4.2 Infinite Series

Definition 4.2.1 *An* ***infinite series****, usually just called a* ***series****, is a formal sum of infinity many terms.*

For instance, $a_1 + a_2 + \dots$ is a series formed by adding the terms of the sequence $\{a_n\}$. This series is denoted by

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$$

Example 4.2.2 *a)* $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = 1 - \frac{1}{1} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$

c) $\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + \dots$

d) $\sum_{n=2}^{\infty} \frac{1}{\ln n} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \frac{1}{\ln 5} + \dots$

Note that it is sometimes necessary or useful to start the sum from some index other than 1 as is the case in c) and d) above.

When necessary, we can change the summation index to start at a different value by an appropriate substitution. For instance, if we use the substitution $n = m - 2$, we can write $\sum_{n=1}^{\infty} a_n$ in the form $\sum_{m=3}^{\infty} a_{m-2}$. Both sums give the same expansion

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots = \sum_{m=3}^{\infty} a_{m-2}$$

We define a sequence s_n, called **the sequence of partial sums** of the series $\sum_{n=1}^{\infty} a_n$ as follows:

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= s_1 + a_2 = a_1 + a_1 \\
s_3 &= s_2 + a_3 = a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= s_{n-1} + a_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{i=1}^{n} a_n \\
&\vdots
\end{aligned}$$

Note that s_n is the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$. The sum of the infinity series is the limit of this sequence of partial sums.

Definition 4.2.3 *The series* $\sum_{n=1}^{\infty} a_n$ ***converges*** *to a sum* s*, and we write*

$$\sum_{n=1}^{\infty} a_n = s,$$

if $\lim_{n\to\infty} s_n = s$*, where* s_n *is the* n*th partial sum of* $\sum_{n=1}^{\infty} a_n$*. Otherwise, the series* ***diverges****.*

Thus a series converges (respectively diverges) if and only if the sequence of partial sums converges (respectively diverges).

Definition 4.2.4 *A series of the form*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$$

whose n*th term is* $a_n = ar^{n-1}$*, is called a* ***geometric series****. The number* a *is the first term. The number* r *is called the* ***common ratio*** *of the series, since it is the value of the ration of the* $(n+1)$*st term to the* n*th term for any* $n \geq 1$*:*

$$\frac{a_{n+1}}{a_n} = \frac{ar^n}{ar^{n-1}} = r, \quad n = 1, 2, 3, \dots.$$

The nth partial sum of the geometric series is calculated as follows:

$$\begin{aligned} s_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ rs_n &= \quad\; ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Subtracting these two equations, we get

$$(1-r)s_n = a - ar^n$$

If $r = 1$ then the nth partial sum of the geometric series is

$$s_n = a + a + a + \cdots + a = na$$

If $r \neq 1$ then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

If $a = 0$, then $s_n = 0$ for every n, and $\lim_{n\to\infty} s_n = 0$. Now suppose that $a \neq 0$. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, so

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

If $r > 1$, then $\lim_{n\to\infty} r^n = \infty$, and $\lim_{n\to\infty} s_n = \infty$ if $a > 0$ or $\lim_{n\to\infty} s_n = -\infty$ if $a < 0$. The same conclusion holds for $r = 1$ since $s_n = na$ in this case. If $r \leq -1$, then $\lim_{n\to\infty} r^n$ does not exist and neither does $\lim_{n\to\infty} s_n$. Hence, we conclude that

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{converges to } 0 & \text{if } a = 0 \\ \text{converges to } \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges to } \infty & \text{if } r \geq 1 \text{ and } a > 0 \\ \text{diverges to } -\infty & \text{if } r \geq 1 \text{ and } a < 0 \\ \text{diverges} & \text{if } r \leq -1 \text{ and } a \neq 0 \end{cases}$$

For $-1 < x < 1$, the function $\frac{1}{1-x}$ can be represented as a sum of a geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

which will be useful later.

Example 4.2.5 *a)* $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} = \frac{1}{1-\frac{1}{2}} = 2$. *Here* $a = 1$ *and* $r = \frac{1}{2}$. *Since* $|r| < 1$, *the series converges.*

b) $\pi - e + \frac{e^2}{\pi} - \frac{e^3}{\pi^2} + \cdots = \sum_{n=1}^{\infty} \pi(-\frac{e}{\pi})^{n-1} = \frac{\pi}{1-(-\frac{e}{\pi})} = \frac{\pi^2}{\pi+e}$. *Here* $a = \pi$ *and* $r = -\frac{e}{\pi}$. *Since* $|-\frac{e}{\pi}| < 1$, *the series converges.*

c) The series $1 + 2^{1/2} + 2 + 2^{3/2} + \cdots = \sum_{n=1}^{\infty} (\sqrt{2})^{n-1}$ *diverges to* ∞ *since* $a = 1 > 0$ *and* $r = \sqrt{2} > 1$

d) The series $1 - 1 + 1 - 1 + 1 - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}$ *diverges since* $r = -1$.

e) Let $x = 0.323232\cdots = 0.\overline{32}$, *then*

$$x = \frac{32}{100} + \frac{32}{100^2} + \frac{32}{100^3} + \cdots = \sum_{n=1}^{\infty} \frac{32}{100} \left(\frac{1}{100}\right)^{n-1} = \frac{32}{100} \left(\frac{1}{1 - \frac{1}{100}}\right) = \frac{32}{99}.$$

This is an alternative way to represent repeating decimals as quotients of integers.

Example 4.2.6 *If money earns interest at a constant effective rate of* 5% *per year, how much should you pay today for an annuity that will pay you*

i) \$1,000 *at the end of each of the next* 10 *years, and*

ii) \$1,000 *at the end of every year forever.*

Solution. The payment of \$1,000 that is due to be received n years from now has present value $\$1,000 \times \left(\frac{1}{1.05}\right)^n$ (since $\$A$ would grow to $\$A(1.05)^n$ in n years.) Thus \$1,000 payment at the end of each of the next n years are worth $\$s_n$ at the present time, where

$$\begin{aligned} s_n &= 1,000\left[\frac{1}{1.05} + \left(\frac{1}{1.05}\right)^2 + \left(\frac{1}{1.05}\right)^3 + \cdots + \left(\frac{1}{1.05}\right)^n\right] \\ &= \frac{1,000}{1.05}\left[1 + \frac{1}{1.05} + \left(\frac{1}{1.05}\right)^2 + \cdots + \left(\frac{1}{1.05}\right)^{n-1}\right] \\ &= \frac{1,000}{1.05}\left[\frac{1-\left(\frac{1}{1.05}\right)^n}{1-\frac{1}{1.05}}\right] \\ &= \frac{1,000}{0.05}\left[1-\left(\frac{1}{1.05}\right)^n\right] \end{aligned}$$

i) The present value of 10 future payments is

$$\$s_{10} = \$\frac{1,000}{0.05}\left[1-\left(\frac{1}{1.05}\right)^{10}\right] = \$7,721.73$$

ii) The present value of future payments continuing forever is

$$\$\lim_{n\to\infty} s_n = \$\lim_{n\to\infty}\left\{\frac{1,000}{0.05}\left[1-\left(\frac{1}{1.05}\right)^n\right]\right\} = \$\frac{1,000}{0,05} = \$20,000$$

■

Telescoping series and Harmonic series

Example 4.2.7 *Show that the series*

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \ldots$$

converges and find its sum.

Solution. Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (by partial fractions) we can write the partial sum s_n in the form

$$\begin{aligned} s_n &= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Therefore, $\lim_{n\to\infty} s_n = 1$ and the series converges to 1; i.e,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

■

This is an example of a **telescoping series**, so called because the partial sums fold up into a simple form when the terms are expanded in partial fractions.

Example 4.2.8 *Show that the* ***harmonic series***

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

diverges to ∞*.*

Solution. If s_n is the nth partial sum of the harmonic series, then

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\ &= \text{sum of areas of rectangles shaded in the figure below} \\ &> \text{area under } y = \frac{1}{x} \text{ from } x = 1 \text{ to } x = n+1 \\ &= \int_1^{n+1} \frac{dx}{x} = \ln(n+1). \end{aligned}$$

Now, $\lim_{n\to\infty} \ln(n+1) = \infty$, therefore, $\lim_{n\to\infty} s_n = \infty$ and the harmonic series diverges to ∞. ■

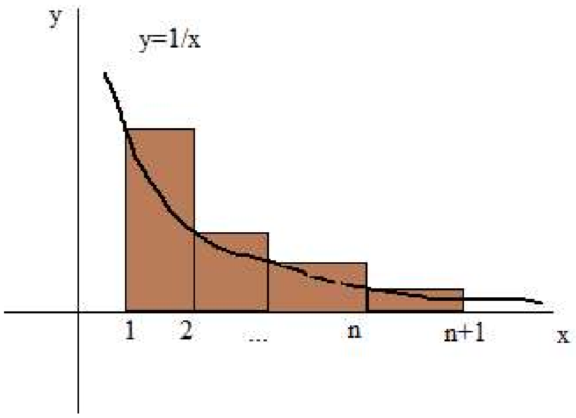

Figure 4.1: A partial sum of the harmonic series

Theorem 4.2.9 *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$.*

Proof. If $s_n = a_1 + a_2 + a_3 + \cdots + a_n$, then $s_n - s_{n-1} = a_n$. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} s_n = s$ exists, and also $\lim\limits_{n\to\infty} s_{n-1} = s$. Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = s - s = 0$$

as desired. ■

Example 4.2.10 *a)* $\sum_{n=1}^{\infty} \frac{n}{2n-1}$ *diverges to infinity since* $\lim\limits_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} > 0$.

b) $\sum_{n=1}^{\infty}(-1)^n n \sin\frac{1}{n}$ *diverges since*

$$\begin{aligned}\lim_{n\to\infty} |(-1)^n n \sin\frac{1}{n}| &= \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \\ &= \lim_{x\to 0^+} \frac{\sin x}{x} \;\; (\textit{where } x = 1/n) \\ &= 1 \neq 0\end{aligned}$$

Theorem 4.2.11 *$\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges for any integer $N \geq 1$.*

Theorem 4.2.12 *If $\{a_n\}$ is ultimately positive, then the series $\sum_{n=1}^{\infty} a_n$ must either converge (if its partial sums are bounded above) or diverge to infinity (if its partial sums are not bounded above).*

Theorem 4.2.13 *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converges to* A *and* B*, respectively, then*

a) $\sum_{n=1}^{\infty} ca_n$ *converges to* cA *(where* c *is any constant);*

b) $\sum_{n=1}^{\infty}(a_n \pm b_n)$ *converges to* $A \pm B$*;*

c) if $a_n \leq b_n$ *for all* $n = 1, 2, 3, \ldots$ *then* $A \leq B$*.*

Example 4.2.14 *Find the sum of the series* $\sum_{n=1}^{\infty} \frac{1+2^{n+1}}{3^n}$

Solution. The given series is the sum of two geometric series

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1/3}{1-(1/3)} = \frac{1}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{4}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{4/3}{1-(2/3)} = 4$$

Thus, its sum is $\frac{1}{2} + 4 = \frac{9}{2}$ by Theorem 4.2.13 b). ■

4.3 Convergence Tests for positive Series

The Integral Test

Theorem 4.3.1 (The integral test) *Suppose that* $a_n = f(n)$*, where* f *is positive, continuous and non-increasing on an interval* $[N, \infty)$ *for some positive integer* N*. Then*

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_N^{\infty} f(t)dt$$

either both converge or diverge to infinity.

Note that this theorem does not say that the sum of the series is equal to the value of the integral.

Example 4.3.2 *Show that the p-series*

$$\sum_{n=1}^{\infty} n^{-p} = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ *and diverges to infinity if* $p \leq 1$*.*

Solution. Observe that if $p > 0$, then $f(x) = x^{-p}$ is positive, continuous, and decreasing on $[1, \infty)$. By the integral test, the series converges for $p > 1$ and diverges for $0 \leq p \leq 1$ by comparison with $\int_1^\infty x^{-p} dx$. If $p < 0$, then $\lim_{n \to \infty} \frac{1}{n^p} \neq 0$, so the series cannot converge in this case. Being a positive series, it must diverge to infinity. ■

Comparison Test

Theorem 4.3.3 (Comparison test) *Let $\{a_n\}$ and $\{b_n\}$ be sequences for which there exists a constant K such that ultimately, $0 \leq a_n \leq K b_n$.*

i) If the series $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ also converges.

ii) If the series $\sum_{n=1}^\infty a_n$ diverges to infinity, then $\sum_{n=1}^\infty b_n$ also diverges to infinity.

Example 4.3.4 *Which of the following series converge? Give reasons for your answers.*

$$a) \sum_{n=1}^\infty \frac{1}{2^n + 1}, \quad b) \sum_{n=1}^\infty \frac{3n+1}{n^3+1}, \quad c) \sum_{n=2}^\infty \frac{1}{\ln n}$$

Solution. In each case, we must find a suitable comparison series that we already know converges or diverges.

a) Since $0 < \frac{1}{2^n+1} < \frac{1}{2^n}$ for $n = 1, 2, 3, \ldots$ and the series $\sum_{n=1}^\infty \frac{1}{2^n}$ is a convergent geometric series, the series $\sum_{n=1}^\infty \frac{1}{2^n+1}$ also converges by comparison.

b) Observe that $\frac{3n+1}{n^3+1}$ behaves like $\frac{3}{n^2}$ for large n, so we would expect to compare with the convergent p-series $\sum_{n=1}^\infty n^{-2}$. We have for $n \geq 1$,

$$\frac{3n+1}{n^3+1} = \frac{3n}{n^3+1} + \frac{1}{n^3+1} < \frac{3n}{n^3} + \frac{1}{n^3} < \frac{3}{n^2} + \frac{1}{n^2} = \frac{4}{n^2}.$$

Thus the given series converges by the comparison test.

c) For $n = 2, 3, 4, \ldots$ we have $0 < \ln n < n$. Thus

$$\frac{1}{\ln n} > \frac{1}{n}$$

since the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ diverges to infinity, the series $\sum_{n=1}^\infty \frac{1}{\ln n}$ also diverges to infinity by comparison.

■

Theorem 4.3.5 (Limit comparison test) *Suppose that $\{a_n\}$ and $\{b_n\}$ are positive sequences and that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L is a positive finite number or $+\infty$.

i) If $L < \infty$ and the series $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

ii) If $L > 0$ and the series $\sum_{n=1}^{\infty} b_n$ diverges to infinity, then $\sum_{n=1}^{\infty} a_n$ also diverges to infinity.

Example 4.3.6 *Which of the following series converge? Give reasons for your answers.*

$$a) \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}, \quad b) \sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}.$$

Solution. Again, we must find a suitable comparison series.

a) The terms of the series decrease like $\frac{1}{\sqrt{n}}$. Observe that

$$L = \lim_{n\to\infty} \frac{\frac{1}{1+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} = 1$$

Since the p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges to infinity ($p = 1/2$), so does the series $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ by the limit comparison test.

b) For large n, the sequence behaves like $\frac{n}{n^3}$, so we compare with the convergent p-series $\sum_{n=1}^{\infty} n^{-2}$. Now,

$$L = \lim_{n\to\infty} \frac{\frac{n+5}{n^3-2n+3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+5n^2}{n^3-2n+3} = 1$$

Since $L < \infty$, the series $\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ also converges by the limit comparison test.

■

Example 4.3.7 *Test the series $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$ for convergence.*

Solution. Since

$$\lim_{n\to\infty}\frac{\frac{1+\sin n}{n^2}}{\frac{1}{n^2}}=\lim_{n\to\infty}(1+\sin n)$$

does not exist, the limit comparison test doesn't give us information. However, since $\sin n \le 1$, we have

$$0\le\frac{1+\sin n}{n^2}\le\frac{2}{n^2}$$

for $n = 1, 2, 3, \ldots$. Thus the given series converges by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$, using the ordinary comparison test. ■

Ratio and Root Tests

Theorem 4.3.8 (Ratio test) *Suppose that $a_n > 0$ (ultimately) and that*

$$\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

exists or is $+\infty$.

a) If $0\le\rho<1$, then the series $\sum_{n=1}^{\infty}a_n$ converges.

b) If $1<\rho\le\infty$ then $\lim\limits_{n\to\infty}a_n=\infty$ and the series $\sum_{n=1}^{\infty}a_n$ diverges to infinity.

c) If $\rho = 1$, this test gives no information, the series may either converge or diverge to infinity.

Example 4.3.9 *Test the following series for convergence.*

$$a)\ \sum_{n=1}^{\infty}\frac{99^n}{n!},\quad b)\ \sum_{n=1}^{\infty}\frac{n^5}{2^n},\quad c)\ \sum_{n=2}^{\infty}\frac{n!}{n^n},\quad d)\ \sum_{n=2}^{\infty}\frac{(2n)!}{(n!)^2}$$

Solution. We use the ratio test for each of these.

a) $\rho=\lim\limits_{n\to\infty}\frac{99^{n+1}}{(n+1)!}\Big/\frac{99^n}{n!}=\lim\limits_{n\to\infty}\frac{99}{n+1}=0<1$. Thus the series $\sum_{n=1}^{\infty}\frac{99^n}{n!}$ converges.

b) $\rho=\lim\limits_{n\to\infty}\frac{(n+1)^5}{2^{n+1}}\Big/\frac{n^5}{2^n}=\lim\limits_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^5=\frac{1}{2}<1$. Hence the series $\sum_{n=1}^{\infty}\frac{n^5}{2^n}$ converges.

c) $\rho=\lim\limits_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\Big/\frac{n!}{n^n}=\lim\limits_{n\to\infty}\frac{(n+1)!n^n}{(n+1)^{n+1}n!}=\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim\limits_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}<1$. Hence the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges.

d) $\rho = \lim_{n\to\infty} \frac{(2(n+1))!}{((n+1)!)^2} / \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4 > 1$. Hence the series $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges to infinity.

■

The root test is similar to the ratio test.

Theorem 4.3.10 (Root test) *Suppose that $a_n > 0$ (ultimately) and that*

$$\sigma = \lim_{n\to\infty} (a_n)^{1/n}$$

exists or is $+\infty$.

a) If $0 \leq \sigma < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

b) If $1 < \sigma \leq \infty$ then $\lim_{n\to\infty} a_n = \infty$ and the series $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

c) If $\sigma = 1$, this test gives no information, the series may either converge or diverge to infinity.