Question One

Part (a)

$r = 4\sin\theta$

$\sqrt{x^2+y^2} = 4\dfrac{y}{\sqrt{x^2+y^2}}$

$x^2+y^2 = 4y.$

$x^2+y^2-4y = 0$

$x^2+y^2-4y+(-2)^2 = (-2)^2$

$x^2+(y-2)^2 = 4.$

Circle of radius 2 centered at $(0,2)$.

(1 mark)

$r = 2\sin\theta$

$\sqrt{x^2+y^2} = 2\dfrac{y}{\sqrt{x^2+y^2}}$

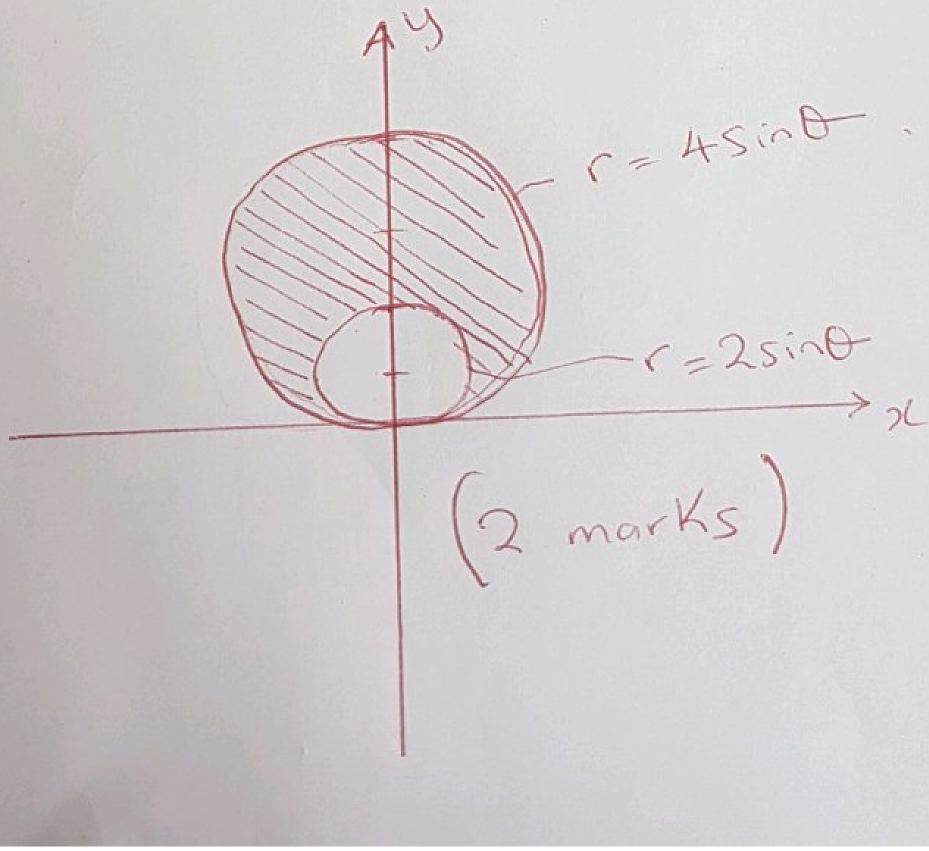
$x^2+y^2 = 2y$

$x^2+y^2-2y = 0$

$x^2+(y-1)^2 = 1$

Circle of radius 1 centered at $(0,1)$.

(1 mark)



(2 marks)

Using a double integral to determine area of the shaded region, we have

$$\text{Area} = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r \, dr\, d\theta. \quad (4 \text{ marks})$$

$$= \int_0^{\pi} \left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta} \cdot d\theta.$$

$$= \int_0^{\pi} \frac{16\sin^2\theta}{2} - \frac{4\sin^2\theta}{2}$$

$$= 6\int_0^{\pi} \sin^2\theta \cdot d\theta.$$

$$= 3\int_0^{\pi} 1 - \cos 2\theta \cdot d\theta$$

$$= 3\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$\text{Area.} = 3\pi. \quad 2 \text{ (marks)}.$$

Notice that this is the answer you would obtain if you used "common sense", that is, area of 4π removing area of π.

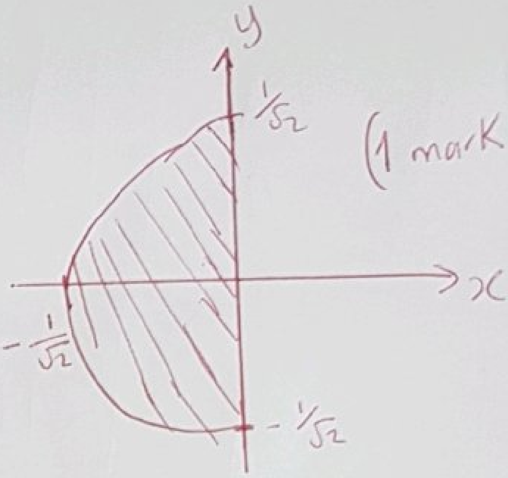
$$4\pi - \pi = 3\pi.$$

Question One

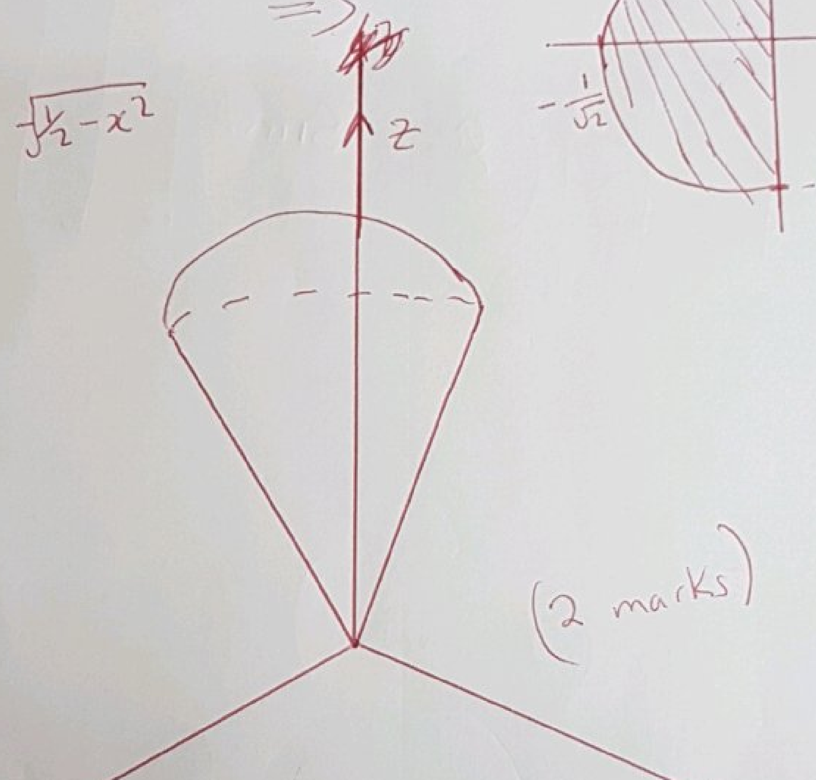
Part (b).

$$\int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 18y \, dz \, dy \, dz.$$

$$\int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \Rightarrow$$



(1 mark).



(2 marks)

Changing to spherical coordinates, we get.

$$\int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 18y \, dz\,dy\,dx = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{\frac{\pi}{4}} \int_{0}^{1} 18(\rho \sin\theta \sin\phi)\, \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

(5 marks)

$$\int_{\pi/2}^{3\pi/2}\int_{0}^{\pi/4}\int_{0}^{1} 18(\rho\sin\theta\sin\phi)\,\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

$$= 18\left[\int_{\pi/2}^{3\pi/2}\sin\theta\, d\theta\right]\times\left[\int_{0}^{\pi/2}\sin^2\phi\right]\times\left[\int_{0}^{1}\rho^3\, d\rho\right]$$

$$= 18\left[-\cos\theta\Big|_{\pi/2}^{3\pi/2}\right]\times\left[\left(\frac{\phi}{2}-\frac{\sin 2\phi}{4}\right)\Big|_{0}^{\pi/2}\right]\left[\frac{\rho^4}{4}\Big|_{0}^{1}\right]$$

$=$

$= 0$ (2 marks)

Question Two

$$F(x,y,z) = (2z^4 - 2y - y^3)\,i + (z - 2x - 3xy^2)\,j + (6 + y + 8xz^3)\,K.$$

Part (a).

$$\frac{\partial f}{\partial x} = 2z^4 - 2y - y^3$$

$$f(x,y,z) = 2xz^4 - 2xy - xy^3 + g(y,z).$$

$$\frac{\partial f}{\partial y} = -2x - 3xy^3 + g_y(y,z) = z - 2x - 3xy^2$$

(6 marks) $\therefore \quad g_y(y,z) = z \quad \Rightarrow \quad g(y,z) = yz + h(z).$

$$f(x,y,z) = 2xz^4 - 2xy - xy^3 + yz + h(z)$$

$$\frac{\partial f}{\partial z} = 8xz^3 + y + h'(z) = 6 + y + 8xz^3$$

(4 marks) $\therefore \quad h'(z) = 6 \quad \Rightarrow \quad h(z) = 6z + C$

Hence

$$f(x,y,z) = 2xz^4 - 2xy - xy^3 + yz + 6z + C$$

Part (b)

$$r(t) = \cos t\, i + \sin t\, j + \frac{2t}{5\pi} k$$

$$r(0) = i$$

$$r\left(\frac{5\pi}{2}\right) = j + k$$

Therefore,

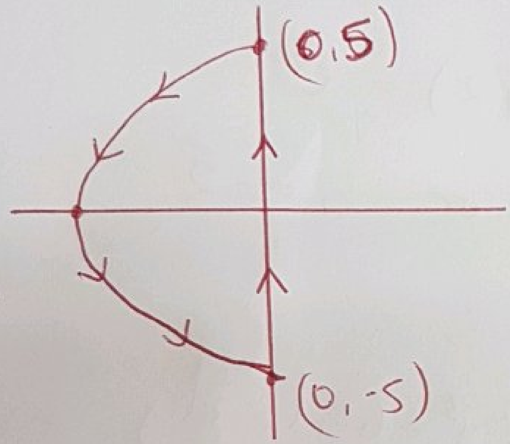
$$\int_C F \cdot dr = f(0,1,1) - f(1,0,0)$$
$$= (7 + c) - c$$
$$= 7.$$

Question Three

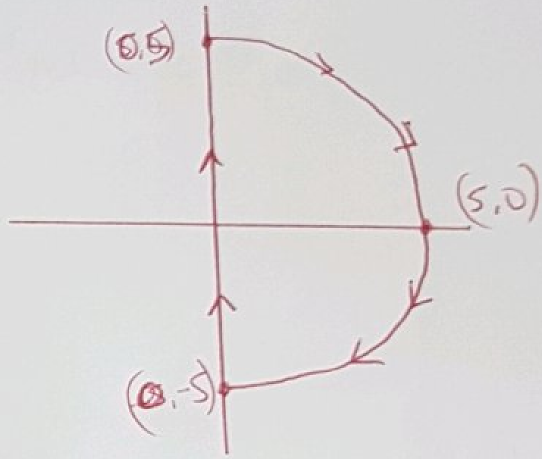
Part (a)

This question was badly phrased ~~and~~ ~~the~~ ~~is~~ because of the unambiguity of the path.

Is it

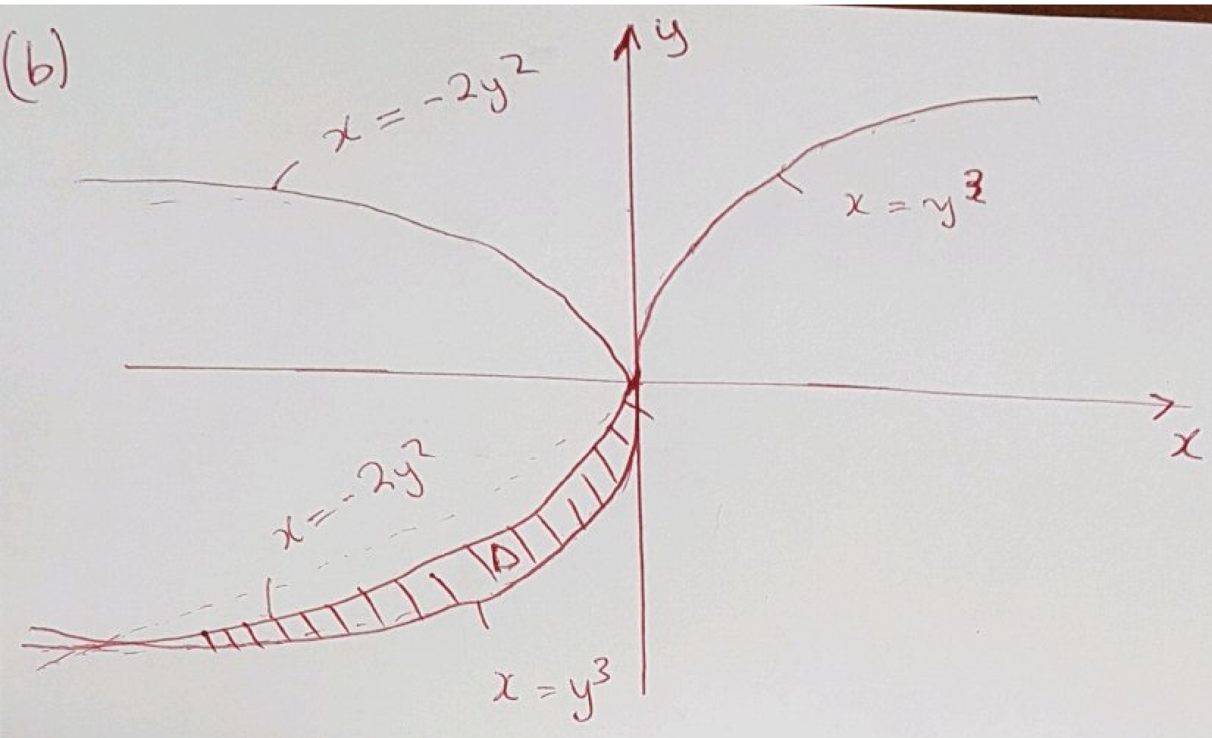


or



I will acept any of these paths and correct evaluation of the line integral either by using Green's theorem or evaluating directly.

(b)



$$y^3 = -2y^2 \Rightarrow y^3 + 2y^2 = 0$$

$$y^2(y+2) = 0$$

$$y = 0 \quad \text{or} \quad y = -2$$

$$\iint_D 10x^2y^3 - 6 \, dA = \int_{-2}^{0} \int_{-2y^2}^{y^3} 10x^2y^3 - 6 \, dx \, dy$$

$$= \int_{-2}^{0} \left[\frac{10x^3y^3}{3} - 6x \right]_{-2y^2}^{y^3} dy$$

$$= \int_{-2}^{0} \left(\frac{10y^{12}}{3} - 6y^3 \right) - \left(-\frac{80}{3}y^9 + 12y^2 \right) d$$

$$\frac{10y^{12}}{3} - 6y^3 + \frac{80}{3}y^9 - 12y^2 \, dy$$

$$= \int_{-2}^{0} \frac{10}{3}y^{12} - 6y^3 + \frac{80}{3}y^9$$

$$= \left[\frac{10\, y^{13}}{39} - \frac{6\, y^{4}}{4} + \frac{8}{3} y^{10} - 4 y^{3}\right]_{-2}^{0}$$

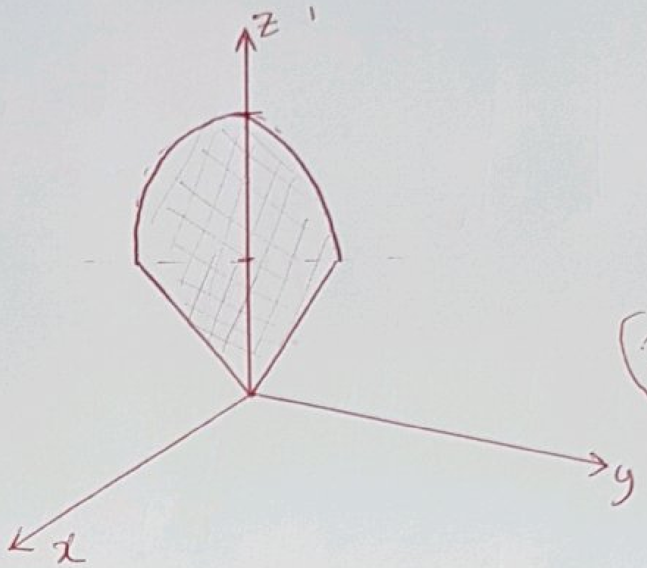
$$= -\left[-\frac{10}{39} 2^{13} - \frac{6\, 2^{4}}{4} + \frac{8}{3} 2^{10} + 32\right]$$

$$= -\left(32 - \frac{10}{39} 2^{13} - 24 + \frac{8}{3} 2^{10}\right)$$

$$= -8 + \frac{10}{39} 2^{13} - \frac{8}{3} 2^{10}$$

Question Four

Part (a)



(2 marks)

$$2 - x^2 - y^2 = \sqrt{x^2+y^2}$$

Let $r = \sqrt{x^2+y^2}$ then

$$2 - r^2 = r$$

$$r^2 + r - 2 = 0$$

$$\Rightarrow (r-1)(r+2) = 0.$$

$r = 1$ or $r = -2$.

$\sqrt{x^2+y^2} = 1$ or $\sqrt{x^2+y^2} = -2$ no solution

(1 mark)

$$x^2+y^2 = 1$$

Thus, the elliptic paraboloid $z = 2 - x^2 - y^2$ and the cone $z = \sqrt{x^2+y^2}$ intersect on the circle $x^2+y^2 = 1$ in the plane $z = 1$.

Using cylindrical coordinates, we have

$$\text{Volume} = \int_0^{2\pi}\int_0^1\int_r^{2-r^2} r \, dz \, dr \, d\theta$$

(5 marks)

$$= \int_0^{2\pi}\int_0^1 \Big[zr\Big]_r^{2-r^2} dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\left(2-r^2-r\right) dr \, d\theta$$

$$= 2\pi \int_0^1 2r - r^2 - r^3 \, dr$$

$$= 2\pi \left[r^2 - \frac{r^3}{3} - \frac{r^4}{4}\right]_0^1$$

$$= 2\pi\left(1 - 1/3 - 1/4\right).$$

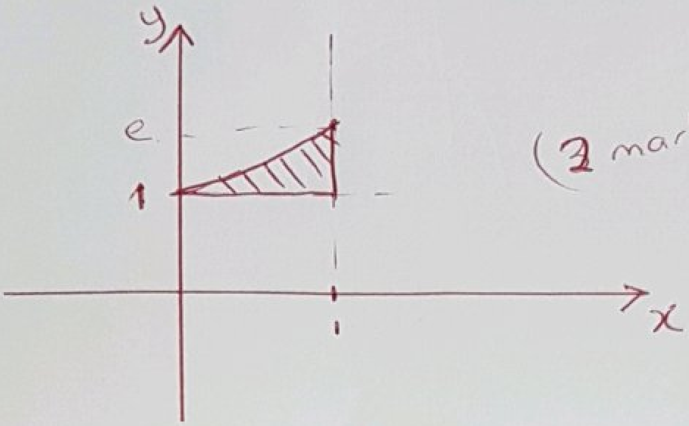
$$= 2\pi\left(\frac{12-4-3}{12}\right)$$

(2 marks)

$$= 5\pi/6$$

Question Four

Part (b).



(3 marks)

$$\text{Area} = \int_0^1 \int_1^{e^x} dy\, dx \quad \text{(5 marks)}$$

$$= \int_0^1 e^x - 1 \, dx$$

$$= \left[e^x - x \right]_0^1$$

$$= (e - 1) - (1)$$

$$= e - 2 \quad \text{(3 marks)}.$$