2.3 Application of derivatives

Linear approximations

Definition 2.3.1 *The* ***linearization*** *of the function f about a is the function L defined by*

$$L(x) = f(a) + f'(a)(x - a).$$

We say that $f(x) \approx L(x)$ provides ***linear approximation*** *for values of f near a.*

Example 2.3.2 *Find the linearization of*

a) $f(x) = \sqrt{1+x}$ *about* $x = 0$

b) $g(t) = \frac{1}{t}$ *about* $t = \frac{1}{2}$

.

Solution

a) We have $f(0) = \sqrt{1+0} = 1$ and since $f'(x) = \frac{1}{2\sqrt{1+x}}$,

$$f'(0) = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}.$$

Therefore, the linearization of f about $x = 0$ is

$$L(x) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

b) We have $g(1/2) = 2$ and since $g'(t) = \frac{-1}{t^2}$, $g'(1/2) = -4$. Therefore, then linearization of g about $t = 1/2$ is

$$L(t) = 2 - 4(t - 1/2) = 4 - 4t.$$

Example 2.3.3 *Use the linearization for $\sqrt{x}$ about $x = 25$ to find an approximate value of $\sqrt{26}$.*

Solution

If $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$. Since $f(25) = 5$ and $f'(25) = \frac{1}{10}$, the linearization of $\sqrt{x}$ about $x = 25$ is

$$L(x) = 5 + \frac{1}{10}(x - 25).$$

Putting $x = 26$, we get

$$\sqrt{26} = f(26) \approx L(26) = 5 + \frac{1}{10}(26 - 25) = 5.1$$

That is $\sqrt{26} \approx 5.1$.

Error Analysis

In any approximation, the error is defined by

$$\text{error} = \text{true value} - \text{approximate value}$$

If $f(x) \approx L(x) = f(a) + f'(a)(x - a)$, then the error $E(x)$ in this approximation is

$$E(x) = f(x) - L(x) = f(x) - f(a) - f'(a)(x - a).$$

The following theorem and it's corollaries give a way to estimate this error if we know bounds for the second derivative of f.

Theorem 2.3.4 *If $f''(t)$ exists for all t in an interval containing a and x, then there exists some point s, $a \le s \le x$ such that the error*

$$E(x) = f(x) - L(x)$$

in the linear approximation $f(x) \approx L(x)$ satisfies

$$E(x) = \frac{f''(s)}{2}(x - a)^2.$$

Corollary 2.3.5 *If $f''(t)$ has constant sign (positive or negative) between a and x, then $E(x)$ in the approximation $f(x) \approx L(x)$ in the theorem above has the same sign. If $f''(t) > 0$ between a and x, then $f(x) > L(x)$; if $f''(t) < 0$ between a and x, then $f(x) < L(x)$.*

Corollary 2.3.6 *If $|f''(t)| < K$ for all t between a and x, then*

$$|E(x)| < \frac{K}{2}(x - a)^2.$$

Corollary 2.3.7 *If $f''(t)$ satisfies $M < f''(t) < N$ for all t between a and x, then*

$$L(x) + \frac{M}{2}(x-a)^2 < f(x) < L(x) + \frac{N}{2}(x-a)^2.$$

If M and N have the same sign, a better approximation of $f(x)$ is given by the midpoint of this interval containing $f(x)$:

$$f(x) \approx L(x) + \frac{M+N}{4}(x-a)^2.$$

For this approximation, then error is less that half the length of the interval:

$$|E(x)| < \frac{N-M}{4}(x-a)^2.$$

Example 2.3.8 *Determine the sign and estimate the size of the error in the approximation $\sqrt{26} \approx 5.1$ in Example 2.3.3 above. Use these to find an interval that you can be sure contains $\sqrt{26}$.*

Solution

For $f(t) = t^{\frac{1}{2}}$, we have $f'(t) = \frac{1}{2}t^{-\frac{1}{2}}$ and $f''(t) = -\frac{1}{4}t^{-\frac{3}{2}}$. For $25 < t < 26$, we have $f''(t) < 0$ and so by Corollary 2.3.5 we have $\sqrt{26} = f(26) < L(26) = 5.1$. Also, $t^{\frac{3}{2}} > 25^{\frac{3}{2}} = 125$, so

$$|f''(t)| = |\frac{1}{4}t^{-3/2}| < \left(\frac{1}{4}\right)\left(\frac{1}{125}\right) = \frac{1}{500} = K \text{ and}$$

$$|E(26)| < \left(\frac{1}{2}\right)\left(\frac{1}{500}\right)(26-25)^2 = \frac{1}{1000} = 0.001$$

Therefore, $f(26) > L(26) - 0.001 = 5.1 - 0.001 = 5.099$, and $\sqrt{26}$ is in the interval $(5.099, 5.1)$.

We can also use Corollary 2.3.7 and the fact that $\sqrt{26} < 5.1$ to find a better (smaller) interval containing $\sqrt{26}$ as follows: If $25 < t < 26$, then

$$125 = 25^{3/2} < t^{3/2} < 26^{3/2} < 5.1^3$$

Thus

$$M = -\frac{1}{4 \times 125} < f''(t) < -\frac{1}{4 \times 5.1^3} = N,$$

$$\sqrt{26} \approx L(26) + \frac{M+N}{4} = 5.1 - \frac{1}{4}\left(\frac{1}{4 \times 125} + \frac{1}{4 \times 5.1^3}\right) \approx 5.0990288,$$

and

$$|E(26)| < \frac{N-M}{4}(x-a)^2 = \frac{1}{16}\left(-\frac{1}{5.1^3} + \frac{1}{125}\right) \approx 0.0000288.$$

Thus $\sqrt{26}$ lie in the interval $(5.09900, 5.09906)$

Example 2.3.9 *Use a suitable linearization to find an approximate value for* $\cos(36°) = \cos(\pi/5)$*. Is the true value greater than or less than your approximation? Estimate the size of the error and give an interval that you can be sure contains* $\cos(36°)$*.*

Solution

Let $f(t) = \cos t$, then $f'(t) = -\sin t$ and $f''(t) = -\cos t$. Now, we know the value of $\cos 30° = \cos(\pi/6)$ and $30°$ is the nearest angle to $36°$. So we use the linearization about the point $t = \pi/6$. Thus

$$L(t) = \cos(\pi/6) - \sin(\pi/6)(t - \pi/6) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(t - \frac{\pi}{6}\right).$$

Since $\pi/5 - \pi/6 = \pi/30$, our approximation is

$$\cos(36°) = \cos(\pi/5) \approx L(\pi/5) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{\pi}{30}\right) \approx 0.81367.$$

Now, if $\pi/6 < t < \pi/5$, then $f''(t) < 0$ and $|f''(t)| < \cos(\pi/6) = \frac{\sqrt{3}}{2}$. Therefore, $\cos(36°) < 0.81367$.

For the error, we have

$$|E(36°)| < \frac{\sqrt{3}}{2}\left(\frac{\pi}{30}\right)^2 < 0.00475.$$

Thus

$$0.81367 - 0.00475 < \cos(36°) < 0.81367$$

and so $\cos(36°)$ lies in the interval $(0.80892, 0.81367)$.

Taylor Polynomials

If $f^{(n)}(x)$ exists in an open interval containing $x = a$, then the n^{th} **Taylor polynomial** for f about a is given by

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The Taylor polynomials about $x = 0$ are called **Maclaurin polynomials**.

Example 2.3.10 *Find the Taylor polynomial* $P_3(x)$ *for* $f(x) = \ln x$ *about* $x = e$*. (Here,* $e \approx 2.72$*)*

Solution

We calculate the first three derivatives of f. We have $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$ and $f'''(x) = \frac{2}{x^3}$. Thus the Taylor polynomial is

$$\begin{aligned} P_3(x) &= f(e) + f'(e)(x-e) + \frac{f''(x)}{2!}(x-e)^2 + \frac{f'''(e)}{3!}(x-e)^3 \\ &= 1 + \frac{1}{e}(x-e) - \frac{2}{2e^2}(x-e^3) + \frac{1}{3e^3}(x-e)^3 \end{aligned}$$

Example 2.3.11 *Find the n^{th}-order Maclaurin polynomial $P_n(x)$ for e^x.*

Solution

Since every derivative of e^x is e^x and so is equal to 1 at $x = 0$, then n^{th}-order Maclaurin polynomial for e^x is

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

The Mean Value Theorem

The Mean Value Theorem and its generalizations is more influential in calculus. It provides the mathematics we need to estimate the amount of error involved in linear approximations.

The key to the Mean Value Theorem is an early version of it, called Rolle's Theorem, which we now state.

Theorem 2.3.12 (Rolle's Theorem) *Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of it's interior (a, b). If $f(a) = f(b) = 0$, then there is at least one number c between a and b at which $f'(c) = 0$.*

Example 2.3.13 *The polynomial $y = x^3 - 4x = f(x)$ is continuous and differentiable for all $x, -\infty < x < \infty$. If we take $a = -2$ and $b = 2$, the hypotheses of Rolle's Theorem are satisfied since $f(-2) = 0 = f(2)$. Thus $f'(x) = 3x^2 - 4$ must be zero at least once between -2 and 2. In fact, we find $3x^2 - 4 = 0$ at $x = c_1 = -\frac{2\sqrt{3}}{3}$ and $x = c_2 = \frac{2\sqrt{3}}{3}$.*

Example 2.3.14 *Show that the equation $x^3 + 3x + 1$ has exactly one real solution.*

Solution

We observe that the function $f(x) = x^3 + 3x + 1$ is differentiable at every value of x and the derivative $f'(x) = 3x^2 + 3$ is never zero. If f has as many as two zeros, by Rolle's Theorem f' would have a zero between them. Hence, f has at most one zero. On the other hand, f has at least one zero because $f(-1) = -3$ is negative, $f(1) = 5$ is positive and f is continuous. That is, the curve of f crosses the x-axis. Therefore, f has exactly one zero.

Theorem 2.3.15 (Mean Value Theorem) *If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of it's interior (a, b), then there is at least one number c between a and b at which*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 2.3.16 *Let $f(x) = x^3$, $a = -2$ and $b = 2$. Show that $f(x)$ satisfies the hypotheses of the Mean Value Theorem. Hence, find c.*

Solution

Since f is continuous on $[-2, 2]$ and is differentiable on $(-2, 2)$, it satisfies the Mean Value Theorem hypotheses.

Since $f'(x) = 3x^2$, $f'(c) = 3c^2$, $f(b) = 2^3 = 8$ and $f(a) = (-2)^3 = -8$ we have by the Mean Value Theorem

$$3c^2 = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{8 - (-8)}{2 - (-2)} = \frac{16}{4} = 4.$$

Solving for c gives $c = \pm\frac{2}{3}\sqrt{3}$. Therefore, there are two values of c between $a = -2$ and $b = 2$ where the tangent the the curve $y = f(x) = x^3$ is parallel to the chord through $A(-2, -8)$ and $B(2, 8)$.

Example 2.3.17 *Show that the function $y = \sqrt{1 - x^2}$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-1, 1]$.*

Solution

The function $y = \sqrt{1 - x^2}$ is continuous at each point of the closed interval, and its derivative

$$y' = \frac{-x}{\sqrt{1 - x^2}}$$

is defined at each point of the interior $(-1, 1)$. Hence, the hypotheses of the Mean Value Theorem are satisfied. The graph has a horizontal tangent at $x = 0$. Notice that the function is not differentiable at $x = -1$ and $x = 1$. It does not need to be for the theorem to apply.

Example 2.3.18 *Show that the Mean Value Theorem does not apply to the function* $f(x) = x^{\frac{2}{3}}$ *on the interval* $[-8, 8]$.

Solution

The derivative of $f(x) = x^{\frac{2}{3}}$ is

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

Suppose that $f(x)$ satisfies the hypotheses of the Mean Value Theorem, then there is a c in $(-8, 8)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{8^{\frac{2}{3}} - (-8)^{\frac{2}{3}}}{8 - (-8)} = \frac{4 - 4}{16} = 0$$

But $f'(x) = \frac{2}{3\sqrt[3]{x}}$ is never zero for any value c in the interval $(-8, 8)$. Observe also that f' does not exist at $x = 0$. Therefore, the Mean Value Theorem does not apply to the given function on a given closed interval.