ODE's

- Power Series method
- Euler equations
- Systems of first Order differential equations
- Laplace transforms.

ODE

An Ode is an equation which involves derivative of a function of a single variable.

Example $y' - y = 0$.

$y(x)$

dependent Variable ; independent Variables.

Order of an ODE

highest order of derivative involved in the ODE $y' - y = 0$ (has order 1). ($y'' - y = 0$) (has order 2)

Types of methods of solving ODE's

1. Separation of Variables
2. integrating factor
3. Variation parameter.
4. Power Series method.

Power Series method

what is a power Series? is a Series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$

more generally,

$\sum_{n=0}^{\infty} a_n (x - x_0)^n$ when it is simply fixed and x_0 is the center of the series

a_n - Coefficients

x_0 = Center of the series

x - Variable.

n = indexing.

Examples

① $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = e^x$

② $\sum_{n=0}^{\infty} n(x-1)^n = 0 + (x-1) + 2(x-1)^2 + 3(x-1)^3 + \ldots$

Analytic functions

There are functions that have a power series representation.

E.g ① polynomial functions $p(x) = x^2 + 2x + 1 + 0x^3 + 0x^4 + \ldots$

② Exponential functions.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ how do you come up with this by using (Taylor Series of functions) (Maclaurin Series of functions)

③ $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{(2n)!}$

④ $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}$

Technique of the power series method

Manipulating power series

① Addition

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x$$

② Differentiate.

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

③ Shifting the index for example

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

Sometimes you shift so as to start the indexing from zero

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n .$$

of which the two are the same

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{let } m = n-1$$
$$n = m+1$$
$$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$$

Idea behind the method of power series

Example -

Find the power series solution to $y' + y = 0$

Solution

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1} n a_n x^{n-1} \qquad y'' = \sum_{n=2} (n-1) n a_n x^{n-2}$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$[(n+1) a_{n+1} - a_n] x^n = 0$$
$$(n+1) a_{n+1} = + a_n$$
$$a_{n+1} = \frac{+ a_n}{(n+1)}$$

$n = 0$ $\qquad a_1 = a_0$

$n = 1$ $\qquad a_2 = \frac{a_1}{2}$ $\qquad a_2 = \frac{a_0}{2}$

$n = 2$ $\qquad a_3 = \frac{a_2}{3} = \frac{a_0}{6}$

$n = 3$ $\qquad a_4 = \frac{a_3}{4} = \left(\frac{a_0}{6}\right)\frac{1}{4} = \frac{a_0}{24}$

$n = 4$ $\qquad a_5 = \frac{a_4}{5} = \left(\frac{a_0}{24}\right)\frac{1}{5} = \frac{a_0}{120}$

$n = m$ $\qquad a_m = \frac{a_0}{m!}$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0 + a_0 x + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!} + \frac{a_0 x^4}{4!}$$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= a_0 e^x$$

Power Series method (method 2)

$$y' - y = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 0.$$

$$\sum_{n=1}^{\infty} \left[n a_n - a_{n-1} \right] x^{n-1} = 0$$

$$n a_n - a_{n-1} = 0$$

$$a_{n-1} = n a_n$$

$$a_n = \frac{a_{n-1}}{n}$$

when $n=1$ $\quad a_1 = \frac{a_0}{1}$ $\quad n = 2$ $\quad a_2 = a_1/2$

which is just the same as the last ~~comp~~ solution

Solution of a Second Order Linear D.E using power method

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$$

when $r(x) = 0$ Homogeneous

$r(x) \neq 0$ Non-Homogeneous

If a_2, a_1 and a_0 are Constant, the ODE has Constant Coefficients

If not, it have Variable Coefficients.

$$y'' + \frac{a_1(x)}{a_2(x)} y' + \frac{a_0(x)}{a_2(x)} y = \frac{r(x)}{a_2(x)}$$

are analytic about $x_0 = 0$, then the above ODE have power series solution about $x = x_0$

Example 1

$x_0 = 0 \qquad y'' + y = 0$

$y'' + y = 0$

$\sum_{n=0}^{\infty} a_n x^n$

From

$\sum_{n=0}^{\infty} a_n (x - x_0)^n$

$\sum_{n=0}^{\infty} a_n (x-0)^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1} n$

$\sum_{n=2}^{\infty} (n-1)n a_n x^{n-2} + \sum_{n=0}^{\infty} a_n (x)^n = 0$

$\sum_{n=0}^{\infty} (n-1+2)(n+2) a_{n+2} x^{x} +$

$\sum_{n=0}^{\infty} a_n x^n = 0$

$[(n+1)(n+2) a_{n+2} = -a_n$

$a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$

$n = 0 \quad a_2 = \frac{-a_0}{2}$

$n = 1$

$a_3 = \frac{-a_1}{6}$

$a_n = 2$

$a_4 = \frac{-a_2}{12} = -\frac{a_2}{12}\left(\frac{-a_0}{2}\right)$

$= \frac{+a_0}{24}$

$n = 3$

$a_5 = \frac{-a_3}{20}\left(\frac{-a_1}{6}\right)$

$= \frac{1}{120} a_1$

$n = 4$

$a_6 = \frac{-a_4}{30} = -a_0$

Example 2

$(x-3)y' + 2y = 0$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1} n a_n x^{\infty}$

~~$(x-3)\sum_{n=1}^{\infty} n a_n x^n$~~

$(x-3)\sum_{n=1}^{\infty}(n-1)$

$(x-3)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n$

$= \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 3n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n$

$= \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n$

Example 2

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(x^2+1)y'' + 4xy' + 6y = 0 \qquad y'' = \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2}$$

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=1}^{\infty} a_n x^n =$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 4n a x^n +$$

$$\sum_{n=0}^{\infty} 6a_n x^n = 0.$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} 4n a x^n +$$

$$\sum_{n=0}^{\infty} 6a_n x^n$$

$$n(n-1)a_n \cancel{x^n} + (n+1)(n+2)a_{n+2}\cancel{x^n} - 4n a_n \cancel{x^n} + 6a_n \cancel{x^n} = 0$$

$$(n+1)(n+2)a_{n+2} = 4na_n - 6a_n - n(n-1)a_n$$

$$a_{n+2} = \frac{4na_n - 6a_n - n(n-1)a_n}{(n+1)(n+2)}$$

$$a_{n+2} = \frac{[4n - 6 - n(n-1)]a_n}{(n+1)(n+2)}$$

Example 3:

iii) $(x^2-1)y'' + 6xy' + 4y = -4$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2}.$$

) $(x+1)y' = 8y$ $\qquad y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$(x+1)\sum_{n=1}^{\infty} n a_n x^{n-1} - 8\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} n a_n x^{n-1} - 8\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 8\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} \cancel{x^n} = \sum_{n=0}^{\infty} 8 a_n \cancel{x^n} - \sum_{n=1}^{\infty} n a_n \cancel{x^n}$$

$$(n+1)a_{n+1} = 8a_n - n a_n$$

$$a_{n+1} = \frac{8a_n - n a_n}{n+1}$$

$$a_{n+1} = \frac{(8-n)a_n}{n+1}$$

when $n=0$

$a_1 = 8a_0$

when $n=1$

$a_2 = \frac{7}{2} a_1 = \frac{7}{2}(8a_0)$

$= 28a_0$

$n=2$

$a_3 = \frac{6}{3} a_2$

$= 2a_2$

when $n=3$

$a_4 = \frac{5}{4} a_3$

Something that can not be solved using power rule

For example, the ODE $x^2y'' + xy' + y = 0$ does not have a power series solution about $x = 0$.

$$y'' + \frac{x}{x^2}y' + \frac{1}{x^2}y = 0$$

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0 \quad \text{because } \frac{1}{x} = x^{-1}$$

and $\frac{1}{x^2} = x^{-2}$ which are not analytic unless

$n = 0, 1, 2 \ldots$

for this we use euler equations

Examples

$y'' - y = 0$ (ii). $y'' - xy = 0$

ii)
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2}.$$

$$= \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$= \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2)\underset{a_{n+2}}{x^n} - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$= 2a_2 + \sum_{n=1}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+1)(n+2) a_{n+2} - a_{n-1}] x^n = 0$$

$$2a_2 = 0 \qquad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$$

$$a_2 = 0$$

$$a_{n+2} = \frac{a_n(n-1)}{(n+2)(n+1)}$$

$n = 1$

$$a_3 = \frac{a_0}{3\times 2}$$

$n = 2$

$$a_4 = \frac{a_1}{4\times 3}$$

$n = 3$

$$a_5 = \frac{a_2}{5\times 4} = 0$$

$$a_2 = a_5 = a_8 = a_{11} = a_{14} = 0$$

$n = 4$

$$a_6 = \frac{a_3}{6\times 5} = \frac{a_0}{6\times 5\times 3\times 2.}$$

$n = 5$

$$a_7 = \frac{a_4}{7\times 6} = \frac{a_1}{7\times 4\times 6\times 3}$$

$$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$y = a_0 + a_1 x + \frac{a_0}{3\times 2}x^3 + \frac{a_1}{4\times 3}x^4 + \frac{a_0}{6\times 5\times 3\times 2}x^6 + \frac{a_1}{7\times 6\times 4\times 3}x^7$$

MAT3110 POWER SERIES TUTORIAL

Q1 y^n

1 $(1+x)y'' - y = x.$

2. $(x^2-1)y'' + 6xy' + 4y = -4$

3. $(x+1)y' = 3y$.