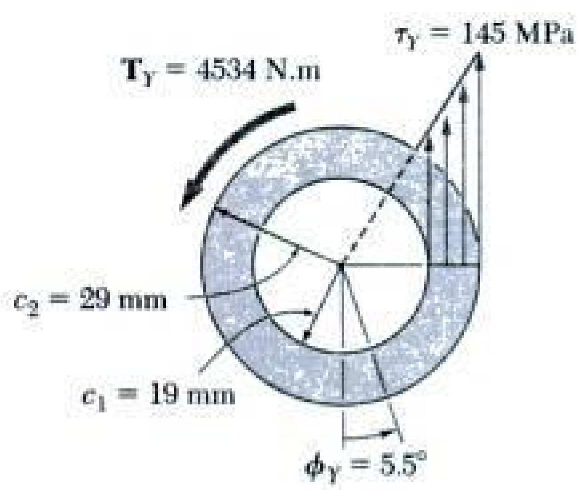
$$J = \tfrac{1}{2}\pi(c_2^4 - c_1^4) = \tfrac{1}{2}\pi[(29 \text{ mm})^4 - (19 \text{ mm})^4 = 906.8(10^3)\text{mm}^4$$



***a.* Onset of Yield.** For $\tau_{max} = \tau_Y = 145$ MPa, we find

$$T_Y = \frac{\tau_Y J}{c_2} = \frac{(145 \text{ MPa})\, 906.8(10^3)\text{mm}^4}{29 \text{ mm}}$$

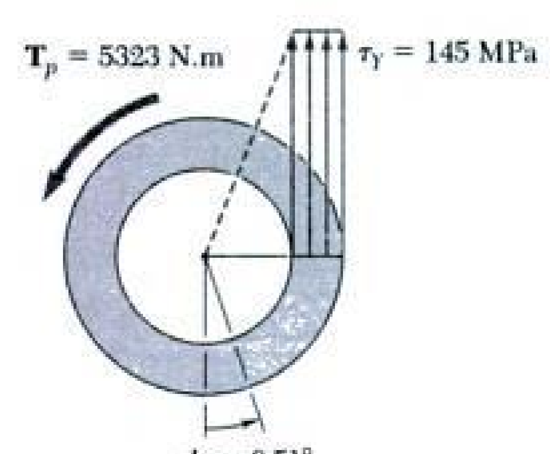
$$T_Y = 4534 \text{ N} \cdot \text{m} \blacktriangleleft$$



Making $\rho = c_2$ and $\gamma = \gamma_Y$ in Eq. (3.2) and solving for ϕ, we obtain the value of ϕ_Y:

$$\phi_Y = \frac{\gamma_Y L}{c_2} = \frac{\tau_Y L}{c_2 G} = \frac{(145 \text{ MPa})\,(1500 \text{ mm})}{(29 \text{ mm})\,(77 \text{ GPa})} = 0.097 \text{ rad}$$

$$\phi_Y = 5.5° \blacktriangleleft$$



***b.* Fully Plastic Deformation.** When the plastic zone reaches the inner surface, the stresses are uniformly distributed as shown. Using Eq. (3.26), we write

$$T_P = 2\pi\tau_Y \int_{c_1}^{c_2} \rho^2\, d\rho = \tfrac{2}{3}\pi\tau_Y(c_2^3 - c_1^3)$$

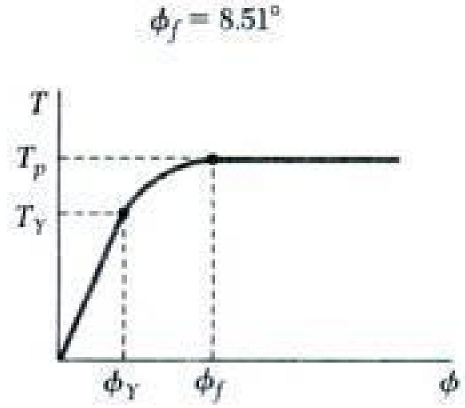
$$= \tfrac{2}{3}\pi(145 \text{ MPa})[(29 \times 10^{-3} \text{m})^3 - (19 \times 10^{-3} \text{m})^3]$$

$$T_p = 5323 \text{ N} \cdot \text{m} \blacktriangleleft$$

When yield first occurs on the inner surface, the deformation is fully plastic; we have from Eq. (3.2):

$$\phi_f = \frac{\gamma_Y L}{c_1} = \frac{\tau_Y L}{c_1 G} = \frac{(145 \text{ MPa})\,(1.5 \text{ m})}{(0.019 \text{ m})\,(77 \text{ GPa})} = 0.148 \text{ rad}$$

$$\phi_f = 8.51° \blacktriangleleft$$



For larger angles of twist the torque remains constant; the T-ϕ diagram of the shaft is as shown.

3.111 A 50-mm-diameter cylinder is made of a brass for which the stress-strain diagram is as shown. Knowing that the angle of twist is 5° in a 725-mm length, determine by approximate means the magnitude T of the torque applied to the shaft.

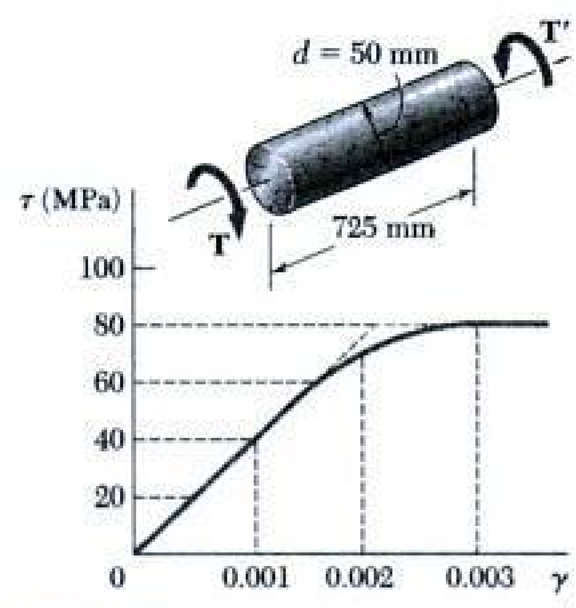


Fig. P3.111

3.112 Three points on the nonlinear stress-strain diagram used in Prob. 3.111 are (0, 0), (0.0015, 55 MPa), and (0.003, 80 MPa). By fitting the polynomial $\tau = A + B\gamma + C\gamma^2$ through these points, the following approximate relation has been obtained.

$$\tau = 46.7 \times 10^9\gamma - 6.67 \times 10^{12}\gamma^2$$

Solve Prob. 3.111 using this relation, Eq. (3.2), and Eq. (3.26).

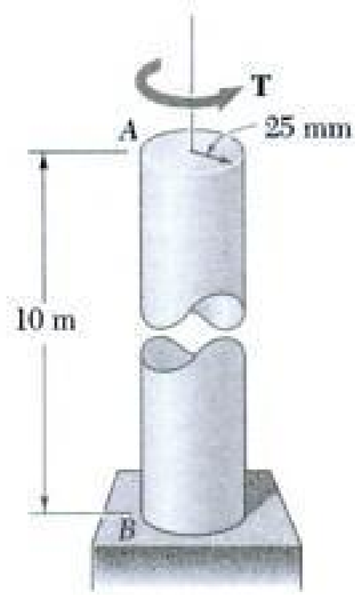


Fig. P3.113

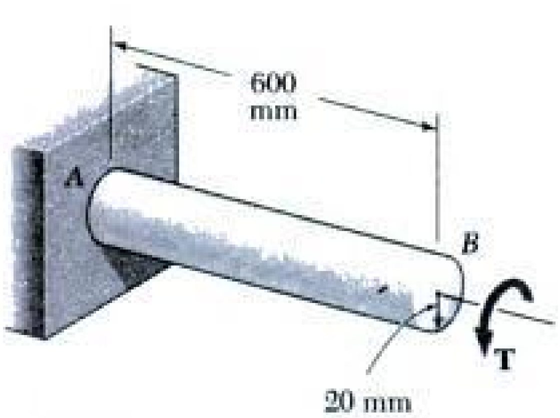


Fig. P3.114

3.113 The solid circular drill rod AB is made of a steel that is assumed to be elastoplastic with $\tau_Y = 160$ MPa and $G = 77$ GPa. Knowing that a torque $T = 5$ kN · m is applied to the rod and then removed, determine the maximum residual shearing stress in the rod.

3.114 The solid circular shaft AB is made of a steel that is assumed to be elastoplastic with $G = 77$ GPa and $\tau_Y = 145$ MPa. The torque **T** is increased until the radius of the elastic core is 6 mm. Determine the maximum residual shearing stress in the shaft after the torque **T** has been removed.

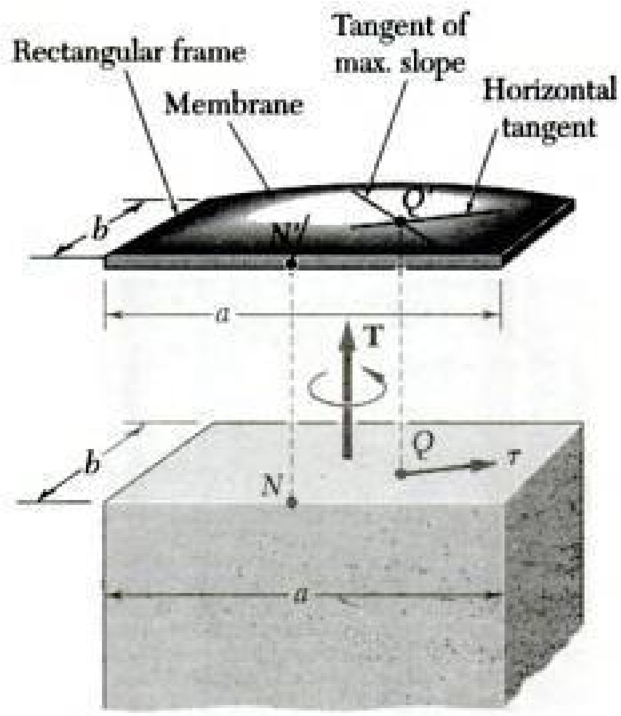


Fig. 3.49

brane depends upon the solution of the same partial differential equation as the determination of the shearing stresses in the bar.† More specifically, if Q is a point of the cross section of the bar and Q' the corresponding point of the membrane (Fig. 3.49), the shearing stress τ at Q will have the same direction as the horizontal tangent to the membrane at Q', and its magnitude will be proportional to the maximum slope of the membrane at Q'.‡ Furthermore, the applied torque will be proportional to the volume between the membrane and the plane of the fixed frame. In the case of the membrane of Fig. 3.49, which is attached to a rectangular frame, the steepest slope occurs at the midpoint N' of the larger side of the frame. Thus, we verify that the maximum shearing stress in a bar of rectangular cross section will occur at the midpoint N of the larger side of that section.

The membrane analogy may be used just as effectively to visualize the shearing stresses in any straight bar of uniform, noncircular cross section. In particular, let us consider several thin-walled members with the cross sections shown in Fig. 3.50, which are subjected to the same torque. Using the membrane analogy to help us visualize the shearing stresses, we note that, since the same torque is applied to each member, the same volume will be located under each membrane, and the maximum slope will be about the same in each case. Thus, for a thin-walled member of uniform thickness and arbitrary shape, the maximum shearing stress is the same as for a rectangular bar with a very large value of a/b and may be determined from Eq. (3.43) with $c_1 = 0.333$.§

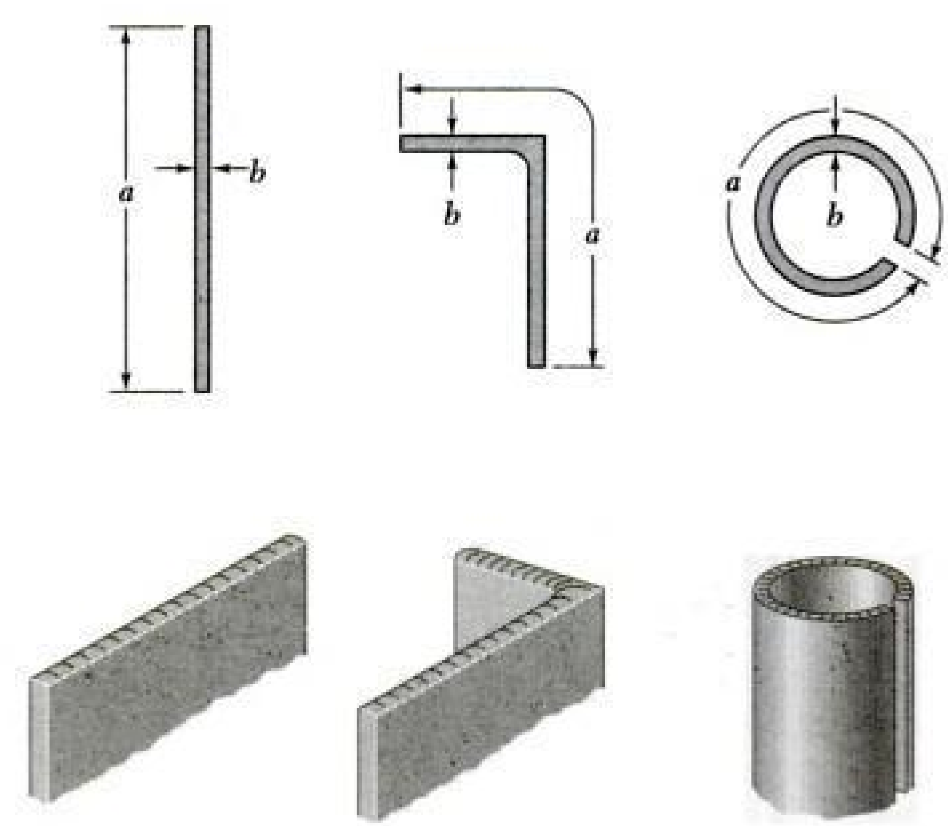
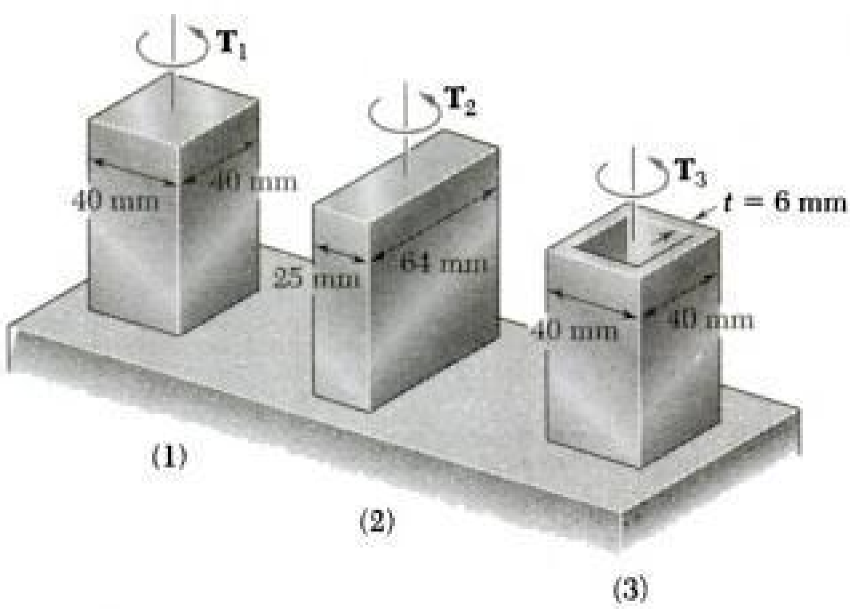


Fig. 3.50

†See ibid. sec. 107.

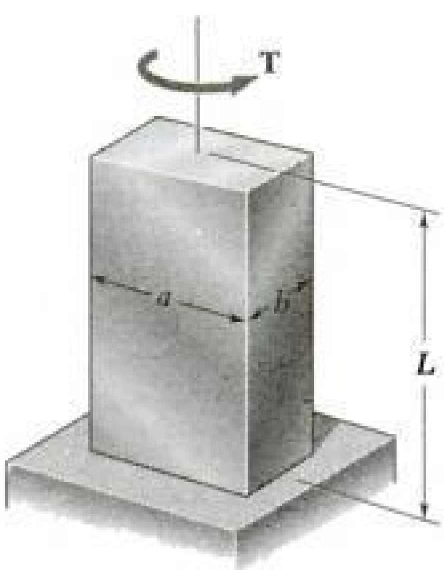
‡This is the slope measured in a direction perpendicular to the horizontal tangent at Q'.

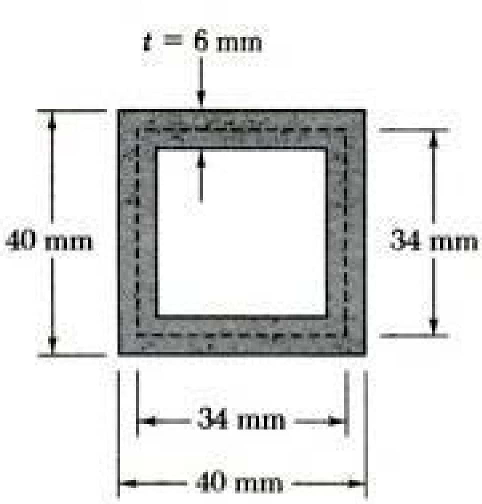
§It could also be shown that the angle of twist may be determined from Eq. (3.44) with $c_2 = 0.333$.



SAMPLE PROBLEM 3.9

Using $\tau_{all} = 40$ MPa, determine the largest torque that may be applied to each of the brass bars and to the brass tube shown. Note that the two solid bars have the same cross-sectional area, and that the square bar and square tube have the same outside dimensions.





SOLUTION

1. Bar with Square Cross Section. For a solid bar of rectangular cross section the maximum shearing stress is given by Eq. (3.43)

$$\tau_{max} = \frac{T}{c_1 ab^2}$$

where the coefficient c_1 is obtained from Table 3.1 in Sec. 3.12. We have

$$a = b = 0.040 \text{ m} \qquad \frac{a}{b} = 1.00 \qquad c_1 = 0.208$$

For $\tau_{max} = \tau_{all} = 40$ MPa, we have

$$\tau_{max} = \frac{T_1}{c_1 ab^2} \qquad 40 \text{ MPa} = \frac{T_1}{0.208(0.040 \text{ m})^3} \qquad T_1 = 532 \text{ N} \cdot \text{m} \blacktriangleleft$$

2. Bar with Rectangular Cross Section. We now have

$$a = 0.064 \text{ m} \qquad b = 0.025 \text{ m} \qquad \frac{a}{b} = 2.56$$

Interpolating in Table 3.1: $c_1 = 0.259$

$$\tau_{max} = \frac{T_2}{c_1 ab^2} \qquad 40 \text{ MPa} = \frac{T_2}{0.259(0.064 \text{ m})(0.025 \text{ m})^2} \qquad T_2 = 414 \text{ N} \cdot \text{m} \blacktriangleleft$$

3. Square Tube. For a tube of thickness t, the shearing stress is given by Eq. (3.53)

$$\tau = \frac{T}{2t\mathcal{A}}$$

where $\mathcal{A}$ is the area bounded by the center line of the cross section. We have

$$\mathcal{A} = (0.034 \text{ m})(0.034 \text{ m}) = 1.156 \times 10^{-3} \text{ m}^2$$

We substitute $\tau = \tau_{all} = 40$ MPa and $t = 0.006$ m and solve for the allowable torque:

$$\tau = \frac{T}{2t\mathcal{A}} \qquad 40 \text{ MPa} = \frac{T_3}{2(0.006 \text{ m})(1.156 \times 10^{-3} \text{ m}^2)} \qquad T_3 = 555 \text{ N} \cdot \text{m} \blacktriangleleft$$

3.139 and 3.140 A 5650 N·m torque is applied to a hollow shaft having the cross section shown. Neglecting the effect of stress concentration, determine the shearing stress at points a and b.

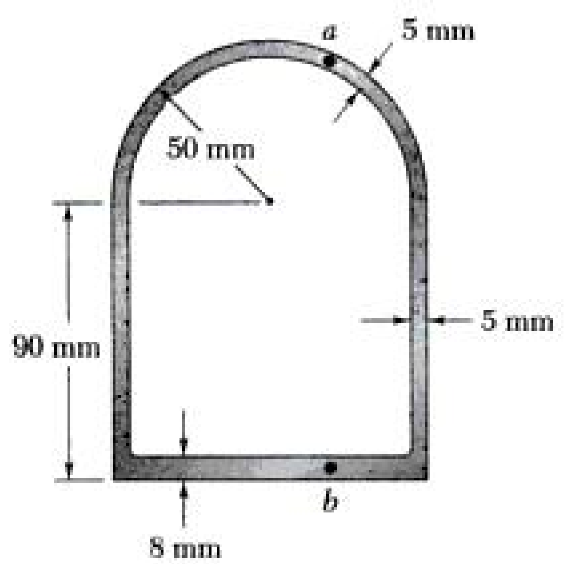


Fig. P3.139

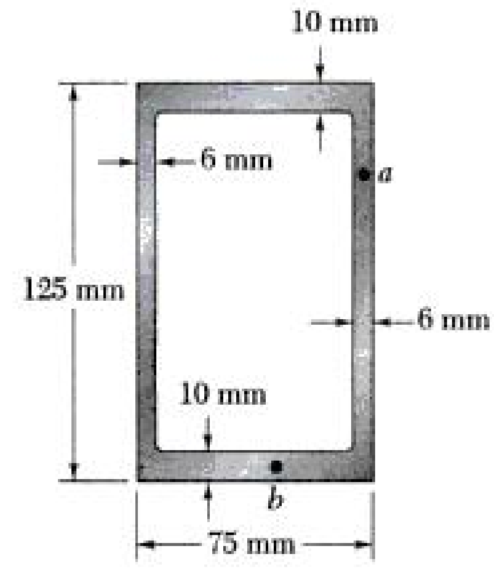


Fig. P3.140

3.141 and 3.142 A hollow member having the cross section shown is formed from sheet metal of 2-mm thickness. Knowing that the shearing stress must not exceed 3 MPa, determine the largest torque that may be applied to the member.

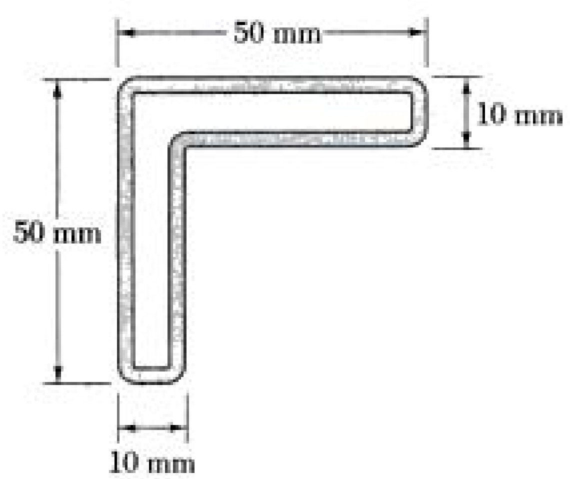


Fig. P3.141

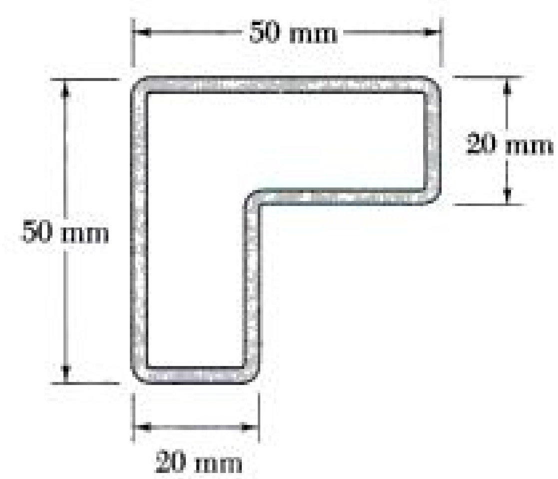


Fig. P3.142

3.143 and 3.144 A hollow member having the cross section shown is to be formed from sheet metal of 1.5 mm thickness. Knowing that a 140 N · m torque will be applied to the member, determine the smallest dimension d that may be used if the shearing stress is not to exceed 5 MPa.

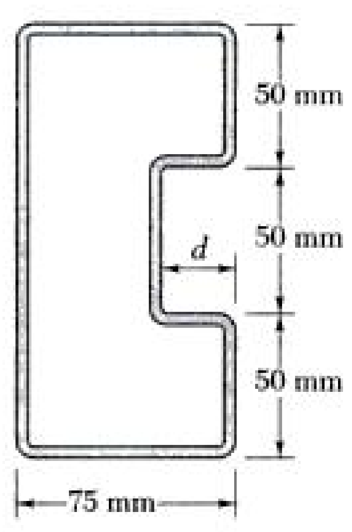


Fig. *P3.143*

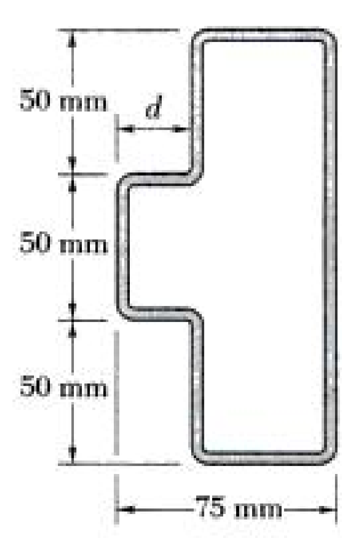


Fig. *P3.144*

Transmission shafts

In Sec. 3.7, we discussed the *design of transmission shafts*. We first observed that the power P transmitted by a shaft is

$$P = 2\pi f T \qquad (3.20)$$

where T is the torque exerted at each end of the shaft and f the *frequency* or speed of rotation of the shaft. The unit of frequency is the revolution per second (s^{-1}) or *hertz* (Hz). If SI units are used, T is expressed in newton-meters (N · m) and P in *watts* (W).

To design a shaft to transmit a given power P at a frequency f, you should first solve Eq. (3.20) for T. Carrying this value and the maximum allowable value of τ for the material used into the elastic formula (3.9), you will obtain the corresponding value of the parameter J/c, from which the required diameter of the shaft may be calculated [Examples 3.06 and 3.07].

Stress concentrations

In Sec. 3.8, we discussed *stress concentrations* in circular shafts. We saw that the stress concentrations resulting from an abrupt change in the diameter of a shaft can be reduced through the use of a *fillet* (Fig. 3.31). The maximum value of the shearing stress at the fillet is

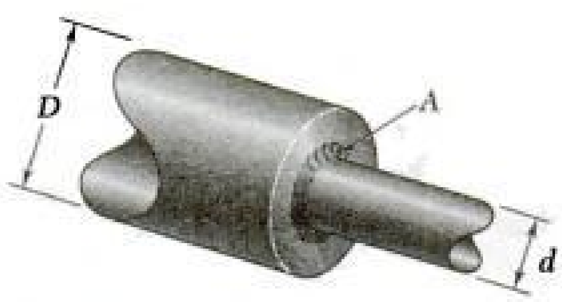


Fig. 3.31

$$\tau_{max} = K\frac{Tc}{J} \qquad (3.25)$$

where the stress Tc/J is computed for the smaller-diameter shaft, and where K is a stress-concentration factor. Values of K were plotted in Fig. 3.32 on p. 167 against the ratio r/d, where r is the radius of the fillet, for various values of D/d.

Plastic deformations

Sections 3.9 through 3.11 were devoted to the discussion of *plastic deformations* and *residual stresses* in circular shafts. We first recalled that even when Hooke's law does not apply, the distribution of *strains* in a circular shaft is always linear [Sec. 3.9]. If the shearing-stress-strain diagram for the material is known, it is then possible to plot the shearing stress τ against the distance π from the axis of the shaft for any given value of τ_{max} (Fig. 3.35). Summing the contributions to the torque of annular elements of radius ρ and thickness $d\rho$ we expressed the torque T as

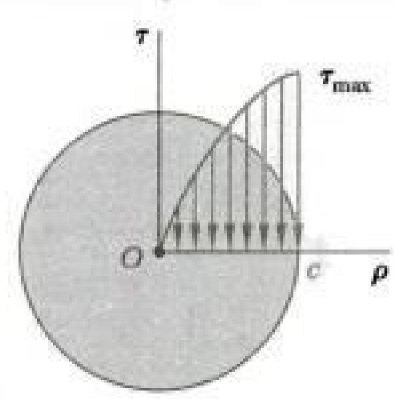


Fig. 3.35

$$T = \int_0^c \rho\tau\,(2\pi\rho\, d\rho) = 2\pi \int_0^c \rho^2 \tau\, d\rho \qquad (3.26)$$

where τ is the function of ρ plotted in Fig. 3.35.

3.154 Two solid steel shafts, each of 30-mm diameter, are connected by the gears shown. Knowing that $G = 77$ GPa, determine the angle through which end A rotates when a torque of magnitude $T = 200$ N · m is applied at A.

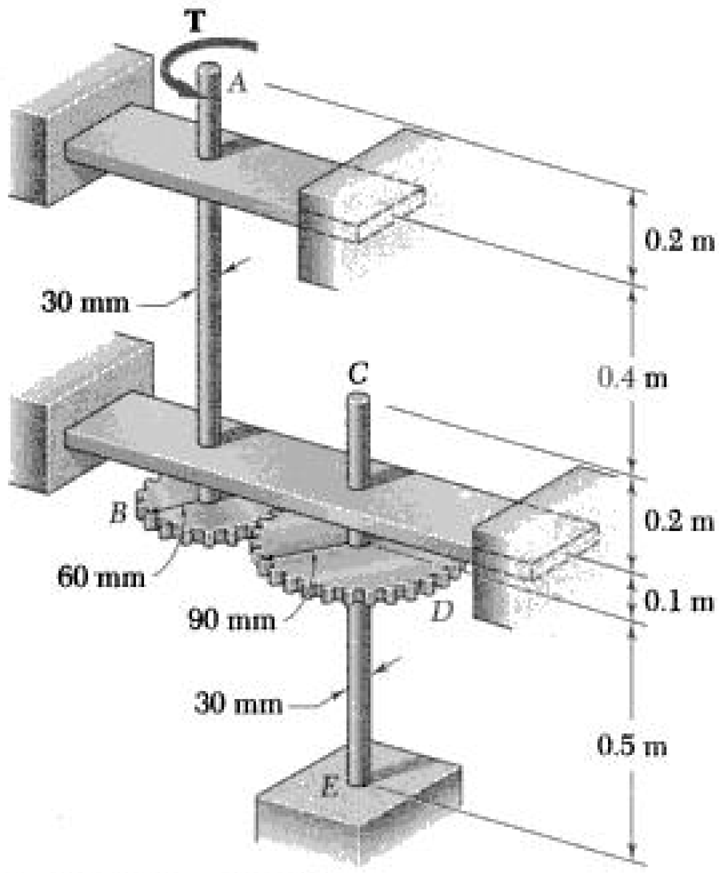


Fig. P3.154 and P3.155

3.155 The angle of rotation of end A of the gear-and-shaft system shown must not exceed 4°. Knowing that the shafts are made of a steel for which $\tau_{all} = 65$ MPa and $G = 77$ GPa, determine the largest torque **T** that can be safely applied at end A.

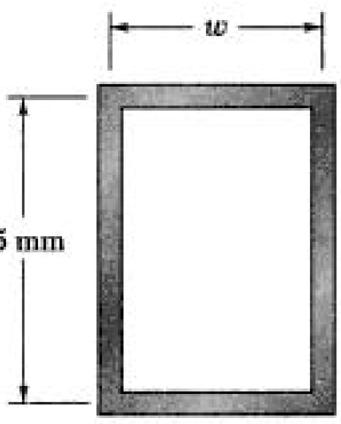


Fig. P3.156

3.156 A sheet-metal strip of width 150 mm and 3 mm thickness is to be formed into a tube of rectangular cross section. Knowing that $\tau_{all} = 28$ MPa, determine the largest torque that may be applied to the tube when (*a*) w = 37.5 mm, (*b*) w = 30 mm, (*c*) w = 25 mm.

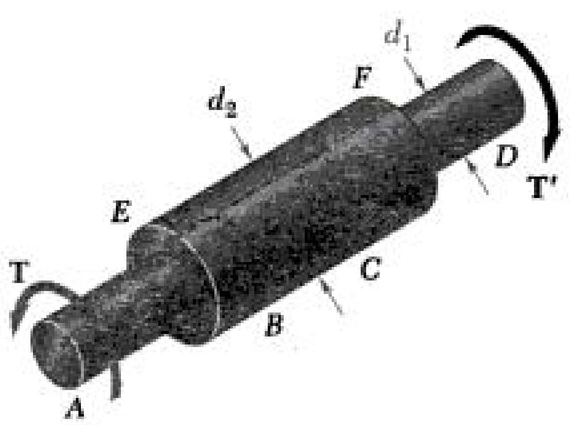


Fig. P3.157

3.157 Two solid brass rods AB and CD are brazed to a brass sleeve EF. Determine the ratio d_2/d_1 for which the same maximum shearing stress occurs in the rods and in the sleeve.

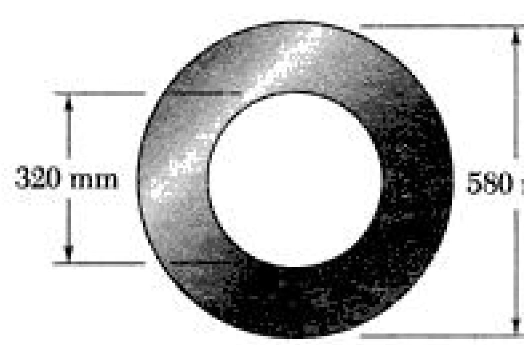


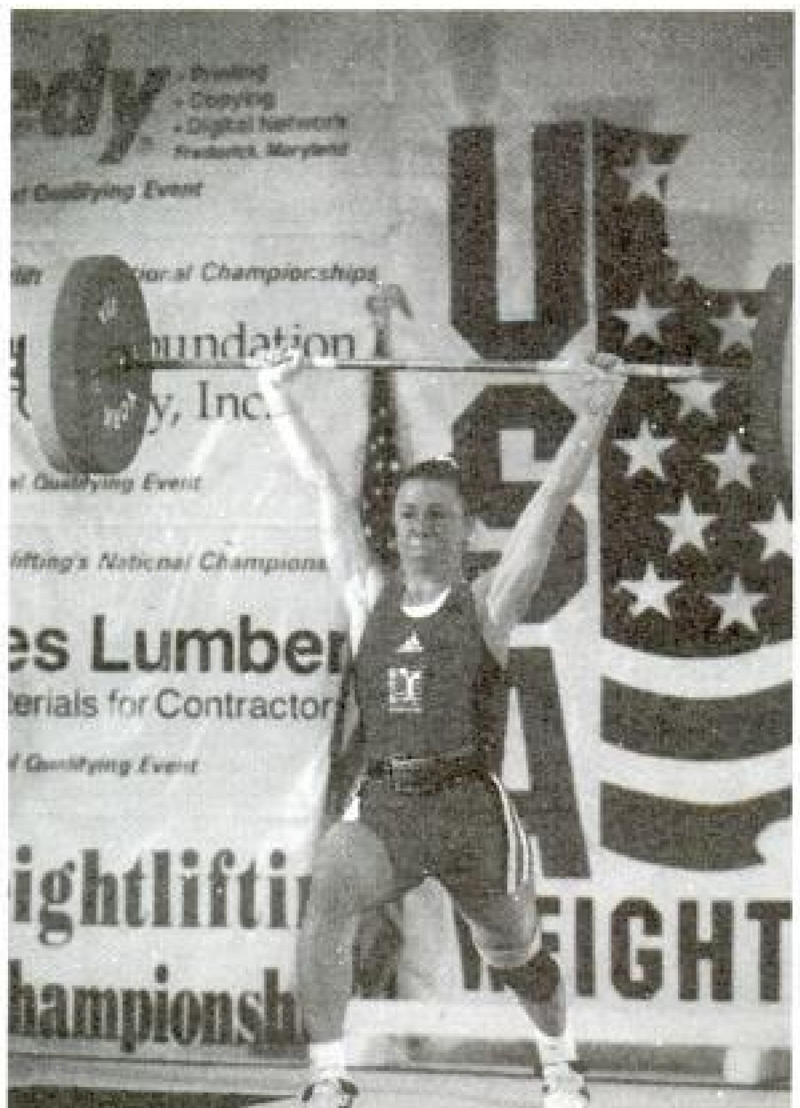
Fig. *P3.158*

3.158 One of the two hollow steel drive shafts of an ocean liner is 75 m long and has the cross section shown. Knowing that $G = 77$ GPa and that the shaft transmits 44 MW to its propeller when rotating at 144 rpm, determine (*a*) the maximum shearing stress in the shaft, (*b*) the angle of twist of the shaft.

C H A P T E R

4

Pure Bending



The athlete shown holds the barbell with his hands placed at equal distances from the weights. This results in pure bending in the center portion of the bar. The normal stresses and the curvature resulting from pure bending will be determined in this chapter.

Denoting by σ_x the normal stress at a given point of the cross section and by τ_{xy} and τ_{xz} the components of the shearing stress, we express that the system of the elementary internal forces exerted on the section is equivalent to the couple **M** (Fig. 4.8).

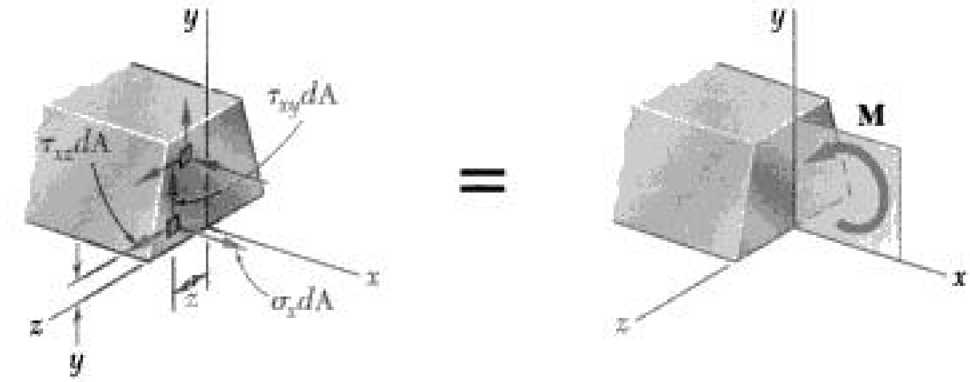


Fig. 4.8

We recall from statics that a couple **M** actually consists of two equal and opposite forces. The sum of the components of these forces in any direction is therefore equal to zero. Moreover, the moment of the couple is the same about *any* axis perpendicular to its plane, and is zero about any axis contained in that plane. Selecting arbitrarily the z axis as shown in Fig. 4.8, we express the equivalence of the elementary internal forces and of the couple **M** by writing that the sums of the components and of the moments of the elementary forces are equal to the corresponding components and moments of the couple **M:**

$$x \text{ components:} \qquad \int \sigma_x \, dA = 0 \qquad (4.1)$$

$$\text{moments about } y \text{ axis:} \qquad \int z\sigma_x \, dA = 0 \qquad (4.2)$$

$$\text{moments about } z \text{ axis:} \qquad \int (-y\sigma_x \, dA) = M \qquad (4.3)$$

Three additional equations could be obtained by setting equal to zero the sums of the y components, z components, and moments about the x axis, but these equations would involve only the components of the shearing stress and, as you will see in the next section, the components of the shearing stress are both equal to zero.

Two remarks should be made at this point: (1) The minus sign in Eq. (4.3) is due to the fact that a tensile stress ($\sigma_x > 0$) leads to a negative moment (clockwise) of the normal force $\sigma_x \, dA$ about the z axis. (2) Equation (4.2) could have been anticipated, since the application of couples in the plane of symmetry of member AB will result in a distribution of normal stresses that is symmetric about the y axis.

Once more, we note that the actual distribution of stresses in a given cross section cannot be determined from statics alone. It is *statically indeterminate* and may be obtained only by analyzing the *deformations* produced in the member.

value of the maximum stress in the critical cross section can then be expressed as

$$\sigma_m = K\frac{Mc}{I} \tag{4.29}$$

where K is the stress-concentration factor, and where c and I refer to the critical section, i.e., to the section of width d in both of the cases considered here. An examination of Figs. 4.31 and 4.32 clearly shows the importance of using fillets and grooves of radius r as large as practical.

Finally, we should point out that, as was the case for axial loading and torsion, the values of the factors K have been computed under the assumption of a linear relation between stress and strain. In many applications, plastic deformations will occur and result in values of the maximum stress lower than those indicated by Eq. (4.29).

EXAMPLE 4.04

Grooves 10 mm deep are to be cut in a steel bar which is 60 mm wide and 9 mm thick (Fig. 4.33). Determine the smallest allowable width of the grooves if the stress in the bar is not to exceed 150 MPa when the bending moment is equal to 180 N · m.

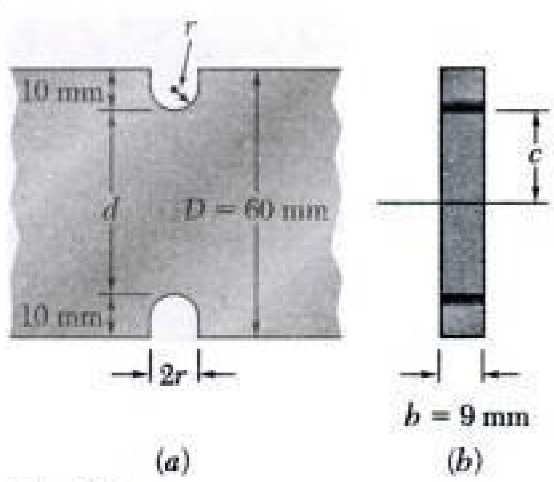


Fig. 4.33

We note from Fig. 4.33a that

$$d = 60 \text{ mm} - 2(10 \text{ mm}) = 40 \text{ mm}$$
$$c = \tfrac{1}{2}d = 20 \text{ mm} \qquad b = 9 \text{ mm}$$

The moment of inertia of the critical cross section about its neutral axis is

$$I = \tfrac{1}{12}bd^3 = \tfrac{1}{12}(9 \times 10^{-3} \text{ m})(40 \times 10^{-3} \text{ m})^3 = 48 \times 10^{-9} \text{ m}^4$$

The value of the stress Mc/I is thus

$$\frac{Mc}{I} = \frac{(180 \text{ N} \cdot \text{m})(20 \times 10^{-3} \text{ m})}{48 \times 10^{-9} \text{ m}^4} = 75 \text{ MPa}$$

Substituting this value for Mc/I into Eq. (4.29) and making $\sigma_m = 150$ MPa, we write

$$150 \text{ MPa} = K(75 \text{ MPa})$$
$$K = 2$$

We have, on the other hand,

$$\frac{D}{d} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.5$$

Using the curve of Fig. 4.32 corresponding to $D/d = 1.5$, we find that the value $K = 2$ corresponds to a value of r/d equal to 0.13. We have, therefore,

$$\frac{r}{d} = 0.13$$

$$r = 0.13d = 0.13(40 \text{ mm}) = 5.2 \text{ mm}$$

The smallest allowable width of the grooves is thus

$$2r = 2(5.2 \text{ mm}) = 10.4 \text{ mm}$$

4.45 and 4.46 A copper strip (E_c = 105 GPa) and an aluminum strip (E_a = 75 GPa) are bonded together to form the composite bar shown. Knowing that the bar is bent about a horizontal axis by a couple of moment 35 N · m, determine the maximum stress in (*a*) the aluminum strip, (*b*) the copper strip.

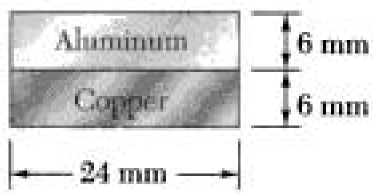


Fig. P4.45

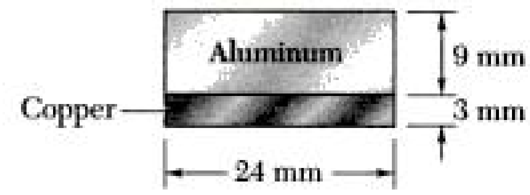


Fig. P4.46

4.47 and 4.48 A 150 × 250 mm timber beam has been strengthened by bolting to it the steel straps shown. The modulus of elasticity is 10 GPa for the wood and 200 GPa for the steel. Knowing that the beam is bent about a horizontal axis by a couple of moment 22 kN · m, determine the maximum stress in (*a*) the wood, (*b*) the steel.

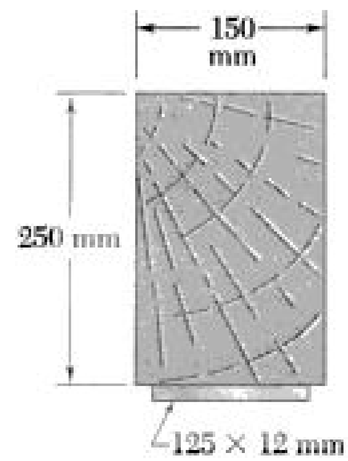


Fig. P4.47

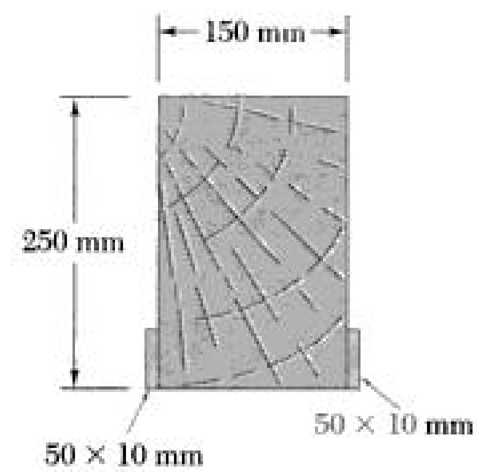


Fig. P4.48

4.49 and 4.50 For the composite bar indicated, determine the radius of curvature caused by the couple of moment 35 N · m.

4.49 Bar of Prob. 4.45.

4.50 Bar of Prob. 4.46.

4.51* and *4.52 For the composite beam indicated, determine the radius of curvature caused by the couple of moment 22 kN · m.

4.51 Beam of Prob. 4.47.

4.52 Beam of Prob. 4.48.

4.53 A concrete slab is reinforced by 16-mm-diameter steel rods placed on 180-mm centers as shown. The modulus of elasticity is 20 GPa for concrete and 200 GPa for steel. Using an allowable stress of 9 MPa for the concrete and of 120 MPa for the steel, determine the largest allowable positive bending moment in a portion of slab 1 m wide.

4.54 Solve Prob. 4.53, assuming that the spacing of the 16-mm-diameter rods is increased to 225 mm on centers.

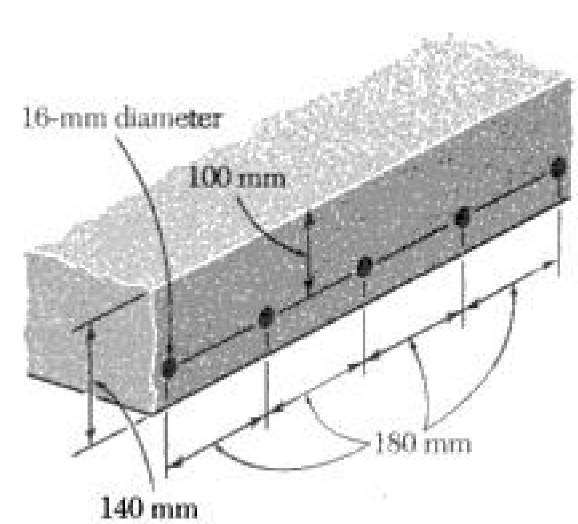


Fig. P4.53

4.73 The allowable stress used in the design of a steel bar is 80 MPa. Determine the largest couple **M** that can be applied to the bar (*a*) if the bar is designed with grooves having semicircular portions of radius $r = 15$ mm, as shown in Fig. P4.73*a*, (*b*) if the bar is redesigned by removing the material above the grooves as shown in Fig. P4.73*b*.

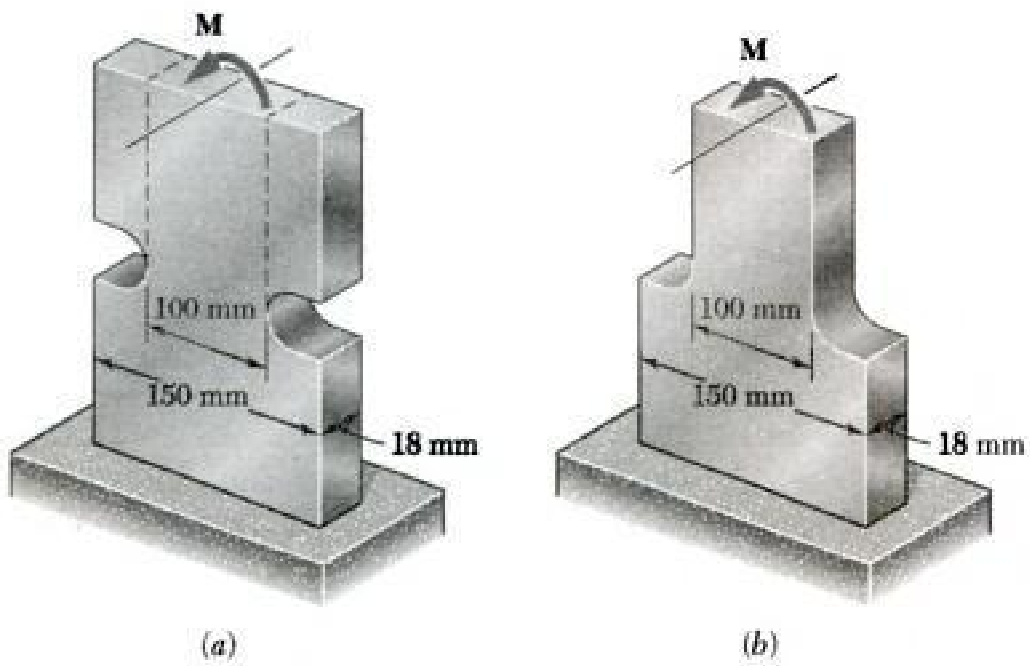


Fig. *P4.73* and *P4.74*

4.74 A couple of moment $M = 2$ kN · m is to be applied to the end of a steel bar. Determine the maximum stress in the bar (*a*) if the bar is designed with grooves having semicircular portions of radius $r = 10$ mm, as shown in Fig. P4.74*a*, (*b*) if the bar is redesigned by removing the material above the grooves as shown in Fig. P4.74*b*.

*4.8. PLASTIC DEFORMATIONS

When we derived the fundamental relation $\sigma_x = -My/I$ in Sec. 4.4, we assumed that Hooke's law applied throughout the member. If the yield strength is exceeded in some portion of the member, or if the material involved is a brittle material with a nonlinear stress-strain diagram, this relation ceases to be valid. The purpose of this section is to develop a more general method for the determination of the distribution of stresses in a member in pure bending, which can be used when Hooke's law does not apply.

We first recall that no specific stress-strain relationship was assumed in Sec. 4.3, when we proved that the normal strain ϵ_x varies linearly with the distance y from the neutral surface. Thus, we can still use this property in our present analysis and write

$$\epsilon_x = -\frac{y}{c}\epsilon_m \qquad (4.10)$$

where y represents the distance of the point considered from the neutral surface, and c the maximum value of y.

4.138 A short length of a rolled-steel column supports a rigid plate on which two loads **P** and **Q** are applied as shown. The strains at two points *A* and *B* on the center lines of the outer faces of the flanges have been measured and found to be

$$\epsilon_A = -400 \times 10^{-6} \text{ mm/mm} \qquad \epsilon_B = -300 \times 10^{-6} \text{ mm/mm}$$

Knowing that $E = 200$ GPa, determine the magnitude of each load.

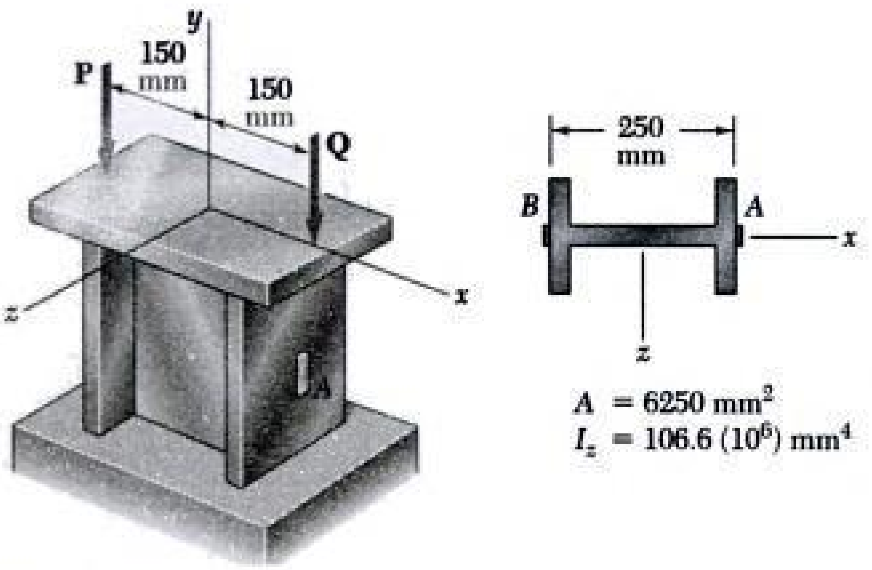


Fig. P4.138

4.139 Solve Prob. 4.138, assuming that the measured strains are

$$\epsilon_A = -350 \times 10^{-6} \text{ mm/mm} \qquad \epsilon_B = -50 \times 10^{-6} \text{ mm/mm}$$

4.140 An eccentric axial force **P** is applied as shown to a steel bar of 25 × 90-mm cross section. The strains at *A* and *B* have been measured and found to be

$$\epsilon_A = +350\ \mu \qquad \epsilon_B = -70\ \mu$$

Knowing that $E = 200$ GPa, determine (*a*) the distance *d*, (*b*) the magnitude of the force **P**.

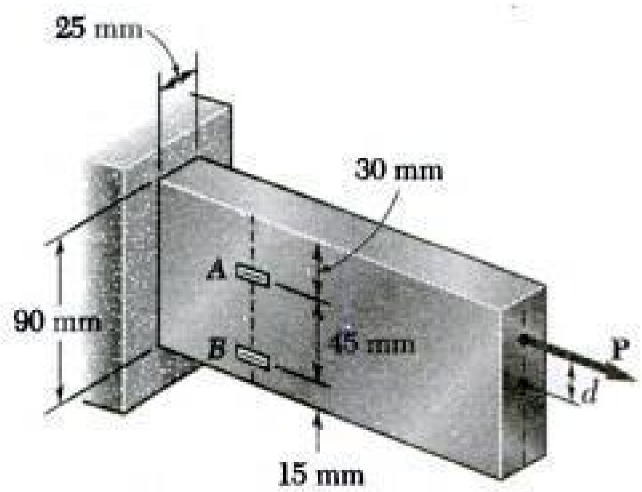


Fig. P4.140

4.141 Solve Prob. 4.140, assuming that the measured strains are

$$\epsilon_A = +600\ \mu \qquad \epsilon_B = +420\ \mu$$

4.142 The shape shown was formed by bending a thin steel plate. Assuming that the thickness *t* is small compared to the length *a* of a side of the shape, determine the stress (*a*) at *A*, (*b*) at *B*, (*c*) at *C*.

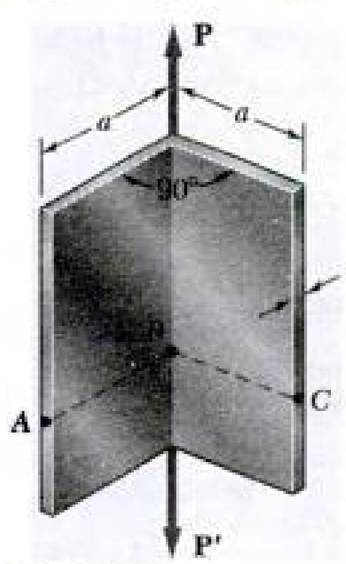


Fig. *P4.142*

Consider first a member with a vertical plane of symmetry, which is subjected to bending couples **M** and **M′** acting in a plane forming an angle u with the vertical plane (Fig. 4.60). The couple vector **M** representing the forces acting on a given cross section will form the same

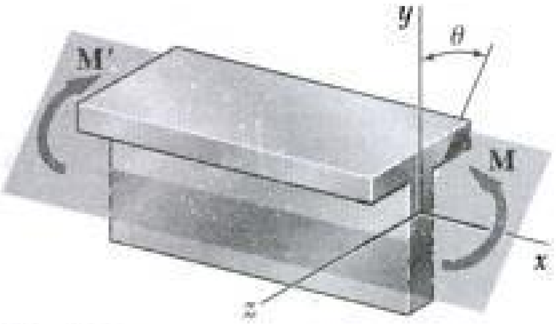


Fig. 4.60

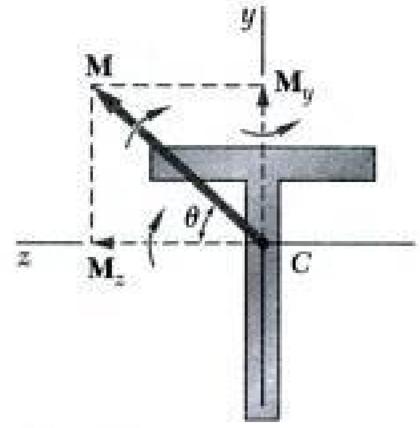


Fig. 4.61

angle u with the horizontal z axis (Fig. 4.61). Resolving the vector **M** into component vectors $\mathbf{M}_z$ and $\mathbf{M}_y$ along the z and y axes, respectively, we write

$$M_z = M\cos\theta \qquad M_y = M\sin\theta \tag{4.52}$$

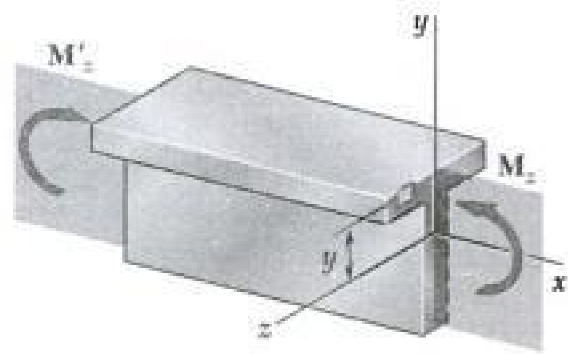


Fig. 4.62

Since the y and z axes are the principal centroidal axes of the cross section, we can use Eq. (4.16) to determine the stresses resulting from the application of either of the couples represented by $\mathbf{M}_z$ and $\mathbf{M}_y$. The couple $\mathbf{M}_z$ acts in a vertical plane and bends the member in that plane (Fig. 4.62). The resulting stresses are

$$\sigma_x = -\frac{M_z y}{I_z} \tag{4.53}$$

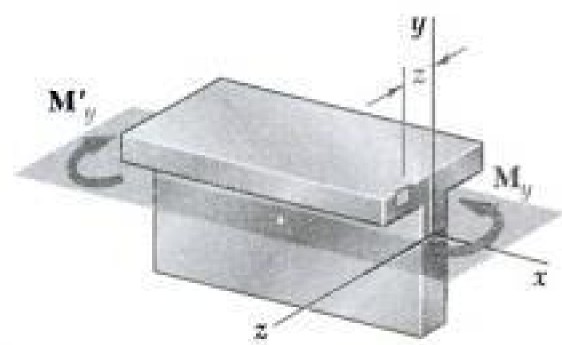


Fig. 4.63

where I_z is the moment of inertia of the section about the principal centroidal z axis. The negative sign is due to the fact that we have compression above the xz plane ($y > 0$) and tension below ($y < 0$). On the other hand, the couple $\mathbf{M}_y$ acts in a horizontal plane and bends the member in that plane (Fig. 4.63). The resulting stresses are found to be

$$\sigma_x = +\frac{M_y z}{I_y} \tag{4.54}$$

where I_y is the moment of inertia of the section about the principal centroidal y axis, and where the positive sign is due to the fact that we have tension to the left of the vertical xy plane ($z > 0$) and compression to its right ($z < 0$). The distribution of the stresses caused by the original couple **M** is obtained by superposing the stress distributions defined by Eqs. (4.53) and (4.54), respectively. We have

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{4.55}$$

EXAMPLE 4.09

A vertical 4.80-kN load is applied as shown on a wooden post of rectangular cross section, 80 by 120 mm (Fig. 4.71). (*a*) Determine the stress at points *A*, *B*, *C*, and *D*. (*b*) Locate the neutral axis of the cross section.

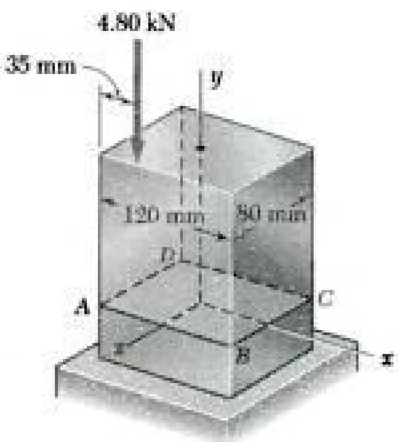


Fig. 4.71

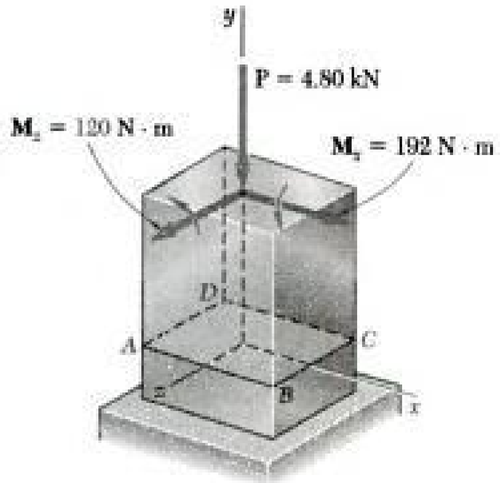


Fig. 4.72

(a) Stresses. The given eccentric load is replaced by an equivalent system consisting of a centric load **P** and two couples $\mathbf{M}_x$ and $\mathbf{M}_z$ represented by vectors directed along the principal centroidal axes of the section (Fig. 4.72). We have

$$M_x = (4.80 \text{ kN})(40 \text{ mm}) = 192 \text{ N} \cdot \text{m}$$
$$M_z = (4.80 \text{ kN})(60 \text{ mm} - 35 \text{ mm}) = 120 \text{ N} \cdot \text{m}$$

We also compute the area and the centroidal moments of inertia of the cross section:

$$A = (0.080 \text{ m})(0.120 \text{ m}) = 9.60 \times 10^{-3} \text{ m}^2$$
$$I_x = \tfrac{1}{12}(0.120 \text{ m})(0.080 \text{ m})^3 = 5.12 \times 10^{-6} \text{ m}^4$$
$$I_z = \tfrac{1}{12}(0.080 \text{ m})(0.120 \text{ m})^3 = 11.52 \times 10^{-6} \text{ m}^4$$

The stress σ_0 due to the centric load **P** is negative and uniform across the section. We have

$$\sigma_0 = \frac{P}{A} = \frac{-4.80 \text{ kN}}{9.60 \times 10^{-3} \text{m}^2} = -0.5 \text{ MPa}$$

The stresses due to the bending couples $\mathbf{M}_x$ and $\mathbf{M}_z$ are linearly distributed across the section, with maximum values equal, respectively, to

$$\sigma_1 = \frac{M_x z_{\max}}{I_x} = \frac{(192 \text{ N} \cdot \text{m})(40 \text{ mm})}{5.12 \times 10^{-6} \text{ m}^4} = 1.5 \text{ MPa}$$

$$\sigma_2 = \frac{M_z x_{\max}}{I_z} = \frac{(120 \text{ N} \cdot \text{m})(60 \text{ mm})}{11.52 \times 10^{-6} \text{ m}^4} = 0.625 \text{ MPa}$$

The stresses at the corners of the section are

$$\sigma_y = \sigma_0 \pm \sigma_1 \pm \sigma_2$$

where the signs must be determined from Fig. 4.72. Noting that the stresses due to $\mathbf{M}_x$ are positive at *C* and *D*, and negative at *A* and *B*, and that the stresses due to $\mathbf{M}_z$ are positive at *B* and *C*, and negative at *A* and *D*, we obtain

$$\sigma_A = -0.5 - 1.5 - 0.625 = -2.625 \text{ MPa}$$
$$\sigma_B = -0.5 - 1.5 + 0.625 = -1.375 \text{ MPa}$$
$$\sigma_C = -0.5 + 1.5 + 0.625 = +1.625 \text{ MPa}$$
$$\sigma_D = -0.5 + 1.5 - 0.625 = +0.375 \text{ MPa}$$

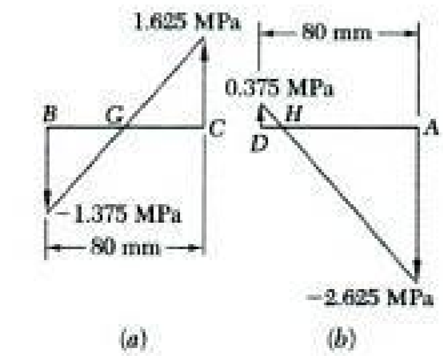


Fig. 4.73

(b) Neutral Axis. We note that the stress will be zero at a point *G* between *B* and *C*, and at a point *H* between *D* and *A* (Fig. 4.73). Since the stress distribution is linear, we write

$$\frac{BG}{80 \text{ mm}} = \frac{1.375}{1.625 + 1.375} \qquad BG = 36.7 \text{ mm}$$

$$\frac{HA}{80 \text{ mm}} = \frac{2.625}{2.625 + 0.375} \qquad HA = 70 \text{ mm}$$

The neutral axis can be drawn through points *G* and *H* (Fig. 4.74).

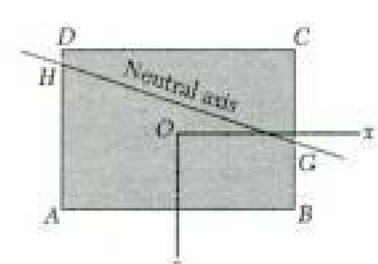


Fig. 4.74

The distribution of the stresses across the section is shown in Fig. 4.75.

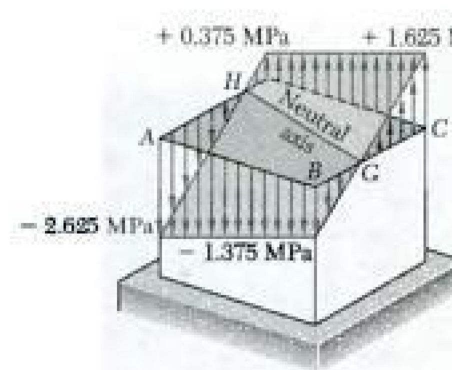


Fig. 4.75

4.152 and 4.153 The couple **M** acts in a vertical plane and is applied to a beam oriented as shown. Determine (*a*) the angle that the neutral axis forms with the horizontal plane, (*b*) the maximum tensile stress in the beam.

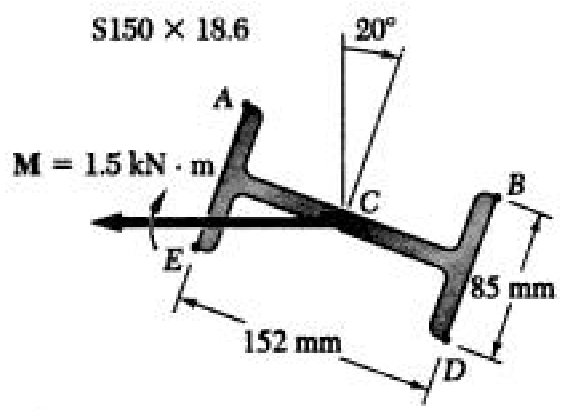


Fig. P4.152

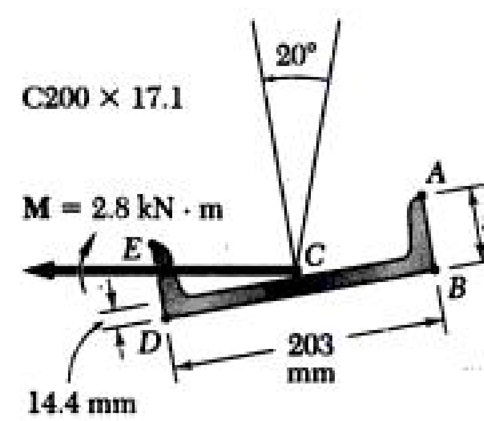
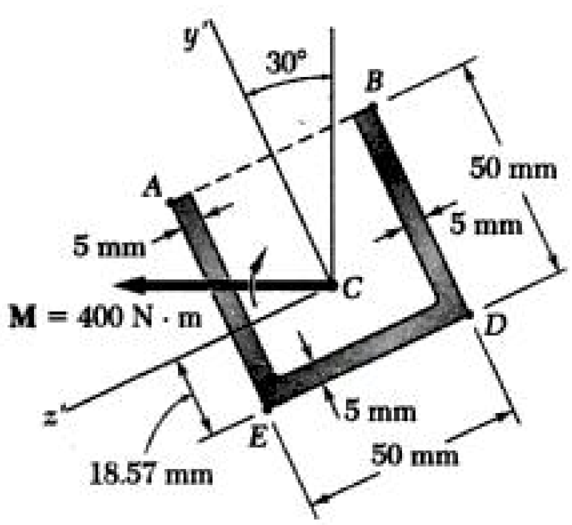


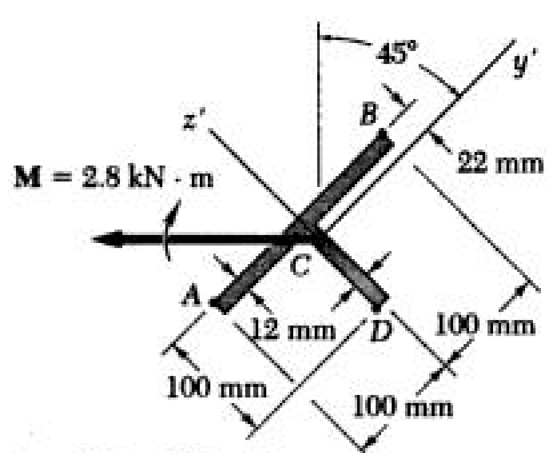
Fig. P4.153

4.154 through 4.156 The couple **M** acts in a vertical plane and is applied to a beam oriented as shown. Determine (*a*) the angle that the neutral axis forms with the horizontal plane, (*b*) the maximum tensile stress in the beam.



$I_{y'} = 281 \times 10^3$ mm^4
$I_{z'} = 176.9 \times 10^3$ mm^4

Fig. *P4.154*



$I_{y'} = 2.8 \times 10^6$ mm^4
$I_{z'} = 8.9 \times 10^6$ mm^4

Fig. P4.155

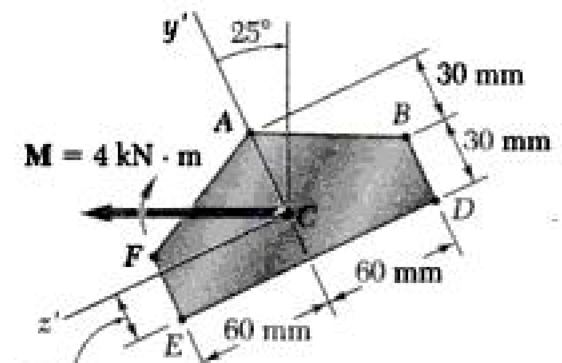
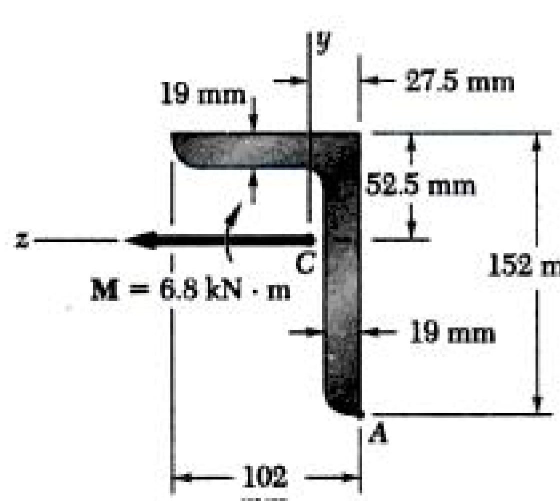
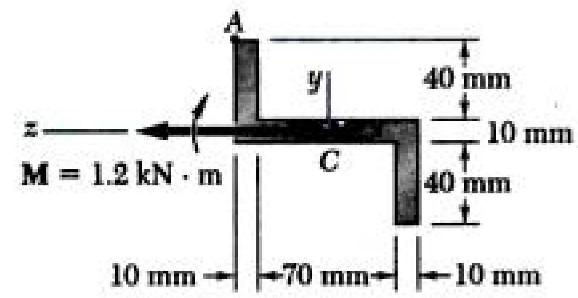


Fig. *P4.156*

***4.157 and 4.158** The couple **M** acts in a vertical plane and is applied to a beam of the cross section shown. Determine the stress at point *A*.



$I_y = 3.65\,(10^6)$ mm^4
$I_z = 10.10\,(10^6)$ mm^4
Fig. P4.157 $I_{yz} = 3.45\,(10^6)$ mm^4



$I_y = 1.894 \times 10^6$ mm^4
$I_z = 0.614 \times 10^6$ mm^4
$I_{yz} = +0.800 \times 10^6$ mm^4

Fig. P4.158

4.178 (*a*) Show that, if a vertical force **P** is applied at point *A* of the section shown, the equation of the neutral axis *BD* is

$$\left(\frac{x_A}{k_z^2}\right)x + \left(\frac{z_A}{k_x^2}\right)z = -1$$

where k_z and k_x denote the radius of gyration of the cross section with respect to the z axis and the x axis, respectively. (*b*) Further show that, if a vertical force **Q** is applied at any point located on line *BD*, the stress at point *A* will be zero.

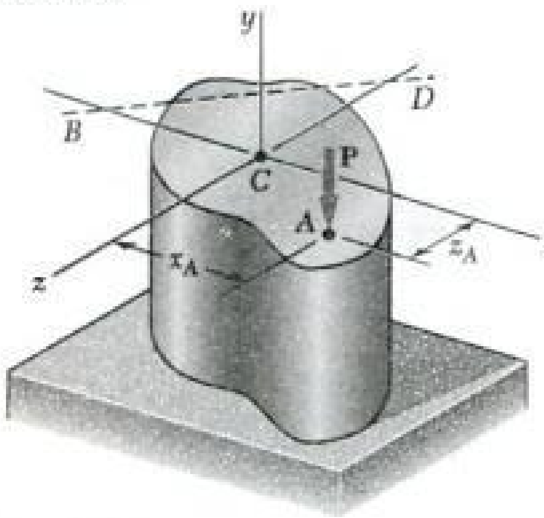


Fig. P4.178

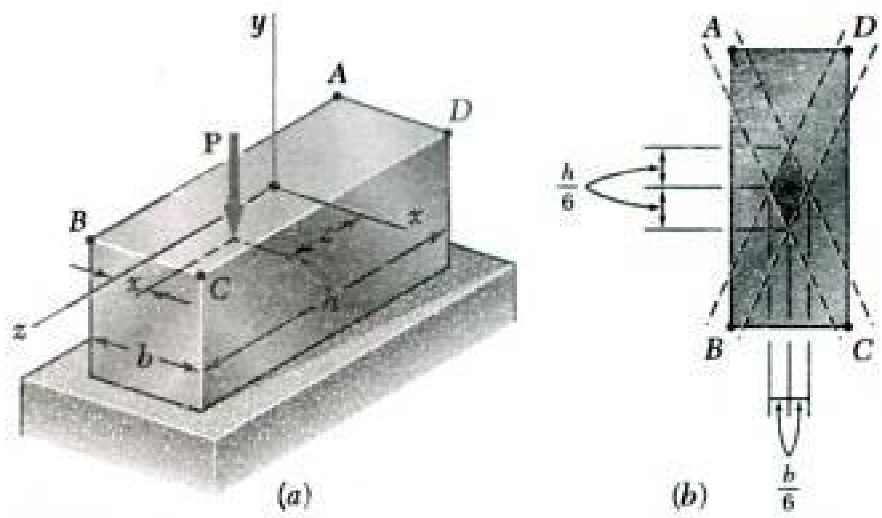


Fig. P4.179

4.179 (*a*) Show that the stress at corner *A* of the prismatic member shown in Fig. P4.179*a* will be zero if the vertical force **P** is applied at a point located on the line

$$\frac{x}{b/6} + \frac{z}{h/6} = 1$$

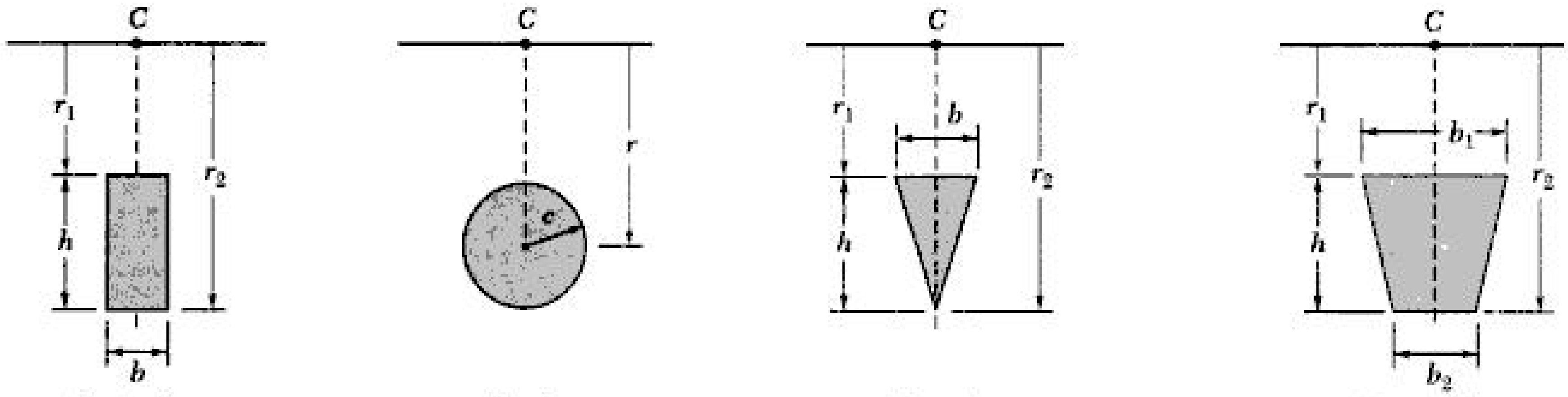
(*b*) Further show that, if no tensile stress is to occur in the member, the force **P** must be applied at a point located within the area bounded by the line found in part *a* and the three similar lines corresponding to the condition of zero stress at *B*, *C*, and *D*, respectively. This area, shown in Fig. P4.179*b*, is known as the *kern* of the cross section.

*4.15. BENDING OF CURVED MEMBERS

Our analysis of stresses due to bending has been restricted so far to straight members. In this section we will consider the stresses caused by the application of equal and opposite couples to members that are initially curved. Our discussion will be limited to curved members of uniform cross section possessing a plane of symmetry in which the bending couples are applied, and it will be assumed that all stresses remain below the proportional limit.

If the initial curvature of the member is small, i.e., if its radius of curvature is large compared to the depth of its cross section, a good approximation can be obtained for the distribution of stresses by assuming the member to be straight and using the formulas derived in Secs. 4.3 and 4.4† However, when the radius of curvature and the dimensions of the cross section of the member are of the same order of magnitude, we must use a different method of analysis, which was first introduced by the German engineer E. Winkler (1835–1888).

†See Prob. 4.185.



Rectangle	Circle	Triangle	Trapizoid
$R = \dfrac{h}{\ln \dfrac{r_2}{r_1}}$	$R = \frac{1}{2}(\bar{r} + \sqrt{\bar{r}^2 - c^2})$	$R = \dfrac{\frac{1}{2}h}{\dfrac{r_2}{h}\ln\dfrac{r_2}{r_1} - 1}$	$R = \dfrac{\frac{1}{2}h^2(b_1 + b_2)}{(b_1 r_2 - b_2 r_1)\ln\dfrac{r_2}{r_1} - h(b_1 - b_2)}$

Fig. 4.79 Radius of neutral surface for various cross-sectional shapes.

Substituting now for σ_x from (4.65) into Eq. (4.3), we write

$$\int \frac{E\,\Delta\theta}{\theta}\,\frac{R - r}{r}\,y\,dA = M$$

or, since $y = R - r$,

$$\frac{E\,\Delta\theta}{\theta}\int \frac{(R - r)^2}{r}\,dA = M$$

Expanding the square in the integrand, we obtain after reductions

$$\frac{E\,\Delta\theta}{\theta}\left[R^2 \int \frac{dA}{r} - 2RA + \int r\,dA\right] = M$$

Recalling Eqs. (4.66) and (4.67), we note that the first term in the brackets is equal to RA, while the last term is equal to $\bar{r}A$. We have, therefore,

$$\frac{E\,\Delta\theta}{\theta}(RA - 2RA + \bar{r}A) = M$$

and, solving for $E\,\Delta\theta/\theta$,

$$\frac{E\,\Delta\theta}{\theta} = \frac{M}{A(\bar{r} - R)} \tag{4.68}$$

Referring to Fig. 4.76, we note that $\Delta\theta > 0$ for $M > 0$. It follows that $\bar{r} - R > 0$, or $R < \bar{r}$, regardless of the shape of the section. Thus, the neutral axis of a transverse section is always located between the centroid of the section and the center of curvature of the member (Fig. 4.78). Setting $\bar{r} - R = e$, we write Eq. (4.68) in the form

$$\frac{E\,\Delta\theta}{\theta} = \frac{M}{Ae} \tag{4.69}$$

PROBLEMS

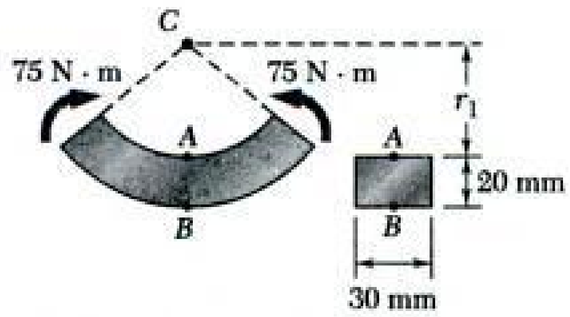


Fig. P4.180 and P4.181

4.180 For the curved bar and loading shown, determine the stress at point A when (*a*) $r_1 = 30$ mm, (*b*) $r_1 = 50$ mm.

4.181 For the curved bar and loading shown, determine the stress at points A and B when $r_1 = 40$ mm.

4.182 For the curved bar and loading shown, determine the stress at point A when (*a*) $h = 60$ mm, (*b*) $h = 75$ mm.

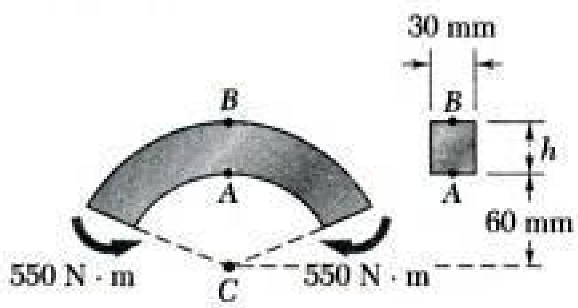


Fig. P4.182 and P4.183

4.183 For the curved bar and loading shown, determine the stress at points A and B when $h = 68$ mm.

4.184 The curved bar shown has a cross section of 40×60 mm and an inner radius $r_1 = 15$ mm. For the loading shown determine the largest tensile and compressive stresses.

4.185 For the curved bar and loading shown, determine the percent error introduced in the computation of the maximum stress by assuming that the bar is straight. Consider the case when (*a*) $r_1 = 20$ mm, (*b*) $r_1 = 200$ mm, (*c*) $r_1 = 2$ m.

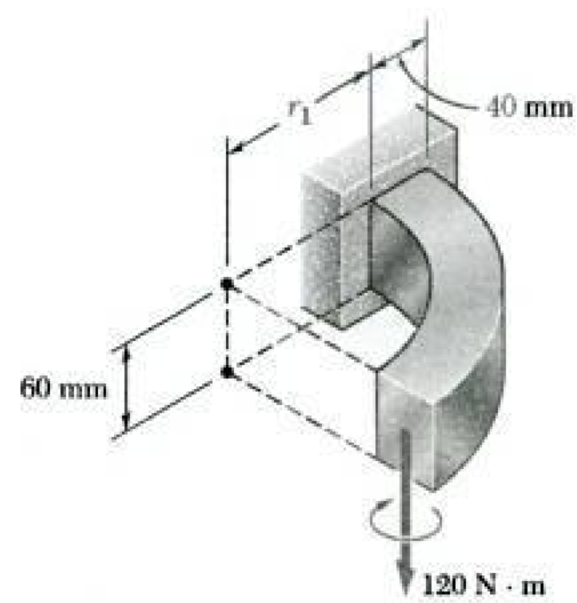


Fig. *P4.184* and *P4.185*

4.186 Steel links having the cross section shown are available with different central angles β. Knowing that the allowable stress is 100 MPa, determine the largest force **P** that can be applied to a link for which $\beta = 90°$.

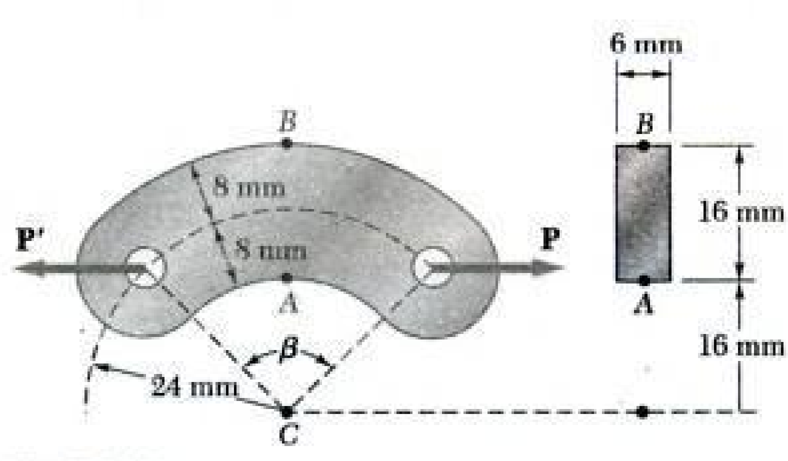


Fig. P4.186

4.187 Solve Prob. 4.186, assuming that $\beta = 60°$.

4.206 Show that if the cross section of a curved beam consists of two or more rectangles, the radius R of the neutral surface can be expressed as

$$R = \frac{A}{\ln\left[\left(\frac{r_2}{r_1}\right)^{b_1}\left(\frac{r_3}{r_2}\right)^{b_2}\left(\frac{r_4}{r_3}\right)^{b_3}\right]}$$

where A is the total area of the cross section.

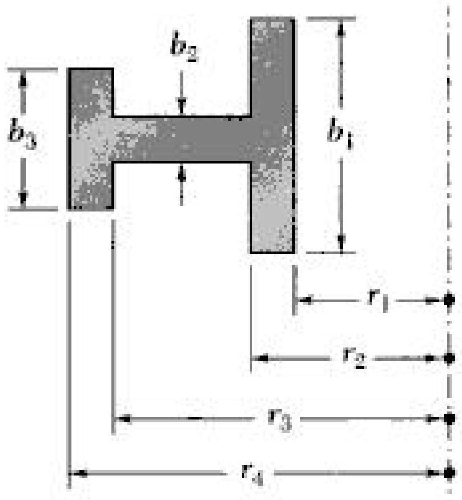


Fig. P4.206

4.207 through 4.209 Using Eq. (4.66), derive the expression for R given in Fig. 4.79 for

***4.207** A circular cross section
4.208 A trapezoidal section
4.209 A triangular cross section

***4.210** For a curved bar of rectangular cross section subjected to a bending couple **M**, show that the radial stress at the neutral surface is

$$\sigma_r = \frac{M}{Ae}\left(1 - \frac{r_1}{R} - \ln\frac{R}{r_1}\right)$$

and compute the value of s, for the curved bar of Examples 4.10 and 4.11. (*Hint*: consider the free-body diagram of the portion of the beam located above the neutral surface.)

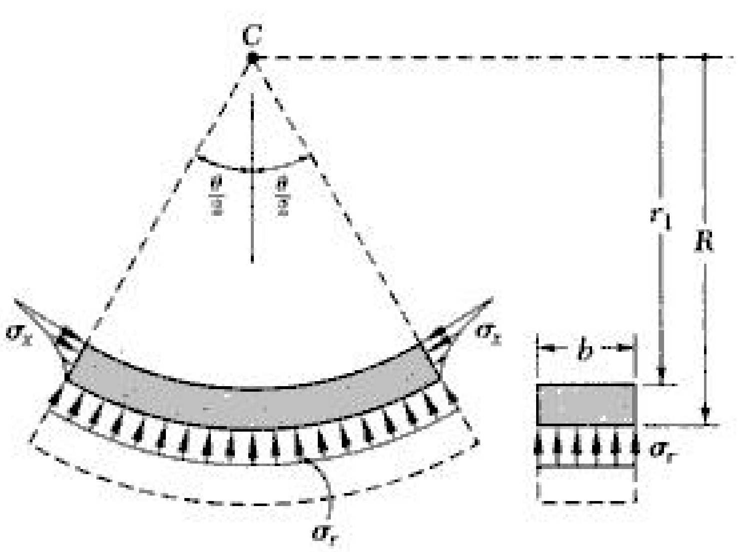


Fig. P4.210

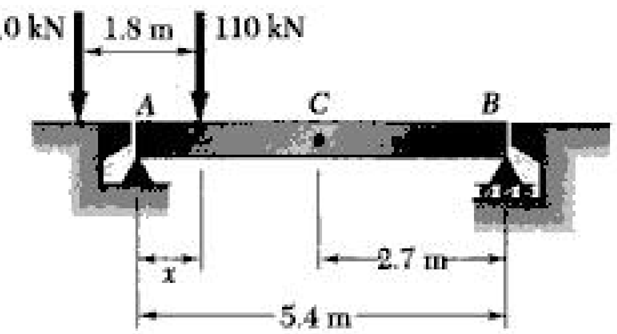


Fig. P5.C4

5.C4 Two 110-kN loads are maintained 1.8 m apart as they are moved slowly across the 5.4 m beam AB. Write a computer program and use it to calculate the bending moment under each load and at the midpoint C of the beam for values of x from 0 to 7.2 m at intervals $\Delta x = 0.45$ m.

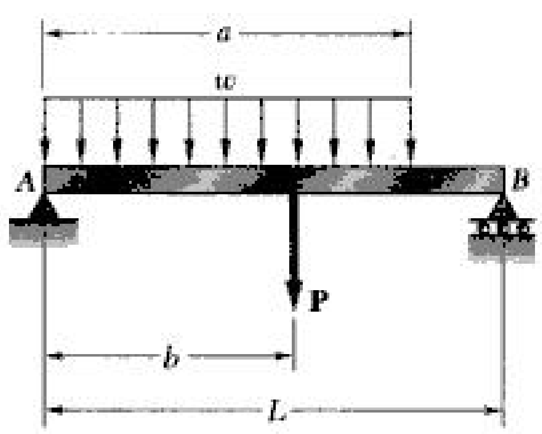


Fig. P5.C5

5.C5 Write a computer program that can be used to plot the shear and bending-moment diagrams for the beam and loading shown. Apply this program with a plotting interval $\Delta L = 0.06$ m to the beam and loading of (*a*) Prob. 5.83, (*b*) Prob. 5.125.

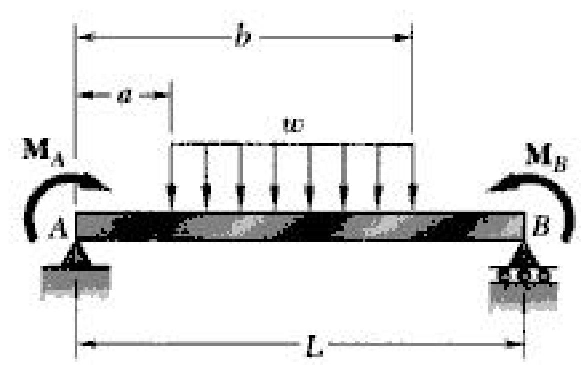


Fig. P5.C6

5.C6 Write a computer program that can be used to plot the shear and bending-moment diagrams for the beam and loading shown. Apply this program with a plotting interval $\Delta L = 0.025$ m to the beam and loading of Prob. 5.124.

6.2. SHEAR ON THE HORIZONTAL FACE OF A BEAM ELEMENT

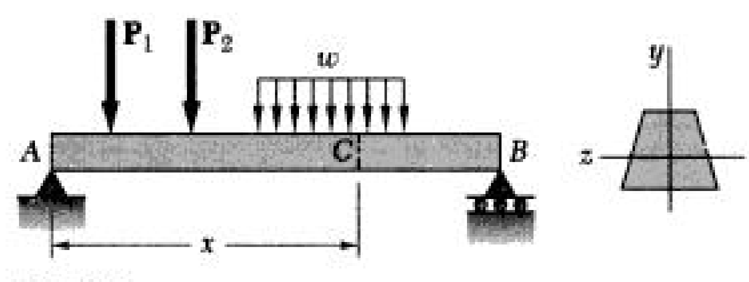


Fig. 6.5

Consider a prismatic beam AB with a vertical plane of symmetry that supports various concentrated and distributed loads (Fig. 6.5). At a distance x from end A we detach from the beam an element $CDD'C'$ of length Δx extending across the width of the beam from the upper surface of the beam to a horizontal plane located at a distance y_1 from the neutral axis (Fig. 6.6). The forces exerted on this element consist of

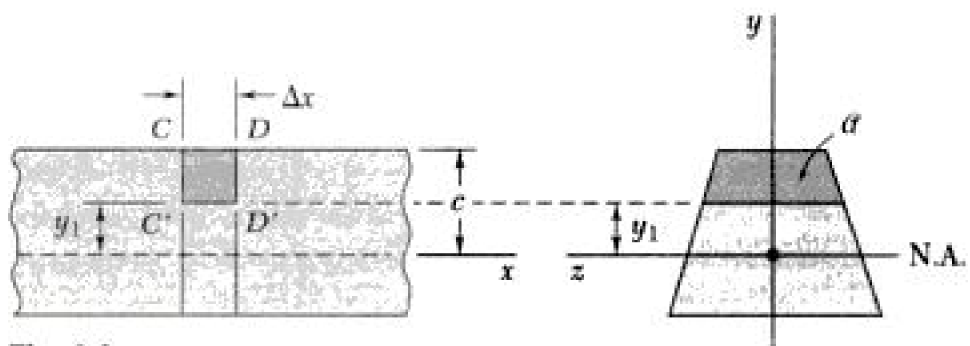


Fig. 6.6

vertical shearing forces $\mathbf{V}'_C$ and $\mathbf{V}'_D$, a horizontal shearing force $\Delta\mathbf{H}$ exerted on the lower face of the element, elementary horizontal normal forces $\sigma_C\, dA$ and $\sigma_D\, dA$, and possibly a load $w\,\Delta x$ (Fig. 6.7). We write the equilibrium equation

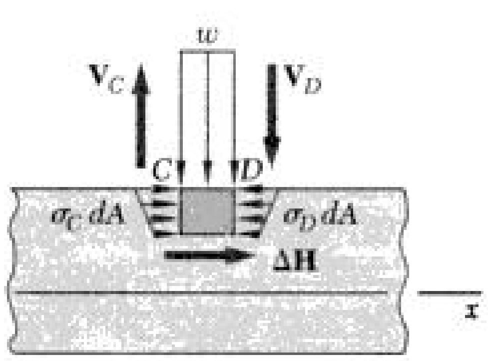


Fig. 6.7

$$\xrightarrow{+}\sum F_x = 0: \qquad \Delta H + \int_{\mathcal{a}} (\sigma_D - \sigma_C)\, dA = 0$$

where the integral extends over the shaded area $\mathcal{a}$ of the section located above the line $y = y_1$. Solving this equation for ΔH and using Eq. (5.2) of Sec. 5.1, $\sigma = My/I$, to express the normal stresses in terms of the bending moments at C and D, we have

$$\Delta H = \frac{M_D - M_C}{I} \int_{\mathcal{a}} y\, dA \tag{6.3}$$

moment with respect to the neutral axis of the shaded area A (Fig. 6.15).

Observing that the distance from the neutral axis to the centroid C' of A is $\bar{y} = \frac{1}{2}(c + y)$, and recalling that $Q = A\bar{y}$, we write

$$Q = A\bar{y} = b(c - y)\tfrac{1}{2}(c + y) = \tfrac{1}{2}b(c^2 - y^2) \qquad (6.8)$$

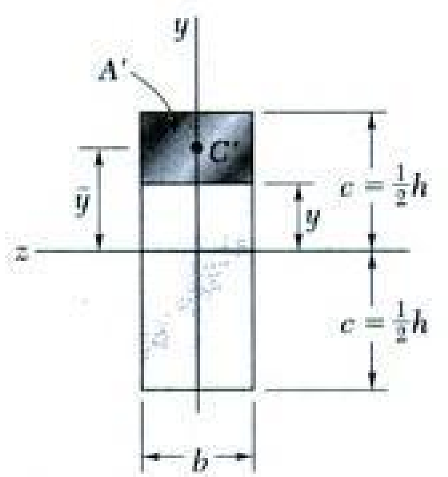


Fig. 6.15

Recalling, on the other hand, that $I = bh^3/12 = \frac{2}{3}bc^3$, we have

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{3}{4}\frac{c^2 - y^2}{bc^3}V$$

or, noting that the cross-sectional area of the beam is $A = 2bc$,

$$\tau_{xy} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right) \qquad (6.9)$$

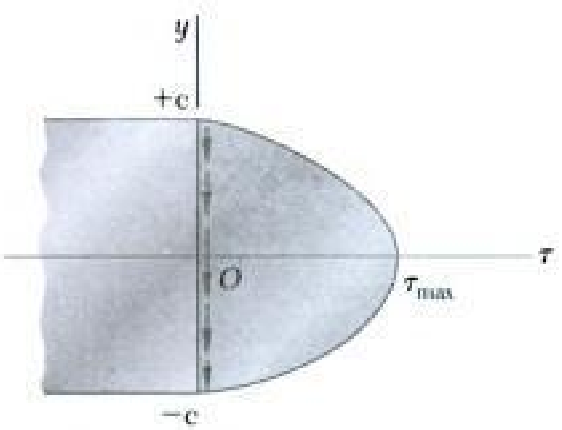


Fig. 6.16

Equation (6.9) shows that the distribution of shearing stresses in a transverse section of a rectangular beam is *parabolic* (Fig. 6.16). As we have already observed in the preceding section, the shearing stresses are zero at the top and bottom of the cross section ($y = \pm c$). Making $y = 0$ in Eq. (6.9), we obtain the value of the maximum shearing stress in a given section of a *narrow rectangular beam:*

$$\tau_{\max} = \frac{3}{2}\frac{V}{A} \qquad (6.10)$$

The relation obtained shows that the maximum value of the shearing stress in a beam of rectangular cross section is 50% larger than the value V/A which would be obtained by wrongly assuming a uniform stress distribution across the entire cross section.

In the case of an *American standard beam* (S-beam) or a *wide-flange beam* (W-beam), Eq. (6.6) can be used to determine the average value of the shearing stress τ_{xy} over a section aa' or bb' of the transverse cross section of the beam (Figs. 6.17*a* and *b*). We write

$$\tau_{\text{ave}} = \frac{VQ}{It} \qquad (6.6)$$

where V is the vertical shear, t the width of the section at the elevation considered, Q the first moment of the shaded area with respect to the neutral axis cc', and I the moment of inertia of the entire cross-sectional area about cc'. Plotting τ_{ave} against the vertical distance y, we obtain the curve shown in Fig. 6.17*c*. We note the discontinuities existing in this curve, which reflect the difference between the values of t corresponding respectively to the flanges $ABGD$ and $A'B'G'D'$ and to the web $EFF'E'$.

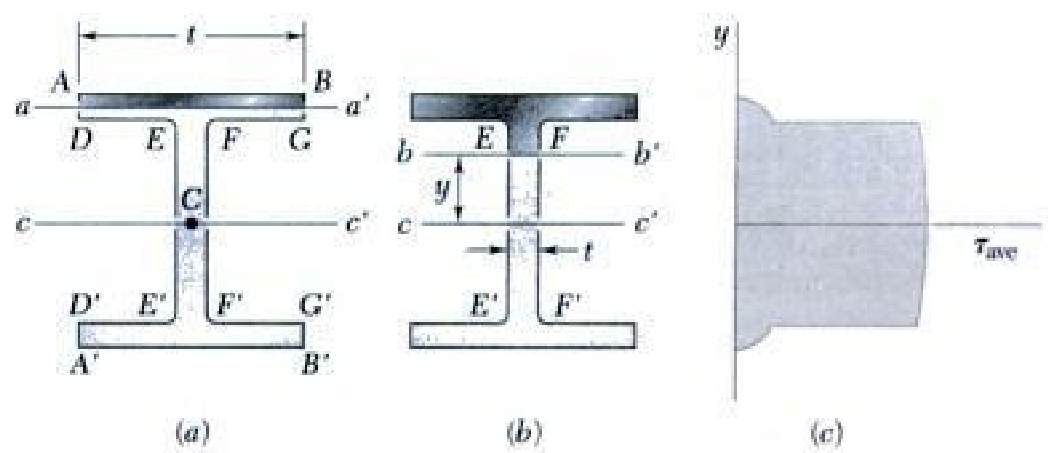
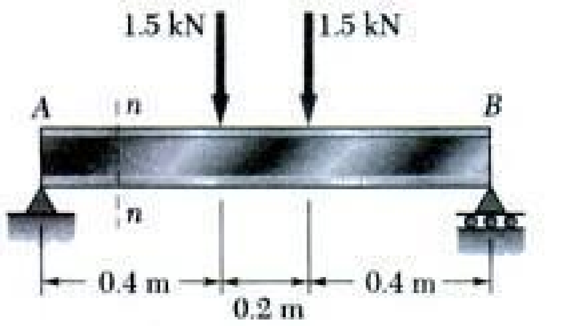
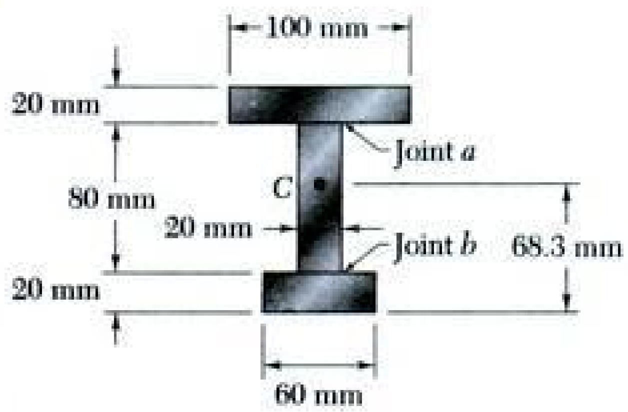


Fig. 6.17



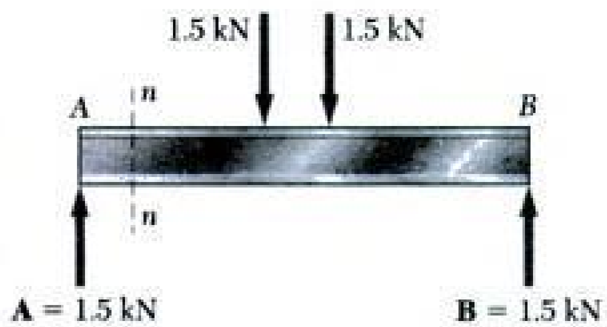
SAMPLE PROBLEM 6.1

Beam AB is made of three planks glued together and is subjected, in its plane of symmetry, to the loading shown. Knowing that the width of each glued joint is 20 mm, determine the average shearing stress in each joint at section n-n of the beam. The location of the centroid of the section is given in the sketch and the centroidal moment of inertia is known to be $I = 8.63 \times 10^{-6}\ \text{m}^4$.



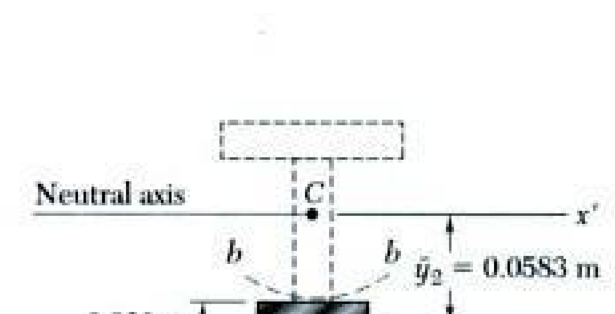
SOLUTION

Vertical Shear at Section *n-n*. Since the beam and loading are both symmetric with respect to the center of the beam, we have $\mathbf{A} = \mathbf{B} = 1.5\ \text{kN}\uparrow$.



Considering the portion of the beam to the left of section n-n as a free body, we write

$$+\uparrow \sum F_y = 0: \qquad 1.5\ \text{kN} - V = 0 \qquad V = 1.5\ \text{kN}$$



Shearing Stress in Joint *a*. We pass the section a-a through the glued joint and separate the cross-sectional area into two parts. We choose to determine Q by computing the first moment with respect to the neutral axis of the area above section a-a.

$$Q = A\bar{y}_1 = [(0.100\ \text{m})(0.020\ \text{m})](0.0417\ \text{m}) = 83.4 \times 10^{-6}\ \text{m}^3$$

Recalling that the width of the glued joint is $t = 0.020$ m, we use Eq. (6.7) to determine the average shearing stress in the joint.

$$\tau_{ave} = \frac{VQ}{It} = \frac{(1500\ \text{N})(83.4 \times 10^{-6}\ \text{m}^3)}{(8.63 \times 10^{-6}\ \text{m}^4)(0.020\ \text{m})} \qquad \tau_{ave} = 725\ \text{kPa} \blacktriangleleft$$

Shearing Stress in Joint *b*. We now pass section b-b and compute Q by using the area below the section.

$$Q = A\bar{y}_2 = [(0.060\ \text{m})(0.020\ \text{m})](0.0583\ \text{m}) = 70.0 \times 10^{-6}\ \text{m}^3$$

$$\tau_{ave} = \frac{VQ}{It} = \frac{(1500\ \text{N})(70.0 \times 10^{-6}\ \text{m}^3)}{(8.63 \times 10^{-6}\ \text{m}^4)(0.020\ \text{m})} \qquad \tau_{ave} = 608\ \text{kPa} \blacktriangleleft$$

6.13 Two steel plates of 10 × 220-mm rectangular cross section are welded to the W250 × 67 beam as shown. Determine the largest allowable vertical shear if the shearing stress in the beam is not to exceed 100 MPa.

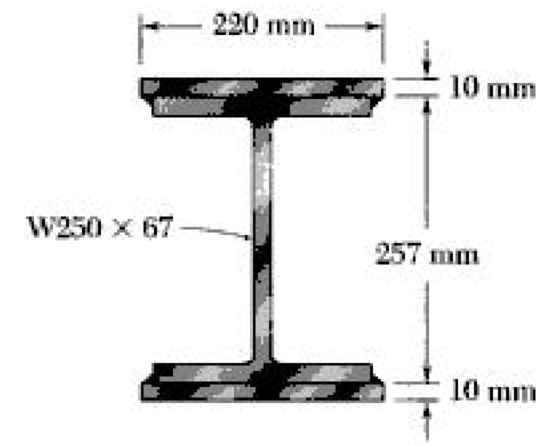


Fig. P6.13

6.14 Solve Prob. 6.13, assuming that the two steel plates are (*a*) replaced by steel plates of 10 × 220-mm rectangular cross section, (*b*) removed.

6.15 Knowing that the allowable shearing stress for the timber used is 825 kPa, check whether the design obtained for the beam indicated is acceptable and, if not, redesign the cross section of the beam. Consider the beam of (*a*) Prob. 5.75, (*b*) Prob. 5.76.

6.16 Knowing that the allowable shearing stress for the timber used is 1.5 MPa, check whether the design obtained for the beam indicated is acceptable and, if not, redesign the cross section of the beam. Consider the beam of (*a*) Prob. 5.77, (*b*) Prob. 5.78.

6.17 Determine the average shearing stress in the web of the beam indicated and check whether the design obtained earlier for that beam is acceptable, knowing that the allowable shearing stress for the steel used is 100 MPa. Consider the beam of (*a*) Prob. 5.81, (*b*) Prob. 5.82.

6.18 Determine the average shearing stress in the web of the beam indicated and check whether the design obtained earlier for that beam is acceptable, knowing that the allowable shearing stress for the steel used is 100 MPa. Consider the beam of (*a*) Prob. 5.83, (*b*) Prob. 5.84.

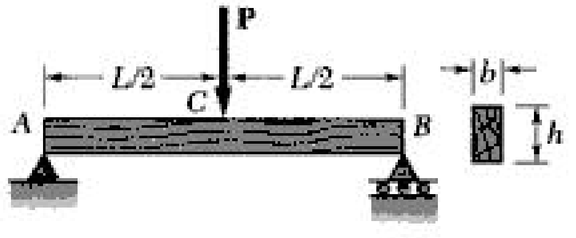


Fig. P6.19

6.19 A simply supported timber beam *AB* of rectangular cross section carries a single concentrated load **P** at its midpoint *C*. (*a*) Show that the ratio τ_m/σ_m of the maximum values of the shearing and normal stresses in the beam is equal to $h/2L$, where h and L are, respectively, the depth and the length of the beam. (*b*) Determine the depth h and width b of the beam, knowing that $L = 2$ m, $P = 40$ kN, $\tau_m = 960$ kPa, and $\sigma_m = 12$ MPa.

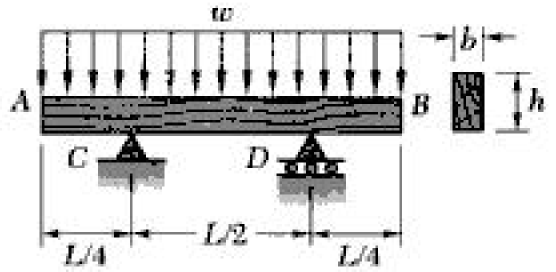


Fig. P6.20

6.20 A timber beam *AB* of length *L* and rectangular cross section carries a uniformly distributed load w and is supported as shown. (*a*) Show that the ratio τ_m/σ_m of the maximum values of the shearing and normal stresses in the beam is equal to $2h/L$, where h and L are, respectively, the depth and the length of the beam. (*b*) Determine the depth h and width b of the beam, knowing that $L = 5$ m, $w = 8$ kN/m, $\tau_m = 1.08$ MPa, and $\sigma_m = 12$ MPa.

6.7. SHEARING STRESSES IN THIN-WALLED MEMBERS

We saw in the preceding section that Eq. (6.4) may be used to determine the longitudinal shear $\Delta \mathbf{H}$ exerted on the walls of a beam element of arbitrary shape and Eq. (6.5) to determine the corresponding shear flow q. These equations will be used in this section to calculate both the shear flow and the average shearing stress in thin-walled members such as the flanges of wide-flange beams (Fig. 6.28) and box beams, or the walls of structural tubes (Fig. 6.29).

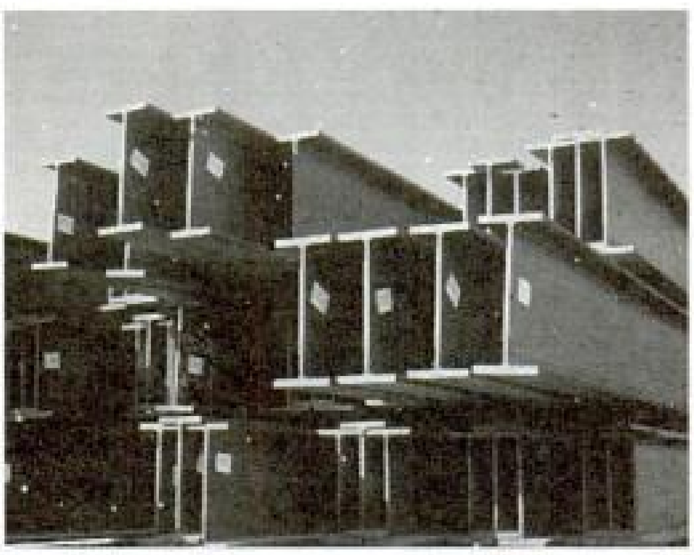


Fig. 6.28

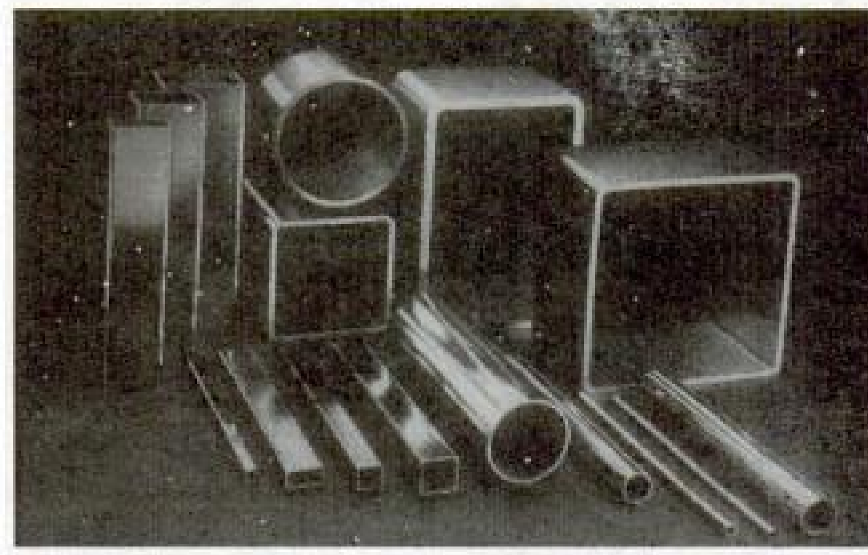


Fig. 6.29

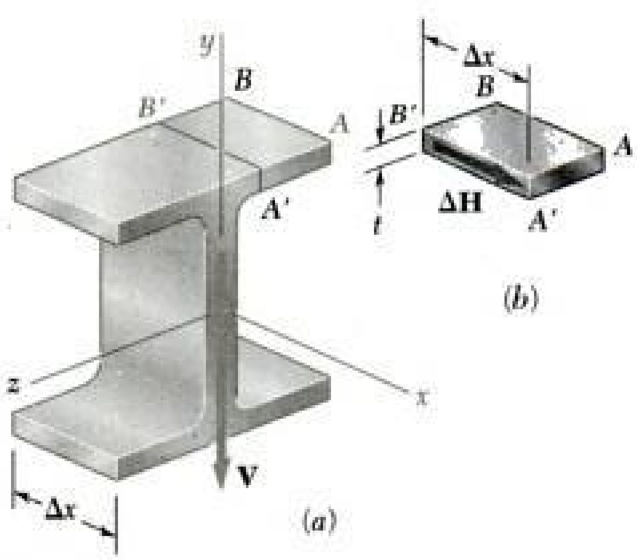


Fig. 6.30

Consider, for instance, a segment of length Δx of a wide-flange beam (Fig. 6.30*a*) and let $\mathbf{V}$ be the vertical shear in the transverse section shown. Let us detach an element $ABB'A'$ of the upper flange (Fig. 6.30*b*). The longitudinal shear $\Delta \mathbf{H}$ exerted on that element can be obtained from Eq. (6.4):

$$\Delta H = \frac{VQ}{I} \Delta x \tag{6.4}$$

Dividing ΔH by the area $\Delta A = t\,\Delta x$ of the cut, we obtain for the average shearing stress exerted on the element the same expression that we had obtained in Sec. 6.3 in the case of a horizontal cut:

$$\tau_{\text{ave}} = \frac{VQ}{It} \tag{6.6}$$

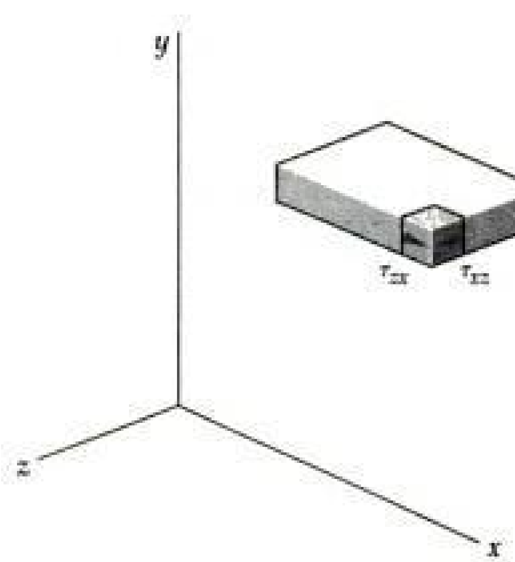
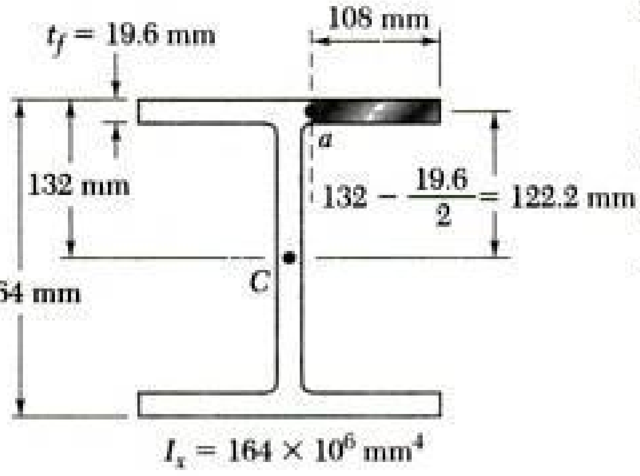


Fig. 6.31

Note that τ_{ave} now represents the average value of the shearing stress τ_{zx} over a vertical cut, but since the thickness t of the flange is small, there is very little variation of τ_{zx} across the cut. Recalling that $\tau_{xz} = \tau_{zx}$ (Fig. 6.31), we conclude that the horizontal component τ_{xz} of the shearing stress at any point of a transverse section of the flange can be obtained from Eq. (6.6), where Q is the first moment of the shaded area about the neutral axis (Fig. 6.32*a*). We recall that a similar result was obtained in Sec. 6.4 for the vertical component τ_{xy} of the shearing stress in the web (Fig. 6.32*b*). Equation (6.6) can be used to determine shear-

SAMPLE PROBLEM 6.3



Knowing that the vertical shear is 220 kN in a W250 ×101 rolled-steel beam, determine the horizontal shearing stress in the top flange at a point a located 108 mm from the edge of the beam. The dimensions and other geometric data of the rolled-steel section are given in Appendix C.

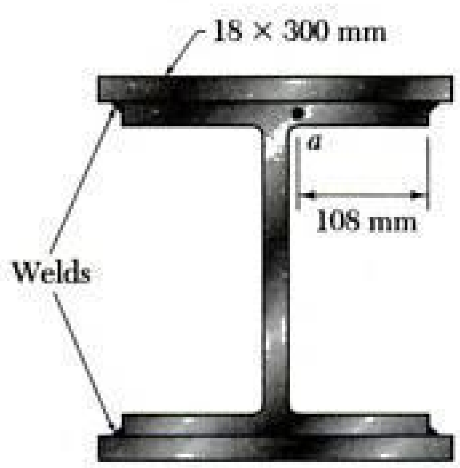
SOLUTION

We isolate the shaded portion of the flange by cutting along the dashed line which passes through point a.

$$Q = (108 \text{ mm})(19.6 \text{ mm})(122.2 \text{ mm}) = 259.3 \times 10^3 \text{ mm}^3$$

$$\tau = \frac{VQ}{It} = \frac{(220 \text{ kN})\, 259.3 \times 10^3 \text{ mm}^3}{(164 \times 10^6 \text{ mm}^4)(19.6 \text{ mm})} \qquad \tau = 17.7 \text{ MPa} \blacktriangleleft$$

SAMPLE PROBLEM 6.4



Solve Sample Prob. 6.3, assuming that 18 × 300 mm plates have been attached to the flanges of the W250 × 101 beam by continuous fillet welds as shown.

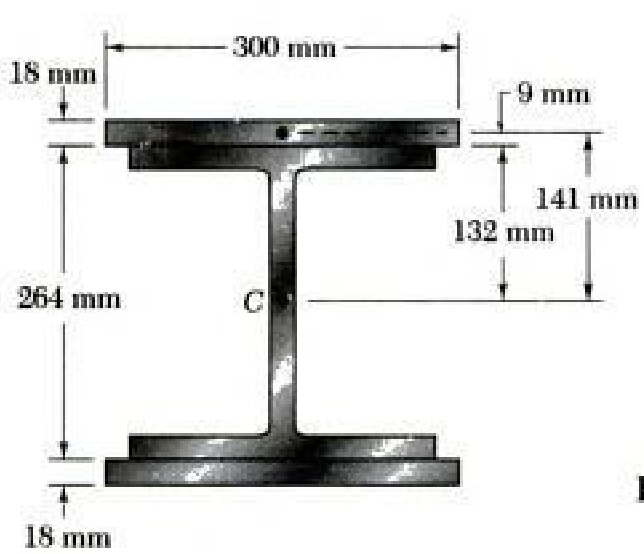
SOLUTION

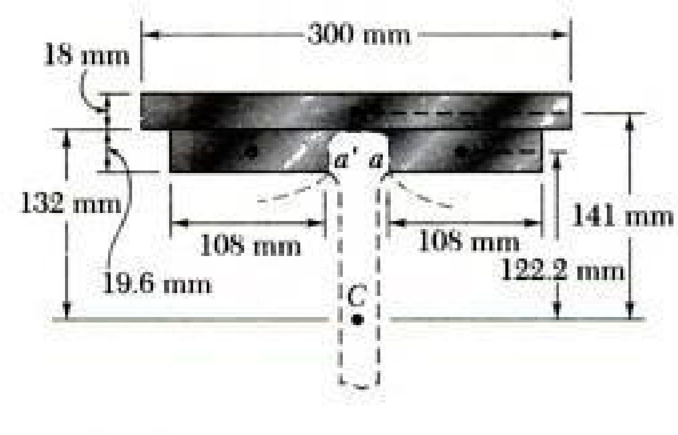
For the composite beam the centroidal moment of inertia is

$$I = 164 \times 10^6 \text{ mm}^4 + 2[\tfrac{1}{12}(300 \text{ mm})(18 \text{ mm})^3 + (300 \text{ mm})(18 \text{ mm})(141 \text{ mm})^2]$$

$$I = 379 \times 10^6 \text{ mm}^4$$

Since the top plate and the flange are connected only at the welds, we find the shearing stress at a by passing a section through the flange at a, *between* the plate and the flange, and again through the flange at the symmetric point a'.





For the shaded area that we have isolated, we have

$$t = 2t_f = 2(19.6 \text{ mm}) = 39.2 \text{ mm}$$

$$Q = 2[(108 \text{ mm})(19.6 \text{ mm})(122.2)] + (300 \text{ mm})(18 \text{ mm})(141 \text{ mm})$$

$$Q = 1.278 \times 10^6 \text{ mm}^3$$

$$\tau = \frac{VQ}{It} = \frac{(220 \text{ kN})(1.278 \times 10^6 \text{ mm}^3)}{(379 \times 10^6 \text{ mm}^4)(39.2 \text{ mm})} \qquad \tau = 18.92 \text{ MPa} \blacktriangleleft$$

6.45 and 6.46 Three planks are connected as shown by bolts of 10 mm diameter spaced every 150 mm along the longitudinal axis of the beam. For a vertical shear of 11 kN, determine the average shearing stress in the bolts.

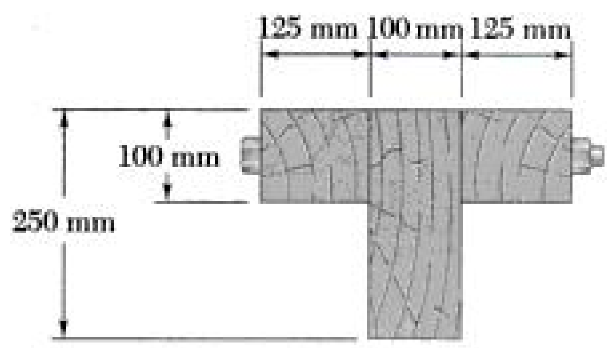


Fig. P6.45

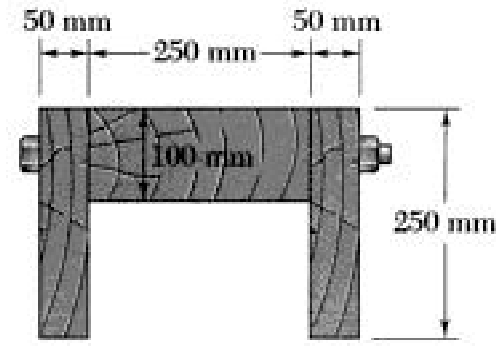


Fig. P6.46

6.47 Three plates, each 12 mm thick, are welded together to form the section shown. For a vertical shear of 100 kN, determine the shear flow through the welded surfaces and sketch the shear flow in the cross section.

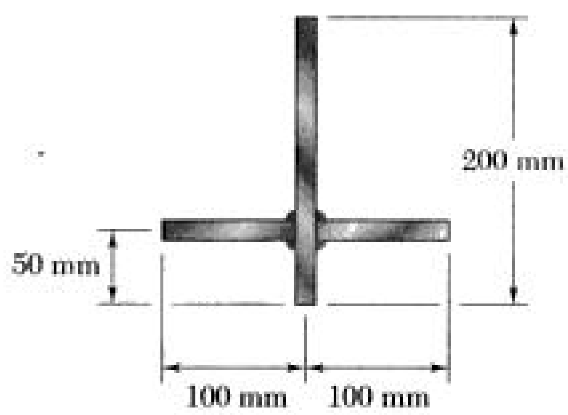


Fig. P6.47

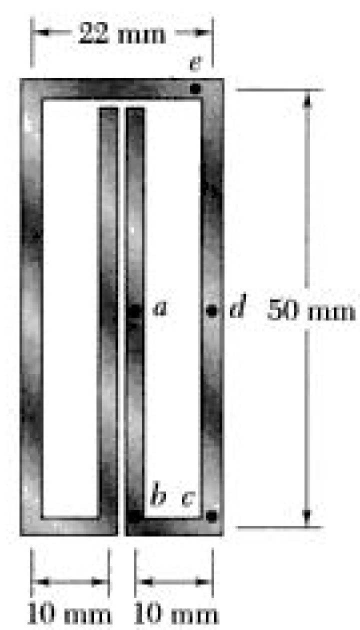


Fig. P6.48

6.48 A plate of 2-mm thickness is bent as shown and then used as a beam. For a vertical shear of 5 kN, determine the shearing stress at the five points indicated and sketch the shear flow in the cross section.

6.49 A plate of 6 mm thickness is corrugated as shown and then used as a beam. For a vertical shear of 5 kN, determine (*a*) the maximum shearing stress in the section, (*b*) the shearing stress at point *B*. Also sketch the shear flow in the cross section.

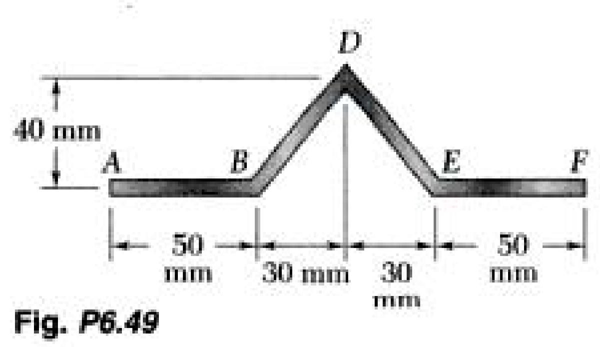


Fig. *P6.49*

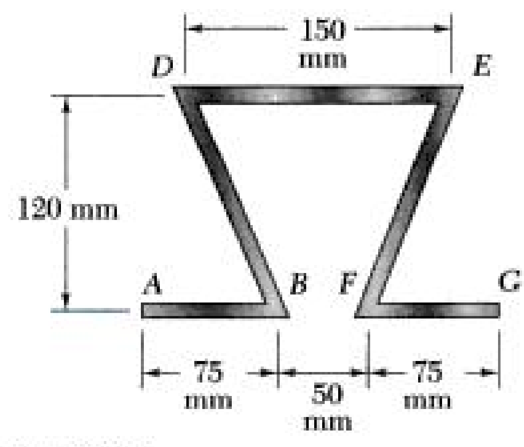


Fig. *P6.50*

6.50 A plate of thickness t is bent as shown and then used as a beam. For a vertical shear of 2.6 kN, determine (*a*) the thickness t for which the maximum shearing stress is 2 MPa, (*b*) the corresponding shearing stress at point *E*. Also sketch the shear flow in the cross section.

Turning our attention to thin-walled members possessing no plane of symmetry, we now consider the case of an angle shape subjected to a vertical load **P**. If the member is oriented in such a way that the load **P** is perpendicular to one of the principal centroidal axes Cz of the cross section, the couple vector **M** representing the bending moment in a given section will be directed along Cz (Fig. 6.58), and the neutral axis will coincide with that axis (cf. Sec. 4.13). Equation (4.16), therefore, is applicable and can be used to compute the normal stresses in the section. We now propose to determine where the load **P** should be applied if Eq. (6.6) is to define the shearing stresses in the section, i.e., if the member is to *bend without twisting*.

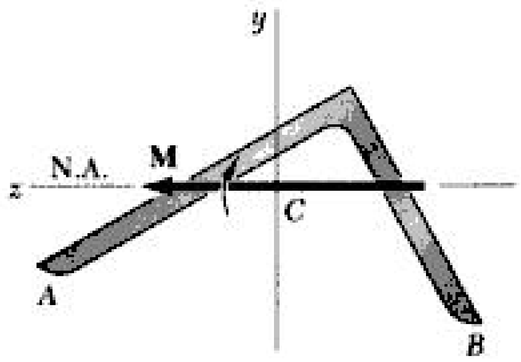


Fig. 6.58

Let us *assume* that the shearing stresses in the section are defined by Eq. (6.6). As in the case of the channel member considered earlier, the elementary shearing forces exerted on the section can be expressed as $dF = q\,ds$, with $q = VQ/I$, where Q represents a first moment with respect to the neutral axis (Fig. 6.59a). We note that the resultant of the

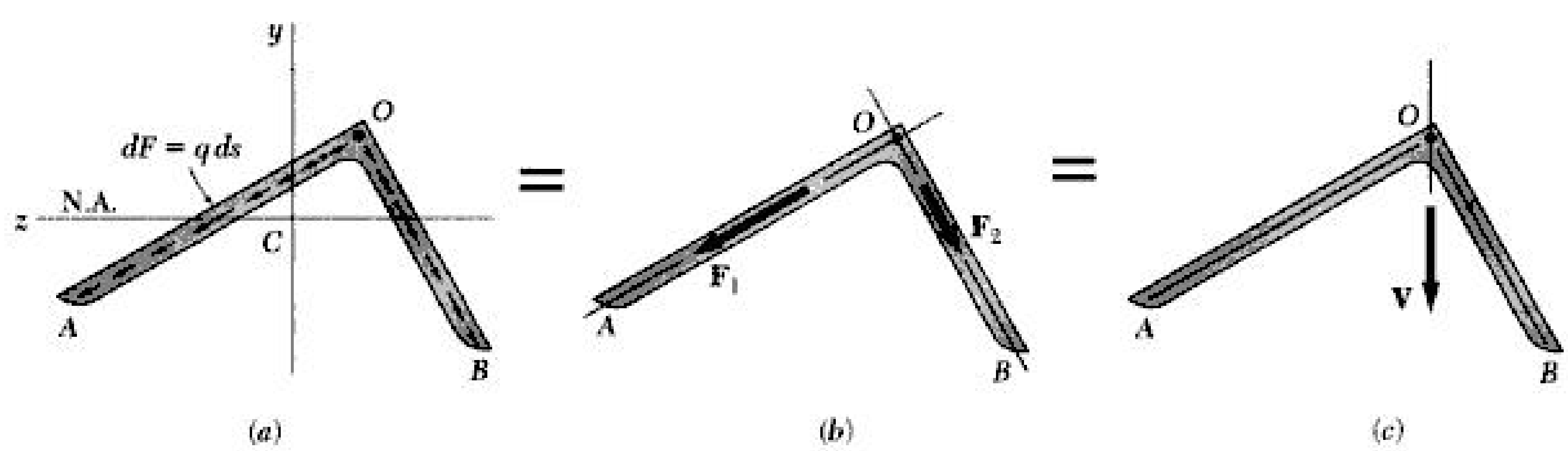


Fig. 6.59

shearing forces exerted on portion OA of the cross section is a force $\mathbf{F}_1$ directed along OA, and that the resultant of the shearing forces exerted on portion OB is a force $\mathbf{F}_2$ along OB (Fig. 6.59b). Since both $\mathbf{F}_1$ and $\mathbf{F}_2$ pass through point O at the corner of the angle, it follows that their own resultant, which is the shear **V** in the section, must also pass through O (Fig. 6.59c). We conclude that the member will not be twisted if the line of action of the load **P** passes through the corner O of the section in which it is applied.

The same reasoning can be applied when the load **P** is perpendicular to the other principal centroidal axis Cy of the angle section. And, since any load **P** applied at the corner O of a cross section can be resolved into components perpendicular to the principal axes, it follows that the member will not be twisted if each load is applied at the corner O of a cross section. We thus conclude that O is the shear center of the section.

Angle shapes with one vertical and one horizontal leg are encountered in many structures. It follows from the preceding discussion that such members will not be twisted if vertical loads are applied along the

6.75 and 6.76 A thin-walled beam of uniform thickness has the cross section shown. Determine the dimension a for which the shear center O of the cross section is located at the point indicated.

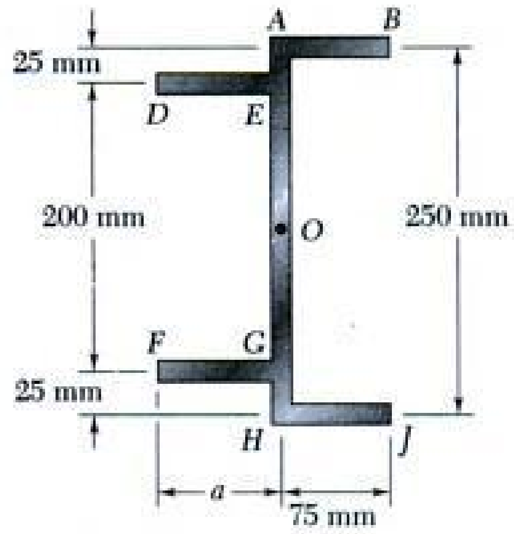


Fig. P6.75

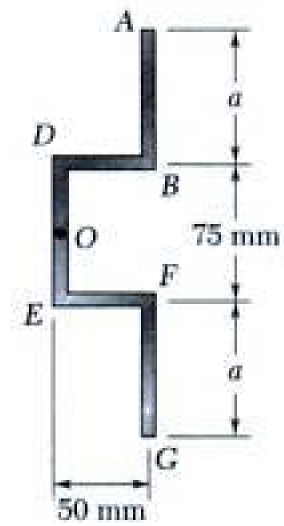


Fig. P6.76

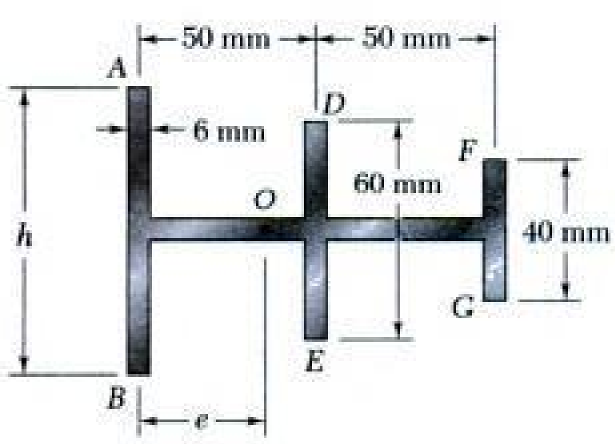


Fig. P6.77 and P6.78

6.77 A thin-walled beam of uniform thickness has the cross section shown. Determine the location of the shear center O of the cross section, knowing that $h = 80$ mm.

6.78 A thin-walled beam of uniform thickness has the cross section shown. Determine the dimension h for which the shear center O of the cross section is located at a distance $e = 25$ mm from the center of the flange AB.

6.79 For the angle shape and loading of Sample Prob. 6.6, check that $\int q\,dz = 0$ along the horizontal leg of the angle and $\int q\,dy = P$ along its vertical leg.

6.80 For the angle shape and loading of Sample Prob. 6.6, (*a*) determine the points where the shearing stress is maximum and the corresponding values of the stress, (*b*) verify that the points obtained are located on the neutral axis corresponding to the given loading.

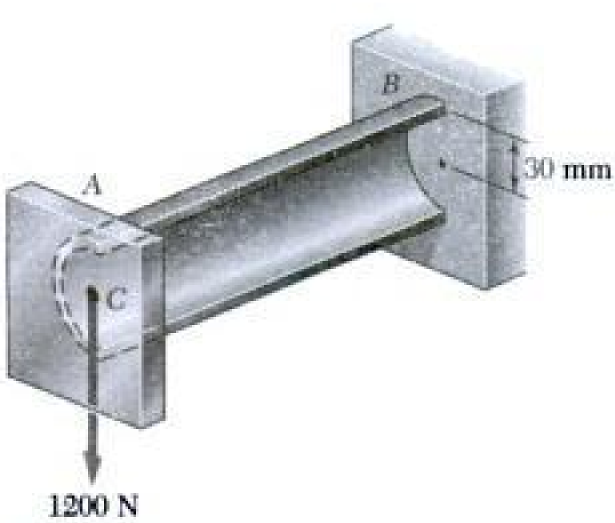


Fig. P6.81

***6.81** A cantilever beam AB, consisting of half of a thin-walled pipe of 30-mm mean radius and 6-mm wall thickness, is subjected to a 1200-N vertical load. Knowing that the line of action of the load passes through the centroid C of the cross section of the beam, determine (*a*) the equivalent force-couple system at the shear center of the cross section, (*b*) the maximum stress in the beam. (*Hint*: The shear center O of this cross section was shown in Prob. 6.74 to be located twice as far from its vertical diameter as its centroid C.)

***6.82** Solve Prob. 6.81, assuming that the thickness of the beam is reduced to 5 mm.

***6.83** The cantilever beam shown at the top of the next page consists of an angle shape of 10 mm thickness. For the given loading, determine the location and magnitude of the largest shearing stress along line $A'B'$ in the horizontal leg of the angle shape. The x' and y' axes are the principal centroidal axes of the cross section and the corresponding moments of inertia are $I_{x'} = 45.2(10^6)$ mm^4 and $I_{y'} = 4.93(10^6)$ mm^4.

6.92 For the beam and loading shown, consider section *n-n* and determine (*a*) the largest shearing stress in that section, (*b*) the shearing stress at point *a*.

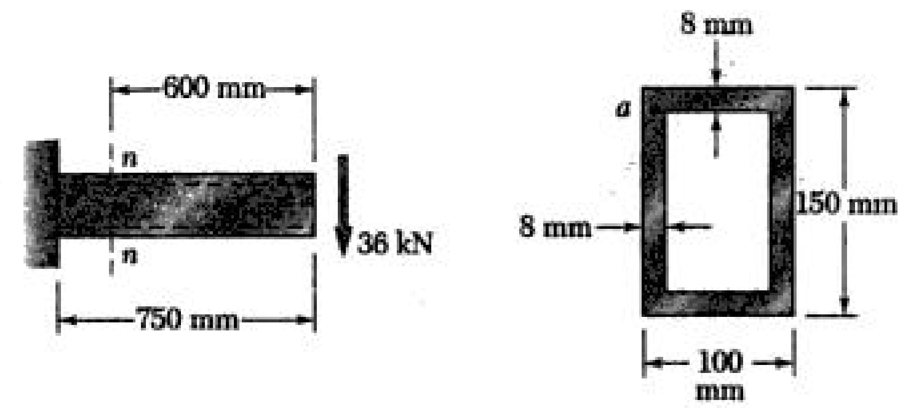


Fig. P6.92

6.93 The built-up timber beam shown is subjected to a 6-kN vertical shear. Knowing that the longitudinal spacing of the nails is $s = 60$ mm and that each nail is 90 mm long, determine the shearing force in each nail.

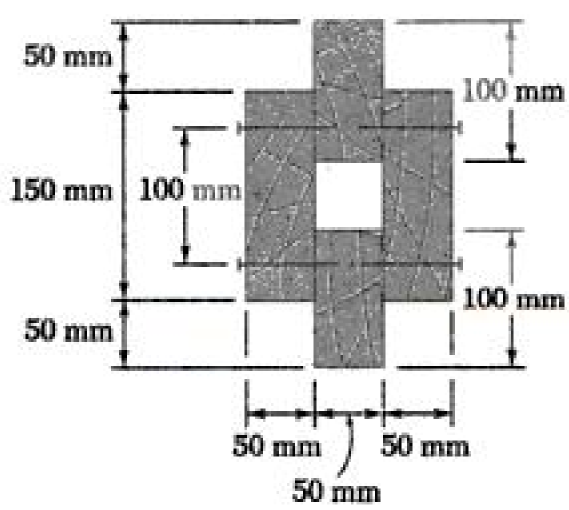
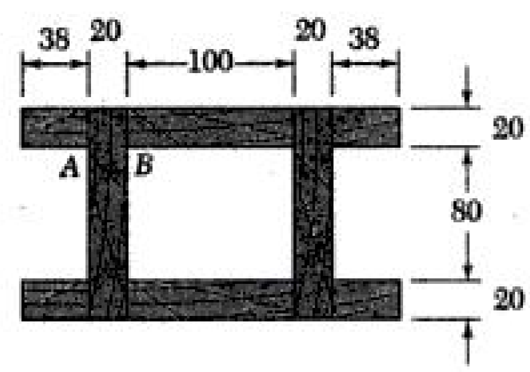


Fig. P6.93



Dimensions in mm

Fig. P6.94

6.94 The built-up beam shown was made by gluing together several wooden planks. Knowing that the beam is subjected to a 5 kN vertical shear, determine the average shearing stress in the glued joint (*a*) at *A*, (*b*) at *B*.

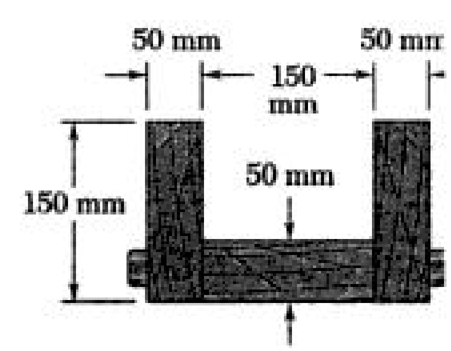


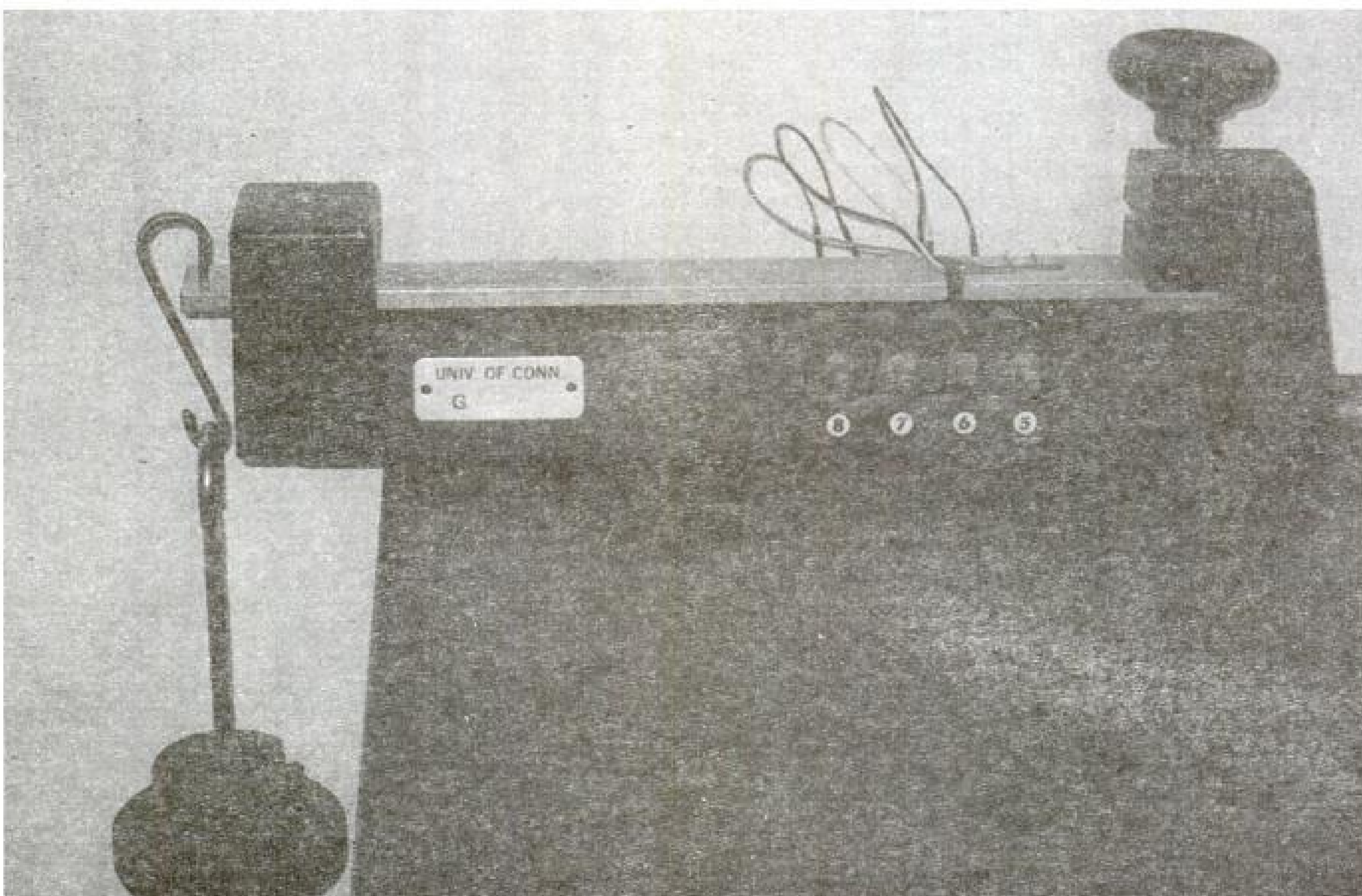
Fig. P6.95

6.95 A beam consists of three planks connected as shown by 10 mm diameter bolts spaced every 300 mm along the longitudinal axis of the beam. Knowing that the beam is subjected to a 11 kN vertical shear, determine the maximum shearing stress in the bolts.

C H A P T E R

7

Transformations of Stress and Strain



In the test setup shown the linear strain in the top surface of the bar is being measured by an electrical strain gage cemented to the top surface. This chapter deals with stresses and strains in structures and machine components.

Consider a square element of a material subjected to plane stress (Fig. 7.17*a*), and let σ_x, σ_y, and τ_{xy} be the components of the stress exerted on the element. We plot a point X of coordinates σ_x and $-\tau_{xy}$, and a point Y of coordinates σ_y and $+\tau_{xy}$ (Fig. 7.17*b*). If τ_{xy} is positive, as assumed in Fig. 7.17*a*, point X is located below the σ axis and point Y above, as shown in Fig. 7.17*b*. If τ_{xy} is negative, X is located above the σ axis and Y below. Joining X and Y by a straight line, we define the point C of intersection of line XY with the σ axis and draw the circle of center C and diameter XY. Noting that the abscissa of C and the radius of the circle are respectively equal to the quantities σ_{ave} and R defined by Eqs. (7.10), we conclude that the circle obtained is Mohr's circle for plane stress. Thus the abscissas of points A and B where the circle intersects the σ axis represent respectively the principal stresses σ_{max} and σ_{min} at the point considered.

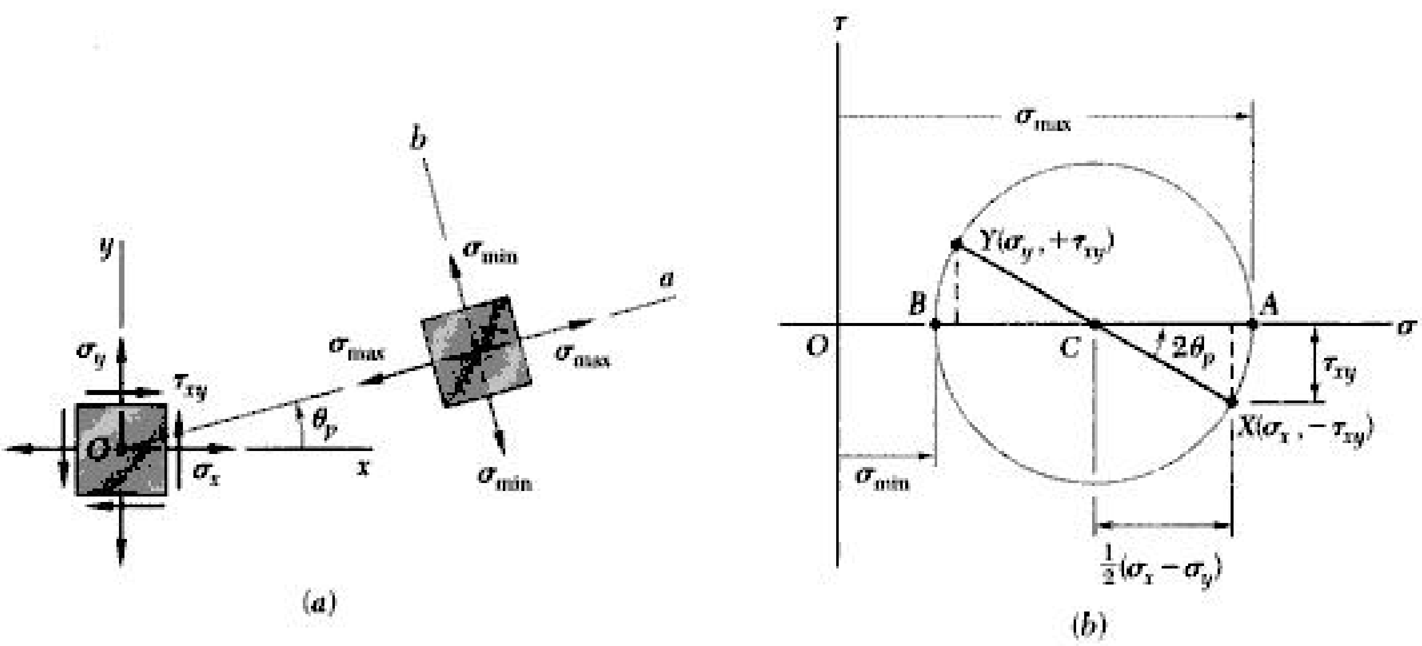
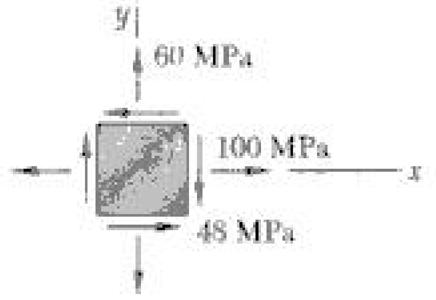


Fig. P7.17

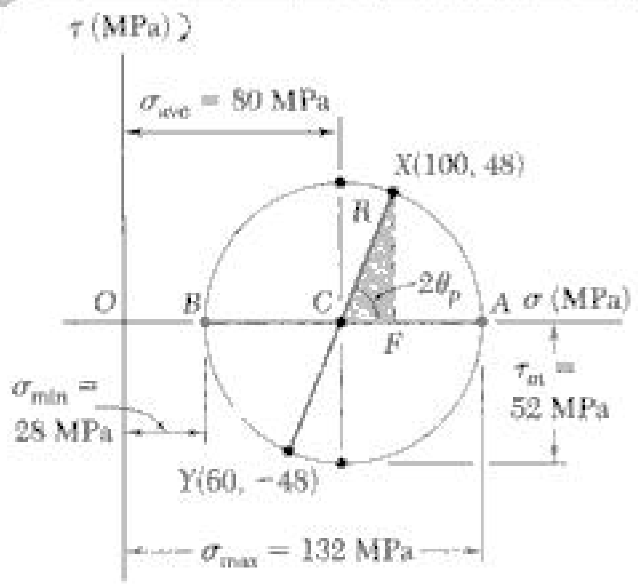
We also note that, since $\tan(XCA) = 2\tau_{xy}/(\sigma_x - \sigma_y)$, the angle XCA is equal in magnitude to one of the angles $2\theta_p$ that satisfy Eq. (7.12). Thus, the angle θ_p that defines in Fig. 7.17*a* the orientation of the principal plane corresponding to point A in Fig. 7.17*b* can be obtained by dividing in half the angle XCA measured on Mohr's circle. We further observe that if $\sigma_x > \sigma_y$ and $\tau_{xy} > 0$, as in the case considered here, the rotation which brings CX into CA is counterclockwise. But, in that case, the angle θ_p obtained from Eq. (7.12) and defining the direction of the normal Oa to the principal plane is positive; thus the rotation bringing Ox into Oa is also counterclockwise. We conclude that the senses of rotation in both parts of Fig. 7.17 are the same; if a counterclockwise rotation through $2\theta_p$ is required to bring CX into CA on Mohr's circle, a counterclockwise rotation through θ_p will bring Ox into Oa in Fig. 7.17*a*.†

†This is due to the fact that we are using the circle of Fig 7.10 rather than the circle of Fig. 7.9 as Mohr's circle.



SAMPLE PROBLEM 7.2

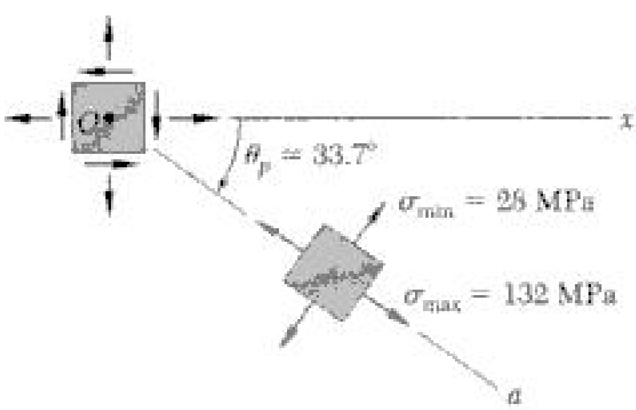
For the state of plane stress shown, determine (*a*) the principal planes and the principal stresses, (*b*) the stress components exerted on the element obtained by rotating the given element counterclockwise through 30°.



SOLUTION

Construction of Mohr's Circle. We note that on a face perpendicular to the x axis, the normal stress is tensile and the shearing stress tends to rotate the element clockwise; thus we plot X at a point 100 units to the right of the vertical axis and 48 units above the horizontal axis. In a similar fashion, we examine the stress components on the upper face and plot point $Y(60, -48)$. Joining points X and Y by a straight line, we define the center C of Mohr's circle. The abscissa of C, which represents σ_{ave}, and the radius R of the circle can be measured directly or calculated as follows:

$$\sigma_{ave} = OC = \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(100 + 60) = 80 \text{ MPa}$$
$$R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52 \text{ MPa}$$



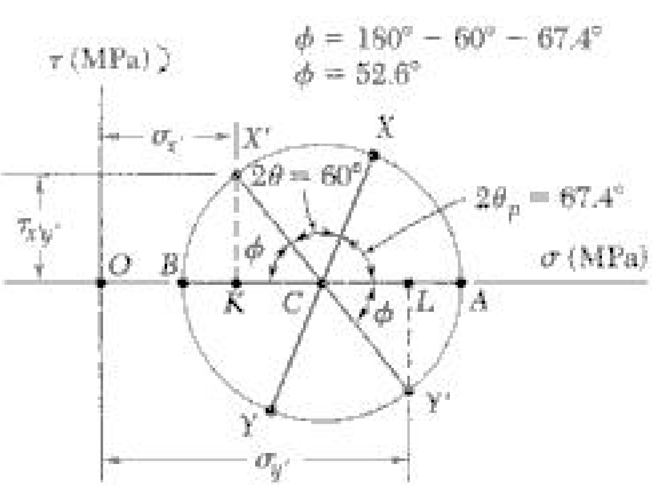
***a.* Principal Planes and Principal Stresses.** We rotate the diameter XY clockwise through $2\theta_p$ until it coincides with the diameter AB. We have

$$\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4 \qquad 2\theta_p = 67.4° \circlearrowright \quad \theta_p = 33.7° \circlearrowright \blacktriangleleft$$

The principal stresses are represented by the abscissas of points A and B:

$$\sigma_{max} = OA = OC + CA = 80 + 52 \qquad \sigma_{max} = +132 \text{ MPa} \blacktriangleleft$$
$$\sigma_{min} = OB = OC - BC = 80 - 52 \qquad \sigma_{min} = +\ 28 \text{ MPa} \blacktriangleleft$$

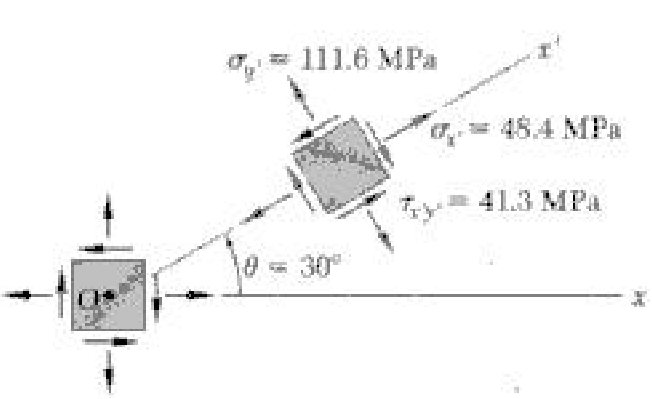
Since the rotation that brings XY into AB is clockwise, the rotation that brings Ox into the axis Oa corresponding to σ_{max} is also clockwise; we obtain the orientation shown for the principal planes.



***b.* Stress Components on Element Rotated 30° ⟲.** Points X' and Y' on Mohr's circle that correspond to the stress components on the rotated element are obtained by rotating XY counterclockwise through $2\theta = 60°$. We find

$$\phi = 180° - 60° - 67.4° \qquad \phi = 52.6° \blacktriangleleft$$
$$\sigma_{x'} = OK = OC - KC = 80 - 52 \cos 52.6° \quad \sigma_{x'} = +\ 48.4 \text{ MPa} \blacktriangleleft$$
$$\sigma_{y'} = OL = OC + CL = 80 + 52 \cos 52.6° \quad \sigma_{y'} = +\ 111.6 \text{ MPa} \blacktriangleleft$$
$$\tau_{x'y'} = KX' = 52 \sin 52.6° \qquad \sigma_{x'y'} = +\ 41.3 \text{ MPa} \blacktriangleleft$$

Since X' is located above the horizontal axis, the shearing stress on the face perpendicular to Ox' tends to rotate the element clockwise.



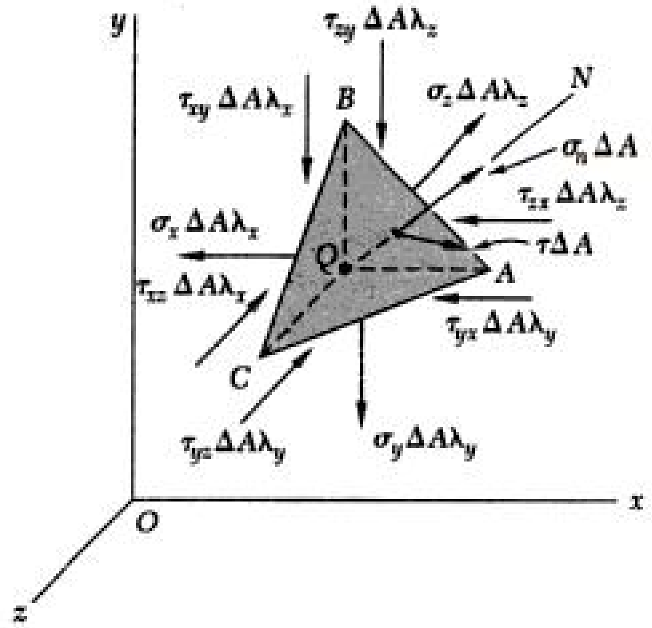


Fig. 7.26

We now express that the sum of the components along QN of all the forces acting on the tetrahedron is zero. Observing that the component along QN of a force parallel to the x axis is obtained by multiplying the magnitude of that force by the direction cosine λ_x, and that the components of forces parallel to the y and z axes are obtained in a similar way, we write

$$\begin{aligned}\sum F_n = 0: \quad & \sigma_n\,\Delta A - (\sigma_x\,\Delta A\,\lambda_x)\lambda_x - (\tau_{xy}\,\Delta A\,\lambda_x)\lambda_y - (\tau_{xz}\,\Delta A\,\lambda_x)\lambda_z \\ & -(\tau_{yx}\,\Delta A\,\lambda_y)\lambda_x - (\sigma_y\,\Delta A\,\lambda_y)\lambda_y - (\tau_{yz}\,\Delta A\,\lambda_y)\lambda_z \\ & -(\tau_{zx}\,\Delta A\,\lambda_z)\lambda_x - (\tau_{zy}\,\Delta A\,\lambda_z)\lambda_y - (\sigma_z\,\Delta A\,\lambda_z)\lambda_z = 0\end{aligned}$$

Dividing through by ΔA and solving for σ_n, we have

$$\sigma_n = \sigma_x\lambda_x^2 + \sigma_y\lambda_y^2 + \sigma_z\lambda_z^2 + 2\tau_{xy}\lambda_x\lambda_y + 2\tau_{yz}\lambda_y\lambda_z + 2\tau_{zx}\lambda_z\lambda_x \quad (7.20)$$

We note that the expression obtained for the normal stress σ_n is a *quadratic form* in λ_x, λ_y, and λ_z. It follows that we can select the coordinate axes in such a way that the right-hand member of Eq. (7.20) reduces to the three terms containing the squares of the direction cosines†. Denoting these axes by a, b, and c, the corresponding normal stresses by σ_a, σ_b, and σ_c, and the direction cosines of QN with respect to these axes by λ_a, λ_b, and λ_c, we write

$$\sigma_n = \sigma_a\lambda_a^2 + \sigma_b\lambda_b^2 + \sigma_c\lambda_c^2 \quad (7.21)$$

The coordinate axes a, b, c are referred to as the *principal axes of stress*. Since their orientation depends upon the state of stress at Q, and thus upon the position of Q, they have been represented in Fig. 7.27 as attached to Q. The corresponding coordinate planes are known as the *principal planes of stress*, and the corresponding normal stresses σ_a, σ_b, and σ_c as the *principal stresses* at Q.‡

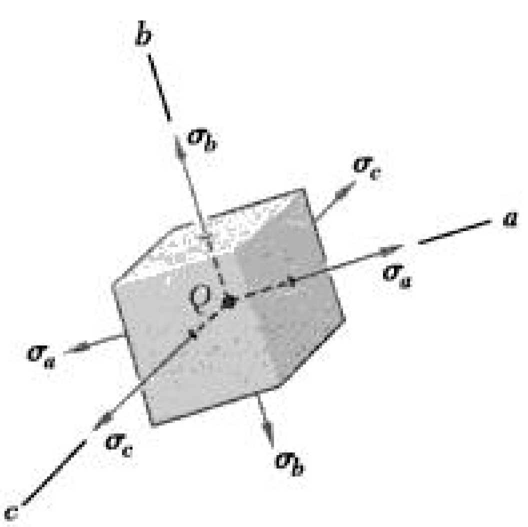


Fig. 7.27

†In Sec. 9 16 of F. P. Beer and E. R. Johnston, *Vector Mechanics for Engineers*, 6th ed., McGraw-Hill Book Company, 1988, a similar quadratic form is found to represent the moment of inertia of a rigid body with respect to an arbitrary axis. It is shown in Sec. 9.17 that this form is associated with a *quadric surface*, and that reducing the quadratic form to terms containing only the squares of the direction cosines is equivalent to determining the principal axes of that surface.

‡For a discussion of the determination of the principal planes of stress and of the principal stresses, see S. P. Timoshenko and J. N. Goodier, Theory of Elasticity, 3d ed., McGraw-Hill Book Company, 1970, sec. 77.

For any other state of stress, the maximum-shearing-stress criterion is more conservative than the maximum-distortion-energy criterion, since the hexagon is located within the ellipse.

A state of stress of particular interest is that associated with yield in a torsion test. We recall from Fig. 7.24 of Sec. 7.4 that, for torsion, $\sigma_{min} = -\sigma_{max}$; thus, the corresponding points in Fig. 7.42 are located on the bisector of the second and fourth quadrants. It follows that yield occurs in a torsion test when $\sigma_a = -\sigma_b = \pm 0.5\sigma_Y$ according to the maximum-shearing-stress criterion, and when $\sigma_a = -\sigma_b = \pm 0.577\sigma_Y$ according to the maximum-distortion-energy criterion. But, recalling again Fig. 7.24, we note that σ_a and σ_b must be equal in magnitude to τ_{max}, that is, to the value obtained from a torsion test for the yield strength τ_Y of the material. Since the values of the yield strength σ_Y in tension and of the yield strength τ_Y in shear are given for various ductile materials in Appendix B, we can compute the ratio τ_Y/σ_Y for these materials and verify that the values obtained range from 0.55 to 0.60. Thus, the maximum-distortion-energy criterion appears somewhat more accurate than the maximum-shearing-stress criterion as far as predicting yield in torsion is concerned.

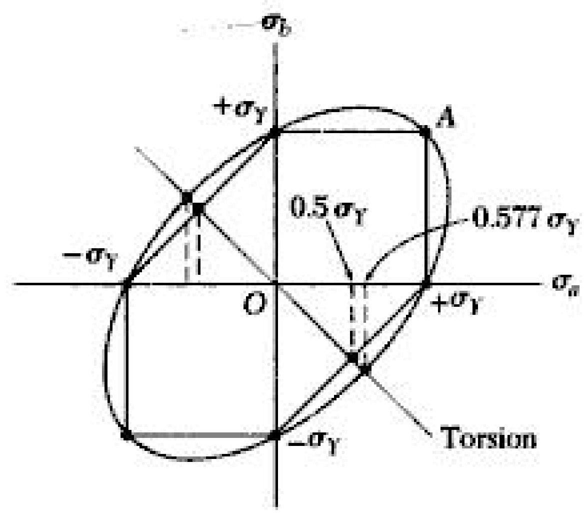


Fig. 7.42

*7.8. FRACTURE CRITERIA FOR BRITTLE MATERIALS UNDER PLANE STRESS

As we saw in Chap. 2, brittle materials are characterized by the fact that, when subjected to a tensile test, they fail suddenly through rupture—or fracture—without any prior yielding. When a structural element or machine component made of a brittle material is under uniaxial tensile stress, the value of the normal stress that causes it to fail is equal to the ultimate strength σ_U of the material as determined from a tensile test, since both the tensile-test specimen and the element or component under investigation are in the same state of stress. However, when a structural element or machine component is in a state of plane stress, it is found convenient to first determine the principal stresses σ_a and σ_b at any given point, and to use one of the criteria indicated in this section to predict whether or not the structural element or machine component will fail.

Maximum-Normal-Stress Criterion. According to this criterion, a given structural component fails when the maximum normal stress in that component reaches the ultimate strength σ_U obtained from the tensile test of a specimen of the same material. Thus, the structural component will be safe as long as the absolute values of the principal stresses σ_a and σ_b are both less than σ_U:

$$|\sigma_a| < \sigma_U \qquad |\sigma_b| < \sigma_U \tag{7.28}$$

The maximum-normal-stress criterion can be expressed graphically as shown in Fig. 7.43. If the point obtained by plotting the values σ_a and σ_b of the principal stresses falls within the square area shown in the figure, the structural component is safe. If it falls outside that area, the component will fail.

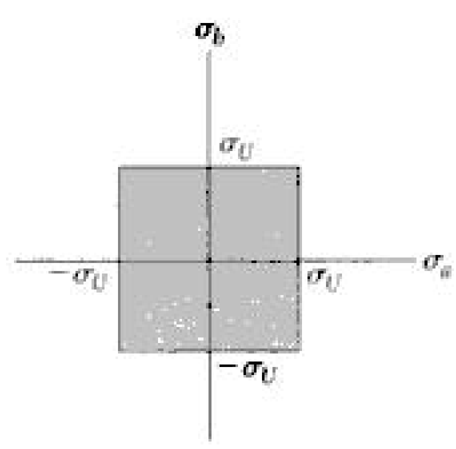


Fig. 7.43

The maximum-normal-stress criterion, also known as *Coulomb's criterion*, after the French physicist Charles Augustin de Coulomb (1736–1806), suffers from an important shortcoming, since it is based

7.72 and 7.73 For the state of stress shown, determine the maximum shearing stress when (*a*) $\sigma_z = +28$ MPa, (*b*) $\sigma_z = -28$ MPa (*c*) $\sigma_z = 0$

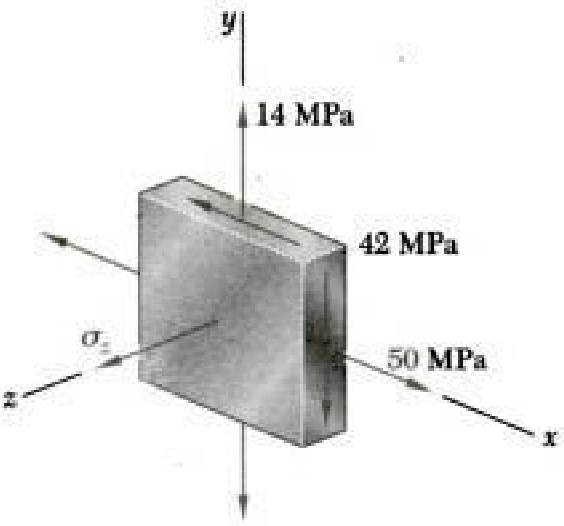


Fig. P7.72

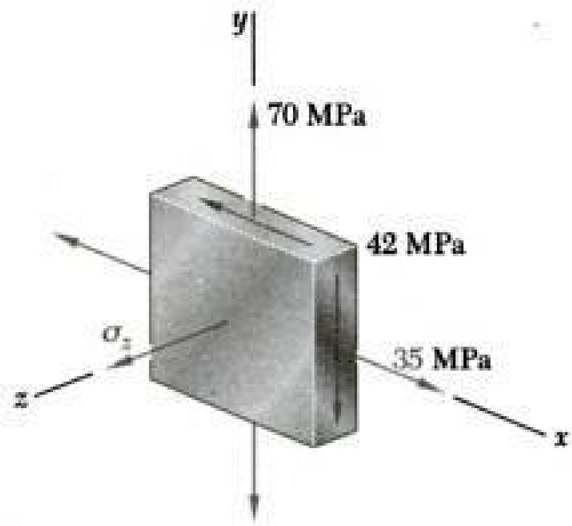


Fig. P7.73

7.74 For the state of stress shown, determine two values of σ_y for which the maximum shearing stress is 75 MPa.

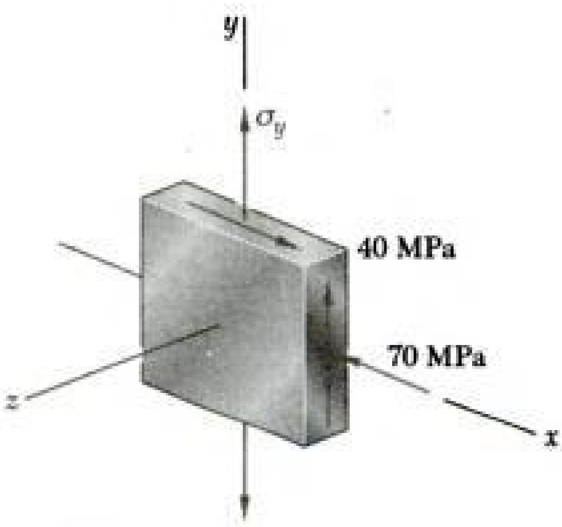


Fig. P7.74

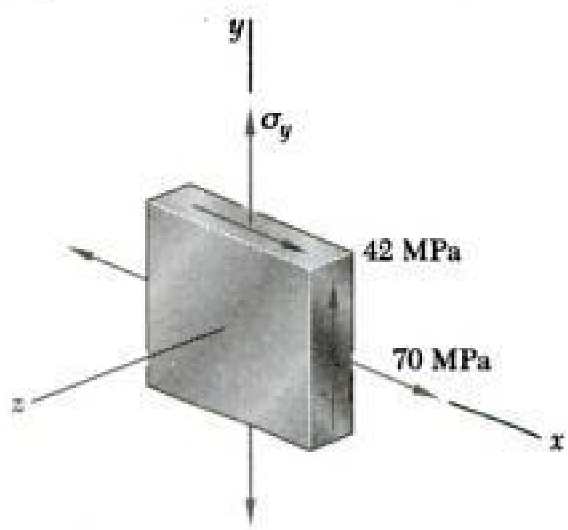


Fig. P7.75

7.75 For the state of stress shown, determine two values of σ_y for which the maximum shearing stress is 52 MPa.

7.76 For the state of stress shown, determine the value of τ_{xy} for which the maximum shearing stress is 80 MPa.

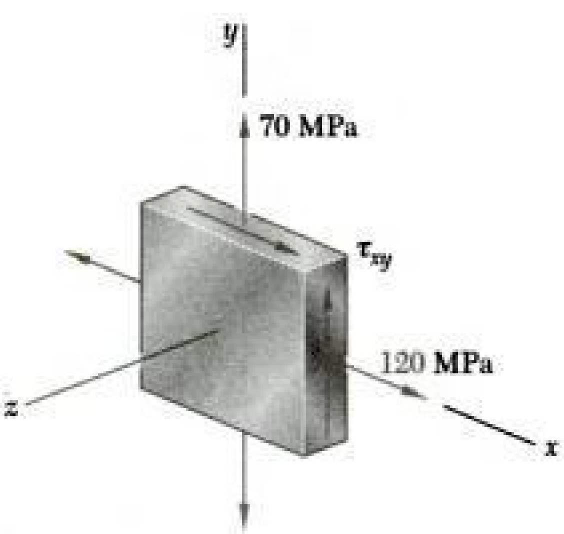


Fig. P7.76

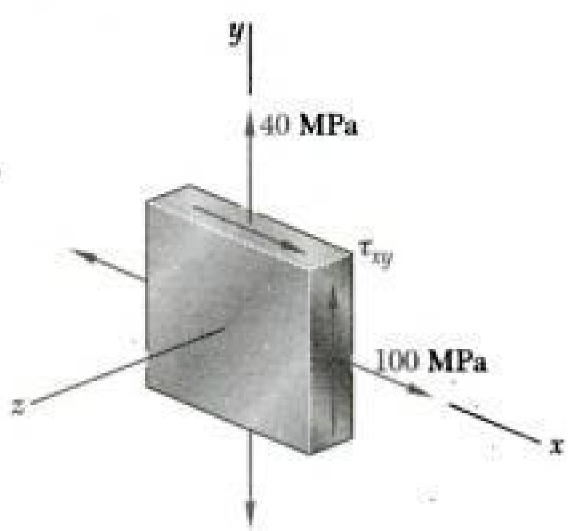


Fig. P7.77

7.77 For the state of stress shown, determine the value of τ_{xy} for which the maximum shearing stress is (*a*) 60 MPa, (*b*) 80 MPa.

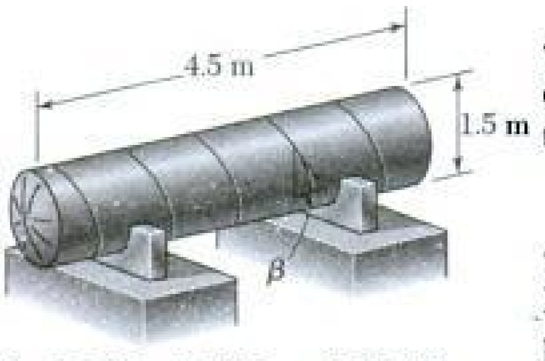


Fig. P7.112, P7.113, and *P7.114*

7.112 The pressure tank shown has a 10 mm wall thickness and butt-welded seams forming an angle $\beta = 20°$ with a transverse plane. For a gage pressure of 580 kPa, determine (*a*) the normal stress perpendicular to the weld, (*b*) the shearing stress parallel to the weld.

7.113 The pressure tank shown has a 10 mm wall thickness and butt-welded seams forming an angle β with a transverse plane. Determine the range of values of β that can be used if tne shearing stress parallel to the weld is not to exceed 9 MPa when the gage pressure is 580 MPa.

7.114 The pressure tank shown has a 10 mm wall thickness and butt-welded seams forming an angle $\beta = 25°$ with a transverse plane. Determine the largest allowable gage pressure, knowing that the allowable normal stress perpendicular to the weld is 120 MPa and the allowable shearing stress parallel to the weld is 70 MPa.

7.115 The pipe shown was fabricated by welding strips of plate along a helix forming an angle β with a transverse plane. Determine the largest value of β that can be used if the normal stress perpendicular to the weld is not to be larger than 85 percent of the maximum stress in the pipe.

Fig. P7.115 and P7.116

7.116 The pipe shown has an outer diameter of 600 mm and was fabricated by welding strips of 10-mm-thick plate along a helix forming an angle $\beta = 25°$ with a transverse plane. Knowing that the ultimate normal stress perpendicular to the weld is 450 MPa and that a factor of safety of 6.0 is desired, determine the largest allowable gage pressure that can be used.

7.117 Square plates, each of 12 mm thickness, can be bent and welded together in either of the two ways shown to form the cylindrical portion of a compressed-air tank. Knowing that the allowable normal stress perpendicular to the weld is 82 MPa, determine the largest allowable gage pressure in each case.

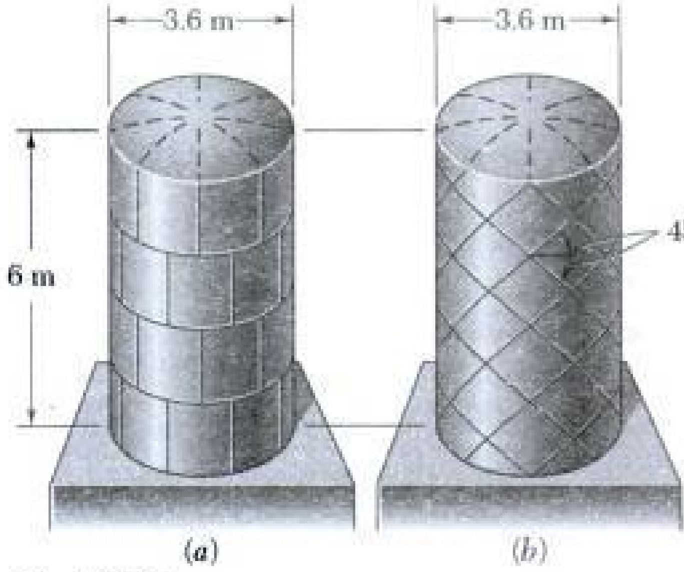


Fig. *P7.117*

The maximum in-plane shearing strain is defined by points D and E in Fig. 7.65a. It is equal to the diameter of Mohr's circle. Recalling the second of Eqs. (7.50), we write

$$\gamma_{\max(\text{in plane})} = 2R = \sqrt{(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2} \qquad (7.53)$$

Finally, we note that the points X' and Y' that define the components of strain corresponding to a rotation of the coordinate axes through an angle θ (Fig. 7.61) are obtained by rotating the diameter XY of Mohr's circle in the same sense through an angle 2θ (Fig. 7.66).

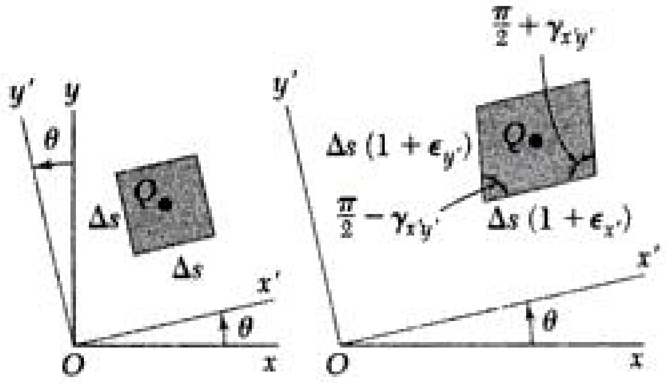


Fig. 7.61 (repeated)

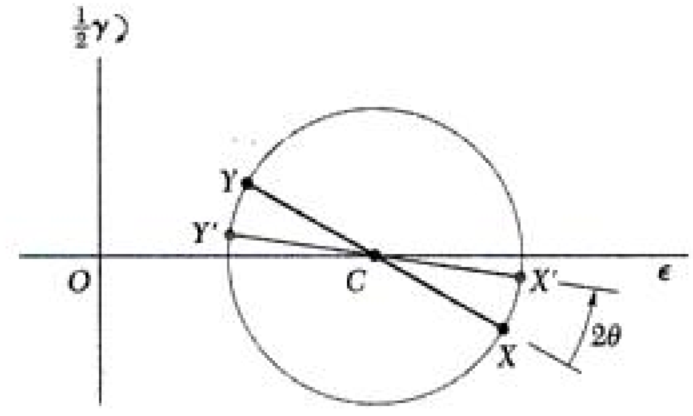


Fig. 7.66

EXAMPLE 7.04

In a material in a state of plane strain, it is known that the horizontal side of a 10 × 10-mm square elongates by 4 μm, while its vertical side remains unchanged, and that the angle at the lower left corner increases by 0.4×10^{-3} rad (Fig. 7.67). Determine (*a*) the principal axes and principal strains, (*b*) the maximum shearing strain and the corresponding normal strain.

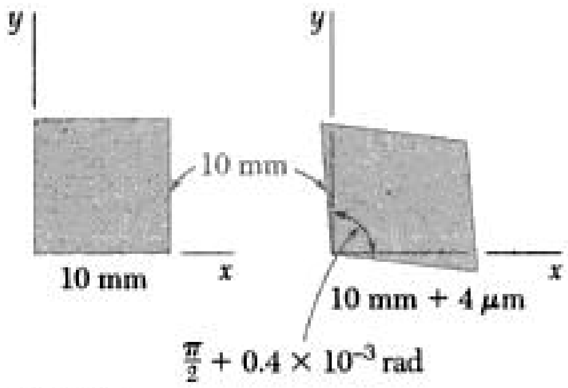


Fig. 7.67

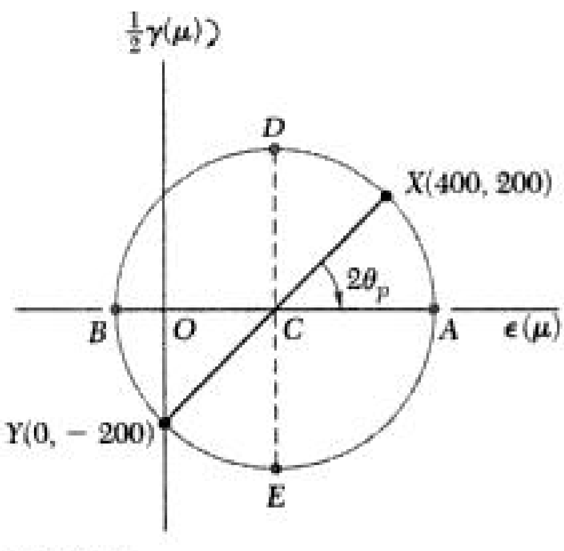


Fig. 7.68

(a) Principal Axes and Principal Strains. We first determine the coordinates of points X and Y on Mohr's circle for strain. We have

$$\epsilon_x = \frac{+4 \times 10^{-6}\text{m}}{10 \times 10^{3}\text{m}} = +400\,\mu \qquad \epsilon_y = 0 \qquad \left|\frac{\gamma_{xy}}{2}\right| = 200\,\mu$$

Since the side of the square associated with ϵ_x rotates *clockwise*, point X of coordinates ϵ_x and $|\gamma_{xy}/2|$ is plotted *above* the horizontal axis. Since $\epsilon_y = 0$ and the corresponding side rotates *counterclockwise*, point Y is plotted directly *below* the origin (Fig. 7.68). Drawing the diameter XY, we determine the center C of Mohr's circle and its radius R. We have

$$OC = \frac{\epsilon_x + \epsilon_y}{2} = 200\,\mu \qquad OY = 200\,\mu$$

$$R = \sqrt{(OC)^2 + (OY)^2} = \sqrt{(200\,\mu)^2 + (200\,\mu)^2} = 283\,\mu$$

The principal strains are defined by the abscissas of points A and B. We write

$$\epsilon_a = OA = OC + R = 200\,\mu + 283\,\mu = 483\,\mu$$
$$\epsilon_b = OB = OC - R = 200\,\mu - 283\,\mu = -83\,\mu$$

strain ϵ_{AB} can be determined accurately and continuously as the load is increased.

The strain components ϵ_x and ϵ_y can be determined at a given point of the free surface of a material by simply measuring the normal strain along x and y axes drawn through that point. Recalling Eq. (7.43) of Sec. 7.10, we note that a third measurement of normal strain, made along the bisector OB of the angle formed by the x and y axes, enables us to determine the shearing strain γ_{xy} as well (Fig. 7.79):

$$\gamma_{xy} = 2\epsilon_{OB} - (\epsilon_x + \epsilon_y) \tag{7.43}$$

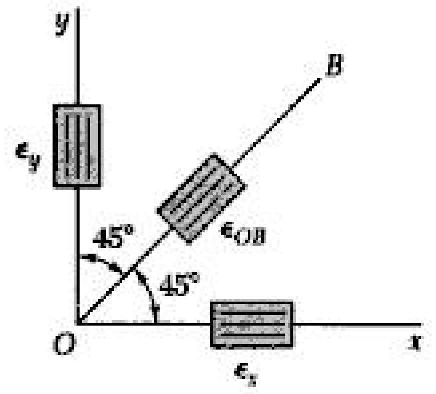


Fig. 7.79

It should be noted that the strain components ϵ_x, ϵ_y, and γ_{xy} at a given point could be obtained from normal strain measurements made along *any three lines* drawn through that point (Fig. 7.80). Denoting respectively by θ_1, θ_2, and θ_3 the angle each of the three lines forms with the x axis, by ϵ_1, ϵ_2, and ϵ_3 the corresponding strain measurements, and substituting into Eq. (7.41), we write the three equations

$$\begin{aligned}
\epsilon_1 &= \epsilon_x \cos^2\theta_1 + \epsilon_y \sin^2\theta_1 + \gamma_{xy}\sin\theta_1\cos\theta_1 \\
\epsilon_2 &= \epsilon_x \cos^2\theta_2 + \epsilon_y \sin^2\theta_2 + \gamma_{xy}\sin\theta_2\cos\theta_2 \\
\epsilon_3 &= \epsilon_x \cos^2\theta_3 + \epsilon_y \sin^2\theta_3 + \gamma_{xy}\sin\theta_3\cos\theta_3
\end{aligned} \tag{7.60}$$

which can be solved simultaneously for ϵ_x, ϵ_y, and γ_{xy}.†

The arrangement of strain gages used to measure the three normal strains ϵ_1, ϵ_2, and ϵ_3 is known as a *strain rosette*. The rosette used to measure normal strains along the x and y axes and their bisector is referred to as a 45° rosette. Another rosette frequently used is the 60° rosette (see Sample Prob. 7.7).

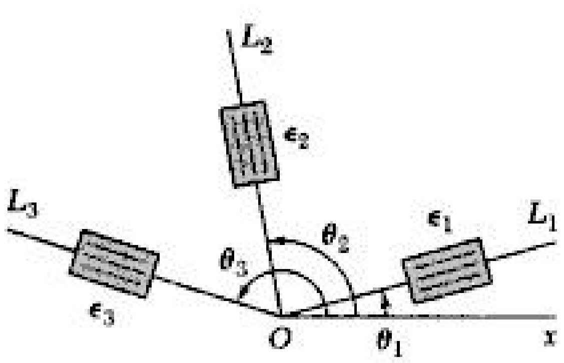


Fig. 7.80

†It should be noted that the free surface on which the strain measurements are made is in a state of *plane stress*, while Eqs. (7.41) and (7.43) were derived for a state of *plane strain*. However, as observed earlier, the normal to the free surface is a principal axis of strain and the derivations given in sec. 7.10 remain valid.

7.147 Using a 45° rosette, the strains ϵ_1, ϵ_2, and ϵ_3 have been determined at a given point. Using Mohr's circle, show that the principal strains are

$$\epsilon_{max,\,min} = \frac{1}{2}(\epsilon_1 + \epsilon_3) \pm \frac{1}{\sqrt{2}}[(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2]^{1/2}$$

(*Hint:* The shaded triangles are congruent.)

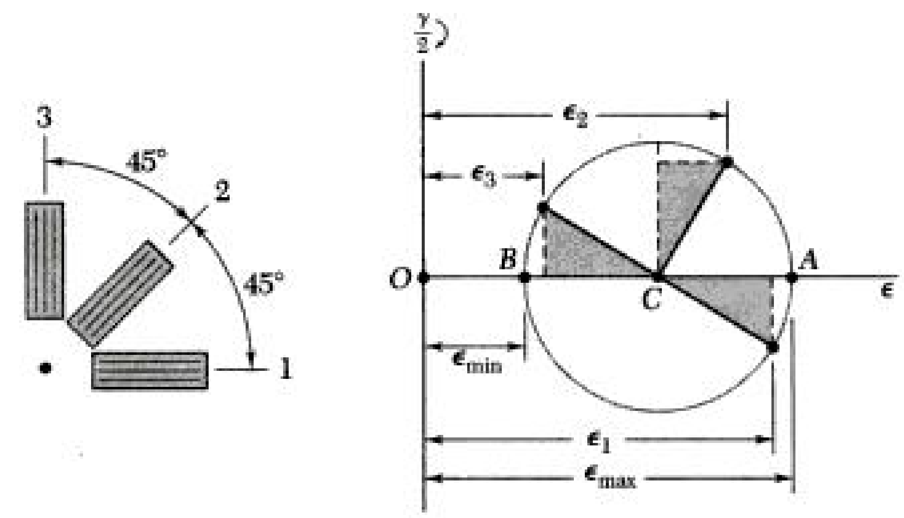


Fig. *P7.147*

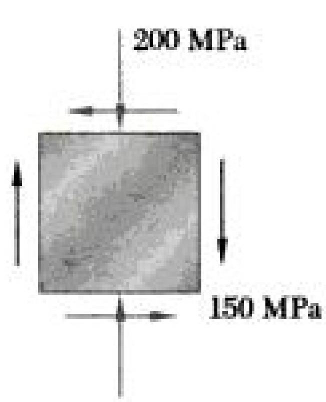


Fig. P7.148

7.148 The given state of plane stress is known to exist on the surface of a machine component. Knowing that E = 200 GPa and G = 77 GPa, determine the direction and magnitude of the three principal strains (*a*) by determining the corresponding state of strain [use Eq. (2.43), page 94, and Eq. (2.38), page 91] and then using Mohr's circle for strain, (*b*) by using Mohr's circle for stress to determine the principal planes and principal stresses and then determining the corresponding strains.

7.149 The following state of strain has been determined on the surface of a cast-iron machine element:

$$\epsilon_1 = -720 \times 10^{-6} \text{ mm/mm} \qquad \epsilon_2 = -400 \times 10^{-6} \text{ mm/mm}$$
$$\gamma = +660 \times 10^{-6} \text{ rad}$$

Knowing that E = 70 GPa and G = 28 GPa, determine the principal planes and the principal stresses (*a*) by determining the corresponding state of plane stress [use Eq. (2.36), page 91; Eq. (2.43), page 94; and the first two equations of Prob. 2.75, page 101] and then using Mohr's circle for stress, (*b*) by using Mohr's circle for strain to determine the orientation and magnitude of the principal strains and then determining the corresponding stresses.

7.150 A single strain gage forming an angle β = 30° with the vertical is used to determine the gage pressure in the cylindrical steel tank shown. The cylindrical wall of the tank is 10 mm thick, has a 900 mm inner diameter, and is made of a steel with E = 200 GPa and ν = 0.30. Determine the pressure in the tank corresponding to a gage reading of 220 $\times$ 10^{-6} mm/mm.

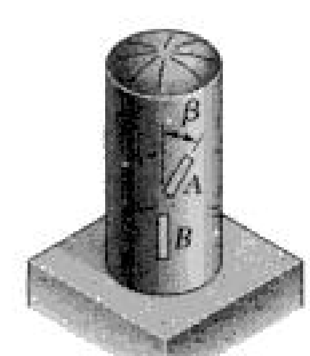


Fig. P7.150

REVIEW PROBLEMS

7.156 The state of stress shown occurs in a steel member made of a grade of steel with a tensile yield strength of 270 MPa. Determine the factor of safety with respect to yield strength, using (*a*) the maximum-shearing-stress criterion, (*b*) the maximum distortion-energy criterion.

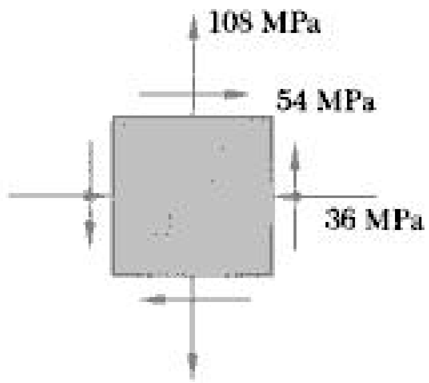


Fig. P7.156

7.157 A spherical pressure tank has a 1.2-m outer diameter and a uniform wall thickness of 10 mm. Knowing that the gage pressure is 1.25 MPa in the tank, determine (*a*) the maximum normal stress, (*b*) the maximum shearing stress, (*c*) the normal strain on the surface of the tank. (Use $E = 200$ GPa and $\nu = 0.30$)

7.158 The strains determined by the use of a rosette attached as shown to the surface of a structural member are:

$$\epsilon_1 = 200 \times 10^{-6} \text{ mm/mm} \qquad \epsilon_2 = 425 \times 10^{-6} \text{ mm/mm}$$
$$\epsilon_3 = 480 \times 10^{-6} \text{ mm/mm}$$

Determine (*a*) the orientation and magnitude of the principal strains in the plane of the rosette, (*b*) the maximum in-plane shearing strain.

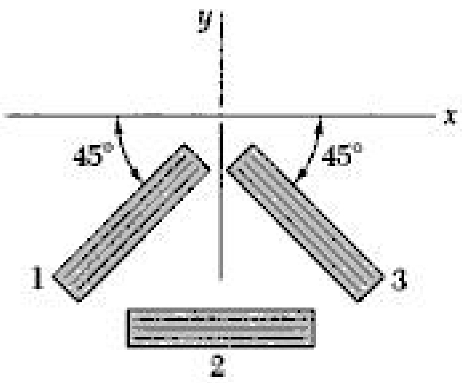


Fig. *P7.158*

7.159 For a state of plane stress it is known that the normal and shearing stresses are directed as shown and that $\sigma_x = 35$ MPa, $\sigma_y = 82$ MPa, and $\sigma_{max} = 124$ MPa. Determine (*a*) the orientation of the principal planes, (*b*) the maximum in-plane shearing stress.

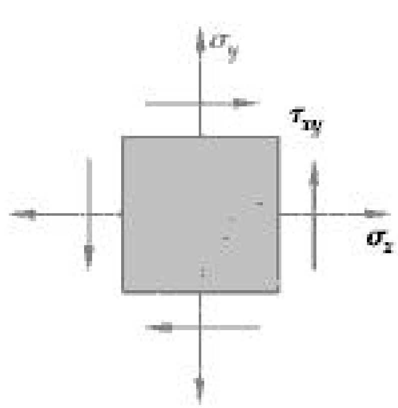


Fig. P7.159

C H A P T E R

8

Principal Stresses under a Given Loading

Due to gravity and wind load, the post supporting the sign shown is subjected simultaneously to compression, bending, and torsion. In this chapter you will learn to determine the stresses created by such combined loadings in structures and machine components.

section, these ratios have been determined at 11 different points, and the orientation of the principal axes has been indicated at each point.†

It is clear that σ_{max} does not exceed σ_m in either of the two sections considered in Fig. 8.8 and that, if it does exceed σ_m elsewhere, it will be in sections close to the load **P**, where σ_m is small compared to τ_m.‡ But, for sections close to the load **P**, Saint-Venant's principle does not apply, Eqs. (8.3) and (8.4) cease to be valid, except in the very unlikely case of a load distributed parabolically over the end section (cf. Sec. 6.5), and more advanced methods of analysis taking into account the effect of stress concentrations should be used. We thus conclude that, for beams of rectangular cross section, and within the scope of the theory presented in this text, the maximum normal stress can be obtained from Eq. (8.1).

In Fig. 8.8 the directions of the principal axes were determined at 11 points in each of the two sections considered. If this analysis were extended to a larger number of sections and a larger number of points in each section, it would be possible to draw two orthogonal systems of curves on the side of the beam (Fig. 8.9). One system would consist of curves tangent to the principal axes corresponding to σ_{max} and the other of curves tangent to the principal axes corresponding to σ_{min}. The curves obtained in this manner are known as the *stress trajectories*. A trajectory of the first group (solid lines) defines at each of its points the direction of the largest tensile stress, while a trajectory of the second group (dashed lines) defines the direction of the largest compressive stress.§

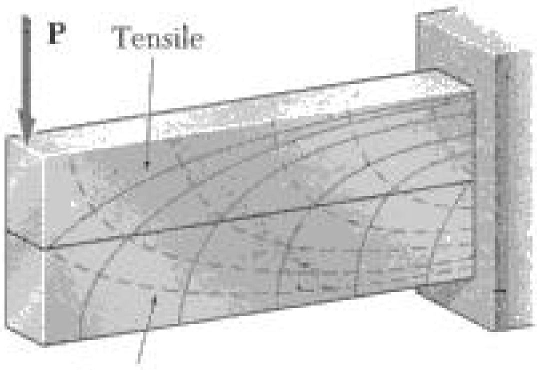


Fig. 8.9. Stress trajectories.

The conclusion we have reached for beams of rectangular cross section, that the maximum normal stress in the beam can be obtained from Eq. (8.1), remains valid for many beams of nonrectangular cross section. However, when the width of the cross section varies in such a way that large shearing stresses τ_{xy} will occur at points close to the surface of the beam, where σ_x is also large, a value of the principal stress σ_{max} larger than σ_m may result at such points. One should be particularly aware of this possibility when selecting W-beams or S-beams, and calculate the principal stress σ_{max} at the junctions b and d of the web with the flanges of the beam (Fig. 8.10). This is done by determining σ_x and τ_{xy} at that point from Eqs. (8.1) and (8.2), respectively, and using either of the methods of analysis of Chap. 7 to obtain σ_{max} (see Sample Prob. 8.1). An alternative procedure consists of using for τ_{xy} the maximum value of the shearing stress in the section, $\tau_{max} = V/A_{web}$, given by Eq. (6.11) of Sec. 6.4. This leads to a slightly larger, and thus conservative, value of the principal stress σ_{max} at the junction of the web with the flanges of the beam (see Sample Prob. 8.2).

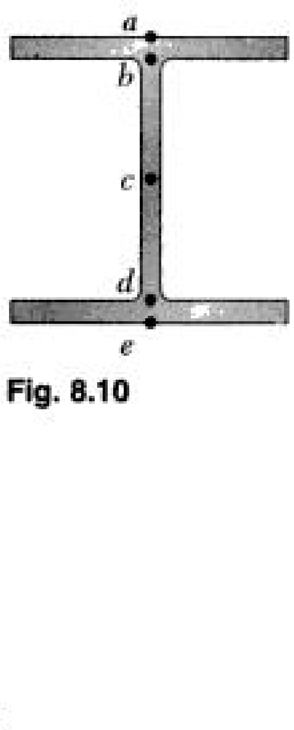
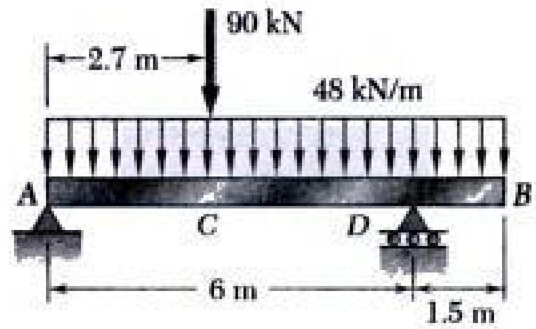


Fig. 8.10

†See Prob. 8.C2, which refers to the program used to obtain the results shown in Fig. 8.8.

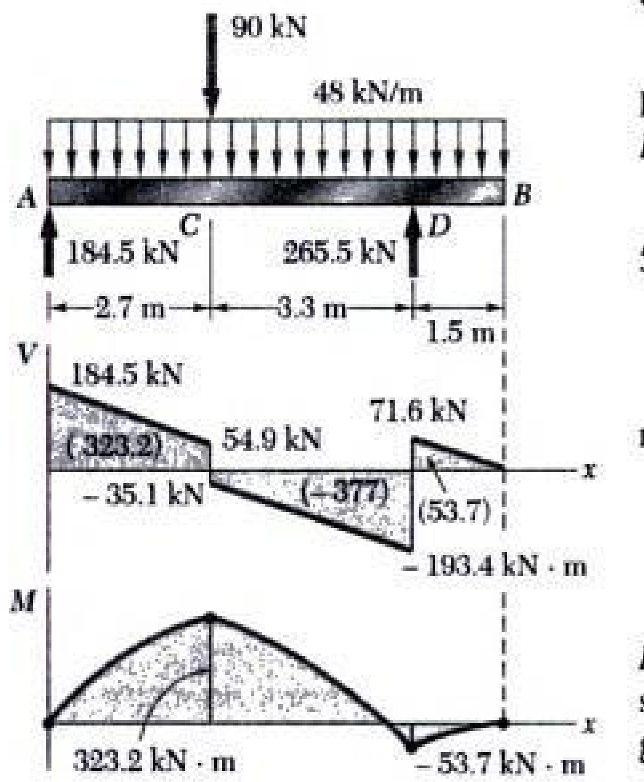
‡As will be verified in Prob. 8.C2, σ_{max} exceeds σ_m if $x \leq 0.544c$.

§A brittle material, such as concrete, will fail in tension along planes that are perpendicular to the tensile-stress trajectories. Thus, to be effective, steel reinforcing bars should be placed so that they intersect these planes. On the other hand, stiffeners attached to the web of a plate girder will be effective in preventing buckling only if they intersect planes perpendicular to the compressive-stress trajectories.



SAMPLE PROBLEM 8.2

The overhanging beam AB supports a uniformly distributed load of 48 kN/m and a concentrated load of 90 kN at C. Knowing that for the grade of steel to be used $\sigma_{all} = 165$ MPa and $\tau_{all} = 100$ MPa, select the wide-flange shape which should be used.



SOLUTION

Reactions at A and D. We draw the free-body diagram of the beam. From the equilibrium equations $\Sigma M_D = 0$ and $\Sigma M_A = 0$ we find the values of R_A and R_D shown in the diagram.

Shear and Bending-Moment Diagrams. Using the methods of Secs. 5.2 and 5.3, we draw the diagrams and observe that

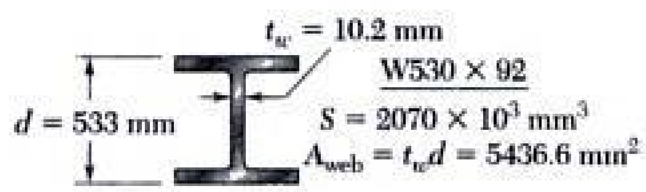
$$|M|_{max} = 323.2 \text{ kN}\cdot\text{m} \qquad |V|_{max} = 193.4 \text{ kN}$$

Section Modulus. For $|M|_{max} = 323.2$ kN · m and $\sigma_{all} = 165$ MPa, the minimum acceptable section modulus of the rolled-steel shape is

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} = \frac{323.2 \text{ kN}\cdot\text{m}}{165 \text{ MPa}} = 1950 \times 10^3 \text{ mm}^3$$

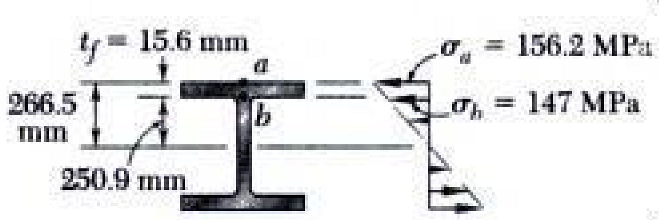
Selection of Wide-Flange Shape. From the table of *Properties of Rolled-Steel Shapes* in Appendix C, we compile a list of shapes which have a section modulus larger than S_{min} and are also the lightest shape in a given depth group.

Shape	S 10^3mm^3
W610 × 101	2530
W530 × 92	2070
W460 × 113	2400
W410 × 114	2200
W360 × 122	2010
W310 × 143	2150



We now select the lightest shape available, namely W530 × 92 ◄

Shearing Stress. Assuming that the maximum shear is uniformly distributed over the web area of a W530 × 92, we write



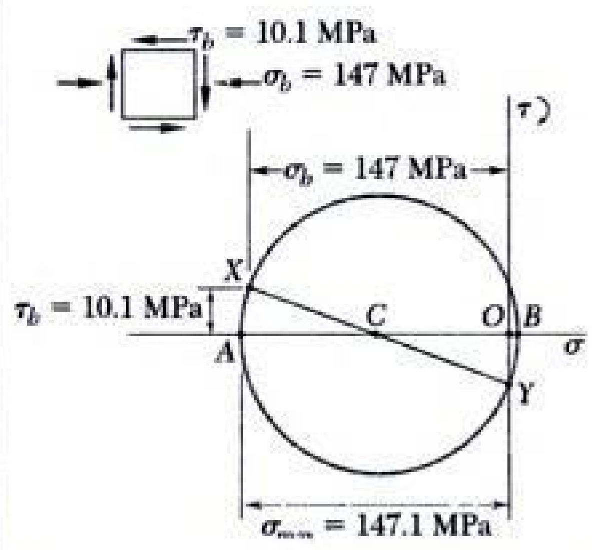
$$\tau_m = \frac{V_{max}}{A_{web}} = \frac{193.4 \text{ kN}}{5436.6 \text{ mm}^2} = 35.5 \text{ MPa} < 100 \text{ MPa} \qquad \text{(OK)}$$

Principal Stress at Point b. We check that the maximum principal stress at point b in the critical section where M is maximum does not exceed $\sigma_{all} = 165$ MPa. We write

$$\sigma_a = \frac{M_{max}}{S} = \frac{323.2 \text{ kN}\cdot\text{m}}{2070 \times 10^{-6} \text{ m}^3} = 156.2 \text{ MPa}$$

$$\sigma_b = \sigma_a \frac{y_b}{c} = (156.2 \text{ MPa}) \frac{250.9 \text{ mm}}{266.5 \text{ mm}} = 147.0 \text{ MPa}$$

$$\tau_b = \frac{V}{A_{web}} = \frac{54.9 \text{ kN}}{5436.6 \times 10^{-6} \text{ m}^2} = 10.1 \text{ MPa}$$



We draw Mohr's circle and find

$$\sigma_{max} = \tfrac{1}{2}\sigma_b + R = \frac{147 \text{ MPa}}{2} + \sqrt{\left(\frac{147}{2}\right)^2 + (10.1 \text{ MPa})^2}$$

$$\sigma_{max} = 147.1 \text{ MPa} \le 165 \text{ MPa (OK)} \; ◄$$

EXAMPLE 8.01

Two forces $\mathbf{P}_1$ and $\mathbf{P}_2$, of magnitude $P_1 = 15$ kN and $P_2 = 18$ kN, are applied as shown to the end *A* of bar *AB*, which is welded to a cylindrical member *BD* of radius $c = 20$ mm (Fig. 8.21). Knowing that the distance from *A* to the axis of member *BD* is $a = 50$ mm and assuming that all stresses remain below the proportional limit of the material, determine (*a*) the normal and shearing stresses at point *K* of the transverse section of member *BD* located at a distance $b = 60$ mm from end *B*, (*b*) the principal axes and principal stresses at *K*, (*c*) the maximum shearing stress at *K*.

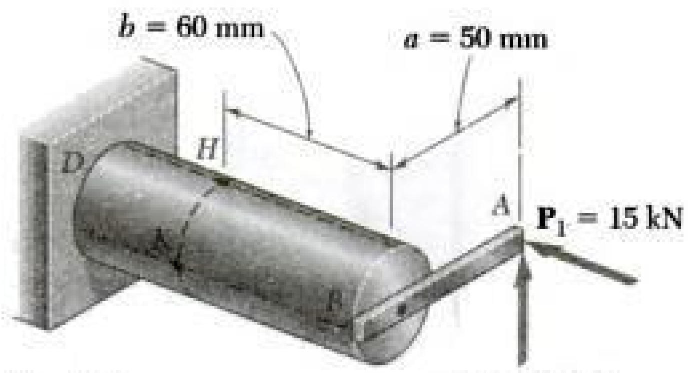


Fig. 8.21

Internal Forces in Given Section. We first replace the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ by an equivalent system of forces and couples applied at the center *C* of the section containing point *K* (Fig. 8.22). This system, which represents the internal forces in the section, consists of the following forces and couples:

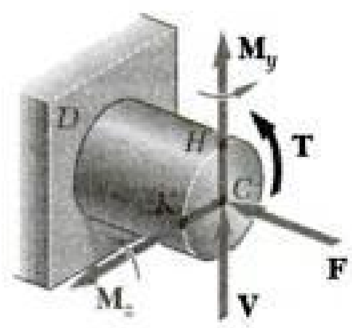


Fig. 8.22

1. A centric axial force $\mathbf{F}$ equal to the force $\mathbf{P}_1$, of magnitude

$$F = P_1 = 15 \text{ kN}$$

2. A shearing force $\mathbf{V}$ equal to the force $\mathbf{P}_2$, of magnitude

$$V = P_2 = 18 \text{ kN}$$

3. A twisting couple $\mathbf{T}$ of torque T equal to the moment of $\mathbf{P}_2$ about the axis of member *BD*:

$$T = P_2 a = (18 \text{ kN})(50 \text{ mm}) = 900 \text{ N} \cdot \text{m}$$

4. A bending couple $\mathbf{M}_y$, of moment M_y equal to the moment of $\mathbf{P}_1$ about a vertical axis through *C*:

$$M_y = P_1 a = (15 \text{ kN})(50 \text{ mm}) = 750 \text{ N} \cdot \text{m}$$

5. A bending couple $\mathbf{M}_z$, of moment M_z equal to the moment of $\mathbf{P}_2$ about a transverse, horizontal axis through *C*:

$$M_z = P_2 a = (18 \text{ kN})(60 \text{ mm}) = 1080 \text{ N} \cdot \text{m}$$

The results obtained are shown in Fig. 8.23.

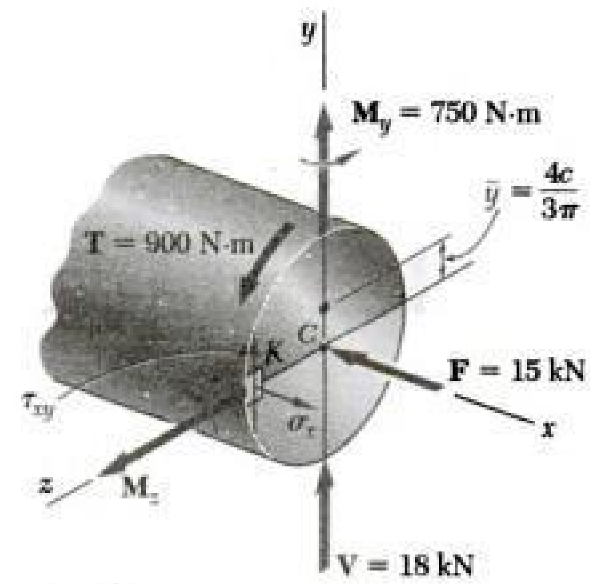


Fig. 8.23

a. Normal and Shearing Stresses at Point K. Each of the forces and couples shown in Fig. 8.23 can produce a normal or shearing stress at point *K*. Our purpose is to compute separately each of these stresses, and then to add the normal stresses and add the shearing stresses. But we must first determine the geometric properties of the section.

Geometric Properties of the Section. We have

$$A = \pi c^2 = \pi(0.020 \text{ m})^2 = 1.257 \times 10^{-3} \text{ m}^2$$
$$I_y = I_z = \tfrac{1}{4}\pi c^4 = \tfrac{1}{4}\pi(0.020 \text{ m})^4 = 125.7 \times 10^{-9} \text{ m}^4$$
$$J_C = \tfrac{1}{2}\pi c^4 = \tfrac{1}{2}\pi(0.020 \text{ m})^4 = 251.3 \times 10^{-9} \text{ m}^4$$

We also determine the first moment *Q* and the width *t* of the area of the cross section located above the *z* axis. Recalling that $\bar{y} = 4c/3\pi$ for a semicircle of radius *c*, we have

$$Q = A'\bar{y} = \left(\frac{1}{2}\pi c^2\right)\left(\frac{4c}{3\pi}\right) = \frac{2}{3}c^3 = \frac{2}{3}(0.020 \text{ m})^3$$
$$= 5.33 \times 10^{-6} \text{ m}^3$$

and

$$t = 2c = 2(0.020 \text{ m}) = 0.040 \text{ m}$$

Normal Stresses. We observe that normal stresses are produced at *K* by the centric force $\mathbf{F}$ and the bending couple $\mathbf{M}_y$, but that the couple $\mathbf{M}_z$ does not produce any stress at *K*, since *K* is located on the neutral axis corresponding to that couple. Determining each sign from Fig. 8.23, we write

$$\sigma_x = -\frac{F}{A} + \frac{M_y c}{I_y} = -11.9 \text{ MPa} + \frac{(750 \text{ N} \cdot \text{m})(0.020 \text{ m})}{125.7 \times 10^{-9} \text{ m}^4}$$
$$= -11.9 \text{ MPa} + 119.3 \text{ MPa}$$
$$\sigma_x = +107.4 \text{ MPa}$$

8.61 For the beam and loading of Prob. 8.59, determine the principal stresses and the maximum shearing stress at points a and b.

8.62 For the beam and loading of Prob. 8.60, determine the principal stresses and the maximum shearing stress at points d and e.

8.63 Two forces are applied to a W200 × 41.7 rolled-steel beam as shown. Determine the principal stresses, principal planes, and maximum shearing stress at point a.

8.64 Two forces are applied to a W200 × 41.7 rolled-steel beam as shown. Determine the principal stresses, principal planes, and maximum shearing stress at point b.

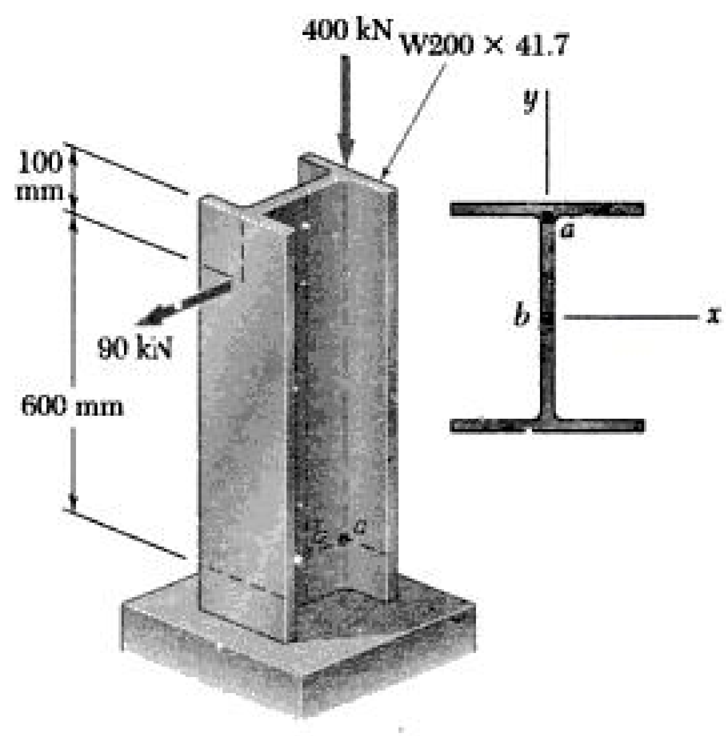


Fig. P8.63 and P8.64

8.65 Two forces, $\mathbf{P}_1$ and $\mathbf{P}_2$, are applied as shown in directions perpendicular to the longitudinal axis of a W310 × 60 beam. Knowing that $P_1 = 25$ kN and $P_2 = 24$ kN, determine the principal stresses and the maximum shearing stress at point a.

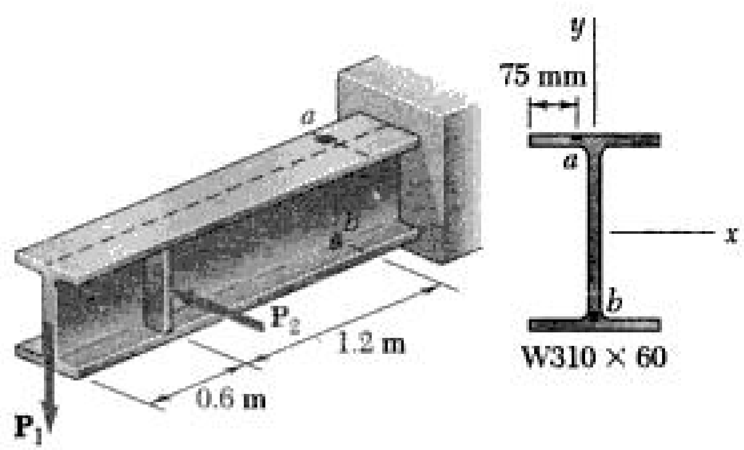


Fig. *P8.65* and *P8.66*

8.66 Two forces, $\mathbf{P}_1$ and $\mathbf{P}_2$, are applied as shown in directions perpendicular to the longitudinal axis of a W310 × 60 beam. Knowing that $P_1 = 25$ kN and $P_2 = 24$ kN, determine the principal stresses and the maximum shearing stress at point b.

8.67 A force **P** is applied to a cantilever beam by means of a cable attached to a bolt located at the center of the free end of the beam. Knowing that **P** acts in a direction perpendicular to the longitudinal axis of the beam, determine (a) the normal stress at point a in terms of P, b, h, l, and β (b) the values of β for which the normal stress at a is zero.

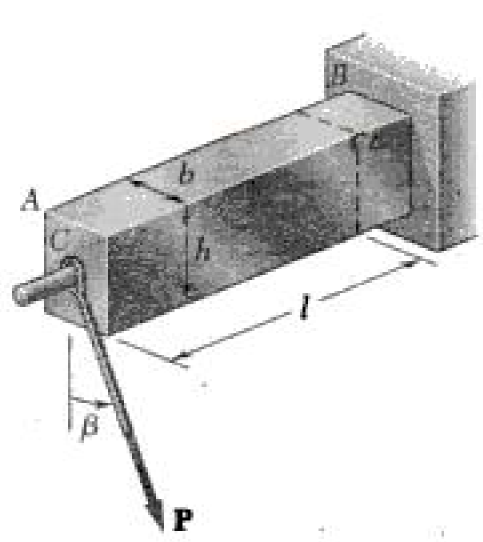


Fig. P8.67

8.80 A vertical force **P** of magnitude 250 N is applied to the crank at point *A*. Knowing that the shaft *BDE* has a diameter of 18 mm, determine the principal stresses and the maximum shearing stress at point *H* located at the top of the shaft, 50 mm to the right of support *D*.

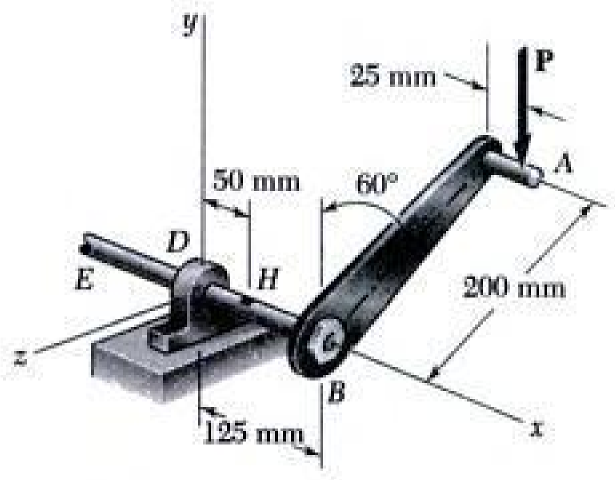


Fig. ***P8.80***

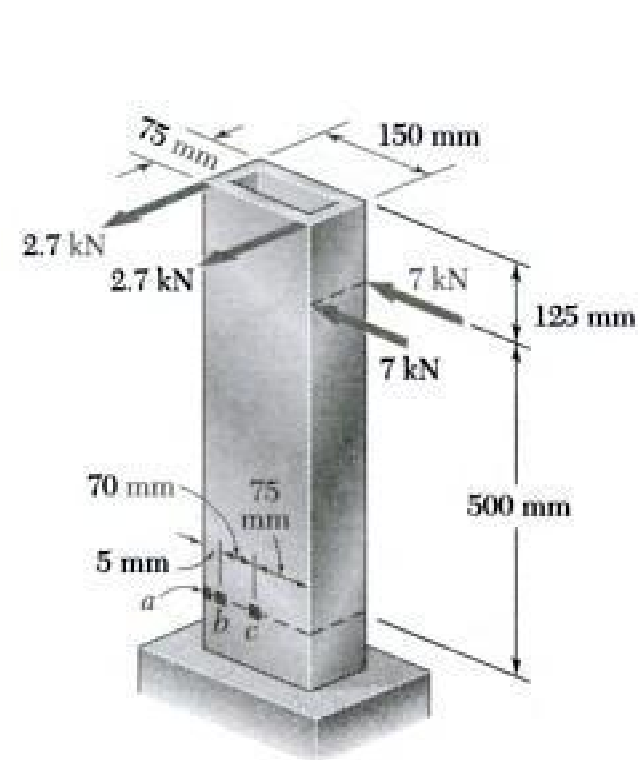


Fig. P8.81

8.81 Knowing that the structural tube shown has a uniform wall thickness of 6 mm, determine the normal and shearing stresses at the three points indicated.

8.82 For the post and loading shown, determine the principal stresses, principal planes, and maximum shearing stress at point *H*.

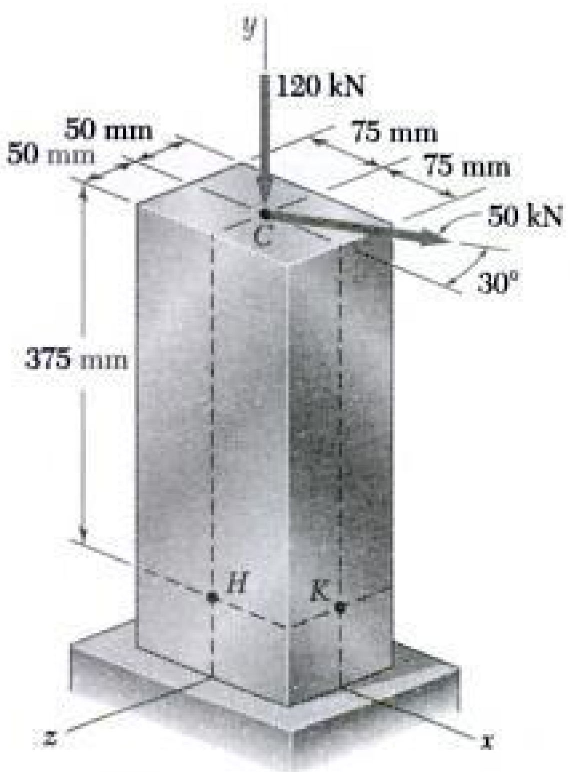


Fig. P8.82 and ***P8.83***

8.83 For the post and loading shown, determine the principal stresses, principal planes, and maximum shearing stress at point *K*.

8.84 Forces are applied at points *A* and *B* of the solid cast-iron bracket shown. Knowing that the bracket has a diameter of 20 mm, determine the principal stresses and the maximum shearing stress (*a*) at point *H*, (*b*) at point *K*.

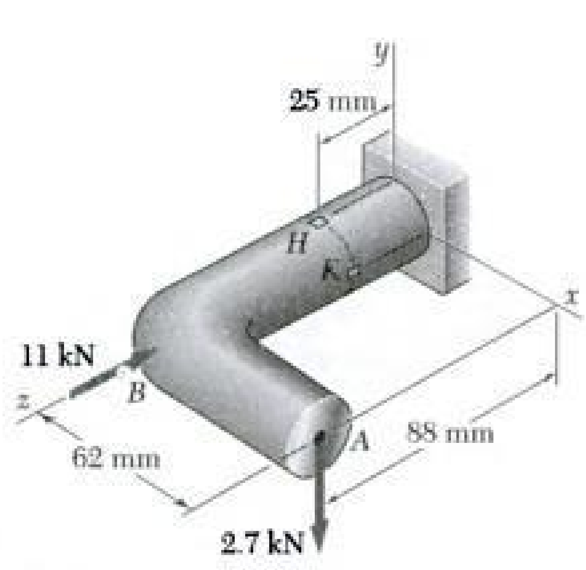


Fig. P8.84

10.84 A square structural tube having the cross section shown is used as a column of 3.1-m effective length to carry a centric load of 129 kN. Knowing that the tubes available for use are made with wall thicknesses of 3.2, 4.8, 6.4, and 7.9 mm, use allowable stress design to determine the lightest tube that can be used. Use $\sigma_Y = 250$ MPa and $E = 200$ GPa.

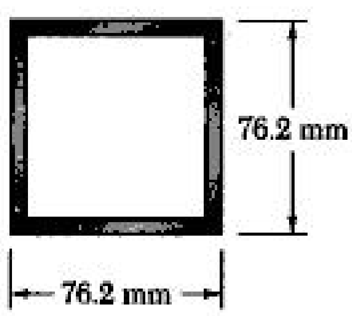


Fig. P10.84

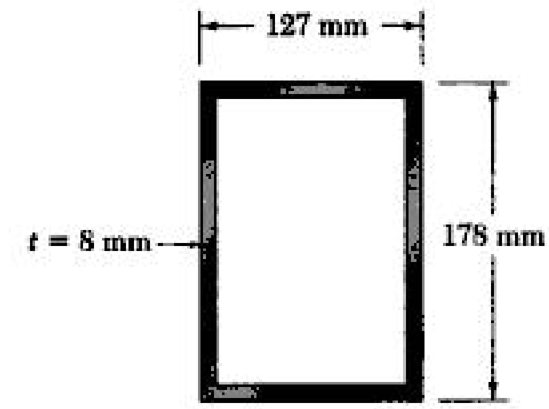


Fig. P10.85

***10.85** A rectangular tube having the cross section shown is used as a column of 4.5-m effective length. Knowing that $\sigma_Y = 250$ MPa and $E = 200$ MPa, use load and resistance factor design to determine the largest centric live load that can be applied if the centric dead load is 140 kN. The dead load factor is $\gamma_D = 1.2$, the live load factor $\gamma_L = 1.6$, and the resistance factor $\phi = 0.85$.

***10.86** A column with a 5.9 m effective length supports a centric load, with ratio of dead to live load equal to 1.35. The dead load factor is $\gamma_D = 1.2$, the live load factor $\gamma_L = 1.6$, and the resistance factor $\phi = 0.85$. Use load and resistance factor design to determine the allowable centric dead and live loads if the column is made of the following rolled-steel shape: (*a*) W250 × 58 (*b*) W360 × 101. Use $E = 200$ GPa and $\sigma_Y = 345$ MPa.

***10.87** The structural tube having the cross section shown is used as a column of 4.5 m effective length to carry a centric dead load of 230 kN and a centric live load of 260 kN. Knowing that the tubes available for use are made with wall thicknesses in increments of 2 mm from 4 mm to 10 mm, use load and resistance factor design to determine the lightest tube that can be used. Use $\sigma_Y = 250$ MPa and $E = 200$ GPa. The dead load factor is $\gamma_D = 1.2$, the live load factor $\gamma_L = 1.6$, and the resistance factor $\phi = 0.85$.

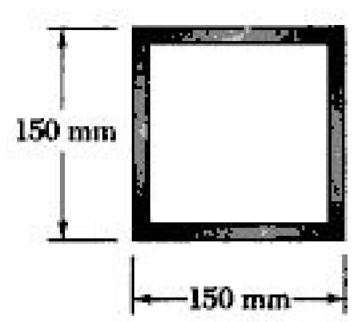


Fig. P10.87

***10.88** A column of 5.5-m effective length must carry a centric dead load of 310 kN and a centric live load of 375 kN. Knowing that $\sigma_Y = 250$ MPa and $E = 200$ GPa, use load and resistance factor design to select the wide-flange shape of 310-mm nominal depth that should be used. The dead load factor is $\gamma_D = 1.2$, the live load factor $\gamma_L = 1.6$ and the resistance factor $\phi = 0.85$.

REVIEW AND SUMMARY FOR CHAPTER 10

This chapter was devoted to the design and analysis of columns, i.e., prismatic members supporting axial loads. In order to gain insight into the behavior of columns, we first considered in Sec. 10.2 the equilibrium of a simple model and found that for values of the load P exceeding a certain value P_{cr}, called the *critical load,* two equilibrium positions of the model were possible: the original position with zero transverse deflections and a second position involving deflections that could be quite large. This led us to conclude that the first equilibrium position was unstable for $P > P_{cr}$, and stable for $P < P_{cr}$, since in the latter case it was the only possible equilibrium position.

Critical load

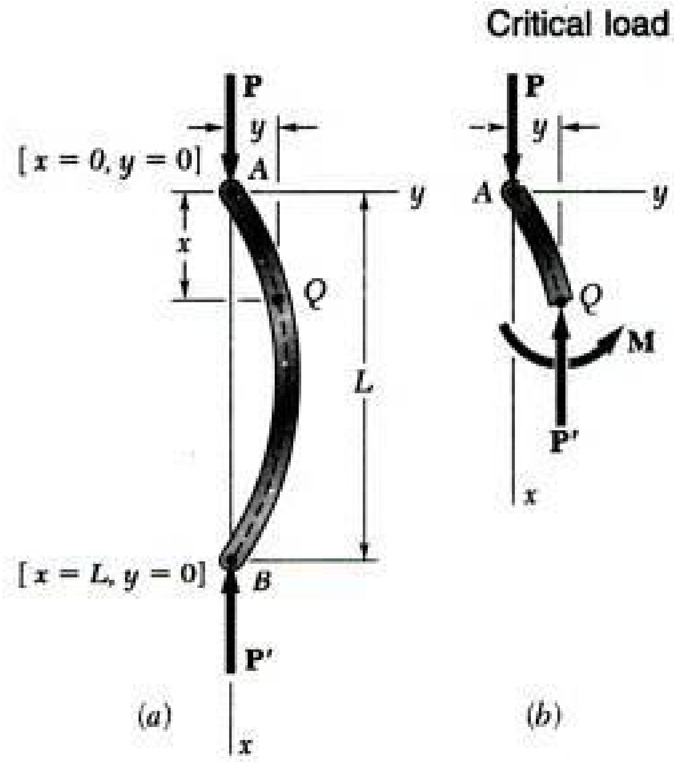


Fig. 10.8

In Sec. 10.3, we considered a pin-ended column of length L and of constant flexural rigidity EI subjected to an axial centric load P. Assuming that the column had buckled (Fig. 10.8), we noted that the bending moment at point Q was equal to $-Py$ and wrote

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y \qquad (10.4)$$

Solving this differential equation, subject to the boundary conditions corresponding to a pin-ended column, we determined the smallest load P for which buckling can take place. This load, known as the *critical load* and denoted by P_{cr}, is given by *Euler's formula:*

Euler's formula

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad (10.11)$$

where L is the length of the column. For this load or any larger load, the equilibrium of the column is unstable and transverse deflections will occur.

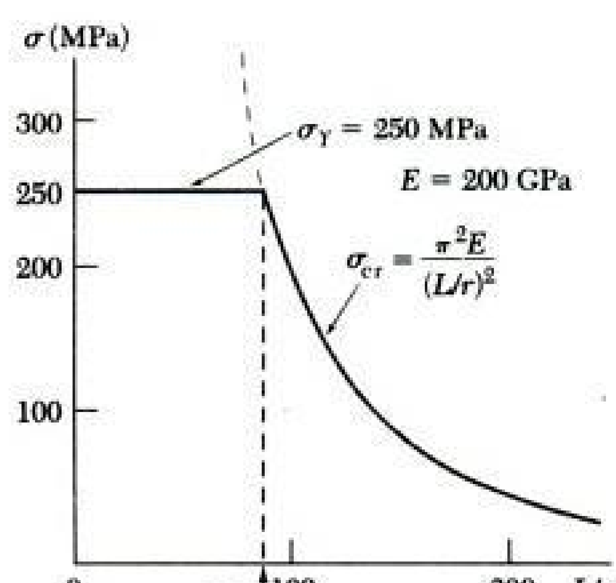


Fig. 10.9

Denoting the cross-sectional area of the column by A and its radius of gyration by r, we determined the critical stress σ_{cr} corresponding to the critical load P_{cr}:

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \qquad (10.13)$$

Slenderness ratio

The quantity L/r is called the *slenderness ratio* and we plotted σ_{cr} as a function of L/r (Fig. 10.9). Since our analysis was based on stresses remaining below the yield strength of the material, we noted that the column would fail by yielding when $\sigma_{cr} > \sigma_Y$.

COMPUTER PROBLEMS

The following problems are designed to be solved with a computer.

10.C1 A solid steel rod having an effective length of 500 mm is to be used as a compression strut to carry a centric load **P**. For the grade of steel used $E = 200$ GPa and $\sigma_Y = 250$ MPa. Knowing that a factor of safety of 2.8 is required and using Euler's formula, write a computer program and use it to calculate the allowable centric load P_{all} for values of the radius of the rod from 6 mm to 24 mm, using 2-mm increments.

10.C2 An aluminum bar is fixed at end A and supported at end B so that it is free to rotate about a horizontal axis through the pin. Rotation about a vertical axis at end B is prevented by the brackets. Knowing that $E = 70$ GPa, use Euler's formula with a factor of safety of 2.5 to determine the allowable centric load **P** for values of b from 20 mm to 38 mm, using 3 mm increments.

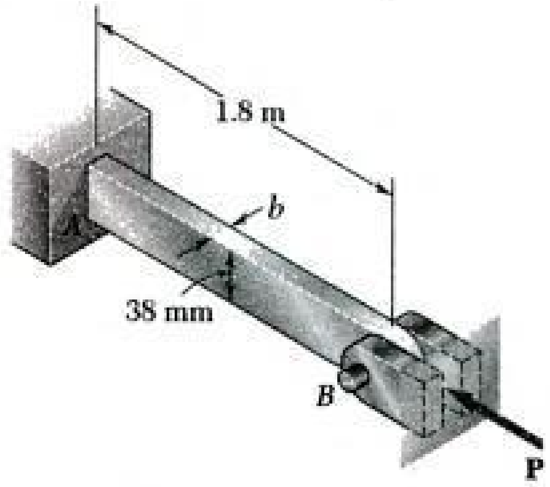


Fig. P10.C2

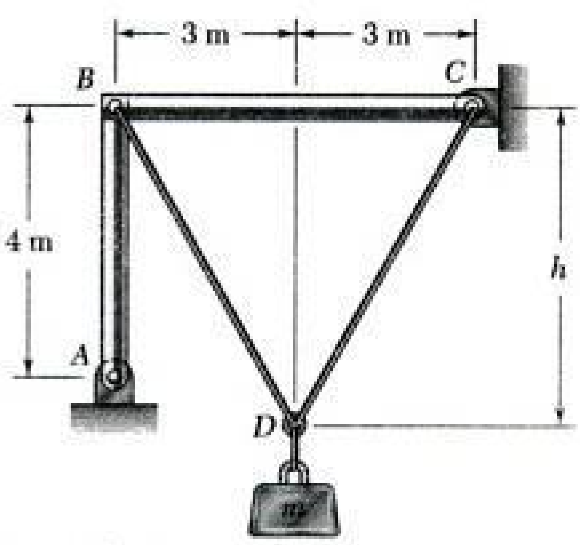


Fig. P10.C3

10.C3 The pin-ended members AB and BC consist of sections of aluminum pipe of 120 mm outer diameter and 10-mm wall thickness. Knowing that a factor of safety of 3.5 is required, determine the mass m of the largest block that can be supported by the cable arrangement shown for values of h from 4 m to 8 m, using 0.25 m increments. Use $E = 70$ GPa and consider only buckling in the plane of the structure.

10.C4 An axial load **P** is applied at a point located on the x axis at a distance $e = 12$ mm from the geometric axis of the W200 × 59 rolled-steel column AB. Using $E = 200$ GPa, write a computer program and use it to calculate for values of P from 110 kN to 330 kN, using 20 kN increments, (a) the horizontal deflection at the midpoint C, (b) the maximum stress in the column.

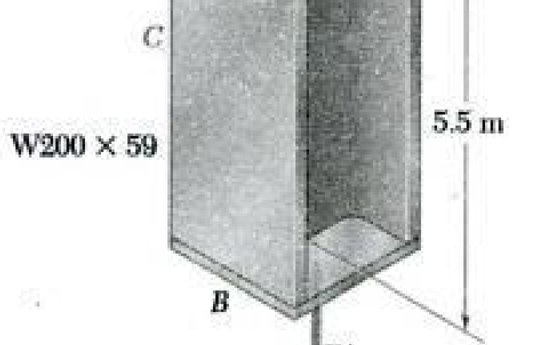


Fig. P10.C4

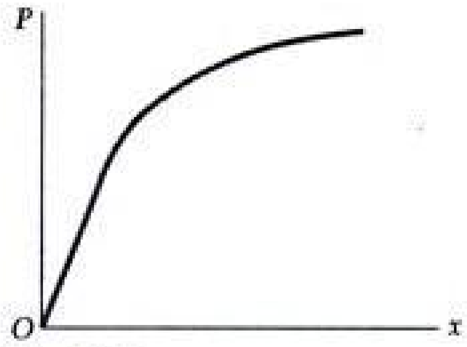


Fig. 11.2

Let us now consider the work dU done by the load $\mathbf{P}$ as the rod elongates by a small amount dx. This *elementary work* is equal to the product of the magnitude P of the load and of the small elongation dx. We write

$$dU = P\,dx \tag{11.1}$$

and note that the expression obtained is equal to the element of area of width dx located under the load-deformation diagram (Fig. 11.3). The *total work* U done by the load as the rod undergoes a deformation x_1 is thus

$$U = \int_0^{x_1} P\,dx$$

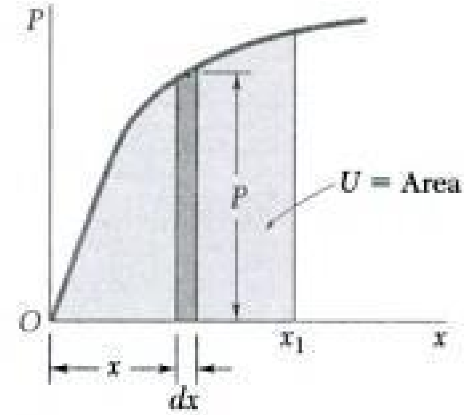


Fig. 11.3

and is equal to the area under the load-deformation diagram between $x = 0$ and $x = x_1$.

The work done by the load $\mathbf{P}$ as it is slowly applied to the rod must result in the increase of some energy associated with the deformation of the rod. This energy is referred to as the *strain energy* of the rod. We have, by definition,

$$\text{Strain energy} = U = \int_0^{x_1} P\,dx \tag{11.2}$$

We recall that work and energy should be expressed in units obtained by multiplying units of length by units of force. Thus, if SI metric units are used, work and energy are expressed in N · m; this unit is called a *joule* (J).

In the case of a linear and elastic deformation, the portion of the load-deformation diagram involved can be represented by a straight line of equation $P = kx$ (Fig. 11.4). Substituting for P in Eq. (11.2), we have

$$U = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2$$

or

$$U = \tfrac{1}{2}P_1x_1 \tag{11.3}$$

where P_1 is the value of the load corresponding to the deformation x_1.

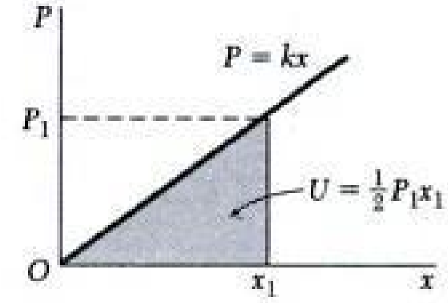


Fig. 11.4

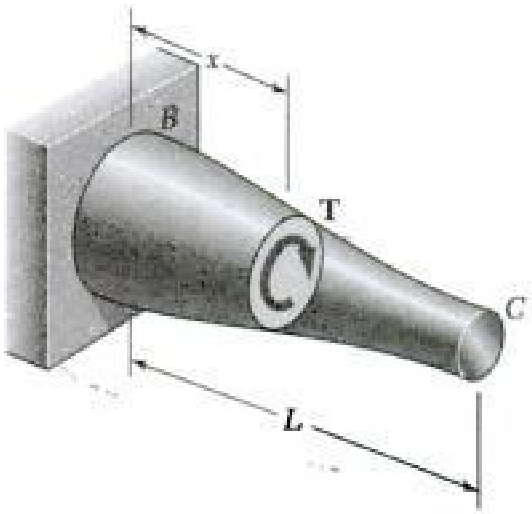


Fig. 11.18

Strain Energy in Torsion. Consider a shaft BC of length L subjected to one or several twisting couples. Denoting by J the polar moment of inertia of the cross section located at a distance x from B (Fig. 11.18), and by T the internal torque in that section, we recall that the shearing stresses in the section are $\tau_{xy} = T\rho/J$. Substituting for τ_{xy} into Eq. (11.20), we have

$$U = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2\rho^2}{2GJ^2} dV$$

Setting $dV = dA\,dx$, where dA represents an element of the cross-sectional area, and observing that $T^2/2GJ^2$ is a function of x alone, we write

$$U = \int_0^L \frac{T^2}{2GJ^2}\left(\int \rho^2\,dA\right) dx$$

Recalling that the integral within the parentheses represents the polar moment of inertia J of the cross section, we have

$$U = \int_0^L \frac{T^2}{2GJ} dx \tag{11.21}$$

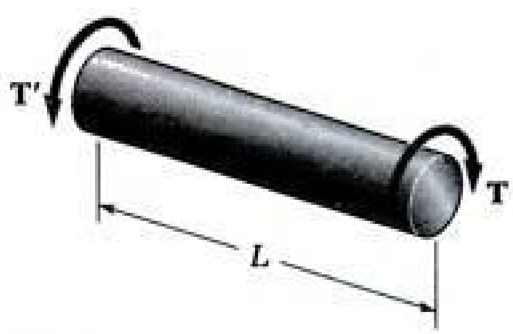


Fig. 11.19

In the case of a shaft of uniform cross section subjected at its ends to equal and opposite couples of magnitude T (Fig. 11.19), Eq. (11.21) yields

$$U = \frac{T^2L}{2GJ} \tag{11.22}$$

EXAMPLE 11.04

A circular shaft consists of two portions BC and CD of the same material and same length, but of different cross sections (Fig. 11.20). Determine the strain energy of the shaft when it is subjected to a twisting couple **T** at end D, expressing the result in terms of T, L, G, the polar moment of inertia J of the smaller cross section, and the ratio n of the two diameters.

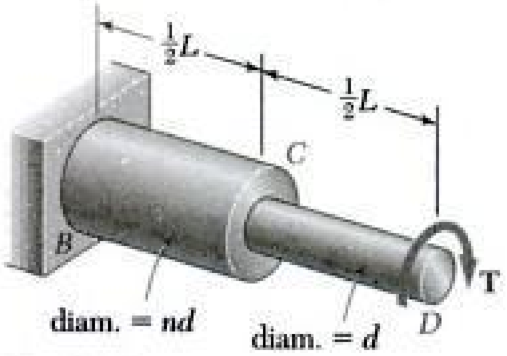


Fig. 11.20

We use Eq. (11.22) to compute the strain energy of each of the two portions of shaft, and add the expressions obtained. Noting that the polar moment of inertia of portion BC is equal to n^4J, we write

$$U_n = \frac{T^2(\frac{1}{2}L)}{2GJ} + \frac{T^2(\frac{1}{2}L)}{2G(n^4J)} = \frac{T^2L}{4GJ}\left(1 + \frac{1}{n^4}\right)$$

or

$$U_n = \frac{1 + n^4}{2n^4}\frac{T^2L}{2GJ} \tag{11.23}$$

We check that, for $n = 1$, we have

$$U_1 = \frac{T^2L}{2GJ}$$

which is the expression given in Eq. (11.22) for a shaft of length L and uniform cross section. We also note that, for $n > 1$, we have $U_n < U_1$; for example, when $n = 2$, we have $U_2 = (\frac{17}{32})\,U_1$. Since the maximum shearing stress occurs in the portion CD of the shaft and is proportional to the torque T, we note as we did earlier in the case of the axial loading of a rod that, for a given allowable stress, increasing the diameter of portion BC of the shaft results in a *decrease* of the overall energy-absorbing capacity of the shaft.

11.34 and 11.35 Using $E = 200$ GPa, determine the strain energy due to bending for the steel beam and loading shown.

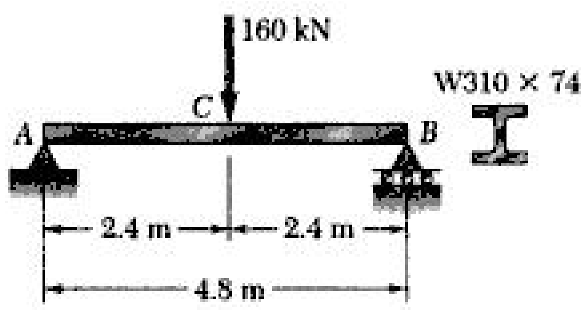


Fig. P11.34

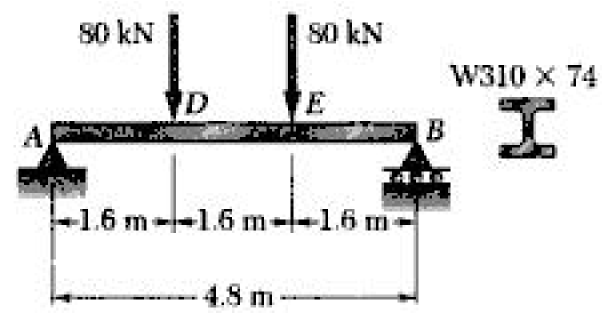


Fig. P11.35

11.36 Using $E = 200$ GPa, determine the strain energy due to bending for the steel beam and loading shown.

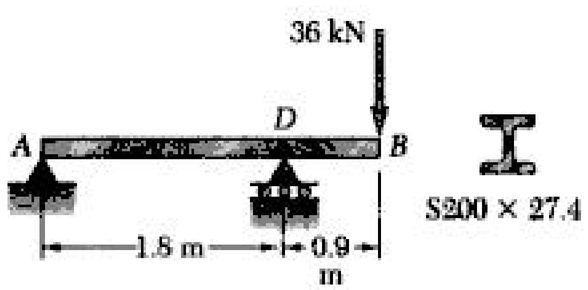


Fig. P11.36

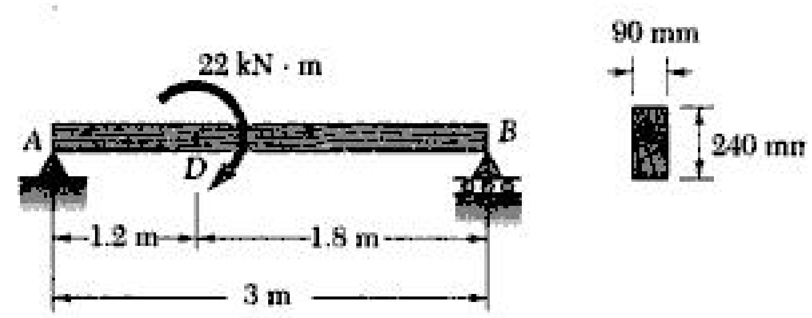


Fig. *P11.37*

11.37 Using $E = 12$ GPa, determine the strain energy due to bending for the timber beam and loading shown.

11.38 Rod AC is made of aluminum ($G = 73$ GPa) and is subjected to a torque **T** applied at end C. Knowing that portion BC of the rod is hollow and has an inner diameter of 16 mm, determine the strain energy of the rod for a maximum shearing stress of 120 MPa.

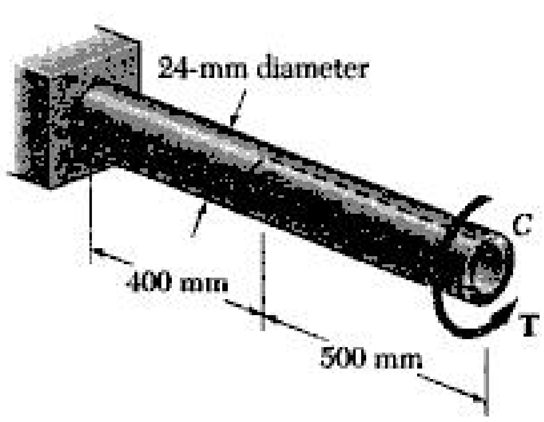


Fig. P11.38

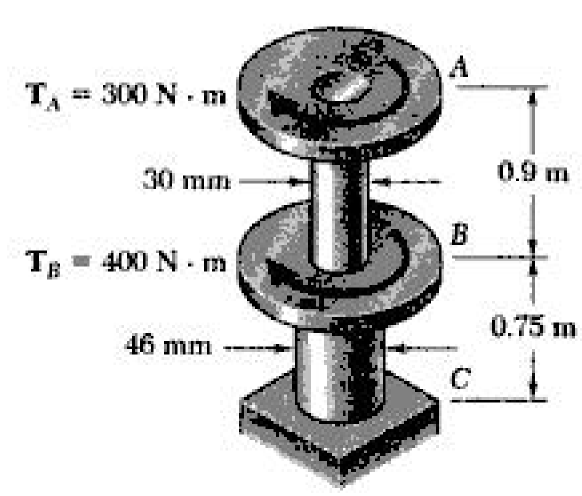


Fig. *P11.39*

11.39 In the assembly shown torques $\mathbf{T}_A$ and $\mathbf{T}_B$ are exerted on disks A and B respectively. Knowing that both shafts are solid and made of aluminum ($G = 73$ GPa), determine the total energy acquired by the assembly.

EXAMPLE 11.10

Determine the deflection of end A of the cantilever beam AB (Fig. 11.34), taking into account the effect of (a) the normal stresses only, (b) both the normal and shearing stresses.

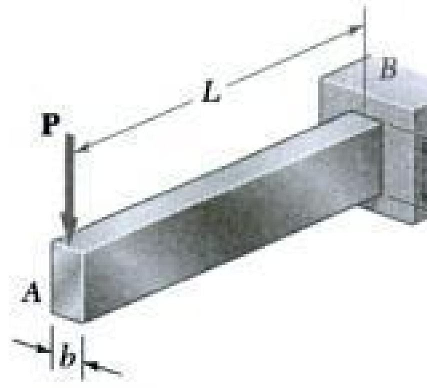


Fig. 11.34

(a) Effect of Normal Stresses. The work of the force **P** as it is slowly applied to A is

$$U = \tfrac{1}{2}Py_A$$

Substituting for U the expression obtained for the strain energy of the beam in Example 11.03, where only the effect of the normal stresses was considered, we write

$$\frac{P^2L^3}{6EI} = \frac{1}{2}Py_A$$

and, solving for y_A,

$$y_A = \frac{PL^3}{3EI}$$

(b) Effect of Normal and Shearing Stresses. We now substitute for U the expression (11.24) obtained in Example 11.05, where the effects of both the normal and shearing stresses were taken into account. We have

$$\frac{P^2L^3}{6EI}\left(1 + \frac{3Eh^2}{10GL^2}\right) = \frac{1}{2}Py_A$$

and, solving for y_A,

$$y_A = \frac{PL^3}{3EI}\left(1 + \frac{3Eh^2}{10GL^2}\right)$$

We note that the relative error when the effect of shear is neglected is the same that was obtained in Example 11.05, i.e., less than $0.9(h/L)^2$. As we indicated then, this is less than 0.9% for a beam with a ratio h/L less than $\frac{1}{10}$.

EXAMPLE 11.11

A torque **T** is applied at the end D of shaft BCD (Fig. 11.35). Knowing that both portions of the shaft are of the same material and same length, but that the diameter of BC is twice the diameter of CD, determine the angle of twist for the entire shaft.

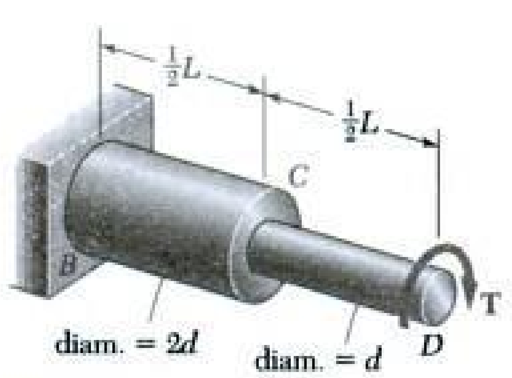


Fig. 11.35

The strain energy of a similar shaft was determined in Example 11.04 by breaking the shaft into its component parts BC and CD. Making $n = 2$ in Eq. (11.23), we have

$$U = \frac{17}{32}\frac{T^2L}{2GJ}$$

where G is the modulus of rigidity of the material and J the polar moment of inertia of portion CD of the shaft. Setting U equal to the work of the torque as it is slowly applied to end D, and recalling Eq. (11.49), we write

$$\frac{17}{32}\frac{T^2L}{2GJ} = \frac{1}{2}T\phi_{D/B}$$

and, solving for the angle of twist $\phi_{D/B}$,

$$\phi_{D/B} = \frac{17TL}{32GJ}$$

11.73 and 11.74 Using the method of work and energy, determine the slope at point D caused by the couple $\mathbf{M}_0$.

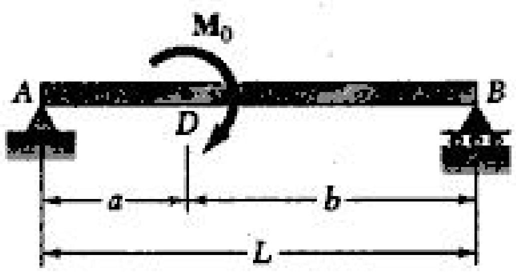


Fig. P11.73

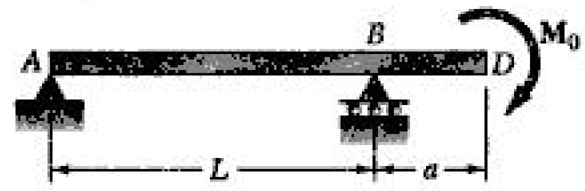


Fig. P11.74

11.75 and *11.76* Using the method of work and energy, determine the deflection at point C caused by the load $\mathbf{P}$.

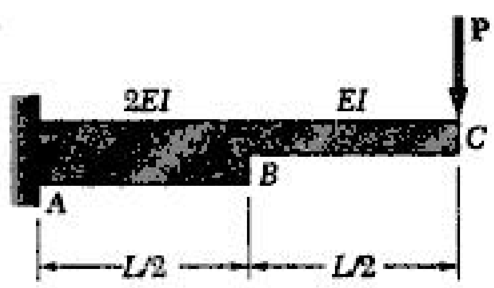


Fig. P11.75

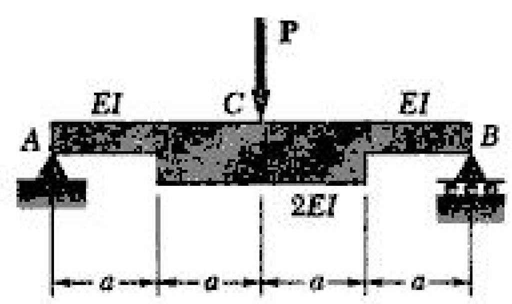


Fig. *P11.76*

11.77 Using the method of work and energy, determine the slope at point B caused by the couple $\mathbf{M}_0$.

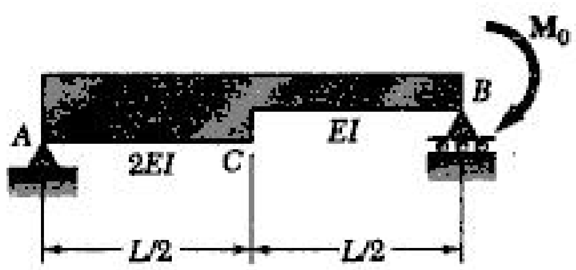


Fig. P11.77

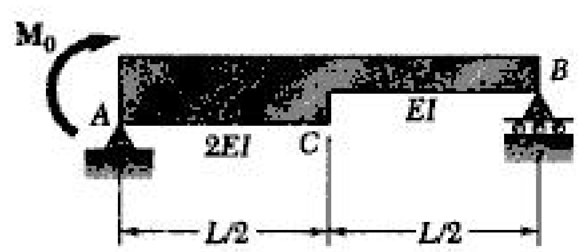


Fig. *P11.78*

11.78 Using the method of work and energy, determine the slope at point A caused by the couple $\mathbf{M}_0$.

11.79 The 12-mm-diameter steel rod ABC has been bent into the shape shown. Knowing that $E = 200$ GPa and $G = 77.2$ GPa, determine the deflection of end C caused by the 150-N force.

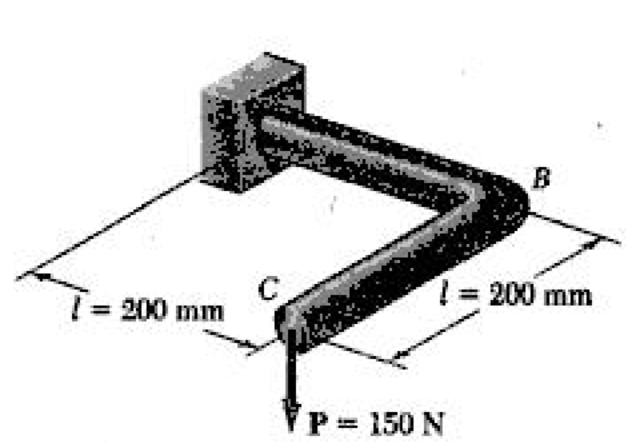


Fig. P11.79

While we are now able to express the strain energy U of a structure subjected to several loads as a function of these loads, we cannot use the method of Sec. 11.10 to determine the deflection of such a structure. Indeed, computing the strain energy U by integrating the strain-energy density u over the structure and substituting the expression obtained into (11.61) would yield only one equation, which clearly could not be solved for the various coefficients a.

*11.12. CASTIGLIANO'S THEOREM

We recall the expression obtained in the preceding section for the strain energy of an elastic structure subjected to two loads $\mathbf{P}_1$ and $\mathbf{P}_2$:

$$U = \tfrac{1}{2}(\mathrm{a}_{11}P_1^2 + 2\mathrm{a}_{12}P_1P_2 + \mathrm{a}_{22}P_2^2) \tag{11.61}$$

where a_{11}, a_{12}, and a_{22} are the influence coefficients associated with the points of application C_1 and C_2 of the two loads. Differentiating both members of Eq. (11.61) with respect to P_1 and recalling Eq. (11.56), we write

$$\frac{\partial U}{\partial P_1} = \mathrm{a}_{11}P_1 + \mathrm{a}_{12}P_2 = x_1 \tag{11.63}$$

Differentiating both members of Eq. (11.61) with respect to P_2, recalling Eq. (11.57), and keeping in mind that $\mathrm{a}_{12} = \mathrm{a}_{21}$, we have

$$\frac{\partial U}{\partial P_2} = \mathrm{a}_{12}P_1 + \mathrm{a}_{22}P_2 = x_2 \tag{11.64}$$

More generally, if an elastic structure is subjected to n loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, the deflection x_j of the point of application of $\mathbf{P}_j$, measured along the line of action of $\mathbf{P}_j$, can be expressed as the partial derivative of the strain energy of the structure with respect to the load $\mathbf{P}_j$. We write

$$x_j = \frac{\partial U}{\partial P_j} \tag{11.65}$$

This is *Castigliano's theorem,* named after the Italian engineer Alberto Castigliano (1847–1884) who first stated it.†

†In the case of an elastic structure subjected to n loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, the deflection of the point of application of $\mathbf{P}_j$, measured along the line of action of $\mathbf{P}_j$, can be expressed as

$$x_j = \sum_k \mathrm{a}_{jk}P_k \tag{11.66}$$

and the strain energy of the structure is found to be

$$U = \tfrac{1}{2}\sum_i \sum_k \mathrm{a}_{ik}P_iP_k \tag{11.67}$$

Differentiating U with respect to P_j, and observing that P_j is found in terms corresponding to either $i = j$ or $k = j$, we write

$$\frac{\partial U}{\partial P_j} = \frac{1}{2}\sum_k \mathrm{a}_{jk}P_k + \frac{1}{2}\sum_i \mathrm{a}_{ij}P_i$$

or, since $\mathrm{a}_{ij} = \mathrm{a}_{ji}$,

$$\frac{\partial U}{\partial P_j} = \frac{1}{2}\sum_k \mathrm{a}_{jk}P_k + \frac{1}{2}\sum_i \mathrm{a}_{ji}P_i = \sum_k \mathrm{a}_{jk}P_k$$

Recalling Eq. (11.66), we verify that

$$x_j = \frac{\partial U}{\partial P_j} \tag{11.65}$$

EXAMPLE 11.16

A load **P** is supported at B by three rods of the same material and the same cross-sectional area A (Fig. 11.52). Determine the force in each rod.

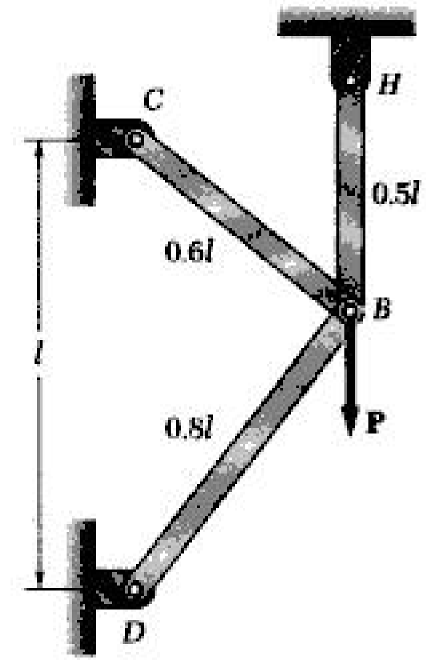


Fig. 11.52

The structure is statically indeterminate to the first degree. We consider the reaction at H as redundant and release rod BH from its support at H. The reaction $\mathbf{R}_H$ is now considered as an unknown load (Fig. 11.53) and will be determined from the condition that the deflection y_H of point H must be zero. By Castigliano's theorem $y_H = \partial U/\partial R_H$, where U is the strain energy of the three-rod system under the load **P** and the redundant reaction $\mathbf{R}_H$. Recalling Eq. (11.72), we write

$$y_H = \frac{F_{BC}(BC)}{AE}\frac{\partial F_{BC}}{\partial R_H} + \frac{F_{BD}(BD)}{AE}\frac{\partial F_{BD}}{\partial R_H} + \frac{F_{BH}(BH)}{AE}\frac{\partial F_{BH}}{\partial R_H} \qquad (11.90)$$

We note that the force in rod BH is equal to R_H and write

$$F_{BH} = R_H \qquad (11.91)$$

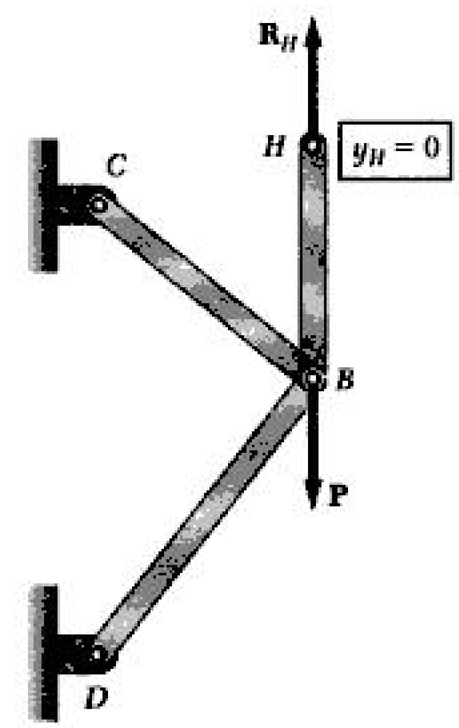


Fig. 11.53

Then, from the free-body diagram of pin B (Fig. 11.54), we obtain

$$F_{BC} = 0.6P - 0.6R_H \qquad F_{BD} = 0.8R_H - 0.8P \qquad (11.92)$$

Differentiating with respect to R_H the force in each rod, we write

$$\frac{\partial F_{BC}}{\partial R_H} = -0.6 \qquad \frac{\partial F_{BD}}{\partial R_H} = 0.8 \qquad \frac{\partial F_{BH}}{\partial R_H} = 1 \qquad (11.93)$$

Substituting from (11.91), (11.92), and (11.93) into (11.90), and noting that the lengths BC, BD, and BH are respectively equal to $0.6l$, $0.8l$, and $0.5l$, we write

$$y_H = \frac{1}{AE}[(0.6P - 0.6R_H)(0.6l)(-0.6) + (0.8R_H - 0.8P)(0.8l)(0.8) + R_H(0.5l)(1)]$$

Setting $y_H = 0$, we obtain

$$1.228R_H - 0.728P = 0$$

and, solving for R_H,

$$R_H = 0.593P$$

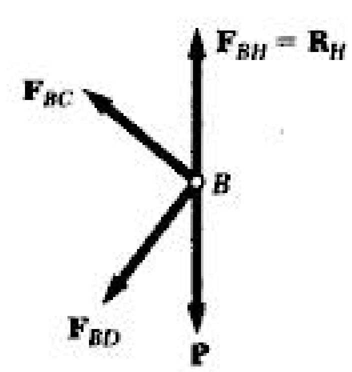


Fig. 11.54

Carrying this value into Eqs. (11.91) and (11.92), we obtain the forces in the three rods:

$$F_{BC} = +0.244P \qquad F_{BD} = -0.326P \qquad F_{BH} = +0.593P$$

PROBLEMS

11.90 through 11.92 Using the information provided in Appendix D, compute the work of the loads as they are applied to the beam (*a*) if the load **P** is applied first, (*b*) if the couple $\mathbf{M}_0$ is applied first.

Fig. 11.90

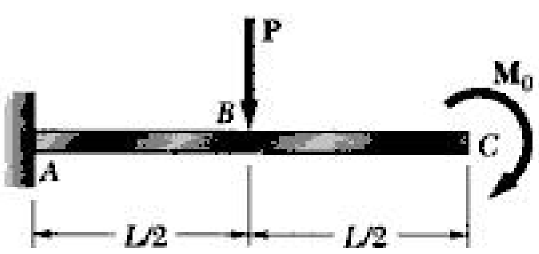


Fig. P11.91

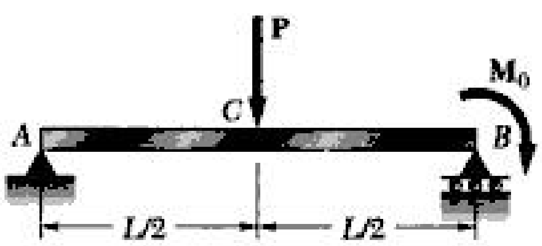


Fig. P11.92

11.93 through 11.95 For the beam and loading shown, (*a*) compute the work of the loads as they are applied successively to the beam, using the information provided in Appendix D, (*b*) compute the strain energy of the beam by the method of Sec. 11.4 and show that it is equal to the work obtained in part *a*.

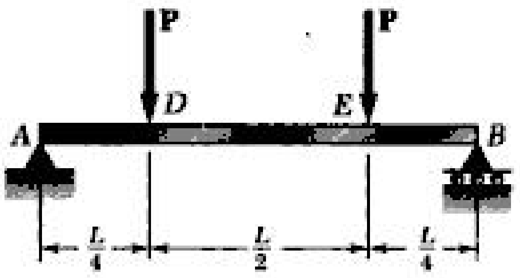


Fig. P11.93

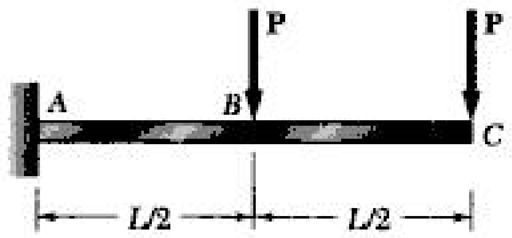


Fig. P11.94

Fig. P11.95

11.96 and 11.97 For the prismatic beam shown, determine the deflection at point *D*.

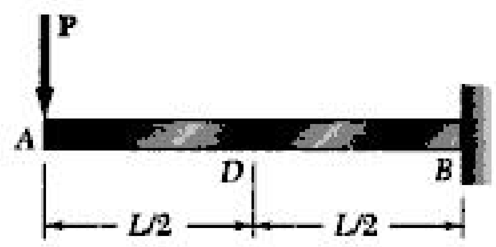


Fig. P11.96 and P11.98

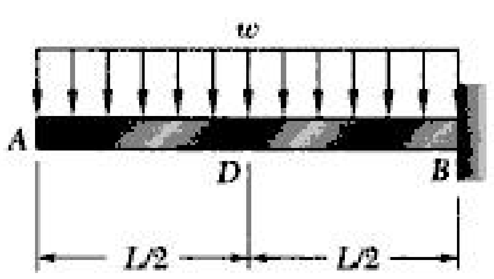


Fig. P11.97 and P11.99

11.98 and 11.99 For the prismatic beam shown, determine the slope at point *D*.

If the normal stress σ remains within the proportional limit of the material, the strain-energy density u is expressed as

$$u = \frac{\sigma^2}{2E}$$

Modulus of resilience

The area under the stress-strain curve from zero strain to the strain ϵ_Y at yield (Fig. 11.9) is referred to as the *modulus of resilience* of the material and represents the energy per unit volume that the material can absorb without yielding. We wrote

$$u_Y = \frac{\sigma_Y^2}{2E} \tag{11.8}$$

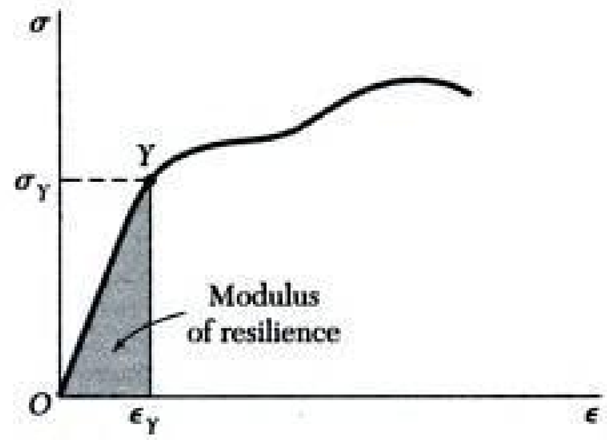


Fig. 11.9

Strain energy under axial load

In Sec. 11.4 we considered the strain energy associated with *normal stresses.* We saw that if a rod of length L and *variable cross-sectional area* A is subjected at its end to a centric axial load **P**, the strain energy of the rod is

$$U = \int_0^L \frac{P^2}{2AE}\,dx \tag{11.13}$$

If the rod is of *uniform cross section* of area A, the strain energy is

$$U = \frac{P^2L}{2AE} \tag{11.14}$$

Strain energy due to bending

We saw that for a beam subjected to transverse loads (Fig. 11.15) the strain energy associated with the normal stresses is

$$U = \int_0^L \frac{M^2}{2EI}\,dx \tag{11.17}$$

where M is the bending moment and EI the flexural rigidity of the beam.

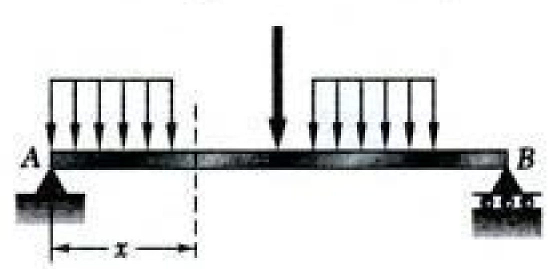


Fig. 11.15

Strain energy due to shearing stresses

The strain energy associated with *shearing stresses* was considered in Sec. 11.5. We found that the strain-energy density for a material in pure shear is

$$u = \frac{\tau_{xy}^2}{2G} \tag{11.19}$$

where τ_{xy} is the shearing stress and G the modulus of rigidity of the material.

Strain energy due to torsion

For a shaft of length L and uniform cross section subjected at its ends to couples of magnitude T (Fig. 11.19) the strain energy was found to be

$$U = \frac{T^2L}{2GJ} \tag{11.22}$$

where J is the polar moment of inertia of the cross-sectional area of the shaft.

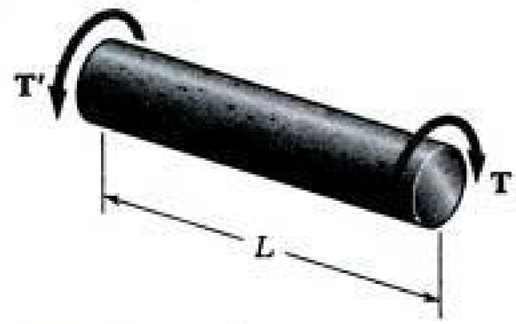


Fig. 11.19

11.142 A single 6-mm-diameter steel pin B is used to connect the steel strip DE to two aluminum strips, each of 20-mm width and 5-mm thickness. The modulus of elasticity is 200 GPa for the steel and 70 GPa for the aluminum. Knowing that for the pin at B the allowable shearing stress is $\tau_{all} = 85$ MPa, determine, for the loading shown, the maximum strain energy that can be acquired by the assembled strips.

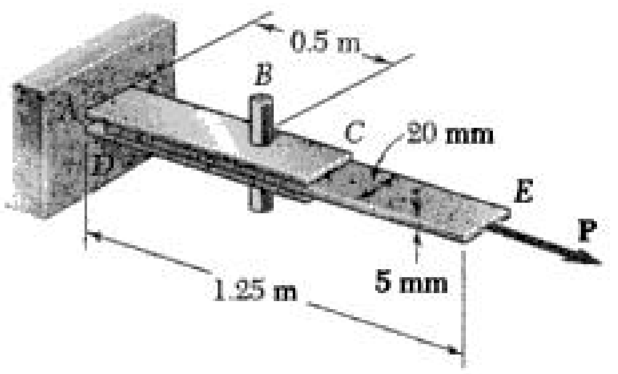


Fig. *P11.142*

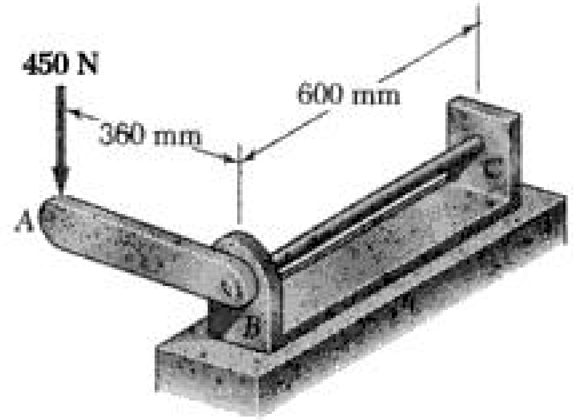


Fig. P11.143

11.143 The 18-mm-diameter steel rod BC is attached to the lever AB and to the fixed support C. The uniform steel lever AB is 9 mm wide and 24 mm deep. Using $E = 200$ GPa, $G = 77$ GPa, and the method of work and energy, determine the deflection of point A.

11.144 The 34 kg collar D is released from rest in the position shown and is stopped by a plate attached at end C of the vertical rod ABC. Knowing that $E = 200$ GPa for both portions of the rod, determine the distance h for which the maximum stress in the rod is 210 MPa.

11.145 The 34 kg collar D is released from rest when $h = 500$ mm and is stopped by a plate attached at end C of the vertical rod ABC. Knowing that $E = 200$ GPa, for both portions of the rod, determine (*a*) the maximum deflection of end C, (*b*) the equivalent static load, (*c*) the maximum stress that occurs in the rod.

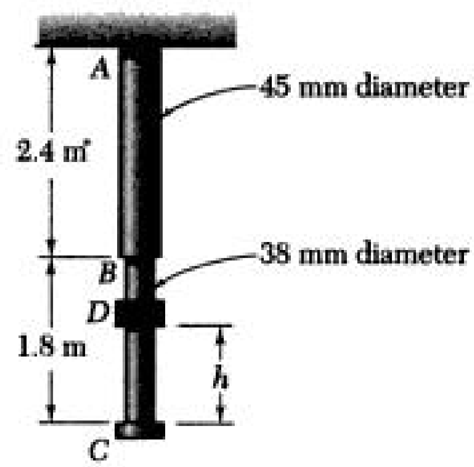


Fig. P11.144 and *P11.145*

11.146 The steel rod BC has a 24-mm diameter and the steel cable $ABDCA$ has a 12-mm diameter. Using $E = 200$ GPa, determine the deflection of point D caused by the 12-kN load.

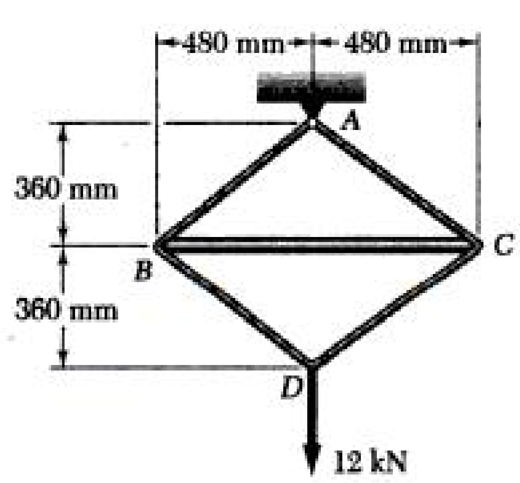


Fig. P11.146

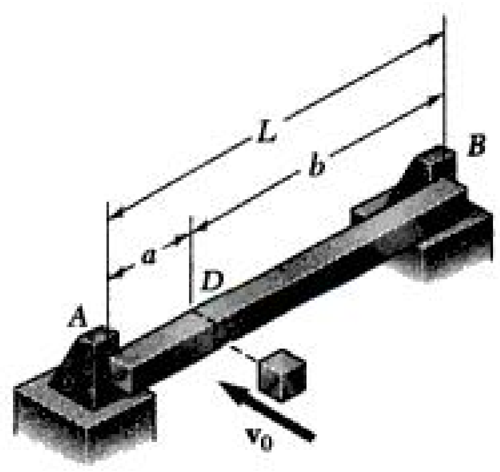


Fig. P11.147

11.147 The simply supported beam AB is struck squarely at D by a block of mass m moving horizontally with a velocity $\mathbf{v}_0$. Show that the resulting maximum normal stress σ_m in the beam due to bending is independent of the location of point D.

When an area possesses an *axis of symmetry*, the first moment of the area with respect to that axis is zero. Indeed, considering the area A of Fig. A.3, which is symmetric with respect to the y axis, we observe that to every element of area dA of abscissa x corresponds an element of area dA' of abscissa $-x$. It follows that the integral in Eq. (A.2) is zero and, thus, that $Q_y = 0$. It also follows from the first of the relations (A.3) that $\bar{x} = 0$. Thus, if an area A possesses an axis of symmetry, its centroid C is located on that axis.

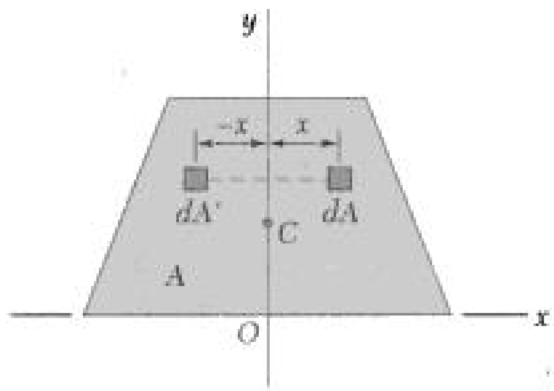


Fig. A.3

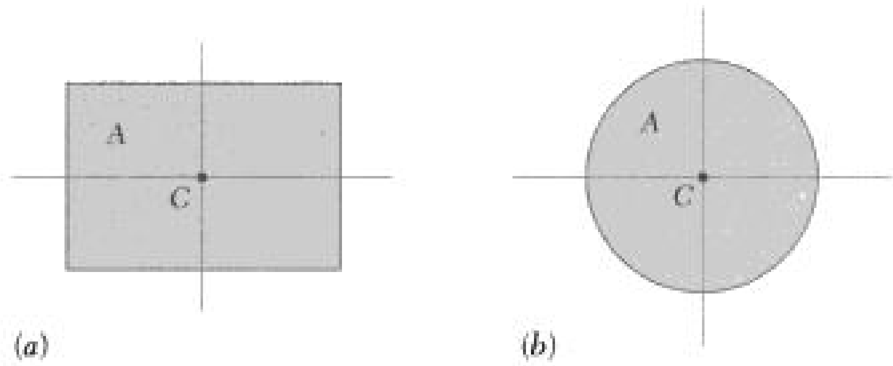


Fig. A.4

Since a rectangle possesses two axes of symmetry (Fig. A.4*a*), the centroid C of a rectangular area coincides with its geometric center. Similarly, the centroid of a circular area coincides with the center of the circle (Fig. A.4*b*).

When an area possesses a *center of symmetry O*, the first moment of the area about any axis through O is zero. Indeed, considering the area A of Fig. A.5, we observe that to every element of area dA of coordinates x and y corresponds an element of area dA' of coordinates $-x$ and $-y$. It follows that the integrals in Eqs. (A.1) and (A.2) are both zero, and that $Q_x = Q_y = 0$. It also follows from Eqs. (A.3) that $\bar{x} = \bar{y} = 0$, that is, the centroid of the area coincides with its center of symmetry.

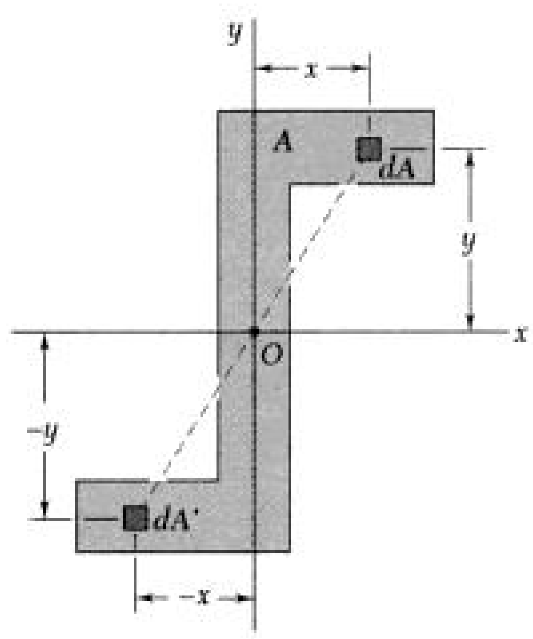


Fig. A.5

When the centroid C of an area can be located by symmetry, the first moment of that area with respect to any given axis can be readily obtained from Eqs. (A.4). For example, in the case of the rectangular area of Fig. A.6, we have

$$Q_x = A\bar{y} = (bh)(\tfrac{1}{2}h) = \tfrac{1}{2}bh^2$$

and

$$Q_y = A\bar{x} = (bh)(\tfrac{1}{2}b) = \tfrac{1}{2}b^2h$$

In most cases, however, it is necessary to perform the integrations indicated in Eqs. (A.1) through (A.3) to determine the first moments and the centroid of a given area. While each of the integrals involved is actually a double integral, it is possible in many applications to select elements of area dA in the shape of thin horizontal or vertical strips, and thus to reduce the computations to integrations in a single variable. This is illustrated in Example A.01. Centroids of common geometric shapes are indicated in a table inside the back cover of this book.

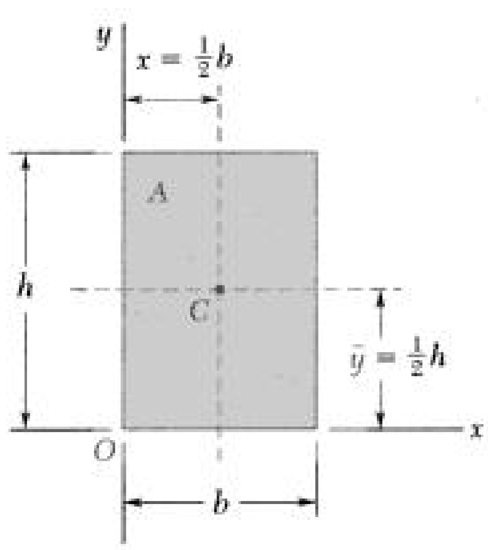


Fig. A.6

resents the first moment $Q_{x'}$ of the area with respect to the x' axis and is equal to zero, since the centroid C of the area is located on that axis. Indeed, we recall from Sec. A.1 that

$$Q_{x'} = A\bar{y}' = A(0) = 0$$

Finally, we observe that the last integral in Eq. (A.15) is equal to the total area A. We have, therefore,

$$I_x = \bar{I}_{x'} + Ad^2 \tag{A.16}$$

This formula expresses that the moment of inertia I_x of an area with respect to an arbitrary x axis is equal to the moment of inertia $\bar{I}_{x'}$ of the area with respect to the centroidal x' axis parallel to the x axis, *plus* the product Ad^2 of the area A and of the square of the distance d between the two axes. This result is known as the *parallel-axis theorem.* It makes it possible to determine the moment of inertia of an area with respect to a given axis, when its moment of inertia with respect to a centroidal axis of the same direction is known. Conversely, it makes it possible to determine the moment of inertia $\bar{I}_{x'}$ of an area A with respect to a centroidal axis x', when the moment of inertia I_x of A with respect to a parallel axis is known, by *subtracting* from I_x the product Ad^2. We should note that the parallel-axis theorem may be used *only if one of the two axes involved is a centroidal axis.*

A similar formula may be derived, which relates the polar moment of inertia J_O of an area with respect to an arbitrary point O and the polar moment of inertia $\bar{J}_C$ of the same area with respect to its centroid C. Denoting by d the distance between O and C, we write

$$J_O = \bar{J}_C + Ad^2 \tag{A.17}$$

A.5. DETERMINATION OF THE MOMENT OF INERTIA OF A COMPOSITE AREA

Consider a composite area A made of several component parts A_1, A_2, and so forth. Since the integral representing the moment of inertia of A may be subdivided into integrals extending over A_1, A_2, and so forth, the moment of inertia of A with respect to a given axis will be obtained by adding the moments of inertia of the areas A_1, A_2, and so forth, with respect to the same axis. The moment of inertia of an area made of several of the common shapes shown in the table inside the back cover of this book may thus be obtained from the formulas given in that table. Before adding the moments of inertia of the component areas, however, the parallel-axis theorem should be used to transfer each moment of inertia to the desired axis. This is shown in Example A.06.

3.17 3.18 kN · m.
3.19 (*a*) 50.3 mm.
(*b*) 63.4 mm.
3.21 (*a*) 72.5 MPa. (*b*) 68.7 MPa.
3.22 (*a*) 59.6 mm. (*b*) 43.9 mm.
3.23 (*a*) 29.6 mm; (*b*) 21.36 mm.
3.25 128.8 N · m.
3.26 (*a*) 37.1 mm. (*b*) 31.7 mm.
3.27 515 N · m.
3.29 (*a*) $T/w = (\tau_m/2\rho g)[c_1^2 + c_2^2)/c_2]$.
(*b*) $T/w = (T/w)_0[1 + (c_1/c_2)^2]$.
3.30 1.0; 1.025; 1.120; 1.200; 1.0.
3.31 (*a*) 4.21°. (*b*) 5.25°.
3.32 (*a*) 428 N · m. (*b*) 10.68°.
3.33 64.5 MPa.
3.35 (*a*) 2.53°. (*b*) 3.42°.
3.37 (*a*) 0.741°. (*b*) 1.573°.
3.38 (*a*) 35.2°. (*b*) 111.6°.
3.39 7.94°.
3.40 4.52°.
3.41 2.53°.
3.44 1.749°.
3.45 (*a*) 82.1 mm. (*b*) 109.4 mm.
3.46 35.2 mm.
3.47 62.9 mm.
3.48 42.1 mm.
3.52 (*a*) 73.6 MPa. (*b*) 34.4 MPa. (*c*) 5.07°.
3.53 4.13°.
3.54 (*a*) 34.37 MPa. (*b*) 50.82 MPa. (*c*) 4.53°.
3.55 7.27°.
3.56 τ_{AB} = 39.6 MPa; τ_{CD} = 31.7 MPa.
3.57 τ_{AB} = 68.8 MPa; τ_{CD} = 14.75 MPa.
3.58 τ_{AB} = 10.34 MPa; τ_{CD} = 48.6 MPa.
3.61 12.24 MPa.
3.62 6.258 mm.
3.65 (*a*) $T/2\pi r_1^2 t$ at $r = r_1$.
3.66 (*a*) 73.7 MPa. (*b*) 0.510°.
3.67 d = 21.4 mm.
3.68 d = 7.36 mm.
3.69 d = 38.8 mm.
3.70 d = 6.69 mm.
3.71 d_2 = 58.2 mm.
3.73 25.6 kW.
3.74 1.89 mm.
3.76 8 mm.
3.77 (*a*) 32.94 MPa. (*b*) 1.47°.
3.78 (*a*) 17.8 mm. (*b*) 5.19°.
3.81 (*a*) 11.95 Hz. (*b*) 23.4 Hz.
3.82 d = 74.0 mm.
3.83 4.91 Hz.
3.84 763.2 rpm.
3.87 313 kW.
3.88 268 kW.
3.89 5.4 mm.
3.90 2084 rpm.
3.91 (*a*) 21.6 MPa. (*b*) 17.9 MPa.
3.94 (*a*) 129.4 MPa; 27 mm.
(*b*) 145 MPa; 23.4 mm.
3.95 (*a*) 125.3 MPa; 19 mm.
(*b*) 145 MPa; 10.3 mm.
3.96 21.2 N · m.
3.97 344 MPa.
3.100 (*a*) 283 N · m. (*b*) 12.95 mm.
3.101 145 MPa; 19.75°.
3.102 (*a*) 1.126 ϕ_Y. (*b*) 1.587 ϕ_Y.
(*c*) 2.15 ϕ_Y.
3.103 145 MPa; 21.03°.
3.104 (*a*) 11.71 kN · m; 3.44°.
(*b*) 14.12 kN · m; 4.82°.
3.105 (*a*) 8.04°. (*b*) 14.89 kN · m.
3.111 2.32 kN · m.
3.112 2.26 kN · m.
3.113 44.9 MPa.
3.114 87.32 MPa.
3.115 (*a*) 40.5 MPa. (*b*) 2.09°.
3.118 6.48°.
3.119 (*b*) $0.221\tau_Y c^3$.
3.120 3.11°.
3.121 1226.6 N · m.
3.122 (*a*) 73.98 MPa; 9.56°.
(*b*) 61.538 MPa; 6.95°.
3.123 (*a*) 186.5 N · m; 8.92°.
(*b*) 224.2 N · m; 7.79°.
3.124 (*a*) 30.8 MPa; 0.535°.
(*b*) 37.9 MPa; 0.684°.
3.125 (*a*) 1.300 kN · m; 0.869°.
(*b*) 1.055 kN · m; 0.902°.
3.126 (*a*) 30.6 mm. (*b*) 30 mm. (*c*) 22.5 mm.
3.129 0.944.
3.130 1.223 mm.
3.131 1.356.
3.132 1.198.
3.133 (*a*) 157.0 kN · m.
(*b*) 8.72°.
3.136 (*a*) 925 N · m. (*b*) 5.79°.
3.137 8.47 MPa.
3.138 18.67 MPa.
3.139 48.2 MPa; 30.1 MPa.
3.140 59.3 MPa; 35.6 MPa.
3.141 8.45 N · m.
3.142 16.85 N · m.
3.145 163.67 MPa; 109.14 MPa.
3.146 (*a*) 12.73 MPa. (*b*) 5.40 N · m.
3.149 (*a*) $1 + t^2/4c_m^2$.
(*b*) 0.25%; 1%; 4%.
3.151 1.73°.
3.154 3.79°.
3.155 211 N · m.
3.156 (*a*) 236.2 N · m. (*b*) 226.8 N · m.
(*c*) 210 N · m.
3.157 1.221.
3.159 (*a*) 8.35 mm. (*b*) 39.34°

5.125 (*a*) 163.35 kN · m; 1.8 m
(*b*) W 410 × 60
5.126 (*a*) 157.5 kN · m; 1.964 m.
(*b*) W 410 × 60
5.129 $|V|_{max}$ = 20.7 kN at x = 0.9 m;
$|M|_{max}$ = 9.72 kN · m at x = 2.7 m.
5.130 $|V|_{max}$ = 89.0 kN for $0 < x < 2$ m;
$|M|_{max}$ = 178.0 kN · m at x = 2 m.
5.131 $|V|_{max}$ = 68.85 kN at x = 0;
$|M|_{max}$ = 51.3 kN · m at x = 1.65 m
5.132 $|V|_{max}$ = 57.5 kN at 1.2 m $< x <$ 1.8 m
$|M|_{max}$ = 52.86 kN · m at x = 1.8 m
5.133 (*a*) $|V|_{max}$ = 13.80 kN at x = 5 m;
$|M|_{max}$ = 16.16 kN · m at x = 2.84 m.
(*b*) 83.3 MPa.
5.134 (*a*) $|V|_{max}$ = 40.0 kN at x = 2 m;
$|M|_{max}$ = 30.0 kN · m at x = 4 m.
(*b*) 40.0 MPa.
5.137 (*a*) $h_0(x/L)^{3/2}$.
(*b*) 150 mm.
5.138 (*a*) $h = h_0\left[\dfrac{(\pi x/2L) - \sin(\pi x/2L)}{(\pi/2) - 1}\right]^{1/2}$
(*b*) 176.7 mm.
5.139 (*a*) $h_0 x/L$.
(*b*) 133.33 kN.
5.140 (*a*) $h_0[(x/L)(1 - x/L)]^{1/2}$.
(*b*) 1185 kN/m.
5.143 1.800 m.
5.144 1.900 m.
5.145 1.824 m; 1.216 m.
5.146 1.52 m; 0.76 m.
5.147 (*a*) $d_0\sqrt[3]{y/L}$. (*b*) 5.09 kN.
5.148 (*a*) $d_0(y/L)^{2/3}$. (*b*) 50.3 mm.
5.151 (*a*) 152.6 MPa. (*b*) 133.6 MPa.
5.152 (*a*) 155.2 MPa. (*b*) 142.4 MPa.
5.153 (*a*) 171.68 MPa (*b*) 119.65.MPa.
5.154 (*a*) 165.20 MPa (*b*) 200.88 MPa.
5.155 (*a*) 4.49 m. (*b*) 211 mm.
5.158 125.4 kN.
5.159 (*a*) x = 0.400 m. (*b*) 156.3 MPa.
5.160 (*a*) x = 0.24 m. (*b*) 150 MPa.
5.163 (*a*) x = 750 mm (*b*) 52.8 kN.
5.164 (*a*) x = 375 mm (*b*) 52.8 N/mm.
5.165 (*a*) 81.92 kN (*b*) 98.1 kN · m.
5.166 136.0 MPa.
5.167 (*a*) 866 mm (*b*) 109.56 MPa.
5.169 (*a*) 37.6 kN · m (*b*) 93.50 MPa.
5.170 (*a*) $V = -12 + 35\langle x - 0.9\rangle^0 - 24\langle x - 2.1\rangle^0 - 12\langle x - 3.3\rangle^0$ kN;
$M = -12x + 35\langle x - 0.9\rangle^1 - 24\langle x - 2.1\rangle^1 - 12\langle x - 3.3\rangle^1$ kN · m.
(*b*) 16.80 kN · m.
5.172 342 mm.
5.173 W 310 × 38.7.
5.175 (*a*) $d_0[(4x/L)(1 - x/L)]^{1/3}$.
(*b*) 32.1 mm.
5.C2 (*a*) 173.2 mm. (*b*) 48.0 mm. (*c*) 203 mm.
5.C4 *For* x = 4.05 m; M_1 = 173.25 kN · m;
M_2 = 206.25 kN · m; M_C = 198 kN · m.

CHAPTER 6

6.1 60 mm.
6.2 2 kN.
6.3 (*a*) 131.4 N (*b*) 0.317 MPa.
6.4 (*a*) 1657.5 N. (*b*) 0.475 MPa.
6.5 180.6 kN.
6.6 209 kN.
6.9 (*a*) 50 MPa. (*b*) 44 MPa.
6.10 (*a*) 23.69 MPa. (*b*) 17.83 MPa.
6.11 (*a*) 920 kPa. (*b*) 765 kPa.
6.12 (*a*) 114.1 MPa. (*b*) 96.9 MPa.
6.15 (*a*) Acceptable. (*b*) Not acceptable; h = 379 mm.
6.16 (*a*) Acceptable. (*b*) Acceptable.
6.17 (*a*) Acceptable. (*b*) Acceptable.
6.18 (*a*) Acceptable. (*b*) Acceptable.
6.21 (*a*) 12.55 MPa. (*b*) 18.82 MPa.
6.22 (*a*) 13 MPa (*b*) 20.94 MPa.
6.23 19.61 MPa.
6.24 23.7 MPa
6.25 356 kN; 254 kN.
6.26 (*a*) Horizontal diameter. (*b*) 1.333.
6.29 336 N.
6.30 (*a*) 239 N. (*b*) 549 N.
6.31 7.64 kN
6.32 11.92 kN
6.35 (*a*) 95.2 MPa. (*b*) 112.9 MPa.
6.36 (*a*) 101.6 MPa. (*b*) 79.9 MPa.
6.37 (*a*) 287 kPa (*b*) 396 kPa
6.38 (*a*) 65.88 mm (*b*) 295 kPa.
6.39 (*a*) 41.4 MPa. (*b*) 41.4 MPa.
6.40 (*a*) 18.23 MPa. (*b*) 14.59 MPa. (*c*) 46.2 MPa.
6.43 255 kN.
6.44 9.05 mm.
6.45 44.5 MPa
6.46 44.5 MPa
6.47 266 kN/m.
6.48 τ_a = 10.76 MPa; τ_b = 0; τ_c = 11.21 MPa;
τ_d = 22.0 MPa; τ_e = 9.35 MPa.
6.51 (*a*) 2.08. (*b*) 2.10.
6.52 (*a*) 2.25. (*b*) 2.12.
6.53 37.4 mm
6.54 (*a*) $(V\sin\theta)/(\pi r_m t)$.
6.59 (*a*) 9.92 MPa (*b*) 9.97 MPa
6.60 (*a*) 6.65 MPa (*b*) 10.9 MPa.
6.61 1.25 a.
6.62 $3(b^2 - a^2)/(6a + 6b + h)$.
6.63 $10a/29$.
6.64 $5a/7$.
6.65 (*a*) 48.7 mm.
(*b*) 0 at *A;* 15.95 and 26.6 MPa at *B;* 73.1 and 43.9 MPa at *D;* 60.5 MPa at midpoint of *DE.*

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