CHAPTER 9

Strength of Materials

James R. Hutchinson

OUTLINE

Mechanics of materials deals with the determination of the internal forces (stresses) and the deformation of solids such as metals, wood, concrete, plastics and composites. In mechanics of materials there are three main considerations in the solution of problems:

1. Equilibrium
2. Force-deformation relations
3. Compatibility

Equilibrium refers to the equilibrium of forces. The laws of statics must hold for the body and all parts of the body. Force-deformation relations refer to the relation of the applied forces to the deformation of the body. If certain forces are applied, then certain deformations will result. Compatibility refers to the compatibility of deformation. Upon loading, the parts of a body or structure must not come apart. These three principles will be emphasized throughout.

AXIALLY LOADED MEMBERS

If a force P is applied to a member as shown in Fig. 9.1(a), then a short distance away from the point of application the force becomes uniformly distributed over the area as shown in Fig. 9.1(b). The force per unit area is called the axis or normal stress and is given the symbol σ. Thus,

$$\sigma = \frac{P}{A} \tag{9.1}$$

The original length between two points A and B is L as shown in Fig. 9.1(c). Upon application of the load P, the length L grows by an amount ΔL. The final length is $L + \Delta L$ as shown in Fig. 9.1(d). A quantity measuring the intensity of deformation and being independent of the original length L is the strain ε, defined as

$$\varepsilon = \frac{\Delta L}{L} = \frac{\delta}{L} \tag{9.2}$$

where ΔL is denoted as δ.

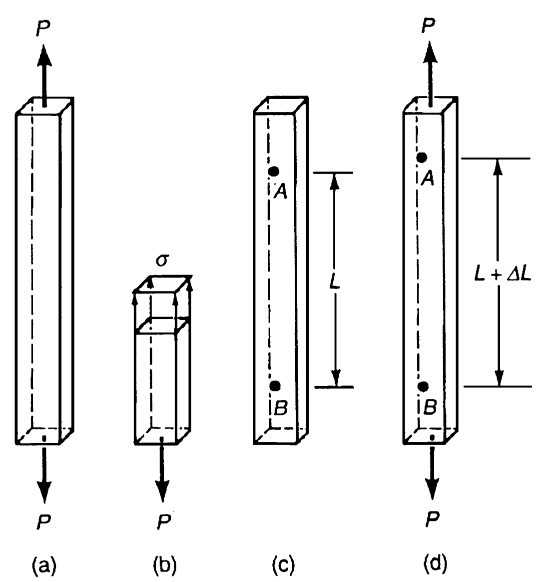


Figure 9.1 Axial member under force P

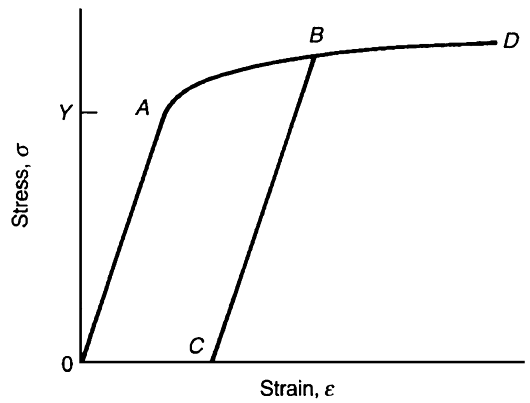


Figure 9.2 Stress-strain curve for a typical material

The relationship between stress and strain is determined experimentally. A typical plot of stress versus strain is shown in Fig. 9.2. On initial loading, the plot is a straight line until the material reaches yield at a stress of *Y*. If the stress remains less than yield then subsequent loading and reloading continues along that same straight line. If the material is allowed to go beyond yield, then during an increase in the load the curve goes from *A* to *D*. If unloading occurs at some point *B*, for example, then the material unloads along the line *BC* which has approximately the same slope as the original straight line from 0 to *A*. Reloading would occur along the line *CB* and then proceed along the line *BD*. It can be seen that if the material is allowed to go into the plastic region (*A* to *D*) it will have a permanent strain offset on unloading.

Modulus of Elasticity

The region of greatest concern is that below the yield point. The slope of the line between 0 and *A* is called the modulus of elasticity and is given the symbol *E*, so

$$\sigma = E\varepsilon \tag{9.3}$$

This is Hooke's Law for axial loading; a more general form will be considered in a later section. The modulus of elasticity is a function of the material alone and not a function of the shape or size of the axial member.

The relation of the applied force in a member to its axial deformation can be found by inserting the definitions of the axial stress [Eq. (9.1)] and the axial strain [Eq. (9.2)] into Hooke's Law [Eq. (9.3)], which gives

$$\frac{P}{A} = E\frac{\delta}{L} \tag{9.4}$$

or

$$\delta = \frac{PL}{AE} \tag{9.5}$$

In the examples that follow, wherever it is appropriate, the three steps of (1) Equilibrium, (2) Force-Deformation, and (3) Compatibility will be explicitly stated.

Example 9.1

The steel rod shown in Exhibit 1 is fixed to a wall at its left end. It has two applied forces. The 3 kN force is applied at the Point B and the 1 kN force is applied at the Point C. The area of the rod between A and B is A_{AB} = 1000 mm^2, and the area of the rod between B and C is A_{BC} = 500 mm^2. Take E = 210 GPa. Find (a) the stress in each section of the rod and (b) the horizontal displacement at the points B and C.

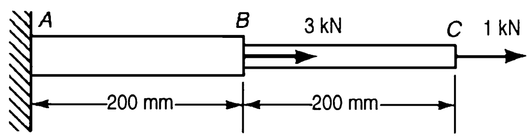


Exhibit 1

Solution—Equilibrium

Draw free-body diagrams for each section of the rod (Exhibit 2). From a summation of forces on the member BC, F_{BC} = 1 kN. Summing forces in the horizontal direction on the center free-body diagram, F_{BA} = 3 + 1 = 4 kN. Summing forces on the left free-body diagram gives $F_{AB} = F_{BA}$ = 4 kN. The stresses then are

$$\sigma_{AB} = 4 \text{ kN/1000 mm}^2 = 4 \text{ MPa}$$
$$\sigma_{BC} = 1 \text{ kN/500 mm}^2 = 2 \text{ MPa}$$

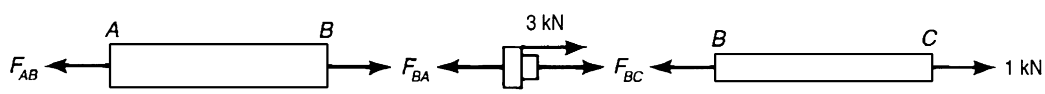


Exhibit 2

Solution—Force-Deformation

$$\delta_{AB} = \left(\frac{PL}{AE}\right)_{AB} = \frac{(4\text{ kN})(200\text{ mm})}{(1000\text{ mm}^2)(210\text{ GPa})} = 0.00381\text{ mm}$$

$$\delta_{BC} = \left(\frac{PL}{AE}\right)_{BC} = \frac{(1\text{ kN})(200\text{ mm})}{(500\text{ mm}^2)(210\text{ GPa})} = 0.001905\text{ mm}$$

Solution—Compatibility

Draw the body before loading and after loading (Exhibit 3).

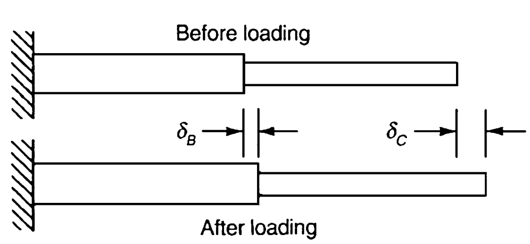


Exhibit 3

It is then obvious that

$$\delta_B = \delta_{AB} = 0.00381 \text{ mm}$$
$$\delta_C = \delta_{AB} + \delta_{BC} = 0.00381 + 0.001905 = 0.00571 \text{ mm}$$

In this first example the problem was statically determinate, and the three steps of Equilibrium, Force-Deformation, and Compatibility were independent steps. The steps are not independent when the problem is statically indeterminate, as the next example will show.

Example 9.2

Consider the same steel rod as in Example 9.1 except that now the right end is fixed to a wall as well as the left (Exhibit 4). It is assumed that the rod is built into the walls before the load is applied. Find (a) the stress in each section of the rod, and (b) the horizontal displacement at the point *B*.

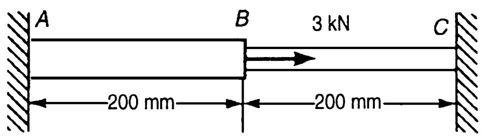


Exhibit 4

Solution—Equilibrium

Draw free-body diagrams for each section of the rod (Exhibit 5). Summing forces in the horizontal direction on the center free-body diagram

$$-F_{AB} + F_{BC} + 3 = 0$$

It can be seen that the forces cannot be determined by statics alone so that the other steps must be completed before the stresses in the rods can be determined.

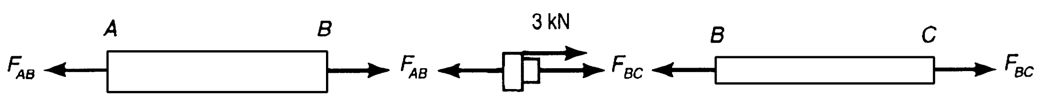


Exhibit 5

Solution—Force-Deformation

$$\delta_{AB} = \left(\frac{PL}{AE}\right)_{AB} = \frac{F_{AB}L}{A_{AB}E}$$
$$\delta_{BC} = \left(\frac{PL}{AE}\right)_{BC} = \frac{F_{BC}L}{A_{BC}}$$

The equilibrium, force-deformation, and compatibility equations can now be solved as follows (see Exhibit 6). The force-deformation relations are put into the compatibility equations:

$$\frac{F_{AB}L}{2A_{BC}E} = -\frac{F_{BC}L}{A_{BC}E}$$

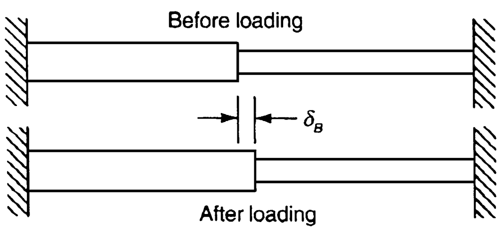


Exhibit 6

Then, $F_{AB} = -2F_{BC}$. Insert this relationship into the equilibrium equation

$$-F_{AB} + F_{BC} + 3 = 0 = 2F_{BC} + F_{BC} + 3; \quad F_{BC} = -1 \text{ kN and } F_{AB} = 2 \text{ kN}$$

The stresses are

$$\sigma_{AB} = 2 \text{ kN/1000 mm}^2 = 2 \text{ MPa (tension)}$$
$$\sigma_{BC} = -1 \text{ kN/500 mm}^2 = -2 \text{ MPa (compression)}$$

The displacement at B is

$$\delta_A = \delta_{AB} = F_{AB}L/(AE) = (2 \text{ kN})(200 \text{ mm})/[(1000 \text{ mm}^2)(210 \text{ GPa})] = 0.001905 \text{ mm}$$

Poisson's Ratio

The axial member shown in Fig. 9.1 also has a strain in the lateral direction. If the rod is in tension, then stretching takes place in the longitudinal direction while contraction takes place in the lateral direction. The ratio of the magnitude of the lateral strain to the magnitude of the longitudinal strain is called Poisson's ratio v.

$$v = -\frac{\text{Lateral strain}}{\text{Longitudinal strain}} \tag{9.6}$$

Poisson's ratio is a dimensionless material property that never exceeds 0.5. Typical values for steel, aluminum, and copper are 0.30, 0.33, and 0.34, respectively.

Example 9.3

A circular aluminum rod 10 mm in diameter is loaded with an axial force of 2 kN. What is the decrease in diameter of the rod? Take E = 70 GN/m^2 and v = 0.33.

Solution

The stress is $\sigma = P/A = 2 \text{ kN}/(\pi 5^2 \text{ mm}^2) = 0.0255 \text{ GN/m}^2 = 25.5 \text{ MN/m}^2$
The longitudinal strain is $\varepsilon_{1on} = \sigma/\text{E} = (25.5 \text{ MN/m}^2)/(70 \text{ GN/m}) = 0.000364$
The lateral strain is $\varepsilon_{1at} = -v\, \varepsilon_{1on} = -0.33(0.000364) = -0.000120$
The decrease in diameter is then $-\text{D}\, \varepsilon_{1at} = -(10 \text{ mm})(-0.000120) = 0.00120 \text{ mm}$

Thermal Deformations

When a material is heated, expansion forces are created. If it is free to expand, the thermal strain is

$$\varepsilon_t = \alpha(t - t_0) \qquad \textbf{(9.7)}$$

where α is the linear coefficient of thermal expansion, t is the final temperature and t_0 is the initial temperature. Since strain is dimensionless, the units of α are °F^{-1} or °C^{-1} (sometimes the units are given as in/in/°F or m/m/°C which amounts to the same thing). The total strain ε_T is equal to the strain from the applied loads plus the thermal strain. For problems where the load is purely axial, this becomes

$$\varepsilon_T = \frac{\sigma}{E} + \alpha(t - t_0) \qquad \textbf{(9.8)}$$

The deformation δ is found by multiplying the strain by the length L

$$\delta = \frac{PL}{AE} + \alpha L(t - t_0) \qquad \textbf{(9.9)}$$

Example 9.4

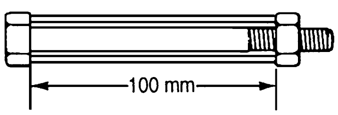


Exhibit 7

A steel bolt is put through an aluminum tube as shown in Exhibit 7. The nut is made just tight. The temperature of the entire assembly is then raised by 60°C. Because aluminum expands more than steel, the bolt will be put in tension and the tube in compression. Find the force in the bolt and the tube. For the steel bolt, take E = 210 GPa, $\alpha = 12 \times 10^{-6}$ °C^{-1} and A = 32 mm^2. For the aluminum tube, take E = 69 GPa, $\alpha = 23 \times 10^{-6}$ °C^{-1} and A = 64 mm^2.

Solution—Equilibrium

Draw free-body diagrams (Exhibit 8). From equilibrium of the bolt it can be seen that $P_B = P_T$.

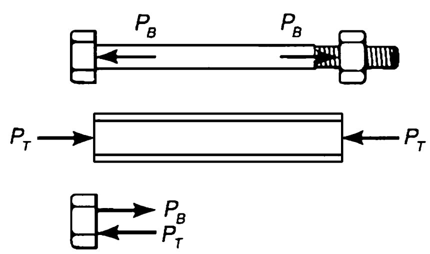


Exhibit 8

Solution—Force-Deformation

Note that both members have the same length and the same force, P.

$$\delta_B = \frac{PL}{A_B E_B} + \alpha_B L(t - t_0)$$

$$\delta_T = -\frac{PL}{A_T E_T} + \alpha_T L(t - t_0)$$

The minus sign in the second expression occurs because the tube is in compression.

Solution—Compatibility

The tube and bolt must both expand the same amount, therefore,

$$\delta_B = \delta_T$$

$$\delta_B = \delta_T = \frac{P \times (100\text{ mm})}{32\text{ mm}^2 \times 210\text{ GPa}} + 12 \times 10^{-6}\frac{1}{°\text{C}} \times 100\text{ mm} \times 60\text{ °C}$$

$$= -\frac{P \times (100\text{ mm})}{64\text{ mm}^2 \times 69\text{ GPa}} + 23 \times 10^{-6}\frac{1}{°\text{C}} \times 100\text{ mm} \times 60\text{ °C}$$

Solving for P gives P = 1.759 kN.

Variable Load

In certain cases the load in the member will not be constant but will be a continuous function of the length. These cases occur when there is a distributed load on the member. Such distributed loads most commonly occur when the member is subjected to gravitation, acceleration or magnetic fields. In such cases, Eq. (9.5) holds only over an infinitesimally small length $L = dx$. Eq. (9.5) then becomes

$$d\delta = \frac{P(x)}{AE}dx \qquad \textbf{(9.10)}$$

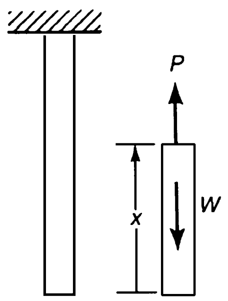
or equivalently

$$\delta = \int_0^L \frac{P(x)}{AE}dx \qquad \textbf{(9.11)}$$

Example 9.5

An aluminum rod is hanging from one end. The rod is 1 m long and has a square cross-section 20 mm by 20 mm. Find the total extension of the rod resulting from its own weight. Take E = 70 GPa and the unit weight γ = 27 kN/m^3.

Solution—Equilibrium



Exihibit 9

Draw a free-body diagram (Exhibit 9). The weight of the section shown in Exhibit 9 is

$$W = \gamma V = \gamma A x = P$$

which clearly yields P as a function of x, and Eq. (9.11) gives

$$\delta = \int_0^L \frac{\gamma A x}{AE}dx = \frac{\gamma}{E}\int_0^L x dx = \frac{\gamma L^2}{2E} = \frac{\left(27\frac{\text{kN}}{\text{m}^3}\right)(1\text{m})^2}{2\left(70\frac{\text{GN}}{\text{m}^2}\right)} = 0.1929\ \mu\text{m}$$

THIN-WALLED CYLINDER

Consider the thin-walled circular cylinder subjected to a uniform internal pressure q as shown in Fig. 9.3. A section of length a, is cut out of the vessel in (a). The cut-out portion is shown in (b). The pressure q can be considered as acting across

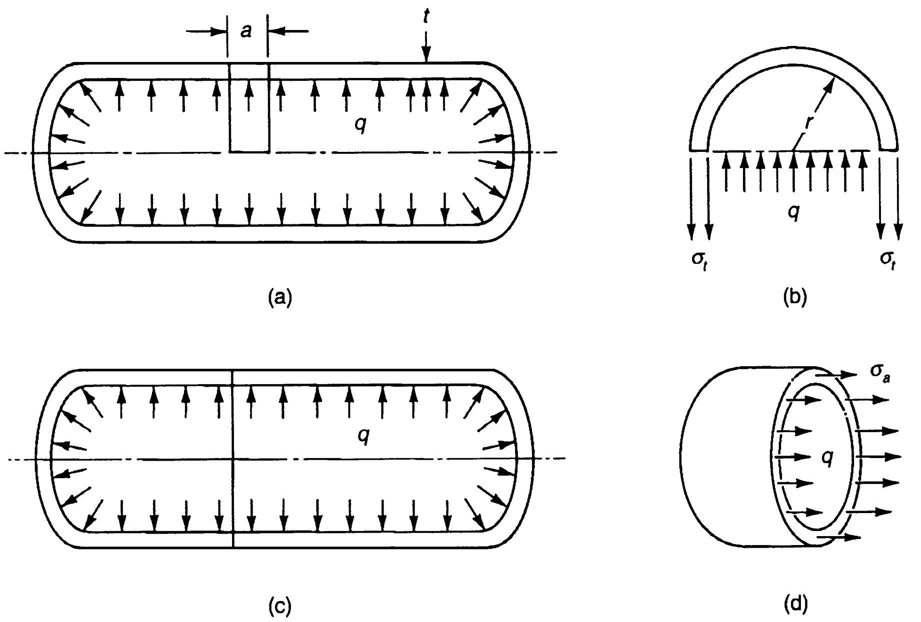


Figure 9.3

the diameter as shown. The tangential stress σ_t is assumed constant through the thickness. Summing forces in the vertical direction gives

$$qDa - 2\sigma_t ta = 0 \tag{9.12}$$

$$\sigma_t = \frac{qD}{2t} \tag{9.13}$$

where D is the inner diameter of the cylinder and t is the wall thickness. The axial stress σ_a is also assumed to be uniform over the wall thickness. The axial stress can be found by making a cut through the cylinder as shown in (c). Consider the horizontal equilibrium for the free-body diagram shown in (d). The pressure q acts over the area πr^2 and the stress σ_a acts over the area πDt which gives

$$\sigma_a \pi Dt = q\pi\left(\frac{D}{2}\right)^2 \tag{9.14}$$

so

$$\sigma_a = \frac{qD}{4t} \tag{9.15}$$

Example 9.6

Consider a cylindrical pressure vessel with a wall thickness of 25 mm, an internal pressure of 1.4 MPa, and an outer diameter of 1.2 m. Find the axial and tangential stresses.

Solution

$$q = 1.4 \text{ MPa}; \; D = 1200 - 50 = 1150 \text{ mm}; \; t = 25 \text{ mm}$$

$$\sigma_t = \frac{qD}{2t} = \frac{1.4 \text{ MPa} \times 1150 \text{ mm}}{2 \times 25 \text{ mm}} = 32.2 \text{ MPa}$$

$$\sigma_a = \frac{qD}{4t} = \frac{1.4 \text{ MPa} \times 1150 \text{ mm}}{4 \times 25 \text{ mm}} = 16.1 \text{ MPa}$$

GENERAL STATE OF STRESS

Stress is defined as force per unit area acting on a certain area. Consider a body that is cut so that its area has an outward normal in the x direction as shown in Fig. 9.4. The force ΔF that is acting over the area ΔA_x can be split into its components ΔF_x, ΔF_y, and ΔF_z. The stress components acting on this face are then defined as

$$\sigma_x = \lim_{\Delta A_x \to 0} \frac{\Delta F_x}{\Delta A_x} \tag{9.16}$$

$$\tau_{xy} = \lim_{\Delta A_x \to 0} \frac{\Delta F_y}{\Delta A_x} \tag{9.17}$$

$$\tau_{xz} = \lim_{\Delta A_x \to 0} \frac{\Delta F_z}{\Delta A_x} \tag{9.18}$$

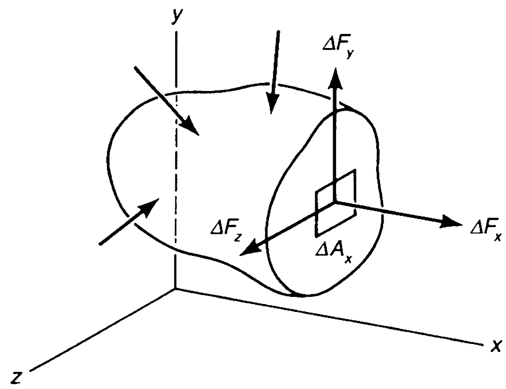


Figure 9.4 Stress on a face

The stress component σ_x is the normal stress. It acts normal to the x face in the x direction. The stress component τ_{xy} is a shear stress. It acts parallel to the x face in the y direction. The stress component τ_{xz} is also a shear stress and acts parallel to the x face in the z direction. For shear stress, the first subscript indicates the *face* on which it acts, and the second subscript indicates the *direction* in which it acts. For normal stress, the single subscript indicates both face and direction. In the general state of stress, there are normal and shear stresses on all faces of an element as shown in Fig. 9.5.

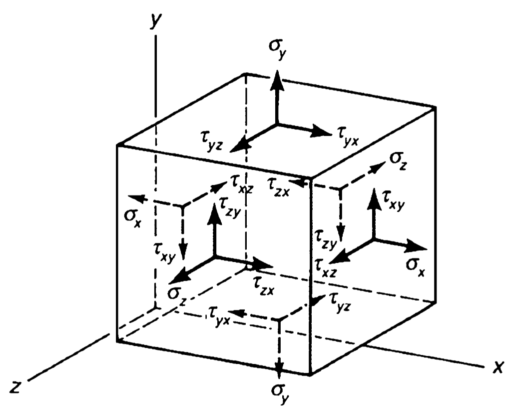


Figure 9.5 Stress at a point (shown in positive directions)

From equilibrium of moments around axes parallel to x, y, and z and passing through the center of the element in Fig. 9.5, it can be shown that the following relations hold

$$\tau_{xy} = \tau_{yx}; \qquad \tau_{yz} = \tau_{zy}; \qquad \tau_{zx} = \tau_{xz} \tag{9.19}$$

Thus, at any point in a body the state of stress is given by six components: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}$, and τ_{zx}. The usual sign convention is to take the components shown in Fig. 9.5 as positive. One way of saying this is that normal stresses are positive in tension. Shear stresses are positive on a positive face in the positive direction. A **positive face** is defined as a face with a positive outward normal.

PLANE STRESS

In elementary mechanics of materials, we usually deal with a state of plane stress in which only the stresses in the x-y plane are non-zero. The stress components σ_z, τ_{xz}, and τ_{yz} are taken as zero.

Mohr's Circle—Stress

In plane stress, the three components σ_x, σ_y, and τ_{xy} define the state of stress at a point, but the components on any other face have different values. To find the components on an arbitrary face, consider equilibrium of the wedges shown in Fig. 9.6.

Summation of forces in the x′ and y′ directions for the wedge shown in Fig. 9.6(a) gives

$$\sum F_{x'} = 0 = \sigma_{x'}\Delta A - \sigma_x \Delta A(\cos\theta)^2 - \sigma_y \Delta A(\sin\theta)^2 - 2\tau_{xy}\Delta A \sin\theta\cos\theta \tag{9.20}$$

$$\sum F_{y'} = 0 = \tau_{x'y'}\Delta A + (\sigma_x - \sigma_y)\Delta A \sin\theta\cos\theta - \tau_{xy}\Delta A\,[(\cos\theta)^2 - (\sin\theta)^2] \tag{9.21}$$

Canceling ΔA from each of these expressions and using the double angle relations gives

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{9.22}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{9.23}$$

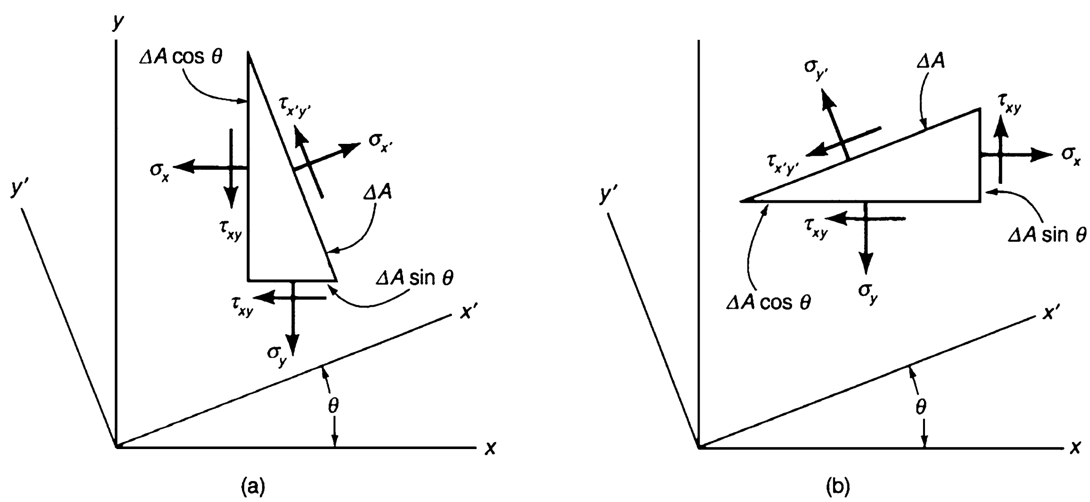


Figure 9.6 Stress on an arbitrary face

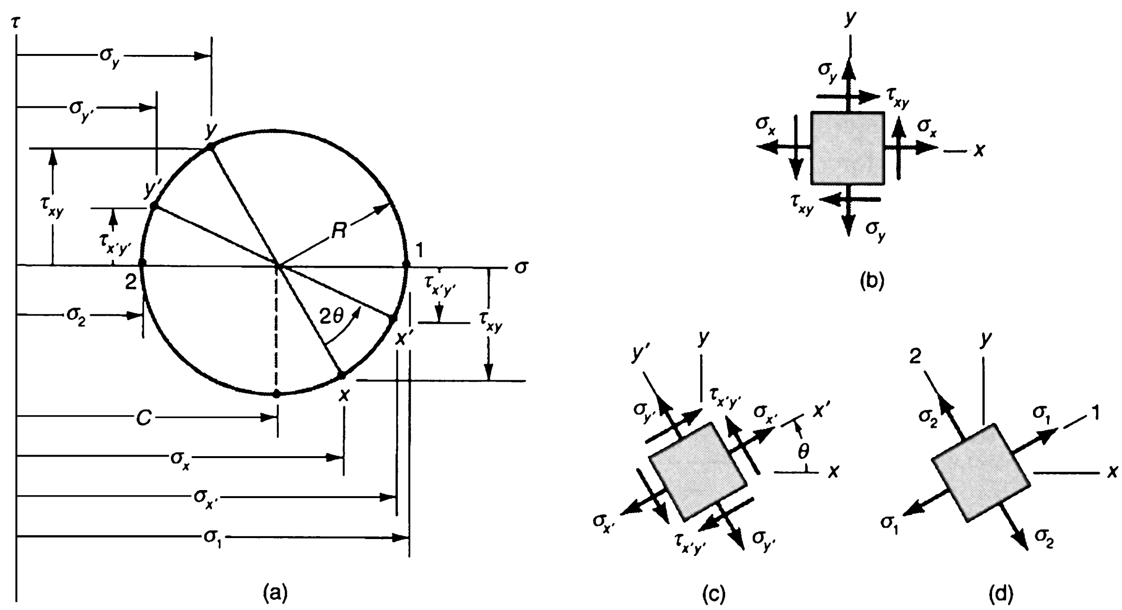


Figure 9.7 Mohr's circle for the stress at a point

Similarly, summation of forces in the y′ direction for the wedge shown in Fig. 9.6(b) gives

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad \textbf{(9.24)}$$

Equations (9.22), (9.23), and (9.24) are the parametric equations of Mohr's circle; Fig. 9.7(a) shows the general Mohr's circle; Fig. 9.7(b) shows the stress on the element in an *x-y* orientation; Fig. 9.7(c) shows the stress in the same element in an *x*′-*y*′ orientation; and Fig. 9.7(d) shows the stress on the element in the 1-2 orientation. Notice that there is always an orientation (for example, a 1-2 orientation) for which the shear stress is zero. The normal stresses σ_1 and σ_2 on these 1-2 faces are the principal stresses, and the 1 and 2 axes are the principal axes of stress. In three-dimensional problems the same is true. There are always three mutually perpendicular faces on which there is no shear stress. Hence, there are always three principal stresses.

To draw Mohr's circle knowing σ_x, σ_y, and τ_{xy},

1. Draw vertical lines corresponding to σ_x and σ_y as shown in Fig. 9.8(a) according to the signs of σ_x and σ_y (to the right of the origin if positive and to the left if negative).

2. Put a point on the σ_x vertical line a distance τ_{xy} below the horizontal axis if τ_{xy} is positive (above if τ_{xy} is negative) as in Fig. 9.8(a). Name this point *x*.

3. Put a point on the σ_y vertical line a distance τ_{xy} in the opposite direction as on the σ_x vertical line also as shown in Fig. 9.8(a). Name this point *y*.

4. Connect the two points *x* and *y*, and draw the circle with diameter *xy* as shown in Fig. 9.8(b).

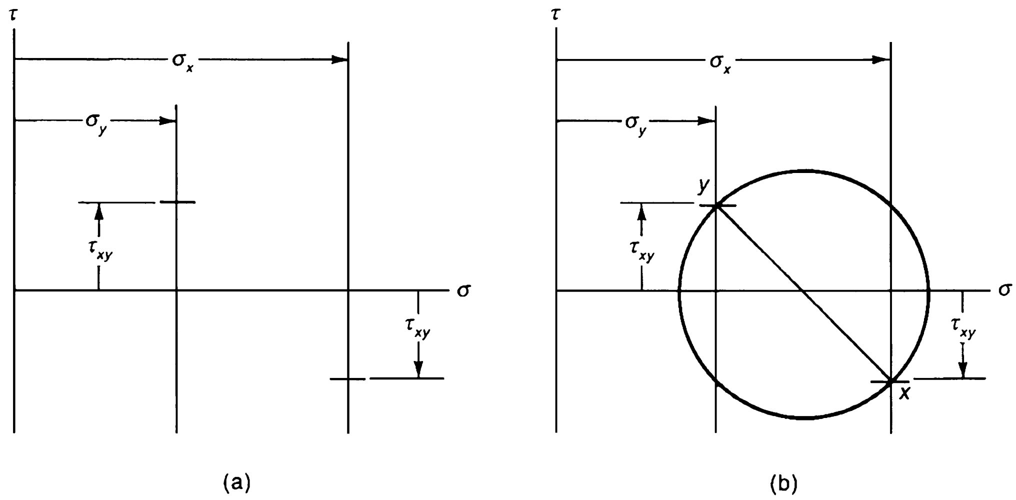


Figure 9.8 Constructing Mohr's circle

Upon constructing Mohr's circle you can now rotate the *xy* diameter through an angle of 2θ to a new position $x'y'$, which can determine the stress on any face at that point in the body as shown in Fig. 9.7. Note that rotations of 2θ on Mohr's circle correspond to θ in the physical plane; also note that the direction of rotation is the same as in the physical plane (that is, if you go clockwise on Mohr's circle, the rotation is also clockwise in the physical plane). The construction can also be used to find the principal stresses and the orientation of the principal axes.

Problems involving stress transformations can be solved with Eqs. (9.22), (9.23), and (9.24), from construction of Mohr's circle, or from some combination. As an example of a combination, it can be seen that the center of Mohr's circle can be represented as

$$C = \frac{\sigma_x + \sigma_y}{2} \tag{9.25}$$

The radius of the circle is

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}} \tag{9.26}$$

The principal stresses then are

$$\sigma_1 = C + R; \qquad \sigma_2 = C - R \tag{9.27}$$

Example 9.7

Given $\sigma_x = -3$ MPa; $\sigma_y = 5$ MPa; $\tau_{xy} = 3$ MPa. Find the principle stresses and their orientation.

Solution

Mohr's circle is constructed as shown in Exhibit 10. The angle 2θ was chosen as the angle between the *y* axis and the 1 axis clockwise from *y* to 1 as shown in the circle. The angle θ in the physical plane is between the y axis and the 1 axis also clockwise from y to 1. The values of σ_1, σ_2, and 2θ can all be scaled from the circle. The values can also be calculated as follows:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \sqrt{\left(\frac{-3-5}{2}\right)^2 + 3^2} = 5 \text{ MPa}$$

$$C = \frac{\sigma_x + \sigma_y}{2} = \frac{-3+5}{2} = 1 \text{ MPa}$$

$$\sigma_1 = C + R = 6 \text{ MPa}$$

$$\sigma_2 = C - R = -4 \text{ MPa}$$

$$2\theta = \tan^{-1}(3/5) = 30.96°; \qquad \theta = 15.48°$$

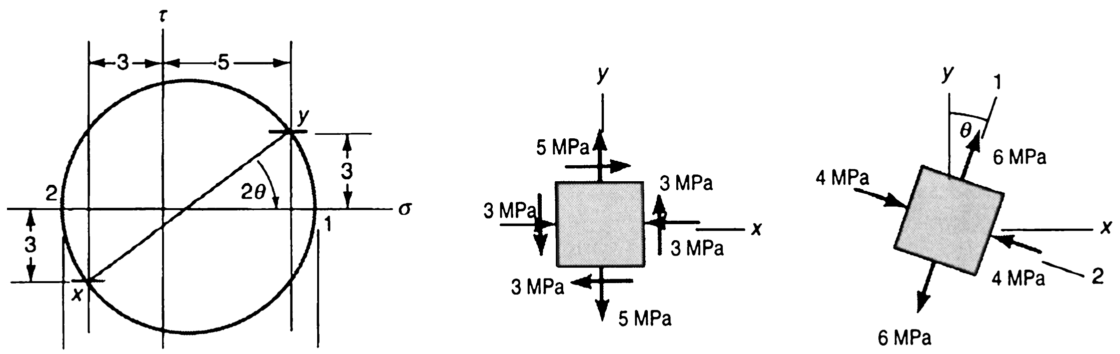


Exhibit 10

STRAIN

Axial strain was previously defined as

$$\varepsilon = \frac{\Delta L}{L} \tag{9.28}$$

In the general case, there are three components of axial strain, ε_x, ε_y, and ε_z. Shear strain is defined as the decrease in angle of two originally perpendicular line segments passing through the point at which strain is defined. In Fig. 9.9, AB is vertical and BC is horizontal. They represent line segments that are drawn before loading. After loading, points A, B, and C move to A', B', and C', respectively. The angle between $A'B'$ and the vertical is α, and the angle between B' and C' and the horizontal is β. The original right angle has been decreased by $\alpha + \beta$, and the shear strain is

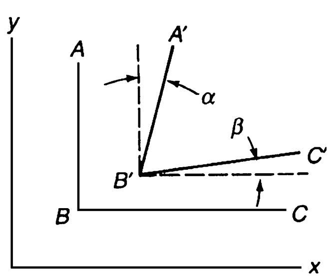


Figure 9.9 Definition of shear strain

$$\gamma_{xy} = \alpha + \beta \tag{9.29}$$

In the general case, there are three components of shear strain, γ_{xy}, γ_{yz}, and γ_{zx}.

Plane Strain

In two dimensions, strain undergoes a similar rotation transformation as stress. The transformation equations are

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \qquad \textbf{(9.30)}$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \qquad \textbf{(9.31)}$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \qquad \textbf{(9.32)}$$

These equations are the same as Eq. (9.22), (9.23), and (9.24) for stress, except that the σ_x has been replaced with ε_x, σ_y with ε_y, and τ_{xy} with $\gamma_{xy}/2$. Therefore, Mohr's circle for strain is treated the same way as that for stress, except for the factor of two on the shear strain.

Example 9.8

Given that $\varepsilon_x = 600\ \mu$; $\varepsilon_y = -200\ \mu$; $\gamma_{xy} = -800\ \mu$, find the principal strains and their orientation. The symbol μ signifies 10^{-6}.

Solution

From the Mohr's circle shown in Exhibit 11, it is seen that $2\theta = 45°$; so, $\theta = 22.5°$ clockwise from x to 1. The principal strains are $\varepsilon_1 = 766\ \mu$ and $\varepsilon_2 = -366\ \mu$.

The principal strains can also be found by computation in the same way as principal stresses,

$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \sqrt{\left(\frac{600+200}{2}\right)^2 + \left(\frac{-800}{2}\right)^2} = 565.7\ \mu$$

$$C = \frac{\varepsilon_x + \varepsilon_y}{2} = \frac{600-200}{2} = 200\ \mu$$

$$\varepsilon_1 = C + R = 766\ \mu$$

$$\varepsilon_2 = C - R = -366\ \mu$$

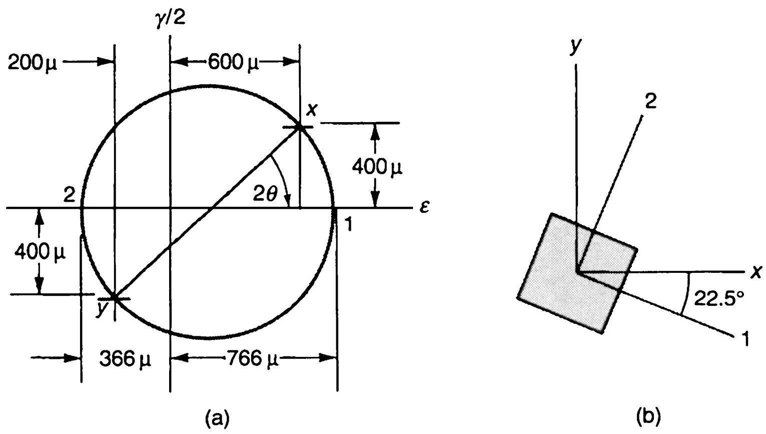


Exhibit 11

HOOKE'S LAW

The relationship between stress and strain is expressed by Hooke's Law. For an isotropic material it is

$$\varepsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y - v\sigma_z) \quad \textbf{(9.33)}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - v\sigma_z - v\sigma_x) \quad \textbf{(9.34)}$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - v\sigma_x - v\sigma_y) \quad \textbf{(9.35)}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \quad \textbf{(9.36)}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz} \quad \textbf{(9.37)}$$

$$\gamma_{zx} = \frac{1}{G}\tau_{zx} \quad \textbf{(9.38)}$$

Further, there is a relationship between E, G, and v which is

$$G = \frac{E}{2(1+v)} \quad \textbf{(9.39)}$$

Thus, for an isotropic material there are only two independent elastic constants. An **isotropic material** is one that has the same material properties in all directions. Notable exceptions to isotropy are wood- and fiber-reinforced composites.

Example 9.9

A steel plate in a state of plane stress is known to have the following strains: $\varepsilon_x = 650\,\mu$, $\varepsilon_y = 250\,\mu$, and $\gamma_{xy} = 400\,\mu$. If $E = 210$ GPa and $v = 0.3$, what are the stress components, and what is the strain ε_z?

Solution

In a state of plane stress, the stresses $\sigma_z = 0$, $\tau_{xz} = 0$ and $\tau_{yz} = 0$. From Hooke's law,

$$\varepsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y - 0)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - v\sigma_x - 0)$$

Inverting these relations gives

$$\sigma_x = \frac{E}{1-v^2}(\varepsilon_x + v\varepsilon_y) = \frac{210\text{ GPa}}{1-0.3^2}[650\,\mu + 0.3(250\,\mu)] = 167.3\text{ Mpa}$$

$$\sigma_y = \frac{E}{1-v^2}(\varepsilon_y + v\varepsilon_x) = \frac{210\text{ GPa}}{1-0.3^2}[250\,\mu + 0.3(650\,\mu)] = 102.7\text{ Mpa}$$

From Hooke's law, the strain γ_{xy} is

$$\gamma_{xy} = \frac{\tau_{xy}}{G}; \qquad G = \frac{E}{2(1+v)}; \qquad \tau_{xy} = \frac{E\gamma_{xy}}{2(1+v)} = \frac{(210\text{ GPa})\,(400\,\mu)}{2(1+0.3)} = 32.3\text{ MPa}$$

The strain in the z direction is

$$\varepsilon_z = \frac{1}{E}(0 - v\sigma_x - v\sigma_y) = \frac{-v}{E}(\sigma_x + \sigma_y)$$
$$= \frac{-0.3}{210 \text{ GPa}}(167.3 \text{ MPa} + 102.7 \text{ MPa}) = -386\,\mu$$

TORSION

Torsion refers to the twisting of long members. Torsion can occur with members of any cross-sectional shape, but the most common is the circular shaft. Another fairly common shaft configuration, which has a simple solution, is the hollow, thin-walled shaft.

Circular Shafts

Fig. 9.10(a) shows a circular shaft before loading; the r-θ-z cylindrical coordinate system is also shown. In addition to the outline of the shaft, two longitudinal lines, two circumferential lines, and two diametral lines are shown scribed on the shaft. These lines are drawn to show the deformed shape loading. Fig. 9.10(b) shows the shaft after loading with a torque T. The **double arrow notation** on T indicates a moment about the z axis in a right-handed direction. The effect of the torsion is that each cross-section remains plane and simply rotates with respect to other cross-sections. The angle ϕ is the twist of the shaft at any position z. The rotation $\phi(z)$ is in the θ direction.

The distance b shown in Fig. 9.10(b) can be expressed as $b = \phi r$ or as $b = \gamma z$. The shear strain for this special case can be expressed as

$$\gamma_{\phi z} = r\frac{\phi}{z} \tag{9.40}$$

For the general case where ϕ is not a linear function of z the shear strain can be expressed as

$$\gamma_{\phi z} = r\frac{d\phi}{dz} \tag{9.41}$$

$d\phi/dz$ is the twist per unit length or the rate of twist.

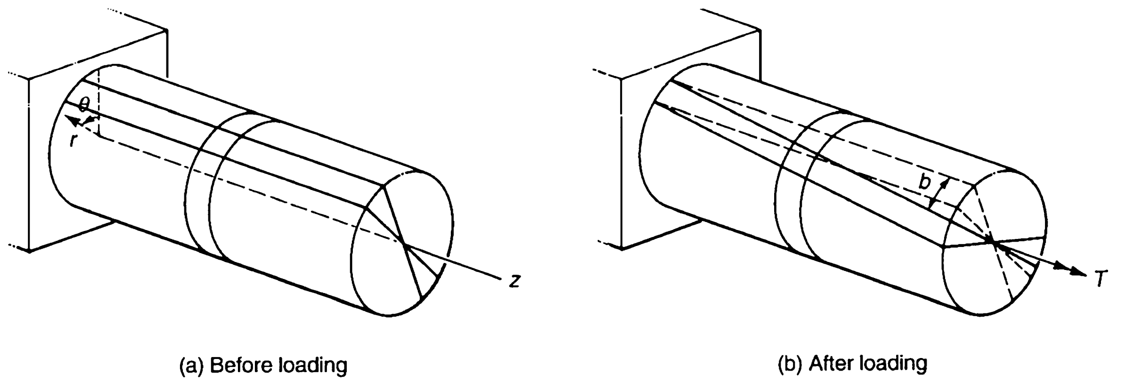


Figure 9.10 Torsion in a circular shaft

The application of Hooke's Law gives

$$\tau_{\phi z} = G\gamma_{\phi z} = Gr\frac{d\phi}{dz} \tag{9.42}$$

The torque at the distance z along the shaft is found by summing the contributions of the shear stress at each point in the cross-section by means of an integration

$$T = \int_A \tau_{\phi z} r\, dA = G\frac{d\phi}{dz}\int_A r^2 dA = GJ\frac{d\phi}{dz} \tag{9.43}$$

where J is the polar moment of inertia of the circular cross-section. For a solid shaft with an outer radius of r_o the polar moment of inertia is

$$J = \frac{\pi r_o^4}{2} \tag{9.44}$$

For a hollow circular shaft with outer radius r_o and inner radius r_i, the polar moment of inertia is

$$J = \frac{\pi}{2}\left(r_o^4 - r_i^4\right) \tag{9.45}$$

Note that the J that appears in Eq. (9.43) is the polar moment of inertia only for the special case of circular shafts (either solid or hollow). For any other cross-section shape, Eq. (9.43) is valid only if J is redefined as a torsional constant *not equal* to the polar moment of inertia. Eq. (9.42) can be combined with Eq. (9.43) to give

$$\tau_{\phi z} = \frac{Tr}{J} \tag{9.46}$$

The maximum shear stress occurs at the outer radius of the shaft and at the location along the shaft where the torque is maximum.

$$\tau_{\phi z\,\max} = \frac{T_{\max} r_o}{J} \tag{9.47}$$

The angle of twist of the shaft can be found by integrating Eq. (9.43)

$$\phi = \int_0^L \frac{T}{GJ}\,dz \tag{9.48}$$

For a uniform circular shaft with a constant torque along its length, this equation becomes

$$\phi = \frac{TL}{GJ} \tag{9.49}$$

Example 9.10

The hollow circular steel shaft shown in Exhibit 12 has an inner diameter of 25 mm, an outer diameter of 50 mm, and a length of 600 mm. It is fixed at the left end and subjected to a torque of 1400 N • m as shown in Exhibit 12. Find the maximum shear stress in the shaft and the angle of twist at the right end. Take G = 84 GPa.

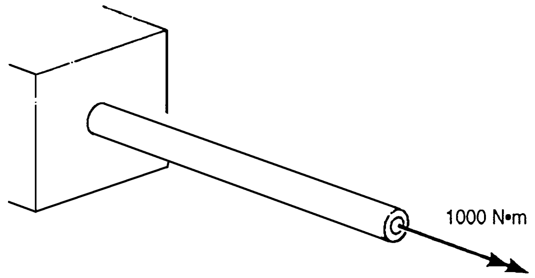


Exhibit 12

Solution

$$J = \frac{\pi}{2}\left(r_o^4 - r_i^4\right) = \frac{\pi}{2}[(25\text{ mm})^4 - (12.5\text{ mm})^4] = 575\times10^3\text{ mm}^4$$

$$\tau_{\theta z\,\max} = \frac{T_{\max} r_o}{J} = \frac{(1400\text{N}\bullet\text{m})(25\text{ mm})}{575\times10^3\text{ mm}^4} = 60.8\text{ MPa}$$

$$\phi = \frac{TL}{GJ} = \frac{(1400\text{N}\bullet\text{m})(600\text{ mm})}{(84\text{ GPa})(575\times10^3\text{ mm}^4)} = 0.01738\text{ rad}$$

Hollow, Thin-Walled Shafts

In hollow, thin-walled shafts, the assumption is made that the shear stress τ_{sz} is constant throughout the wall thickness t. The shear flow q is defined as the product of τ_{sz} and t. From a summation of forces in the z direction, it can be shown that q is constant—even with varying thickness. The torque is found by summing the contributions of the shear flow. Fig. 9.11 shows the cross-section of the thin-walled tube of nonconstant thickness. The z coordinate is perpendicular to the plane of the paper. The shear flow q is taken in a counter-clockwise sense. The torque produced by q over the element ds is

$$dT = qr\,ds$$

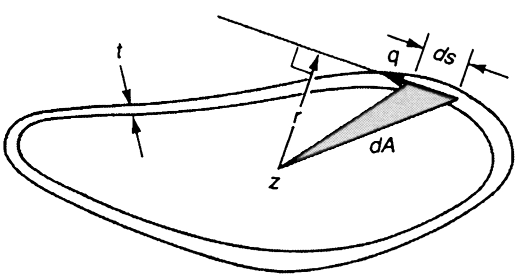


Figure 9.11 Cross-section of thin-walled tube

The total torque is, therefore,

$$T = \oint qr\,ds = q\oint r\,ds \tag{9.50}$$

The area dA is the area of the triangle of base ds and height r,

$$dA = \tfrac{1}{2}(\text{base})(\text{height}) = \frac{r\,ds}{2} \tag{9.51}$$

so that

$$\oint r\,ds = 2A_m \tag{9.52}$$

where A_m is the area enclosed by the wall (including the hole). It is best to use the centerline of the wall to define the boundary of the area, hence A_m is the mean area. The expression for the torque is

$$T = 2A_m q \tag{9.53}$$

and from the definition of q the shear stress can be expressed as

$$\tau_{sz} = \frac{T}{2A_m t} \tag{9.54}$$

Example 9.11

A torque of 10 kN•m is applied to a thin-walled rectangular steel shaft whose cross-section is shown in Exhibit 13. The shaft has wall thicknesses of 5 mm and 10 mm. Find the maximum shear stress in the shaft.

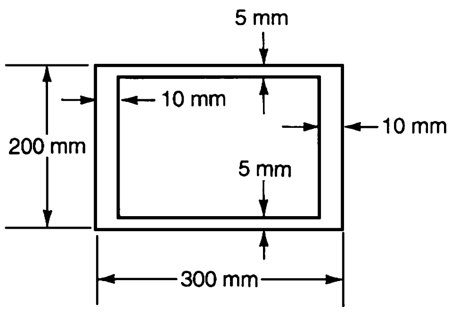


Exhibit 13

Solution

$$A_m = (200 - 5)(300 - 10) = 56{,}550 \text{ mm}^2$$

The maximum shear stress will occur in the thinnest section, so $t = 5$ mm.

$$\tau_{sz} = \frac{T}{2A_m t} = \frac{10 \text{ kN} \bullet \text{m}}{2(56{,}550 \text{ mm}^2)(5 \text{ mm})} = 17.68 \frac{\text{MN}}{\text{m}^2}$$

BEAMS

Shear and Moment Diagrams

Shear and moment diagrams are plots of the shear forces and bending moments, respectively, along the length of a beam. The purpose of these plots is to clearly show maximums of the shear force and bending moment, which are important in the design of beams. The most common sign convention for the shear force and bending moment in beams is shown in Fig. 9.12. One method of determining the shear and moment diagrams is by the following steps:

1. Determine the reactions from equilibrium of the entire beam.
2. Cut the beam at an arbitrary point.

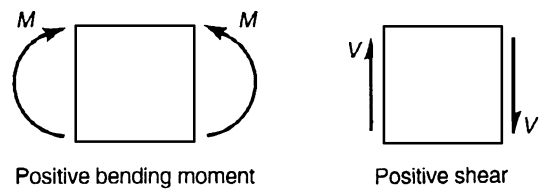


Figure 9.12 Sign convention for bending moment and shear

3. Show the unknown shear and moment on the cut using the positive sign convention shown in Fig. 9.12.
4. Sum forces in the vertical direction to determine the unknown shear.
5. Sum moments about the cut to determine the unknown moment.

Example **9.12**

For the beam shown in Exhibit 14, plot the shear and moment diagram.

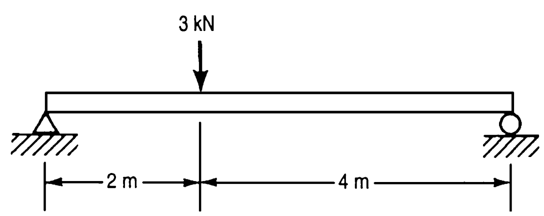


Exhibit 14

Solution

First, solve for the unknown reactions using the free-body diagram of the beam shown in Exhibit 15(a). To find the reactions, sum moments about the left end,

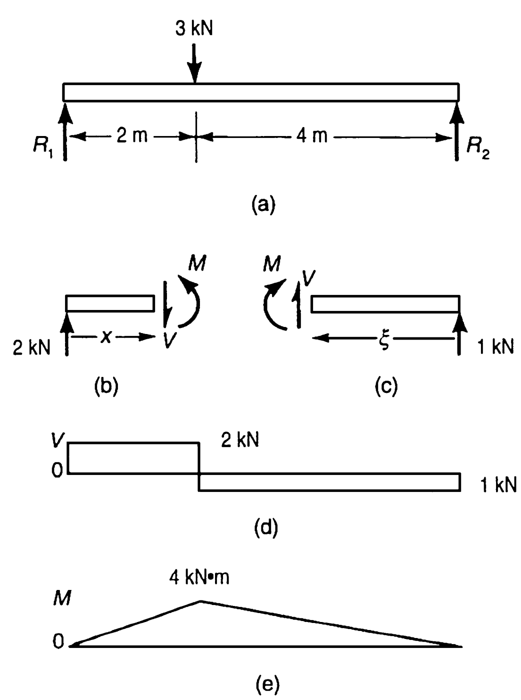


Exhibit 15

which gives

$$6R_2 - (3)(2) = 0 \quad \text{or} \quad R_2 = 6/6 = 1 \text{ kN}$$

Sum forces in the vertical direction to get

$$R_1 + R_2 = 3 = R_1 + 1 \quad \text{or} \quad R_1 = 2 \text{ kN}$$

Cut the beam between the left end and the load as shown in Exhibit 15(b). Show the unknown moment and shear on the cut using the positive sign convention shown in Fig. 9.12. Sum the vertical forces to get

$$V = 2 \text{ kN (independent of } x)$$

Sum moments about the cut to get

$$M = R_1 x = 2x$$

Repeat the procedure by making a cut between the right end of the beam and the 3-kN load, as shown in Exhibit 15(c). Again, sum vertical forces and sum moments about the cut to get

$$V = 1 \text{ kN (independent of } \xi\text{), and } M = 1\xi$$

The plots of these expressions for shear and moment give the shear and moment diagrams shown in Exhibit 15(d) and 15(e).

It should be noted that the shear diagram in this example has a jump at the point of the load and that the jump is equal to the load. This is always the case. Similarly, a moment diagram will have a jump equal to an applied concentrated moment. In this example, there was no concentrated moment applied, so the moment was everywhere continuous.

Another useful way of determining the shear and moment diagram is by using differential relationships. These relationships are found by considering an element of length Δx of the beam. The forces on that element are shown in Fig. 9.13. Summation of forces in the y direction gives

$$q\Delta x + V - V - \frac{dV}{dx}\Delta x = 0 \qquad \textbf{(9.55)}$$

which gives

$$\frac{dV}{dx} = q \qquad \textbf{(9.56)}$$

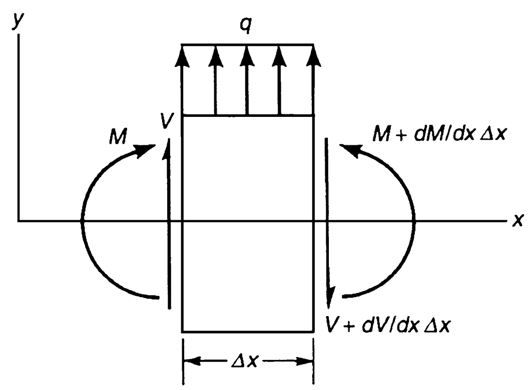


Figure 9.13

Summing moments and neglecting higher order terms gives

$$-M + M + \frac{dM}{dx}\Delta x - V\Delta x = 0 \tag{9.57}$$

which gives

$$\frac{dM}{dx} = V \tag{9.58}$$

Integral forms of these relationships are expressed as

$$V_2 - V_1 = \int_{x_1}^{x_2} q\,dx \tag{9.59}$$

$$M_2 - M_1 = \int_{x_1}^{x_2} V\,dx \tag{9.60}$$

Example 9.13

The simply supported uniform beam shown in Exhibit 16 carries a uniform load of w_0. Plot the shear and moment diagrams for this beam.

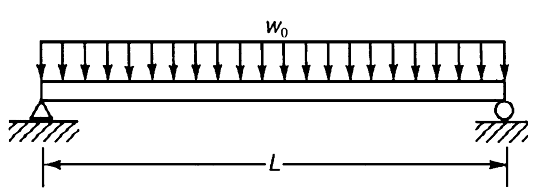


Exhibit 16

Solution

As before, the reactions can be found first from the free-body diagram of the beam shown in Exhibit 17(a). It can be seen that, from symmetry, $R_1 = R_2$. Summing vertical forces then gives

$$R = R_1 = R_2 = \frac{w_0 L}{2}$$

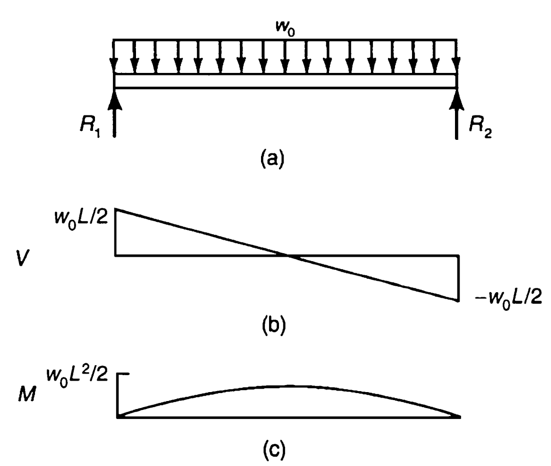


Exhibit 17

The load $q = -w_0$, so Eq. (9.59) reads

$$V = V_0 - \int_0^x w_0\,dx = \frac{w_0 L}{2} - w_0 x$$

Noting that the moment at $x = 0$ is zero, Eq. (9.60) gives

$$M = M_0 - \int_0^x \left(\frac{w_0 L}{2} - w_0 x \right) dx = 0 + \frac{w_0 L x}{2} - \frac{w_0 x^2}{2} = \frac{w_0 x}{2}(L - x)$$

It can be seen that the shear diagram is a straight line, and the moment varies parabolically with x. Shear and moment diagrams are shown in Exhibit 17(b) and Exhibit 17(c). It can be seen that the maximum bending moment occurs at the center of the beam where the shear stress is zero. The maximum bending moment always has a relative maximum at the place where the shear is zero because the shear is the derivative of the moment, and relative maxima occur when the derivative is zero.

Often it is helpful to use a combination of methods to find the shear and moment diagrams. For instance, if there is no load between two points, then the shear diagram is constant, and the moment diagram is a straight line. If there is a uniform load, then the shear diagram is a straight line, and the moment diagram is parabolic. The following example illustrates this method.

Example 9.14

Draw the shear and moment diagrams for the beam shown in Exhibit 18(a).

Solution

Draw the free-body diagram of the beam as shown in Exhibit 18(b). From a summation of the moments about the right end,

$$10\,R_1 = (4)(7) + (3)(2) = 34; \quad \text{so } R_1 = 3.4\,\text{kN}$$

From a summation of forces in the vertical direction,

$$R_2 = 7 - 3.4 = 3.6\,\text{kN}$$

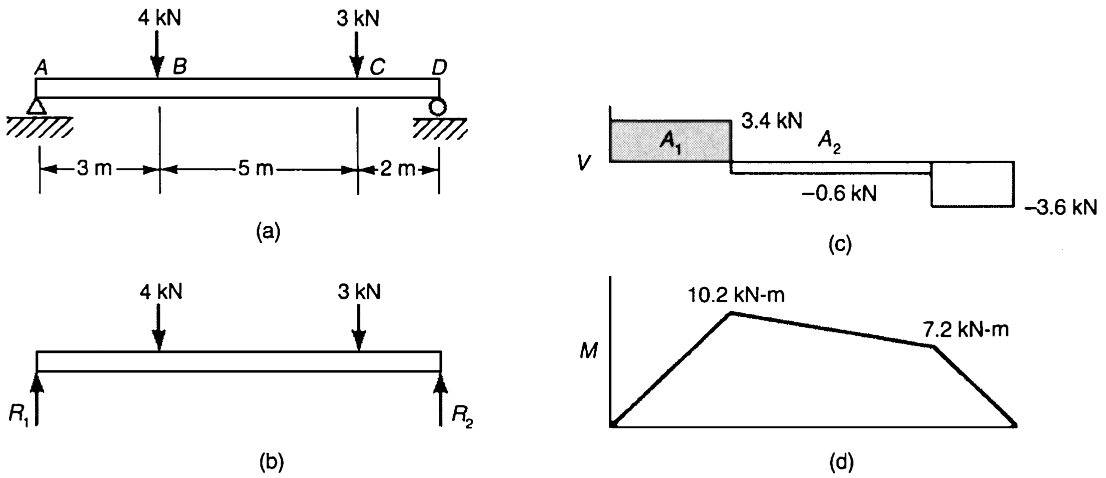


Exhibit 18

The shear in the left portion is 3.4 kN, the shear in the right portion is −3.6 kN and the shear in the center portion is 3.4 − 4 = −0.6 kN. This is sufficient information to draw the shear diagram shown in Exhibit 18(c). The moment at *A* is zero, so the moment at *B* is the shaded area A_1 and the moment at *C* is $A_1 - A_2$.

$$M_B = A_1 = (3.4\,\text{kN})(3\,\text{m}) = 10.2\,\text{kN} \bullet \text{m}$$
$$M_C = A_1 - A_2 = (3.4\,\text{kN})(3\,\text{m}) - (0.6\,\text{kN})(5\,\text{m}) = 7.2\,\text{kN} \bullet \text{m}$$

The moments at *A* and *D* are zero, and the moment diagram consists of straight lines between the points *A*, *B*, *C*, and *D*. There is, therefore, enough information to plot the moment diagram shown in Exhibit 18(d).

Stresses in Beams

The basic assumption in elementary beam theory is that the beam cross-section remains plane and perpendicular to the neutral axis as shown in Fig. 9.14 when the beam is loaded. This assumption is strictly true only for the case of pure bending (constant bending moment and no shear) but gives good results even when shear is taking place. Figure 9.14 shows a beam element before as well as after loading. It can be seen that there is a line of length *ds* that does not change length upon deformation. This line is called the neutral axis. The distance *y* is measured from this neutral axis. The strain in the *x* direction is $\Delta L/L$. The change in length $\Delta L = -y d\phi$ and the length is *ds*, so

$$\varepsilon_x = -y\frac{d\phi}{ds} = -\frac{y}{\rho} = -\kappa y \qquad \textbf{(9.61)}$$

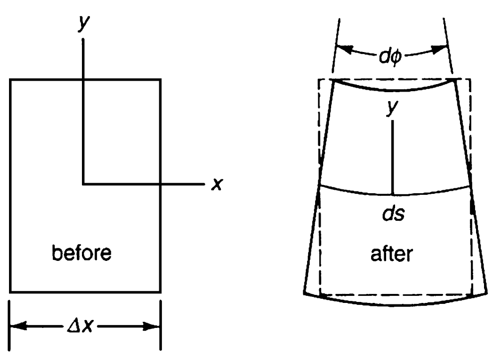


Figure 9.14

where ρ is the radius of curvature of the beam and κ is the curvature of the beam. Assuming that σ_y and σ_z are zero, Hooke's Law yields

$$\sigma_x = -E\kappa y \qquad \textbf{(9.62)}$$

The axial force and bending moment can be found by summing the effects of the normal stress σ_x,

$$P = \int_A \sigma_x\, dA = -E\kappa \int_A y\, dA \qquad \textbf{(9.63)}$$

$$M = -\int_A y\sigma_x\, dA = E\kappa \int_A y^2 dA = EI\kappa \qquad \textbf{(9.64)}$$

where *I* is the moment of inertia of the beam cross-section. If the axial force is zero (as is the usual case) then the integral of *y dA* is zero. That means that *y* is measured from the centroidal axis of the cross-section. Since y is also measured from the neutral axis, the neutral axis coincides with the centroidal axis. From Eq. (9.62) and (9.64), the bending stress σ_x can be expressed as

$$\sigma_x = -\frac{My}{I} \tag{9.65}$$

The maximum bending stress occurs where the magnitude of the bending moment is a maximum and at the maximum distance from the neutral axis. For symmetrical beam sections the value of $y_{\max} = \pm C$ where C is the distance to the extreme fiber so the maximum stress is

$$\sigma_x = \pm\frac{MC}{I} = \pm\frac{M}{S} \tag{9.66}$$

where S is the section modulus ($S = I/C$).

Example 9.15

A 100 mm × 150 mm wooden cantilever beam is 2 m long. It is loaded at its tip with a 4-kN load. Find the maximum bending stress in the beam shown in Exhibit 19. The maximum bending moment occurs at the wall and is $M_{\max}$ = 8 kN • m.

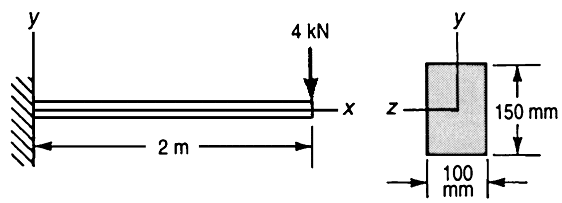


Exhibit 19

Solution

$$I = \frac{bh^3}{12} = \frac{100(150)^3}{12} = 28.1\times 10^6 \text{ mm}^4$$

$$\sigma_{x\max} = \frac{|M|_{\max} c}{I} = \frac{(8\,\text{kN}\bullet\text{m})(75\,\text{mm})}{28.1\times 10^6\text{ mm}^4} = 21.3 \text{ MPa}$$

Shear Stress

To find the shear stress, consider the element of length Δx shown in Fig. 9.15(a). A cut is made in the beam at $y = y_1$. At that point the beam has a thickness b. The shaded cross-sectional area above that cut is called A_1. The bending stresses acting on that element are shown in Fig. 9.15(b). The stresses are slightly larger at the right side than at the left side so that a force per unit length q is needed for equilibrium. Summation of forces in the x direction for the free-body diagram shown in Fig. 9.15(b) gives

$$-F = q\Delta x = \int_{A_1} \sigma\, dA - \int_{A_1}\left(\sigma + \frac{d\sigma}{dx}\Delta x\right) dA = -\int_{A_1}\frac{d\sigma}{dx}\Delta x\, dA \tag{9.67}$$

From the expression for the bending stress ($\sigma = -My/I$) it follows that

$$\frac{d\sigma}{dx} = -\left(\frac{dM}{dx}\right)\frac{y}{I} = -V\frac{y}{I} \tag{9.68}$$

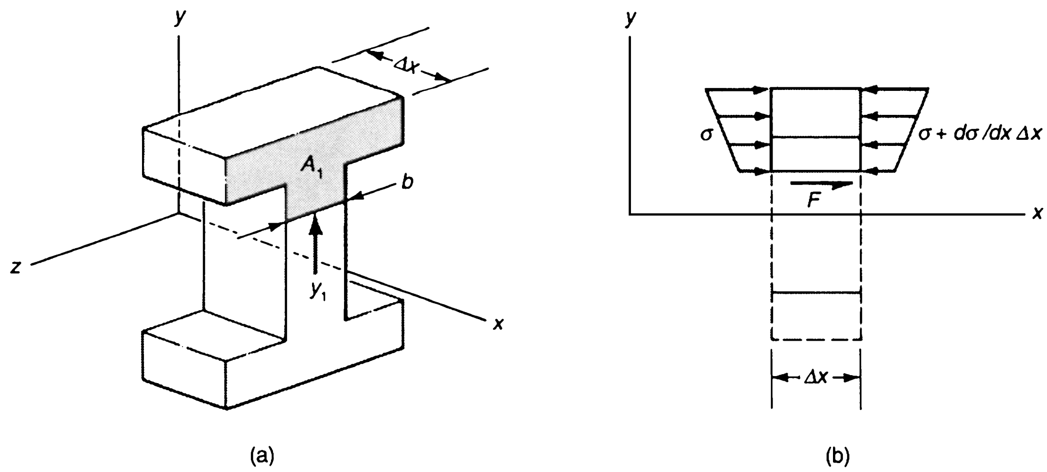


Figure 9.15 Shear stress in beams

Substituting Eq. (9.68) into Eq. (9.67) gives

$$q = \frac{V}{I}\int_{A_1} y\,dA = \frac{VQ}{I} \tag{9.69}$$

If the shear stress τ is assumed to be uniform over the thickness b then $\tau = q/b$ and the expression for shear stress is

$$\tau = \frac{VQ}{Ib} \tag{9.70}$$

where V is the shear in the beam, Q is the moment of area above (or below) the point in the beam at which the shear stress is sought, I is the moment of inertia of the entire beam cross-section, and b is the thickness of the beam cross-section at the point where the shear stress is sought. The definition of Q from Eq. (9.69) is

$$Q = \int_{A_1} y\,dA = A_1 y \tag{9.71}$$

Example **9.16**

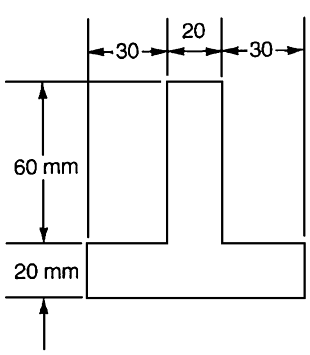


Exhibit 20

The cross-section of the beam shown in Exhibit 20 has an applied shear of 10 kN. Find (a) the shear stress at a point 20 mm below the top of the beam and (b) the maximum shear stress from the shear force.

Solution

The section is divided into two parts by the dashed line shown in Exhibit 21(a). The centroids of each of the two sections are also shown in Exhibit 21(a). The centroid of the entire cross-section is found as follows

$$\bar{y} = \frac{\sum_{n=1}^{N} \bar{y}_n A_n}{\sum_{n=1}^{N} A_n} = \frac{(60)(20)(30+20)+(80)(20)(10)}{(60)(20)+(80)(20)} = 27.14 \text{ mm (from bottom)}$$

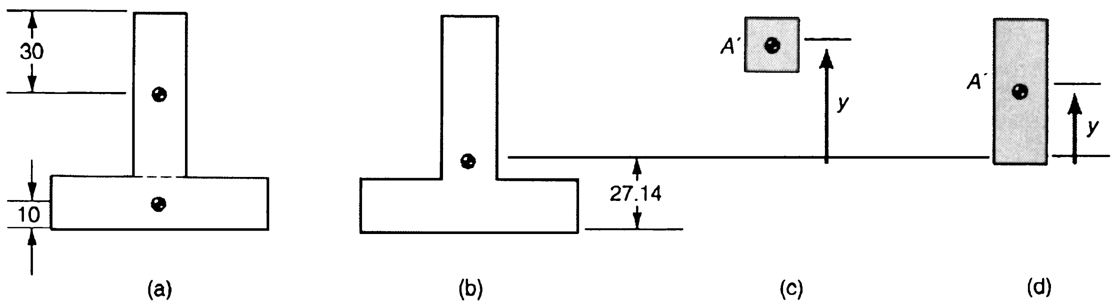


Exhibit 21

Exhibit 21(b) shows the location of the centroid.

The moment of inertia of the cross-section is found by summing the moments of inertia of the two sections taken about the centroid of the entire section. The moment of inertia of each part is found about its own centroid; then the parallel axis theorem is used to transfer it to the centroid of the entire section.

$$
\begin{aligned}
I &= \sum_{n=1}^{N} I_n + A_n \bar{y}_n \\
&= \frac{(20)(60)^3}{12} + (20)(60)(50-27.14)^2 + \frac{(80)(20)^3}{12} - (20)(80)(27.14-10)^2 \\
&= 1.510 \times 10^6 \text{ mm}^4
\end{aligned}
$$

For the point 20 mm below the top of the beam, the area A' and the distance y are shown in Exhibit 21(c). The distance y is from the neutral axis to the centroid of A'. The value of Q is then

$$
\begin{aligned}
Q &= \int_{A'} y\,dA = A'y = (20)(20)(70-27.14) = 17{,}140\,\text{mm}^3 \\
\tau &= \frac{VQ}{Ib} = \frac{(10\text{kN})(17{,}140\,\text{mm}^3)}{(1.510\times 10^6\,\text{mm}^4)(20\,\text{mm})} = 0.00568\frac{\text{kN}}{\text{mm}^2} = 5.68\,\text{MPa}
\end{aligned}
$$

The maximum Q will be at the centroid of the cross-section. Since the thickness is the same everywhere, the maximum shear stress will appear at the centroid. The maximum moment of area Q_{max} is

$$
\begin{aligned}
Q &= \int_{A'} y dA = A' y_1 = (20)(80-27.14)\frac{(80+27.14)}{2} = 56{,}600\,\text{mm}^3 \\
\tau &= \frac{VQ}{IB} = \frac{(10\text{kN})(56{,}600\,\text{mm}^3)}{(1.510\times 10^6\,\text{mm}^4)(20\,\text{mm})} = 0.01875\frac{\text{kN}}{\text{mm}^2} = 18.75\,\text{MPa}
\end{aligned}
$$

Deflection of Beams

The beam deflection in the y direction will be denoted as y, while most modern texts use v for the deflection in the y direction. The *FE Supplied-Reference Handbook* uses the older notation. The main assumption in the deflection of beams is that the slope of the beam is small. The slope of the beam is dy/dx. Since the slope is small, the slope is equal to the angle of rotation in radians.

$$
\frac{dy}{dx} = \text{ rotation in radians} \qquad \textbf{(9.72)}
$$

Because the slope is small it also follows that

$$\kappa = \frac{1}{\rho} \approx \frac{d^2 y}{dx^2} \quad \textbf{(9.73)}$$

From Eq. (9.62) this gives

$$\frac{d^2 y}{dx^2} = \frac{M}{EI} \quad \textbf{(9.74)}$$

This equation, together with two boundary conditions, can be used to find the beam deflection. Integrating twice with respect to x gives

$$\frac{dy}{dx} = \int \frac{M}{EI} dx + C_1 \quad \textbf{(9.75)}$$

$$y = \iint \frac{M}{EI} dx + C_1 x + C_2 \quad \textbf{(9.76)}$$

where the constants C_1 and C_2 are determined from the two boundary conditions. Appropriate boundary conditions are on the displacement y or on the slope dy/dx. In the common problems of uniform beams, the beam stiffness EI is a constant and can be removed from beneath the integral sign.

Example 9.17

The uniform cantilever beam shown in Exhibit 22(a) has a constant, uniform, downward load w_0 along its length. Find the deflection and slope of this beam.

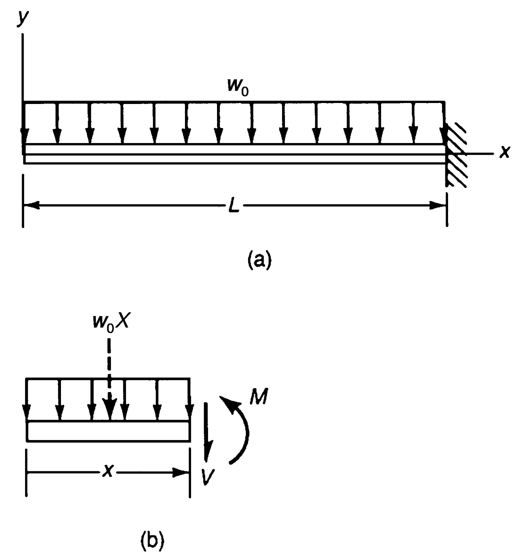


Exhibit 22

Solution

The moment is found by drawing the free-body diagram shown in Exhibit 23(b). The uniform load is replaced with the statically equivalent load $w_0 x$ at the position $x/2$. Moments are then summed about the cut giving

$$M = -w_0 \frac{x^2}{2}$$

Integrating twice with respect to x,

$$\frac{dy}{dx} = \int \frac{M}{EI} dx + C_1 = \frac{1}{EI} \int \left(-w_0 \frac{x^2}{2} \right) dx + C_1 = -\frac{1}{6} \frac{w_0 x^3}{EI} + C_1$$
$$y = \int \left(-\frac{1}{6} \frac{w_0 x^3}{EI} \right) dx + C_1 x + C_2 = -\frac{1}{24} \frac{w_0 x^4}{EI} + C_1 x + C_2$$

At $x = L$ the displacement and slope must be zero so that

$$y(L) = 0 = -\frac{1}{24} \frac{w_0 L^4}{EI} + C_1 L + C_2$$
$$\frac{dy}{dx}(L) = 0 = -\frac{1}{6} \frac{w_0 L^3}{EI} + C_1$$

Therefore,

$$C_1 = \frac{1}{6} \frac{w_0 L^3}{EI}; \qquad C_2 = -\frac{1}{8} \frac{w_0 L^4}{EI}$$

Inserting C_1 and C_2 into the previous expressions gives

$$y = -\frac{w_0}{24EI}(x^4 - 4xL^3 + 3L^4)$$
$$\frac{dy}{dx} = \frac{w_0}{6EI}(L^3 - x^3)$$

Fourth-Order Beam Equation

The second-order beam Eq. (9.74) can be combined with the differential relationships between the shear, moment, and distributed load. Differentiate Eq. (9.74) with respect to x, and use Eq. (9.58).

$$\frac{d}{dx}\left(EI \frac{d^2 y}{dx^2} \right) = \frac{dM}{dx} = V \tag{9.77}$$

Differentiate again with respect to x and use Eq. (9.56).

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 y}{dx^2} \right) = \frac{dV}{dx} = q \tag{9.78}$$

For a uniform beam (that is, constant EI) the fourth-order beam equation becomes

$$EI \frac{d^4 y}{dx^4} = q \tag{9.79}$$

This equation can be integrated four times with respect to x. Four boundary conditions are required to solve for the four constants of integration. The boundary

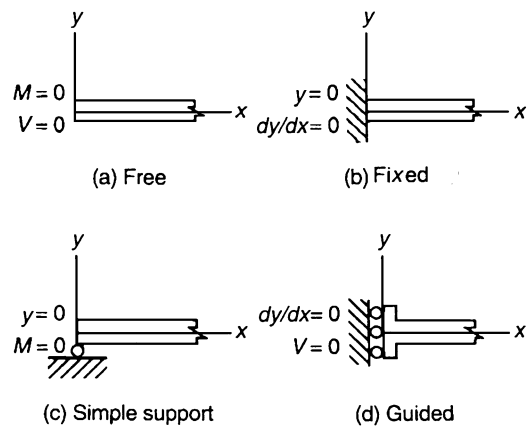


Figure 9.16 Boundary conditions for beams

conditions are on the displacement, slope, moment, and/or shear. Fig. 9.16 shows the appropriate boundary conditions on the end of a beam, even with a distributed loading. If there is a concentrated force or moment applied at the end of a beam, that force or moment enters the boundary condition. For instance, an upward load of P at the left end for the free or guided beam would give $V(0) = P$ instead of $V(0) = 0$.

Example 9.18

Consider the uniformly loaded uniform beam shown in Exhibit 24. The beam is clamped at both ends. The uniform load w_0 is acting downward. Find an expression for the displacement as a function of x.

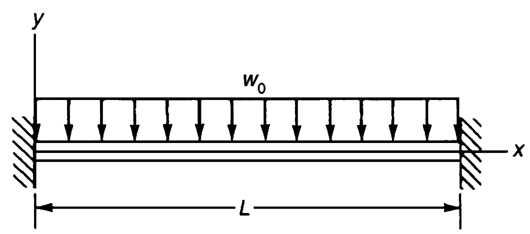


Exhibit 24

Solution

The differential equation is

$$EI\frac{d^4y}{dx^4} = q = -w_0(\text{constant})$$

Integrate four times with respect to x.

$$V = EI\frac{d^3y}{dx^3} = -w_0x + C_1$$
$$M = EI\frac{d^2y}{dx^2} = -w_0\frac{x^2}{2} + C_1x + C_2$$
$$EI\frac{dy}{dx} = -w_0\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2x + C_3$$
$$EIy = -w_0\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4$$

The four constants of integration can be found from four boundary conditions. The boundary conditions are

$$y(0)=0; \quad \frac{dy}{dx}(0)=0; \quad y(L)=0; \quad \frac{dy}{dx}(L)=0$$

These lead to the following:

$$EIy(0)=0=C_4$$

$$EI\frac{dy}{dx}(0)=0=C_3$$

$$EIy(L)=0=-w_0\frac{L^4}{24}+C_1\frac{L^3}{6}+C_2\frac{L^2}{2}$$

$$EI\frac{dy}{dx}(L)=0=-w_0\frac{L^3}{6}+C_1\frac{L^2}{2}+C_2L$$

Solving the last two equations for C_1 and C_2 gives

$$C_1=\frac{1}{2}w_0L; \qquad C_2=-\frac{1}{12}w_0L^2$$

Inserting these values into the equation for y gives

$$y=-\frac{w_0x^2}{EI}\left(\frac{1}{24}x^2-\frac{1}{12}xL+\frac{1}{24}L^2\right)$$

Some solutions for uniform beams with various loads and boundary conditions are shown in Table 9.1.

Table 9.1 Deflection and slope formulas for beams

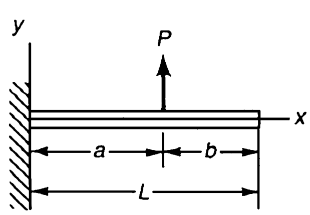
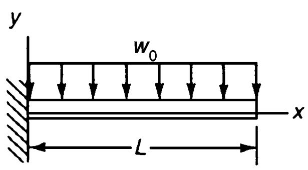
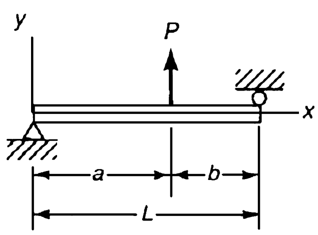
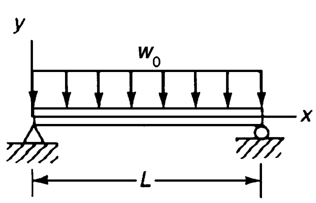
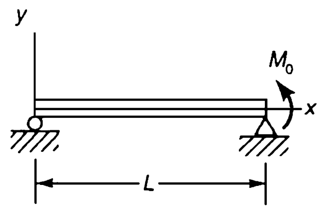
Beam	Deflection, v	Slope, v'
1. 	For $0 \le x \le a$ $y=\frac{Px^2}{6EI}(3a-x)$ For $a \le x \le L$ $y=\frac{Pa^2}{6EI}(3x-a)$	For $0 \le x \le a$ $\frac{dy}{dx}=\frac{px}{2EI}(2a-x)$ For $a \le x \le L$ $\frac{dy}{dx}=\frac{Pa^2}{2EI}a$
2. 	$y=-\frac{w_0x^2}{24EI}(x^2-4Lx+6L^2)$	$\frac{dy}{dx}=-\frac{w_0x}{6EI}(x^2-12Lx+12L^2)$
3. 	For $0 \le x \le a$ $y=\frac{Pbx}{6LEI}(L^2-b^2-x^2)$ For $a \le x \le L$ $y=\frac{Pa(L-x)}{6LEI}(2Lx-a^2-x^2)$	For $0 \le x \le a$ $\frac{dy}{dx}=\frac{Pb}{6LEI}(L^2-b^2-3x^2)$ For $a \le x \le L$ $\frac{dy}{dx}=\frac{Pa}{6LEI}(2L^2+a^2-6Lx+3x^2)$
4. 	$y=-\frac{w_0x}{24EI}(L^3-2Lx^2+x^3)$	$\frac{dy}{dx}=-\frac{w_0}{24EI}(L^3-6Lx^2+4x^3)$

Table 9.1 Deflection and slope formulas for beams (*Continued*)

Beam	Deflection, v	Slope, v'
5. 	$y = -\frac{M_0 x}{6EIL}(L^2 - x^2)$	$\frac{dy}{dx} = -\frac{M_0}{6EIL}(L^2 - 3x^2)$

Superposition

In addition to the use of second-order and fourth-order differential equations, a very powerful technique for determining deflections is the use of superposition. Because all of the governing differential equations are linear, solutions can be directly superposed. Use can be made of tables of known solutions, such as those in Table 9.1, to form solutions to many other problems. Some examples of superposition follow.

Example 9.19

Find the maximum displacement for the simply supported uniform beam loaded by two equal loads placed at equal distances from the ends as shown in Exhibit 25.

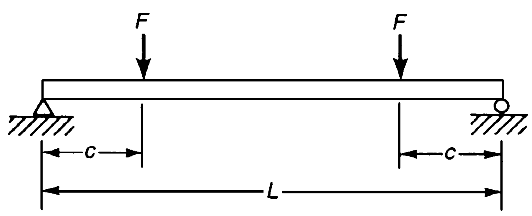


Exhibit 25

Solution

The solution can be found by superposition of the two problems shown in Exhibit 26. From the symmetry of this problem, it can be seen that the maximum deflection will be at the center of the span. The solution for the beam shown in Exhibit 26(a) is found as Case 3 in Table 9.1. In Exhibit 26(a) the center of the span is to the left of the load F so that the formula from the table for $0 \le x \le a$ is chosen. In the formula, $x = L/2$, $c = b$, and $P = -F$ so that

$$y_a\left(\frac{L}{2}\right) = \frac{Pbx}{6LEI}(L^2 - b^2 - x^2) = -\frac{Fc\left(\frac{L}{2}\right)}{6LEI}\left[L^2 - c^2 - \left(\frac{L}{2}\right)^2\right] = \frac{Fc}{48EI}(3L^2 - 4c^2)$$

The central deflection of the beam in Exhibit 26(b) will be the same, so the maximum downward deflection, Δ, will be

$$\delta = -2y_a\left(\frac{L}{2}\right) = \frac{Fc}{24EI}(3L^2 - 4c^2)$$

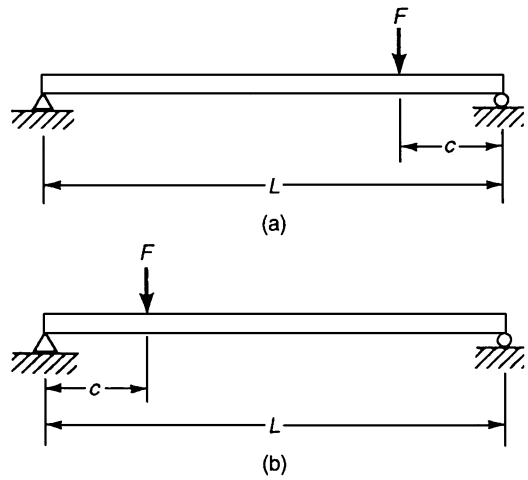


Exhibit 26

Example 9.20

Find an expression for the deflection of the uniformly loaded, supported, cantilever beam shown in Exhibit 27.

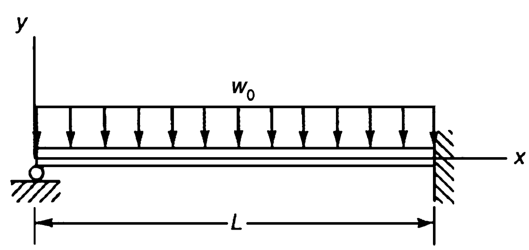


Exhibit 27

Solution

Superpose Case 4 and 5 as shown in Exhibit 28 so that the moment M_0 is of the right magnitude and direction to suppress the rotation at the right end. The rotation

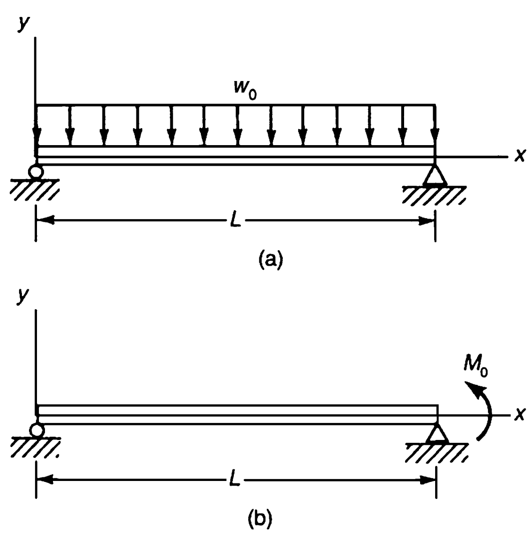


Exhibit 28

for each case from Table 9.1 is

$$\left(\frac{dy}{dx}\right)_4\bigg|_{x=L} = -\frac{w_0}{24EI}(L^3 - 6L^3 + 4L^3) = \frac{w_0 L^3}{24EI}$$

$$\left(\frac{dy}{dx}\right)_5\bigg|_{x=L} = -\frac{M_0}{6EIL}(L^2 - 3L^2) = \frac{M_0 L}{3EI}$$

Setting the rotation at the end equal to zero gives

$$\left(\frac{dy}{dx}\right)_4\bigg|_{x=L} + \left(\frac{dy}{dx}\right)_5\bigg|_{x=L} = 0 = -\frac{w_0 L^3}{24EI} + \frac{M_0 L}{3EI}$$

$$M_0 = -\frac{w_0 L^2}{8}$$

Substituting this expression into the formulas in the table and adding gives

$$y = -\frac{w_0 x}{24EI}(L^3 - 2Lx^2 + x^3) + \frac{w_0 L^2}{8}\frac{x}{6EI}(L^2 - x^2) = -\frac{w_0 x}{48EI}(L^3 - 3Lx^2 + 2x^3)$$

COMBINED STRESS

In many cases, members can be loaded in a combination of bending, torsion, and axial loading. In these cases, the solution of each portion is exactly as before; the effects of each are simply added. This concept is best illustrated by an example.

Example 9.21

In Exhibit 29, there is a thin-walled, aluminum tube *AB*, which is attached to a wall at *A*. The tube has a rectangular cross-section member *BC* attached to it. A vertical load is placed on the member *BC* as shown. The aluminum tube has an outer diameter of 50 mm and a wall thickness of 3.25 mm. Take P = 900 N, a = 450 mm, and b = 400 mm. Find the state of stress at the top of the tube at the point *D*. Draw Mohr's circle for this point, and find the three principal stresses.

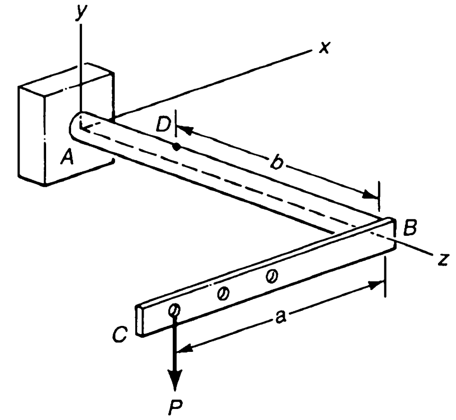


Exhibit 29

Solution

Cut the tube at the Point D. Draw the free-body diagram as in Exhibit 30(a). From that free-body diagram, a summation of moments at the cut about the z axis gives

$$T = Pa = (900 \text{ N})(450 \text{ mm}) = 405 \text{ N} \bullet \text{m}$$

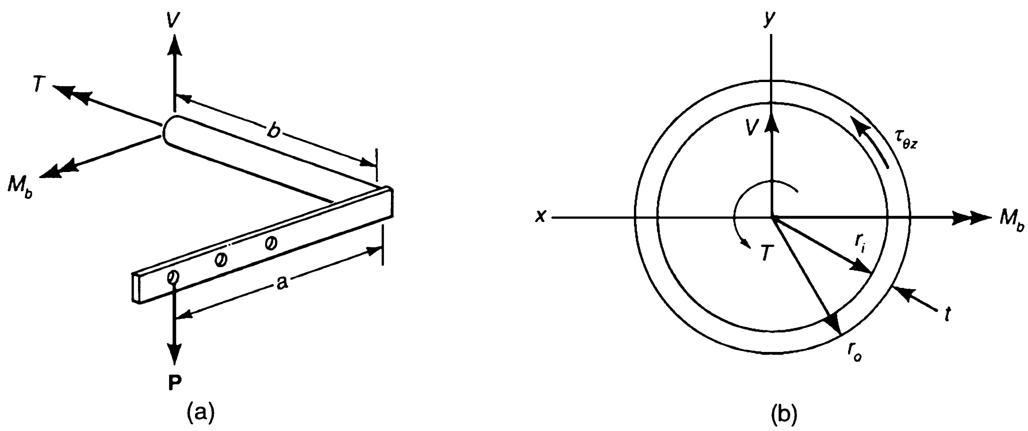


Exhibit 30

A summation of moments at the cut about an axis parallel with the x axis gives

$$M_b = Pb = (900 \text{ N})(400 \text{ mm}) = 360 \text{ N} \bullet \text{m}$$

A summation of vertical forces gives

$$V = P$$

Exhibit 30(b) shows the force and moments acting on the cross-section. The bending and shearing stresses caused by these loads are

$$\sigma_z = \frac{M_b y}{I_{xx}} \qquad (\text{from } M_b)$$

$$\tau_{\theta z} = \frac{Tr}{I_z} \qquad (\text{from } T)$$

$$\tau_{\theta z} = \frac{VQ}{I_{xx} b} \qquad (\text{from } V)$$

The shearing stress attributed to V will be zero at the top of the beam and can be neglected. The moments of inertia are

$$I_{xx} = \frac{\pi\left(r_o^4 - r_i^4\right)}{4} = \frac{\pi(25^4 - 21.75^4)}{4} = 131 \times 10^3 \text{ mm}^4$$

$$I_z = \frac{\pi\left(r_o^4 - r_i^4\right)}{2} = 2I_{xx} = 262 \times 10^3 \text{ mm}^4$$

At the top of the tube $r = 25$ mm and $y = 25$ mm, so the stresses are

$$\sigma_z = \frac{M_b y}{I_{xx}} = \frac{(360 \text{ N} \bullet \text{m})(25 \text{ mm})}{131 \times 10^3 \text{ mm}^4} = 68.7 \text{ MPa}$$

$$\tau_{\theta z} = \frac{Tr}{I_z} = \frac{(405 \text{ N} \bullet \text{m})(25 \text{ mm})}{262 \times 10^3 \text{ mm}^4} = 38.6 \text{ MPa}$$

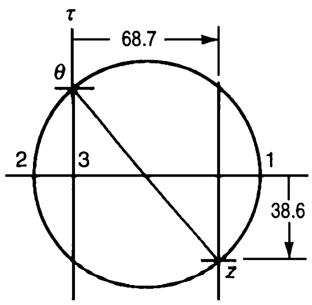


Exhibit 31

The Mohr's circle plot for this is shown in Exhibit 31.

$$R=\sqrt{\left(\frac{\sigma_z-\sigma_\theta}{2}\right)^2+\tau_{\theta z}^2}=\sqrt{\left(\frac{68.7-0}{2}\right)^2+38.6^2}=51.7\,\text{MPa}$$

$$C=\frac{\sigma_z\sigma_\theta}{2}=\frac{68.7}{2}=34.4\,\text{MPa}$$

$$\sigma_1=C+R=86.1\,\text{MPa}$$

$$\sigma_2=C-R=-17.3\,\text{MPa}$$

Because this is a state of plane stress, the third principal stress is

$$\sigma_3 = 0$$

COLUMNS

Buckling can occur in slender columns when they carry a high axial load. Fig. 9.17(a) shows a simply supported slender member with an axial load. The beam is shown in the horizontal position rather than in the vertical position for convenience. It is assumed that the member will deflect from its normally straight configuration as shown. The free-body diagram of the beam is shown in Fig. 9.17(b). Figure 9.17(c) shows the free-body diagram of a section of the beam. Summation of moments on the beam section in Fig. 9.17(c) yields

$$M + Py = 0 \qquad \textbf{(9.80)}$$

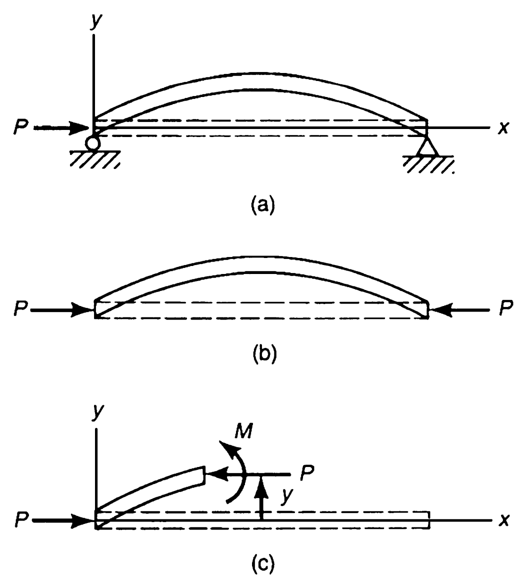


Figure 9.17 Buckling of simply supported column

Since M is equal to EI times the curvature, the equation for this beam can be expressed as

$$\frac{d^2y}{dx^2}+\lambda y=0 \qquad \textbf{(9.81)}$$

where

$$\lambda^2=\frac{P}{EI} \qquad \textbf{(9.82)}$$

The solution satisfying the boundary conditions that the displacement is zero at either end is

$$v = \sin(\lambda x), \text{ where } \lambda = n\pi/L \ n = 1, 2, 3\ldots \quad \textbf{(9.83)}$$

The lowest value for the load P is the buckling load, so $n = 1$ and the critical buckling load, or Euler buckling load, is

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{L^2} \quad \textbf{(9.84)}$$

For other than simply supported boundary conditions, the shape of the deflected curve will always be some portion of a sine curve. The simplest shape consistent with the boundary conditions will be the deflected shape. Fig. 9.18 shows a sine curve and the beam lengths that can be selected from the sine curve. The critical buckling load can be redefined as

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{(kL)^2} = \frac{\pi^2 E}{(kL/r)^2} \quad \textbf{(9.85)}$$

where the radius of gyration r is defined as $\sqrt{I/A}$. The ratio L/r is called the slenderness ratio.

From Fig. 9.18, it can be seen that the values for L_e and k are as follows:

For simple supports: $L = L_e$; $L_e = L$; $k = 1$

For a cantilever: $L = 0.5L_e$; $L_e = 2L$; $k = 2$

For both ends clamped: $L = 2L_e$; $L_e = 0.5L$; $k = 0.5$

For supported-clamped: $L = 1.43L_e$; $L_e = 0.7L$; $k = 0.7$

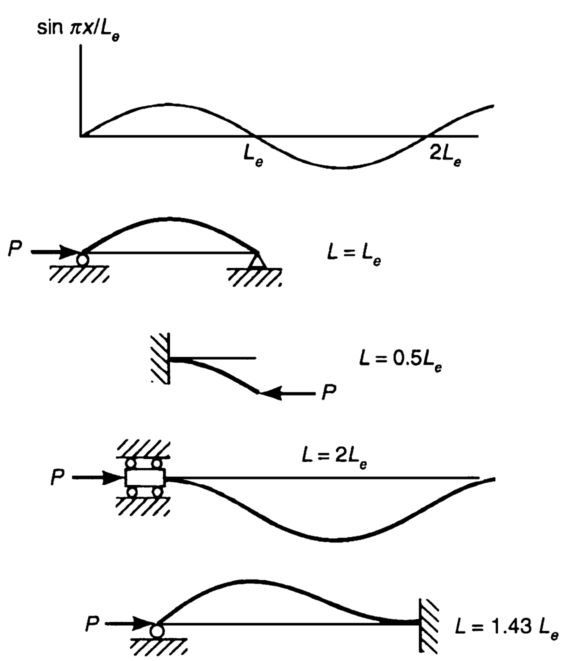


Figure 9.18 Buckling of columns with various boundary conditions

In dealing with buckling problems, keep in mind that the member must be slender before buckling is the mode of failure. If the beam is not slender, it will fail by yielding or crushing before buckling can take place.

Example 9.22

A steel pipe is to be used to support a weight of 130 kN as shown in Exhibit 32. The pipe has the following specifications: $OD = 100$ mm, $ID = 90$ mm, $A = 1500$ mm^2, and $I = 1.7 \times 10^6$ mm^4. Take $E = 210$ GPa and the yield stress $Y = 250$ MPa. Find the maximum length of the pipe.

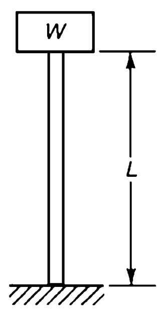


Exhibit 32

Solution

First, check to make sure that the pipe won't yield under the applied weight. The stress is

$$\sigma = \frac{P}{A} = \frac{130\,\text{kN}}{1500\,\text{mm}^2} = 86.7\,\text{MPa} < Y$$

This stress is well below the yield, so buckling will be the governing mode of failure. This is a cantilever column, so the constant k is 2. The critical load is

$$P_{\text{cr}} = \frac{\pi^2 EI}{(2L)^2}$$

Solving for L gives

$$L = \pi\sqrt{\frac{EI}{4P}} = \pi\sqrt{\frac{(210\,\text{GPa})(1.7\times10^6\,\text{mm}^4)}{4(130\,\text{kN})}} = 2.60\,\text{m}$$

The maximum length is 2.6 m.

SELECTED SYMBOLS AND ABBREVIATIONS

Symbol or Abbreviation	Description
σ	stress
ε	strain
v	Poisson's ratio
kip	kilopound
E	modulus of elasticity
δ	deformation
W	weight
P	load
P, p	pressure
I	moment of inertia
τ	shear stress
T	torque
A	area
M	moment
V	shear
L	length
F	force

PROBLEMS

9.1 The stepped circular aluminum shaft in Exhibit 9.1 has two different diameters: 20 mm and 30 mm. Loads of 20 kN and 12 kN are applied at the end of the shaft and at the step. The maximum stress is most nearly

a. 23.4 MPa c. 28.3 MPa
b. 26.2 MPa d. 30.1 MPa

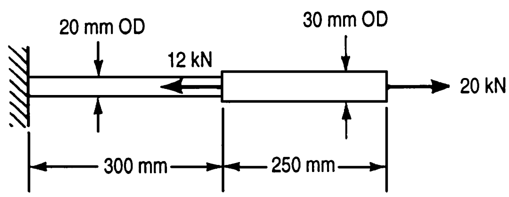


Exhibit 9.1

9.2 For the same shaft as in Problem 9.1 take $E = 69$ GPa. The end deflection is most nearly

a. 0.18 mm c. 0.35 mm
b. 0.21 mm d. 0.72 mm

9.3 The shaft in Exhibit 9.3 is the same aluminum stepped shaft considered in Problems 9.1 and 9.2, except now the right-hand end is also built into a wall. Assume that the member was built in before the load was applied. The maximum stress is most nearly

a. 12.2 MPa c. 13.1 MPa
b. 12.7 MPa d. 15.2 MPa

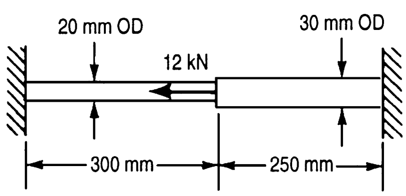


Exhibit 9.3

9.4 For the same shaft as in Problem 9.3 the deflection of the step is most nearly

a. 0.038 mm c. 0.064 mm
b. 0.042 mm d. 0.086 mm

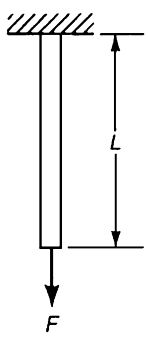


Exhibit 9.5

9.5 The uniform rod shown in Exhibit 9.5 has a force F at its end which is equal to the total weight of the rod. The rod has a unit weight γ. The total deflection of the rod is most nearly

a. 1.00 $\gamma L^2/E$ c. 1.50 $\gamma L^2/E$
b. 1.25 $\gamma L^2/E$ d. 1.75 $\gamma L^2/E$

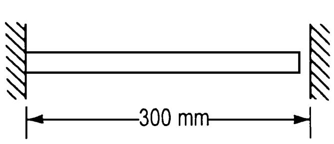


Exhibit 9.6

9.6 At room temperature, 22°C, a 300-mm stainless steel rod (Exhibit 9.6) has a gap of 0.15 mm between its end and a rigid wall. The modulus of elasticity $E = 210$ GPa. The coefficient of thermal expansion $\alpha = 17 \times 10^{-6}$/°C. The area of the rod is 650 mm^2. When the temperature is raised to 100 °C, the stress in the rod is most nearly

a. 175 MPa (tension) c. −17.5 MPa (compression)
b. 0 MPa d. −175 MPa (compression)

9.7 A steel cylindrical pressure vessel is subjected to a pressure of 21 MPa. Its outer diameter is 4.6 m, and its wall thickness is 200 mm. The maximum principal stress in this vessel is most nearly

a. 183 MPa c. 362 MPa
b. 221 MPa d. 432 MPa

9.8 A pressure vessel shown in Exhibit 9.8 is known to have an internal pressure of 1.4 MPa. The outer diameter of the vessel is 300 mm. The vessel is made of steel; $v = 0.3$ and $E = 210$ GPa. A strain gage in the circumferential direction on the vessel indicates that, under the given pressure, the strain is 200×10^{-6}. The wall thickness of the pressure vessel is most nearly

a. 3.2 mm c. 6.4 mm
b. 4.3 mm d. 7.8 mm

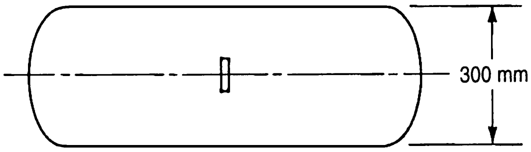


Exhibit 9.8

9.9 An aluminum pressure vessel has an internal pressure of 0.7 MPa. The vessel has an outer diameter of 200 mm and a wall thickness of 3 mm. Poisson's ratio is 0.33 and the modulus of elasticity is 69 GPa for this material. A strain gage is attached to the outside of the vessel at 45° to the longitudinal axis as shown in Exhibit 9.9. The strain on the gage would read most nearly

a. 40×10^{-6} c. 80×10^{-6}
b. 60×10^{-6} d. 160×10^{-6}

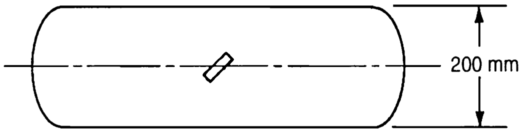


Exhibit 9.9

9.10 If $\sigma_x = -3$ MPa, $\sigma_y = 5$ MPa, and $\tau_{xy} = -3$ MPa, the maximum principal stress is most nearly

a. 4 MPa c. 6 MPa
b. 5 MPa d. 7 MPa

9.11 Given that $\sigma_x = 5$ MPa, $\sigma_y = -1$ MPa, and the maximum principal stress is 7 MPa, the shear stress τ_{xy} is most nearly

a. 1 MPa c. 3 MPa
b. 2 MPa d. 4 MPa

9.12 Given $\varepsilon_x = 800$ μ, $\varepsilon_y = 200$ μ, and $\gamma_{xy} = 400$ μ, the maximum principal strain is most nearly

a. 840 μ c. 900 μ
b. 860 μ d. 960 μ

9.13 A steel plate in a state of plane stress has the same strains as in Problem 9.12: $\varepsilon_x = 800\,\mu$, $\varepsilon_y = 200\,\mu$, and $\gamma_{xy} = 400\,\mu$. Poisson's ratio $v = 0.3$ and the modulus of elasticity $E = 210$ GPa. The maximum principal stress in the plane is most nearly
a. 109 MPa c. 173 MPa
b. 132 MPa d. 208 MPa

9.14 A stepped steel shaft shown in Exhibit 9.14 has torques of 10 kN • m applied at the end and at the step. The maximum shear stress in the shaft is most nearly
a. 760 MPa c. 870 MPa
b. 810 MPa d. 930 MPa

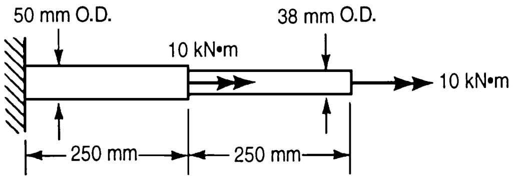


Exhibit 9.14

9.15 The shear modulus for steel is 83 MPa. For the same shaft as in Problem 9.14, the rotation at the end of the shaft is most nearly
a. 0.014° c. 1.4°
b. 0.14° d. 14°

9.16 The same stepped shaft as in Problems 9.14 and 9.15 is now built into a wall at its right end before the load is applied (Exhibit 9.16). The maximum stress in the shaft is most nearly
a. 130 MPa c. 230 MPa
b. 200 MPa d. 300 MPa

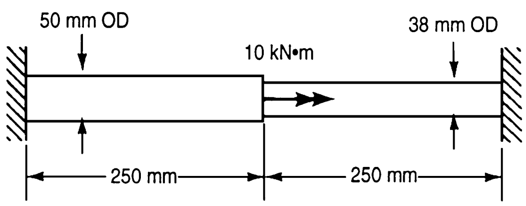


Exhibit 9.16

9.17 For the same shaft as in Problem 9.16, the rotation of the step is most nearly
a. 0.2° c. 1.8°
b. 1.1° d. 2.1°

9.18 A strain gage shown in Exhibit 9.18 is placed on a circular steel shaft which is being twisted with a torque T. The gage is inclined 45° to the axis. If the strain reads $\varepsilon_{45} = 245\ \mu$, the torque is most nearly
a. 1000 N•m c. 1570 N•m
b. 1230 N•m d. 2635 N•m

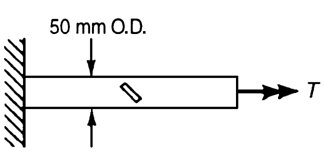


Exhibit 9.18

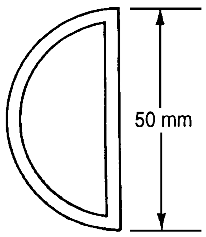


Exhibit 9.19

9.19 A shaft whose cross section is in the shape of a semicircle is shown in Exhibit 9.19 and has a constant wall thickness of 3 mm. The shaft carries a torque of 300 N • m. Neglecting any stress concentrations at the corners, the maximum shear stress in the shaft is most nearly

a. 32 MPa c. 59 MPa
b. 48 MPa d. 66 MPa

9.20 The maximum magnitude of shear in the beam shown in Exhibit 9.20 is most nearly

a. 40 kN c. 60 kN
b. 50 kN d. 75 kN

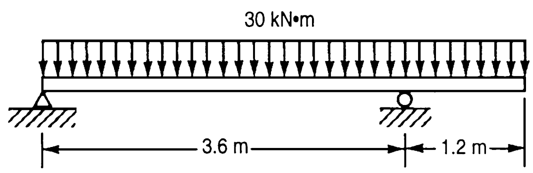


Exhibit 9.20

9.21 For the same beam as in Problem 9.20, the magnitude of the largest bending moment is most nearly

a. 21.0 kN • m c. 38.4 kN • m
b. 26.3 kN • m d. 42.1 kN • m

9.22 The shear diagram shown in Exhibit 9.22 is for a beam that has zero moments at either end. The maximum concentrated force on the beam is most nearly

a. 60 kN upward c. 0
b. 30 kN upward d. 30 kN downward

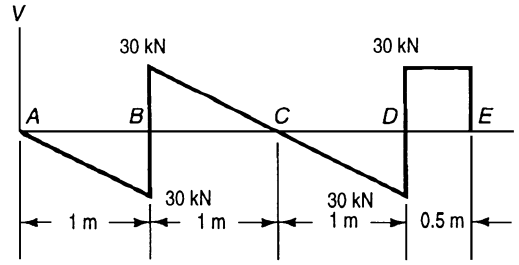


Exhibit 9.22

9.23 For the same beam as in Problem 9.22 the largest magnitude of the bending moment is most nearly

a. 0 c. 12 kN • m
b. 8 kN • m d. 15 kN • m

9.24 The 4-m long, simply supported beam shown in Exhibit 9.24 has a section modulus Z = 1408 × 10^3 mm^3. The allowable stress in the beam is not to exceed 100 MPa. The maximum load, *w* (including its own weight), that the beam can carry is most nearly:

a. 50 kN • m c. 60 kN • m
b. 40 kN • m d. 70 kN • m

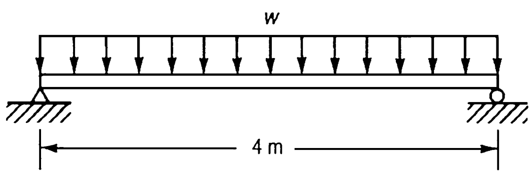


Exhibit 9.24

9.25 The standard wide flange beam shown in Exhibit 9.25 has a moment of inertia about the z axis of $I = 365 \times 10^6\ \text{mm}^4$. The maximum bending stress is most nearly

a. 4.5 MPa c. 6.5 MPa
b. 5.0 MPa d. 8 MPa

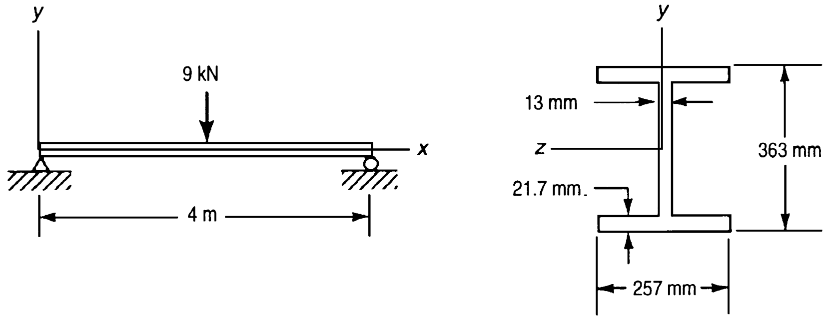


Exhibit 9.25

9.26 For the same beam as in Problem 9.25, the maximum shear stress τ_{xy} in the web is most nearly

a. 1 MPa c. 2.0 MPa
b. 1.5 MPa d. 2.5 MPa

9.27 The deflection at the end of the beam shown in Exhibit 9.27 is most nearly

a. 0.330 FL^3/EI (downward) c. 0.410 FL^3/EI (downward)
b. 0.380 FL^3/EI (downward) d. 0.440 FL^3/EI (downward)

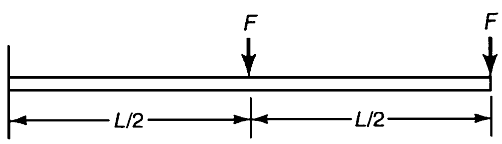


Exhibit 9.27

9.28 A uniformly loaded beam (Exhibit 9.28) has a concentrated load wL at its center that has the same magnitude as the total distributed load w. The maximum deflection of this beam is most nearly

a. 0.029 wL^4/EI (downward) c. 0.043 wL^4/EI (downward)
b. 0.034 wL^4/EI (downward) d. 0.056 wL^4/EI (downward)

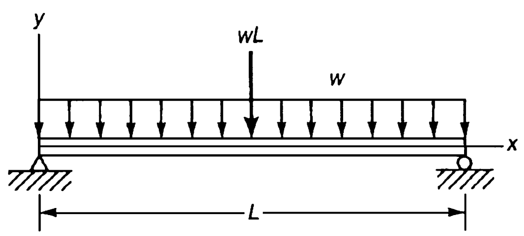


Exhibit 9.28

9.29 The reaction at the center support of the uniformly loaded beam shown in Exhibit 9.29 is most nearly

a. $0.525\,wL$ c. $0.575\,wL$
b. $0.550\,wL$ d. $0.625\,wL$

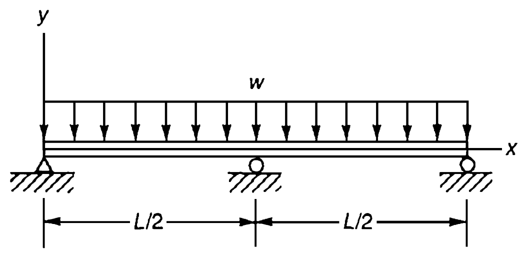


Exhibit 9.29

9.30 A solid circular rod has a diameter of 25 mm (Exhibit 9.30). It is fixed into a wall at *A* and bent 90° at *B*. The maximum bending stress in the section *BC* is most nearly

a. 21.7 MPa c. 32.6 MPa
b. 29.3 MPa d. 45.7 MPa

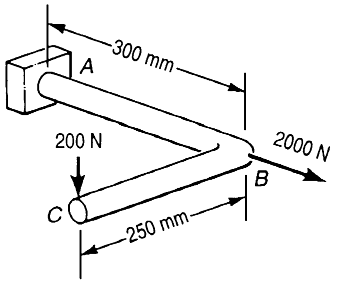


Exhibit 9.30

9.31 For the same member as in Problem 9.30 the maximum bending stress in the section *AB* is most nearly

a. 21 MPa c. 31 MPa
b. 25 MPa d. 39 MPa

9.32 For the same member as in Problem 9.30 the maximum shear stress due to torsion in the section *AB* is most nearly

a. 15.2 MPa c. 17.4 MPa
b. 16.3 MPa d. 18.5 MPa

9.33 For the same member as in Problem 9.30, the maximum stress due to the axial force in the section *AB* is most nearly

a. 4 MPa c. 6 MPa
b. 5 MPa d. 8 MPa

9.34 For the same member as in Problem 9.30, the maximum principal stress in the section *AB* is most nearly

a. 17 MPa
b. 27 MPa
c. 39 MPa
d. 44 MPa

9.35 A truss is supported so that it can't move out of the plane (Exhibit 9.35). All members are steel and have a square cross section 25 mm by 25 mm. The modulus of elasticity for steel is 210 GPa. The maximum load *P* that can be supported without any buckling is most nearly

a. 14 kN
b. 25 kN
c. 34 kN
d. 51 kN

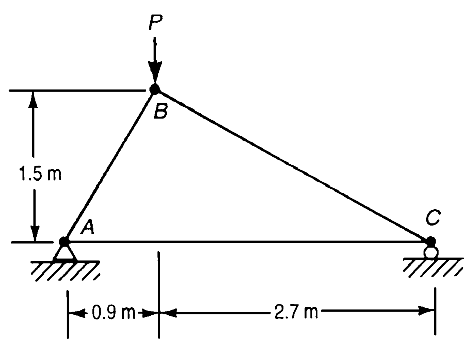


Exhibit 9.35

9.36 A beam is pinned at both ends (Exhibit 9.36). In the *x-y* plane it can rotate about the pins, but in the *x-z* plane the pins constrain the end rotation. In order to have buckling equally likely in each plane, the ratio *b*/*a* is most nearly

a. 0.5
b. 1.0
c. 1.5
d. 2.0

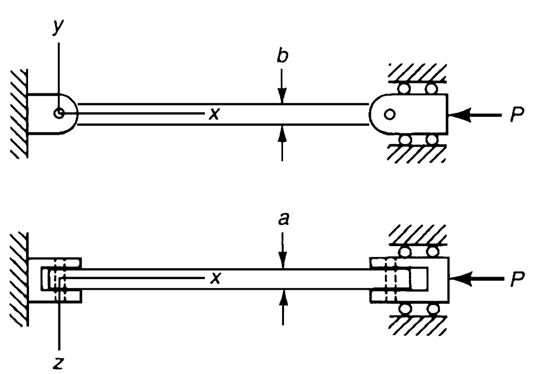


Exhibit 9.36

SOLUTIONS

9.1 c. Draw free-body diagrams. Equilibrium of the center free-body diagram gives

$$F_1 = 20 - 12 = 8 \text{ kN}$$

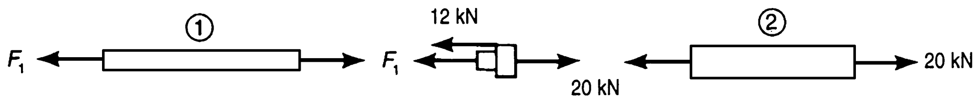


Exhibit 9.1a

The areas are

$$A_1 = \pi r^2 = \pi(10 \text{ mm})^2 = 314 \text{ mm}^2$$
$$A_2 = \pi r^2 = \pi(15 \text{ mm})^2 = 707 \text{ mm}^2$$

The stresses are

$$\sigma_1 = \frac{P}{A} = \frac{8\,\text{kN}}{314\,\text{mm}^2} = 25.5\,\text{MPa}$$

$$\sigma_2 = \frac{P}{A} = \frac{20\,\text{kN}}{707\,\text{mm}^2} = 28.3\,\text{MPa}$$

9.2 b. The force-deformation equations give

$$\delta_1 = \frac{P_1 L_1}{A_1 E_1} = \frac{(8\,\text{kN})(300\,\text{mm})}{(314\ \text{mm}^2)(69\,\text{GPa})} = 0.1107\,\text{mm}$$

$$\delta_2 = \frac{P_2 L_2}{A_2 E_2} = \frac{(20\,\text{kN})(250\,\text{mm})}{(707\,\text{mm}^2)(69\,\text{GPa})} = 0.1025\,\text{mm}$$

Compatibility of deformation gives

$$\delta_{\text{end}} = \delta_1 + \delta_2 = 0.1107 + 0.1025 = 0.213\,\text{mm}$$

9.3 c. Draw the free-body diagrams. From the center free-body diagram, summation of forces yields

$$F_2 = 12 \text{ kN} + F_1$$

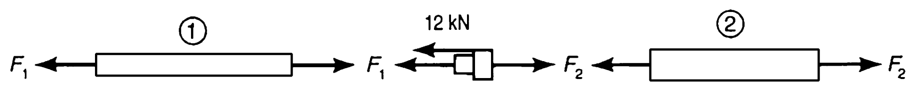


Exhibit 9.3a

Force-deformation relations are

$$\delta_1 = \frac{P_1 L_1}{A_1 E_1} = \frac{(F_1\,\text{kN})(300\,\text{mm})}{(314\,\text{mm}^2)(69\,\text{GPa})} = 0.01384\,F_1$$

$$\delta_2 = \frac{P_2 L_2}{A_2 E_2} = \frac{(F_2\,\text{kN})(200\,\text{mm})}{(707\,\text{mm}^2)(69\,\text{GPa})} = 0.00410\,F_2$$

Compatibility gives

$$\delta_{\text{end}} = 0 = \delta_1 + \delta_2 = 0.01384\ F_1 + 0.00410\ F_2$$

Substitution of the equilibrium relation, $F_2 = 12 + F_1$, into the above equation gives

$$0 = 0.01384\,F_1 + 0.00410\,(12 + F_1)$$
$$F_1 = -2.74 \text{ kN},\ F_2 = 9.26 \text{ kN}$$

The stresses are

$$\sigma_1 = \frac{P}{A} = \frac{(-2.74\,\text{kN})}{(314\,\text{mm}^2)} = -8.73\,\text{MPa}; \qquad \sigma_2 = \frac{P}{A} = \frac{(9.26\,\text{kN})}{(707\,\text{mm}^2)} = 13.10\,\text{MPa}$$

9.4 a. The same three-step process as in Problem 9.3 must be carried out. Since this process has already been completed, the results can be used. The deflection can be expressed as

$$\delta = \delta_1 = -\delta_2 = 0.01384\,F_1 = 0.01384\,(-2.74) = -0.0380 \text{ mm}$$

9.5 c. Draw a free-body diagram. Summation of forces in the vertical direction gives

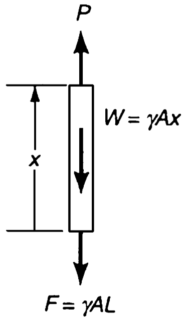


Exhibit 9.5a

$$P = \gamma AL + \gamma Ax$$
$$\delta = \int_0^L \frac{P}{AE}\,dx = \int_0^L \frac{\gamma A(L+x)}{AE}\,dx = \frac{\gamma}{E}\left(L^2 + \frac{L^2}{2}\right) = \frac{3\gamma L^2}{2E}$$

9.6 d. A force will develop in the rod if it attempts to grow more than 0.15 mm. Assuming that it does grow that amount, the displacement is

$$\delta = \frac{PL}{AE} + \alpha L(t - t_o) = 0.15\,\text{mm}$$
$$= \frac{P(300\,\text{mm})}{(650\,\text{mm})(210\,\text{GPa})} + \left(17\times10^{-6}\,\frac{1}{°\text{C}}\right)(300\,\text{mm})(100\,°\text{C} - 22\,°\text{C})$$
$$0.15 \text{ mm} = 0.00220\,P + 0.3978$$
$$P = -112.7 \text{ kN}$$
$$\sigma = \frac{P}{A} = \frac{-112.7\,\text{kN}}{650\,\text{mm}^2} = -173.5\,\text{MPa}$$

9.7 b. In a cylindrical pressure vessel the three principal stresses are

$$\sigma_t = qD/2t; \quad \sigma_a = qD/4t; \quad \sigma_r \approx 0$$

The maximum is σ_t, which gives

$$\sigma_t = \frac{qD}{2t} = \frac{(21\,\text{MPa})(4600\,\text{mm} - 400\,\text{mm})}{2(200\,\text{mm})} = 221\,\text{MPa}$$

9.8 b. The stresses in the pressure vessel are, as in the last problem,

$$\sigma_t = \frac{qD}{2t}; \qquad \sigma_a = \frac{qD}{4t}; \qquad \sigma_r \approx 0$$

The wall thickness is usually thin enough so that it can be assumed that

$$D_i \approx D_o$$

The tangential strain can be found from Hooke's law:

$$\varepsilon_t = \frac{1}{E}(\sigma_t - v\sigma_\theta - v\sigma_r) = \frac{1}{E}\left(\frac{qD}{2t} - v\frac{qD}{4t} - v0\right) = 0.425\frac{qD}{Et}$$

The thickness is then

$$t = 0.425\frac{pD}{E\varepsilon_t} = 0.425\frac{(1.4\,\text{MPa})(300\,\text{mm})}{(210\,\text{GPa})(200\times10^{-6})} = 4.25\,\text{mm}$$

9.9 d. Draw Mohr's circle for the stress. At 45° in the physical plane (90° on Mohr's circle) the two normal stresses are $3qD/8t$. Hooke's law gives

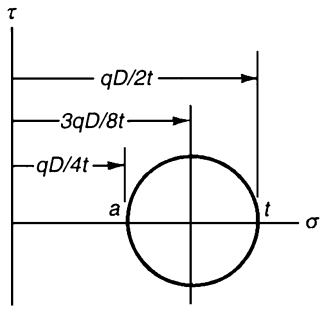


Exhibit 9.9a

$$\varepsilon_{45} = \frac{1}{E}(\sigma_{45} - v\sigma_{-45} - v\sigma_r) = \frac{1}{E}\left(\frac{3qD}{8t} - v\frac{3qd}{8t} - v0\right)$$
$$= \frac{(1-v)}{E}\left(\frac{3qD}{8t}\right)$$
$$\varepsilon_{45} = \frac{(1-0.33)}{(69\,\text{GPa})}\,\frac{3(0.7\,\text{MPa})(200\,\text{mm} - 6\,\text{mm})}{8(3\,\text{mm})} = 164.8\times10^{-6}$$

9.10 c. Draw Mohr's circle. The maximum principal stress is 6 MPa. As an alternative,

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-3-5}{2}\right)^2 + (-3)^2} = 5$$
$$C = \frac{\sigma_x + \sigma_y}{2} = \frac{-3+5}{2} = 1$$
$$\sigma_1 = R + C = 6\ \text{MPa}$$

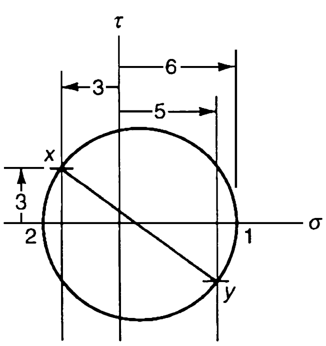


Exhibit 9.10a

9.11 d. Draw Mohr's circle. The center of the circle is

$$C = \frac{\sigma_x + \sigma_y}{2} = \frac{5-1}{2} = 2$$

The radius is then $R = 7 - 2 = 5$. The shear stress can be found from the Mohr's circle or from the expression

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}; \qquad \tau_{xy}^2 = R^2 - \left(\frac{\sigma_x - \sigma_y}{2}\right)^2$$

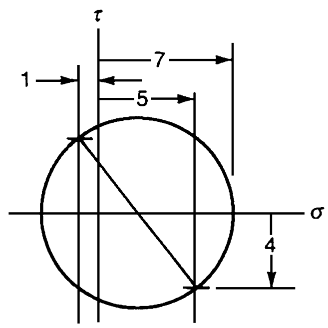


Exhibit 9.11a

In either case the shear stress τ_{xy} = 4 MPa.

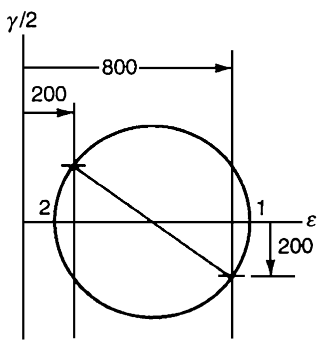


Exhibit 9.12a

9.12 b. Draw Mohr's circle. ε_1 can be scaled from the circle or computed as follows,

$$R=\sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}=\sqrt{\left(\frac{800-200}{2}\right)^2+\left(\frac{400}{2}\right)^2}=361\,\mu$$

$$C=\frac{\varepsilon_x+\varepsilon_y}{2}=\frac{800+200}{2}=500\,\mu$$

$$\varepsilon_1=C+R=500\,\mu+361\,\mu=861\,\mu$$

$$\varepsilon_2=C-R=500\,\mu-361\,\mu=139\,\mu$$

ε_1 is the maximum.

9.13 d. Problems of this type can be done by using Hooke's law first and then Mohr's circle or by using Mohr's circle first and then applying Hooke's law. Since Mohr's circle was already drawn for this problem in the previous solution, the second approach will be followed. The principal strains were found to be the following: ε_1 = 861 μ; ε_2 = 139 μ. Hooke's law in plane stress is

$$\varepsilon_1=\frac{1}{E}(\sigma_1-v\sigma_2);\qquad \varepsilon_2=\frac{1}{E}(\sigma_2-v\sigma_1)$$

Inverting these relationships gives

$$\sigma_1=\frac{E}{1-v^2}(\varepsilon_1+v\varepsilon_2);\qquad \sigma_2=\frac{E}{1-v^2}(\varepsilon_2+v\varepsilon_1)$$

The maximum principal stress is σ_1, which is

$$\sigma_1=\frac{E}{1-v^2}(\varepsilon_1+v\varepsilon_2)=\frac{210\,\text{GPa}}{1-0.3^2}[861\times10^{-6}+0.3(139\times10^{-6})]=208\text{ MPa}$$

9.14 d. Draw free-body diagrams. The torque in shaft 1 is T_1 = 10 + 10 = 20 kN • m. The torque in shaft 2 is T_2 = 10 kN • m.

$$\tau_1=\frac{T_1r_1}{J_1}=\frac{(20\,\text{kN}\bullet\text{m})(25\,\text{mm})}{0.5\pi(25\,\text{mm})^4}=815\text{ MPa}$$

$$\tau_2=\frac{T_2r_2}{J_2}=\frac{(10\,\text{kN}\bullet\text{m})(19\,\text{mm})}{0.5\pi(19\,\text{mm})^4}=928\text{ MPa}$$

The largest stress is 928 MPa.

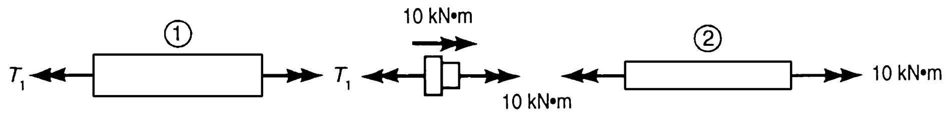


Exhibit 9.14a

9.15 d. From the force-deformation relations,

$$\phi_1 = \frac{T_1 L_1}{GJ_1} = \frac{(20\,\text{kN} \bullet \text{m})(250\,\text{mm})}{(83\,\text{GPa})0.5\pi(25\,\text{mm})^4} = 0.982\,\text{rad} = 5.63°$$

$$\phi_2 = \frac{T_2 L_2}{GJ_2} = \frac{(10\,\text{kN} \bullet \text{m})(250\,\text{mm})}{(83\,\text{GPa})0.5\pi(19\,\text{mm})^4} = 0.1471\,\text{rad} = 8.43°$$

From compatibility,

$$\phi = \phi_1 + \phi_2 = 5.63° + 8.43° = 14.06°$$

9.16 d. Draw the free-body diagrams. Equilibrium of the center free body gives

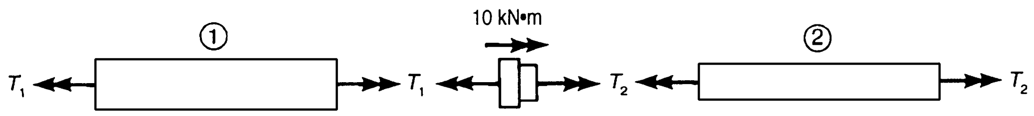


Exhibit 9.16a

$$T_1 = T_2 + 10$$

The force-deformation relations are

$$\phi_1 = \frac{T_1 L_1}{GJ_1} = \frac{(10 + T_2)(250\,\text{mm})}{(83\,\text{GPa})0.5\pi(25\,\text{mm})^4} = 49.1 \times 10^{-3} + 4.91 \times 10^{-3} T_2$$

$$\phi_2 = \frac{T_2 r_2}{J_2} = \frac{T_2(250\,\text{mm})}{(83\,\text{GPa})\ 0.5\pi(19\,\text{mm})^4} = 14.7 \times 10^{-3}\, T_2$$

Compatibility requires that

$$\phi_1 + \phi_2 = 0 = 49.1 \times 10^{-3} + (4.91 \times 10^{-3} + 14.7 \times 10^{-3})T_2$$

Solving for the torques gives

$$T_2 = \frac{5.305}{2.207} = -2.50\,\text{kN} \bullet \text{m}$$

$$T_1 = T_2 + 10 = -2.50 + 10 = 7.50\,\text{kN} \bullet \text{m}$$

The stresses then are

$$\tau_1 = \frac{T_1 r_1}{J_1} = \frac{(7.50\,\text{kN} \bullet \text{m})(25\,\text{mm})}{0.5\pi(25\,\text{mm})^4} = 306\,\text{MPa}$$

$$\tau_2 = \frac{T_2 r_2}{J_2} = \frac{(-2.5\,\text{kN} \bullet \text{m})(19\,\text{mm})}{0.5\pi(19\,\text{mm})^4} = -232\,\text{MPa}$$

9.17 d. The same three-step process as in Problem 9.16 must be carried out. Since this process has already been completed, the results can be used. The rotation can be expressed as

$$\phi = \phi_1 = -\phi_2 = \frac{T_1 L_1}{G J_1} = \frac{(7.50\,\text{kN} \bullet \text{m})(250\,\text{mm})}{(83\,\text{GPa})0.5\pi(25\,\text{mm})^4} = 0.0368 \text{ rad} = 2.11°$$

9.18 a. For a torsion problem, the shear strain is

$$\gamma_{\phi z} = \frac{\tau_{\phi z}}{G} = \frac{Tr}{GJ}$$

Other shear strains in the $r - \phi$ orientation are zero. Mohr's circle for this state of strain is shown in Exhibit 9.18(a). From Mohr's circle,

$$\varepsilon_{45} = \frac{\gamma_{\phi z}}{2} = \frac{\tau_{\phi z}}{2G} = \frac{Tr}{2GJ}$$

$$T = \frac{2GJ\varepsilon_{45}}{r} = \frac{2(83\,\text{GPa})[0.5\pi(25\,\text{mm})^4](245 \times 10^{-6})}{25\,\text{mm}} = 998\,\text{N} \bullet \text{m}$$

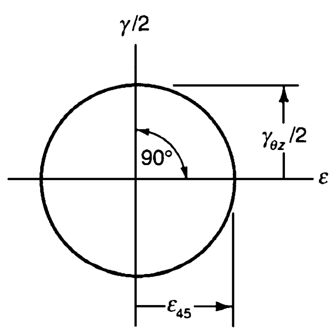


Exhibit 9.18a

9.19 d. In thin-walled shafts the shear stress is

$$\tau_{sz} = \frac{T}{2At}$$

The area A is the cross-sectional area of the shaft including the hole so,

$$A = \frac{\pi r^2}{2} = \frac{\pi(25\,\text{mm} - 3\,\text{mm})^2}{2} = 760 \text{ mm}^2$$

$$\tau_{sz} = \frac{T}{2At} = \frac{300\,\text{N} \bullet \text{m}}{2(760\,\text{mm}^2)(3\,\text{mm})} = 65.7\,\text{MPa}$$

9.20 c. Draw the free-body diagram of the beam, replacing the distributed load with its statically equivalent loads. Summation of moments about the left end gives

$$0 = -3.6\, R_2 + (108)(1.8) + (36)(4.2)$$
$$R_2 = 96 \text{ kN}$$

Summation of forces in the vertical direction gives

$$R_1 = 144 - 96 = 48 \text{ kN}$$

FundEng_Index.book Page 325 Wednesday, November 28, 2007 4:42 PM

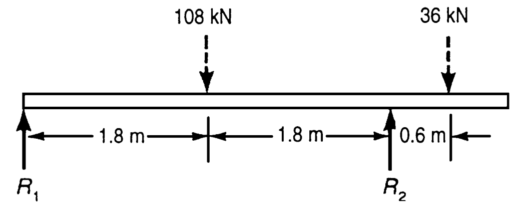


Exhibit 9.20a

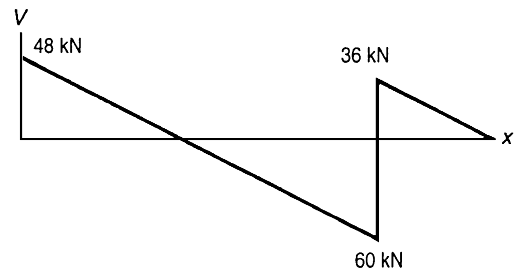


Exhibit 9.20b

This is enough information to plot the shear diagram (Exhibit 9.20b). The largest magnitude of shear is 60 kN.

9.21 c. The maximum bending moment occurs where the shear is zero. From the shear diagram, the distance to the zero from the left end can be found by similar triangles.

$$\frac{48}{x} = \frac{108}{3.6}; \qquad x = \frac{(48)(3.6)}{108} = 1.6\,\text{m}$$

The areas of the shear diagrams are the changes in moment (Exhibit 9.21a).

$$A_1 = \frac{(48\,\text{kN})(1.6\,\text{m})}{2} = 38.4\,\text{kN} \bullet \text{m}$$
$$A_2 = \frac{(36\,\text{kN})(1.2\,\text{m})}{2} = 21.6\,\text{kN} \bullet \text{m}$$

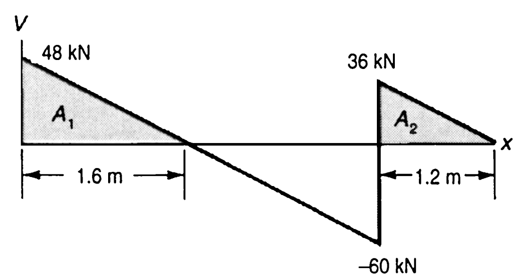


Exhibit 9.21a

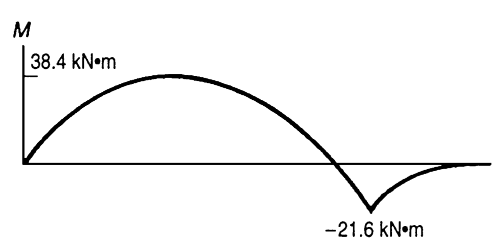


Exhibit 9.21b

The moment diagram is shown in Exhibit 9.21b. The maximum bending moment is 38.4 kN • m.

9.22 a. There is a jump at *B* and *D* of 60 kN upward and a downward jump of 30kN at *E*. These jumps correspond to concentrated forces.

9.23 d. The areas of the shear diagrams (Exhibit 9.23) are the changes in moment. Since the moments are zero on either end,

$$M_B = A_1 = \frac{(30\,\text{kN})(1\,\text{m})}{2} = 15\ \text{kN} \bullet \text{m}$$

$$M_C = A_1 + A_2 = \frac{(30\,\text{kN})(1\,\text{m})}{2} + \frac{(30\,\text{kN})(1\,\text{m})}{2} = 0$$

$$\text{M}_D = A_3 = (30\,\text{kN})(-0.5\,\text{m}) = -15\ \text{kN} \bullet \text{m}$$

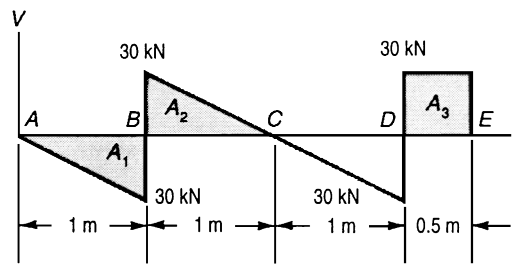


Exhibit 9.23

The largest magnitude of the bending moment is therefore 15 kN • m.

9.24 d. It is obvious that each support will carry half of the load, so the reactions are *wL*/2. The shear diagram is shown in Exhibit 9.24a. The maximum bending moment is

$$M = A_1 = \frac{1}{2}\left(\frac{wL}{2}\right)\left(\frac{L}{2}\right) = \frac{wL^2}{8}$$

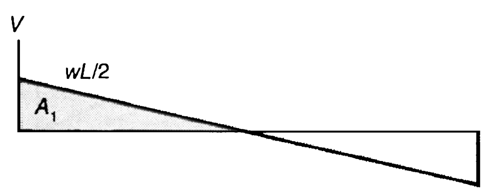


Exhibit 9.24a

The maximum bending stress is

$$\sigma_{\max} = \frac{M}{Z} = \frac{wL^2}{8Z} = 100\,\text{MPa}$$

$$w = \frac{(100\,\text{MPa})\,8Z}{L^2} = \frac{(100\,\text{MPa})(8)(1408\times 10^3\,\text{mm}^3)}{(4\,\text{m})^2} = 70.4\,\text{kN/m}$$

9.25 a. Draw the free-body diagram and the shear and moment diagrams as shown in Exhibit 9.25a. The maximum bending stress is

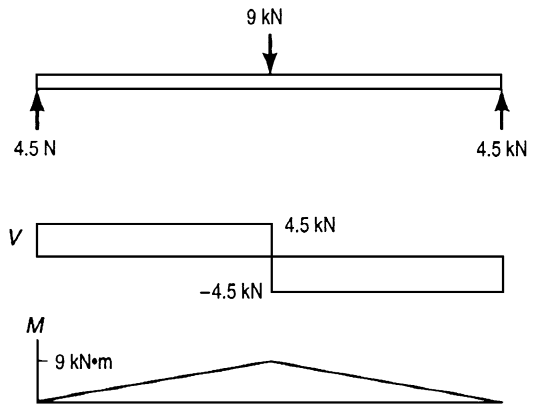


Exhibit 9.25a

$$\sigma_{max} = \frac{M_{max}c}{I} = \frac{(9\,\text{kN}\bullet\text{m})\left(\frac{363\,\text{mm}}{2}\right)}{(365\times10^6\ \text{mm}^4)} = 4.48\ \text{MPa}$$

9.26 a. From the previous problem, the maximum shear in the beam is 4.5 kN. The maximum shearing stress will take place at the centroid (Exhibit 9.26), so a cut must be made there in order to calculate *Q*. The moment of the area *Q* is, therefore,

$$Q = A_1\bar{y}_1 + A_2\bar{y}_2$$

$$Q = (257\,\text{mm})(21.7\,\text{mm})\left(\frac{363\,\text{mm}}{2} - \frac{21.7\,\text{mm}}{2}\right) + \cdots + \left(\frac{363\,\text{mm}}{2} - 21.7\,\text{mm}\right)$$

$$\times (13\,\text{mm})\left(\frac{\frac{363\,\text{mm}}{2} - 21.7\,\text{mm}}{2}\right)$$

$$Q = 1.117\times10^6\,\text{mm}^3$$

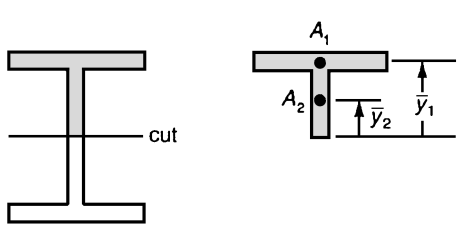


Exhibit 9.26

The maximum shear stress is then

$$\tau = \frac{VQ}{Ib} = \frac{(4.5\,\text{kN})(1.117\times10^6\ \text{mm}^3)}{(365\times10^6\,\text{mm}^4)(13\,\text{mm})} = 1.060\,\text{MPa}$$

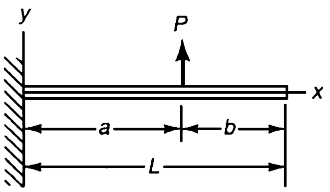


Exhibit 9.27a

9.27 d. From Table 9.1, Beam Type 1 (Exhibit 9.27a), for $a \le x \le L$

$$y = \frac{Pa^2}{6EI}(3x - a)$$

For the load at the half-way point, $a = L/2$, $x = L$, and $P = -F$. For the load at the end, $a = L$, $x = L$, and $P = -F$. Therefore,

$$y = \frac{-F\left(\frac{L}{2}\right)^2}{6EI}\left[3L - \left(\frac{L}{2}\right)\right] + \frac{-FL^2}{6EI}(3L - L) = -0.4375\frac{FL^3}{EI}$$

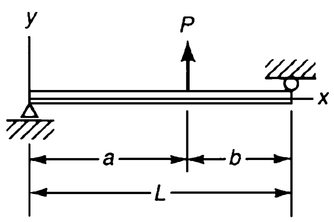


Exhibit 9.28a

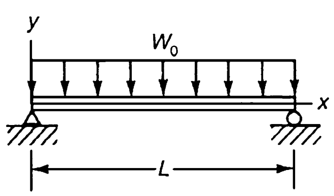


Exhibit 9.28b

9.28 b. The maximum deflection for this beam will take place at the center of the beam. This problem can be solved with the superposition of the following cases from Table 9.1 (Exhibits 9.28a and b).

For $0 \le x \le a$,

$$y = \frac{Pbx}{6LEI}(L^2 - b^2 - x^2)$$

For this problem, $P = -wL$, $a = b = L/2$, and $x = L/2$.

$$y = -\frac{wx}{24EI}(L^3 - 2Lx^2 + x^3)$$

For this problem, $x = L/2$. The total deflection is, therefore,

$$y = \frac{Pbx}{6LEI}(L^2 - b^2 - x^2) - \frac{wx}{24EI}(L^3 - 2Lx^2 + x^3)$$

$$y = \frac{(-wL)\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)}{6LEI}\left[L^2 - \left(\frac{L}{2}\right)^2 - \left(\frac{L}{2}\right)^2\right] - \frac{w\left(\frac{L}{2}\right)}{24EI}\left[L^3 - 2L\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^3\right]$$

$$v = -0.0339\frac{wL^4}{EI} = -\frac{13wL^4}{384EI}$$

9.29 d. This problem can be solved from superposition of the same two cases as used in Problem 9.28. For the concentrated load solution, $b = L/2$ and P is left as an unknown. In both, $x = L/2$. The center support means the beam does not deflect in the center. Therefore,

$$y = 0 = \frac{P\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)}{6LEI}\left[L^2 - \left(\frac{L}{2}\right)^2 - \left(\frac{L}{2}\right)^2\right] - \frac{w\left(\frac{L}{2}\right)}{24EI}\left[L^3 - 2L\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^3\right]$$

$$y = 0 = \frac{PL^3}{48EI} - \frac{5wL^4}{384EI}$$

$$P = \frac{5}{8}wL$$

9.30 c. Draw the free-body diagram (Exhibit 9.30a). From a summation of moments about the cut at *B*, the maximum bending moment in *BC* is the moment $M = 200\ \text{N} \times 250\ \text{mm}$ or 50 kN • m. The maximum bending stress is

$$\sigma = \frac{Mc}{I} = \frac{(50\,\text{kN}\bullet\text{mm})(12.5\,\text{mm})}{0.25\pi(12.5\,\text{mm})^4} = 32.6\ \text{MPa}$$

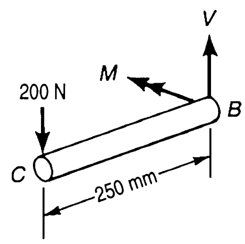


Exhibit 9.30a

9.31 d. Draw the free-body diagram (Exhibit 9.31). The maximum stresses in section *AB* will occur at *A*. Summation of forces in the vertical direction gives V_A = 200 N. Summation of forces along the direction of the rod *AB* gives *P* = 2000 N. Summation of moments along the rod *AB* gives T_A = 200 N × 250 mm or 50 kN • mm. Summation of moments at the cut perpendicular to the rod *AB* gives M_A = 200 N × 300 mm = 600 kN • mm. The maximum bending stress is

$$\sigma = \frac{Mc}{I} = \frac{(60\,\text{kN}\bullet\text{mm})(12.5\,\text{mm})}{0.25\pi(12.5\,\text{mm})^4} = 39.1\,\text{MPa}$$

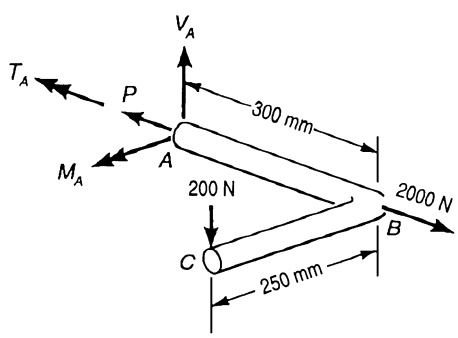


Exhibit 9.31

9.32 b. From the free-body diagram in Problem 9.31, the maximum torque is 50 kN • mm. The maximum shear stress is, therefore,

$$\tau_{\max} = \frac{T_{\max} r_0}{J} = \frac{(50\,\text{kN}\bullet\text{mm})(12.5\,\text{mm})}{0.5\pi(12.5\,\text{mm})^4} = 16.30\ \text{MPa}$$

9.33 a. From the free-body diagram in Problem 9.31, the maximum axial force is 2000 N. The maximum stress due to this force is, therefore,

$$\sigma = \frac{P}{A} = \frac{2000\,\text{N}}{\pi(12.5\,\text{mm})^2} = 4.07\,\text{MPa}$$

9.34 d. The stresses were found in the previous three problems. There is an axial stress due to both bending and axial loads. This stress is

$$\sigma = 39.1 \text{ MPa} + 4.07 \text{ MPa} = 43.2 \text{ MPa}$$

The shear stress is 16.3 MPa. These are the only non-zero stresses. The maximum principal stress can be calculated as follows,

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{43.2 - 0}{2}\right)^2 + 16.3^2} = 22.0 \text{ MPa}$$

$$C = \frac{\sigma_x + \sigma_y}{2} = \frac{43.2 + 0}{2} = 21.6 \text{ MPa}$$

$$\sigma_1 = C + R = 22.0 + 21.6 = 43.6 \text{ MPa}$$

9.35 a. Draw the free-body diagram of the joint B (Exhibit 9.35a). Summation of forces in the vertical direction gives

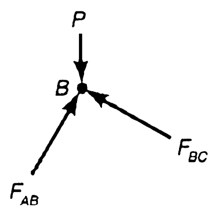


Exhibit 9.35a

$$P = F_{AB}\frac{1.5}{\sqrt{1.5^2 + 0.9^2}} + F_{BC}\frac{1.5}{\sqrt{2.7^2 + 1.5^2}}$$

Summation of forces in the horizontal direction gives

$$F_{AB}\frac{0.9}{\sqrt{1.5^2 + 0.9^2}} = F_{BC}\frac{2.7}{\sqrt{2.7^2 + 1.5^2}}$$

Solving for F_{AB} and F_{BC} gives

$$F_{AB} = 0.875P; \quad F_{BC} = 0.515P$$

The member AC is in tension and does not need to be considered. The moment of inertia for both members is

$$I = \frac{bh^3}{12} = \frac{(25\,\text{mm})(25\,\text{mm})^3}{12} = 32{,}600\,\text{mm}^4$$

The critical buckling load for member AB is

$$P_{\text{cr}} = F_{AB} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(210\,\text{GPa})(32{,}600^4)}{(1.5\,\text{m})^2 + (0.9\,\text{m})^2} = 22.0 \text{ kN}$$

The load P for buckling to occur in AB is

$$P = \frac{22\,\text{kN}}{0.875} = 25.2 \text{ kN}$$

The critical buckling load for member BC is

$$P_{\text{cr}} = F_{BC} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(210\,\text{GPa})(32{,}600\,\text{mm}^4)}{[(2.7\,\text{m})^2 + (1.5\,\text{m})^2]} = 7.07 \text{ kN}$$

The load P for buckling to occur in BC is

$$P = \frac{7.07\,\text{kN}}{0.515} = 13.7\,\text{kN}$$

9.36 d. To buckle in the x-y plane the critical buckling load is

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 E\left(\frac{ab^3}{12}\right)}{L^2}$$

To buckle in the x-z plane the critical buckling load is

$$P_{cr} = \frac{4\pi^2 EI}{L^2} = \frac{4\pi^2 E\left(\frac{ba^3}{12}\right)}{L^2}$$

Equating these two representations of P_{cr} gives

$$ab^3 = 4ba^3; \quad b^2 = 4a^2; \quad \frac{b}{a} = 2$$