Chapter 4

Infinite Series

4.1 Infinite sequence

Definition 4.1.1 *A **sequence** (or an **infinite sequence**) is an ordered list having the first element (term) but no last element (term)*

For this course, the terms of the sequence will always be real numbers, though what we will discuss can be applied to complex terms.

Example 4.1.2 *a)* $\{1, 2, 3, 4, 5, \dots\}$ *the sequence of positive integers.*

b) $\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \dots\}$ *the sequence of positive integer powers of* $-\frac{1}{2}$.

We can look at a sequence as a function with domain the positive integers, e.g, the sequence $a_1, a_2, \dots, a_n, \dots$ is a function f defined by $f(n) = a_n$.

We can specify a sequence in three ways:

i) We list the first few terms followed by $\dots$ if the pattern is obvious.

ii) We can provide a formula for the general term a_n as a function of n

iii) We can provide a formula for calculating a_n as a function of earlier terms $a_1, a_2, \dots, a_{n-1}$ and specify enough of the beginning terms so that the process of computing higher terms can begin.

Example 4.1.3 *Let* n *be a positive integer.*

a) $\{n\} = \{1, 2, 3, 4, 5, \dots\}$

b) $\{\left(-\frac{1}{2}\right)^n\} = \{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \dots\}$.

c) $\{\frac{n-1}{n}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$.

d) $\{(-1)^{n-1}\} = \{\cos(n-1)\pi\} = \{1, -1, 1, -1, 1, \dots\}$.

e) $\{\frac{n^2}{2^n}\} = \{\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \frac{36}{64}, \frac{49}{128} \dots\}$.

f) $\{\left(1+\frac{1}{n}\right)^n\} = \{2, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4, \dots\}$.

g) $\{\cos(n\pi/2)\} = \{0, -\frac{1}{2}, 0, \frac{1}{4}, 0 - \frac{1}{6}, 0, \frac{1}{8}, \dots\}$.

h) $a_1 = 1, a_{n+1} = \sqrt{6 + a_n}, (n = 1, 2, 3, 4, \dots)$. *Here,* $\{a_n\} = \{1, \sqrt{7}, \sqrt{6+\sqrt{7}}, \dots\}$ *and we see that there is no obvious formula for* a_n *as a function of* n*. But we can still calculate* a_n *when we know the earlier values* $a_2, a_3, \dots, a_{n-1}$

i) $a_1 = 1, a_2 = 1, a_{n+2} = a_n + a_{n+1}, (n = 1, 2, 3, 4, \dots)$. *Here,*

$$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

This is called the Fibonacci sequence, each term after the second is the sum of the previous two terms.

In h) and i), the sequence $\{a_n\}$ is said to be defined **recursively** of **inductively**.

Definition 4.1.4 *a) The sequence* $\{a_n\}$ *is* ***bounded below*** *by* L*, and* L *is a* ***lower bound*** *for* $\{a_n\}$*, if* $a_n \geq L$ *for every* $n = 1, 2, 3, \dots$.

b) The sequence $\{a_n\}$ *is* ***bounded above*** *by* M*, and* M *is a* ***upper bound*** *for* $\{a_n\}$*, if* $a_n \leq M$ *for every* $n = 1, 2, 3, \dots$.

c) The sequence a_n *is* ***bounded*** *if it is both bounded above and bounded below. In this case there is a constant* K *such that* $|a_n| \leq K$ *for all* n*. The constant* K *can be taken to be the larger of* $|L|$ *and* $|M|$.

Definition 4.1.5 *The sequence* $\{a_n\}$ *is* ***positive*** *if it is bounded below by zero, i,e,* $a_n \geq 0$*; and is* ***negative*** *if* $a_n \leq 0$ *for every* n.

Definition 4.1.6 *a) The sequence* $\{a_n\}$ *is* ***increasing*** *if* $a_{n+1} \geq a_n$*; and is* ***decreasing*** *if* $a_{n+1} \leq a_n$ *for every* $n = 1, 2, 3, \dots$.

b) The sequence is said to be ***monotonic*** *if it is either increasing or decreasing.*

Definition 4.1.7 *The sequence* $\{a_n\}$ *is* ***alternating*** *if* $a_n a_{n+1} < 0$ *for every* n*; i.e, any two consecutive terms have opposite sign.*

Example 4.1.8 *a) The sequence* $\{n\} = \{1, 2, 3, 4, 5, \dots\}$ *is positive, increasing and bounded below. A lower bound is* 1 *or any number less than* 1*. The sequence is not bounded above.*

b) The sequence $\{(-\frac{1}{2})^n\} = \{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \dots\}$ *is bounded and alternating. Here* $-\frac{1}{2}$ *is a lower bound and* 1 *is an upper bound.*

c) The sequence $\{\frac{n-1}{n}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ *is positive, bounded and increasing. Here,* 0 *is the lower bound and* 1 *is the upper bound.*

d) $\{(-1)^n n\} = \{-1, 2, -3, 4, -5, \dots\}$ *is alternating but not bounded either above or below.*

Example 4.1.9 *If* $a_n = \frac{n}{n^2+1}$*, show that the sequence* $\{a_n\}$ *is decreasing.*

Solution

Since $a_n = f(n)$, where $f(x) = \frac{x}{x^2+1}$ and $f'(x) = \frac{(x^2+1)(1)-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$, the function $f(x)$ is decreasing on $[1, \infty)$; therefore, $\{a_n\}$ is a decreasing sequence.

Example 4.1.10 *a) The sequence* $\{\frac{n^2}{2^n}\} = \{\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \frac{36}{64}, \frac{49}{128} \dots\}$ *is positive and therefore bounded below. It seems clear that from the fourth term on, all the terms are getting smaller. However,* $a_2 > a_1$ *and* $a_3 > a_2$*, Since* $a_{n+1} < a_n$ *only if* $n \geq 3$*, we say that this sequence is* ***ultimately*** *decreasing.*

b) The sequence $\{n - 100\} = \{-99, -98, \cdots - 2, -1, 0, 1, 2, 3, \dots\}$ *is ultimately positive even though the first 99 terms are negative.*

c) The sequence $\{(-1)^n + \frac{4}{n}\} = \{3, 3, \frac{1}{3}, 2, -\frac{1}{5}, \frac{5}{3}, -\frac{3}{7}, \frac{3}{2}, \dots\}$ *is ultimately alternating even though the first few terms do not alternate.*

Convergence of Sequences

Definition 4.1.11 *The sequence* $\{a_n\}$ *is said to* ***converge*** *to a limit* L*, and we write*

$$\lim_{n\to\infty} a_n = L,$$

if for every positive real number ϵ *there exists an integer* N *such that if* $n \geq N$*, then* $|a_n - L| < \epsilon$*. Otherwise, the sequence* ***diverges****.*

Every sequence $\{a_n\}$ must either converge to a real number L or diverge. If $\lim\limits_{n\to\infty} a_n = \infty$, we say that the sequence diverges to ∞; if $\lim\limits_{n\to\infty} a_n = -\infty$, we say that the sequence diverges to $-\infty$. If the limit doesn't exist (but is not ∞ or $-\infty$), we can only say that the sequence diverges.

Example 4.1.12 **a)** $\{\frac{n-1}{n}\}$ *converges to* 1*;* $\lim\limits_{n\to\infty} \frac{n-1}{n} = \lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1$

b) $\{n\} = \{1, 2, 3, 4, \ldots\}$ *diverges to* ∞*.*

c) $\{-n\} = \{-1, -2, -3, -4, \ldots\}$ *diverges to* $-\infty$

d) $\{(-1)^n\} = \{-1, 1, -1, 1, -1, 1, \ldots\}$ *simply diverges.*

e) $\{(-1)^n n\} = \{-1, 2, -3, 4, -5, \ldots\}$ *diverges (but not to* ∞ *or* $-\infty$ *even though* $\lim\limits_{n\to\infty} a_n = \infty$*).*

The following rules (seen for limits of functions) hold: If $\{a_n\}$ and $\{b_n\}$ converge, then

a) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$.

b) $\lim\limits_{n\to\infty} (a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$.

c) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ assuming $\lim\limits_{n\to\infty} b_n \neq 0$.

d) If $a_n \leq b_n$ ultimately, then $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.

e) If $a_n \leq b_n \leq c_n$ ultimately, and $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, then $\lim\limits_{n\to\infty} b_n = L$ (Squeeze Theorem).

Example 4.1.13 *Calculate the limits of the sequences*
a) $\{\frac{2n^2-n-1}{5n^2+n-3}\}$ *b)* $\{\frac{\cos n}{n}\}$ *c)* $\{\sqrt{n^2+2n} - n\}$

Solution

a) We divide the numerator and denominator by the highest power of n, i.e, by n^2;

$$\lim_{n\to\infty} \frac{2n^2 - n - 1}{5n^2 + n - 3} = \lim_{n\to\infty} \frac{2 - (1/n) - (1/n^2)}{5 + (1/n) - 3/n^2} = \frac{2 - 0 - 0}{5 + 0 - 0} = \frac{2}{5}$$

since $\lim\limits_{n\to\infty} 1/n = 0$ and $\lim\limits_{n\to\infty} 1/n^2 = 0$. The sequence converges and its limit is $2/5$.

b) Since $|\cos n| \leq 1$ or $(-1 \leq \cos n \leq 1)$ for every n, we have

$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}.$$

Since $\lim\limits_{n\to\infty} -1/n = 0$ and $\lim\limits_{n\to\infty} 1/n = 0$, it follows (from the squeeze theorem for sequences) that

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0.$$

Thus the given sequence converges to 0.

c) We multiply the numerator and the denominator by the conjugate of the expression:

$$\begin{aligned} \lim_{n\to\infty} (\sqrt{n^2+2n} - n) &= \lim_{n\to\infty} \frac{(\sqrt{n^2+2n} - n)(\sqrt{n^2+2n} + n)}{\sqrt{n^2+2n} + n}) \\ &= \lim_{n\to\infty} \frac{n^2 + 2n + n\sqrt{n^2+2n} - n\sqrt{n^2+2n} - n^2}{\sqrt{n^2+2n} + n} \\ &= \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+2n} + n} \text{ (Divide by } n\text{)} \\ &= \lim_{n\to\infty} \frac{2}{\sqrt{1 + 2/n} + 1} = 1 \end{aligned}$$

Example 4.1.14 *Evaluate* $\lim\limits_{n\to\infty} n\tan^{-1}(\frac{1}{n})$.

Solution

We use l'Hopitals rule:

$$\begin{aligned} \lim_{n\to\infty} n\tan^{-1}\frac{1}{n} &= \lim_{n\to\infty} \frac{\tan^{-1}\frac{1}{n}}{\frac{1}{n}} \quad \left[\frac{0}{0}\right] \\ &= \lim_{n\to\infty} \frac{\frac{1}{1+(1/n^2)}\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \end{aligned}$$

Theorem 4.1.15 *If* $\{a_n\}$ *converges, then* $\{a_n\}$ *is bounded.*

Proof. Suppose $\lim\limits_{n\to\infty} a_n = L$. According to Definition 4.1.11, for $\epsilon = 1$, there exist a number N such that $|a_n - L| < 1$ for $n > N$. Therefore, $|a_n| < 1 + |L|$

for such n. If K is the largest of the numbers $|a_1|, |a_2|, \ldots, |a_N|$ and $1 + |L|$, then $|a_n| < K$ for every $n = 1, 2, 3, \ldots$. Hence $\{a_n\}$ is bounded. ■

The converse of this theorem is false. The sequence $\{(-1)^n\}$ is bounded but does not converge. But a bounded, ultimately monotonic sequence is convergent.

Example 4.1.16 *Let a_n be defined recursively by*

$$a_1 = 1, \ a_{n+1} = \sqrt{6 + a_n} \quad (n = 1, 2, 3, \ldots).$$

Show that $\lim\limits_{n\to\infty} a_n$ exists and find its value.

Solution. Observe that $a_2 = \sqrt{6+1} = \sqrt{7} > a_1$. If $a_{k+1} > a_k$, then $a_{k+2} = \sqrt{6 + a_{k+1}} > \sqrt{6 + a_k} = a_{k+1}$, so $\{a_n\}$ is increasing, by induction. Now, observe that $a_1 = 1 < 3$. If $a_k < 3$, then $a_{k+1} = \sqrt{6 + a_k} < \sqrt{6+3} = 3$, so $a_n < 3$ for every n, by induction. Since $\{a_n\}$ is increasing and bounded above, $\lim\limits_{n\to\infty} a_n = L$ exists. Since $\sqrt{6+n}$ is a continuous function of n, we have

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + \lim_{n\to\infty} a_n} = \sqrt{6 + L}$$

Thus $L^2 = 6 + L$ or $L^2 - L - 6 = 0$ or $(L-3)(L+2) = 0$ so that the roots of the quadratic $L = 3$ and $L = -2$. Since $a_n \geq 1$ for every n, we must have $L \geq 1$. Therefore, $L = 3$ and $\lim\limits_{n\to\infty} a_n = 3$. ■

Theorem 4.1.17 *If $\{a_n\}$ is (ultimately) increasing, then either it is bounded above and therefore convergent or it is not bounded above and diverges to infinity.*

Theorem 4.1.18 *a) If $|x| < 1$, then $\lim\limits_{n\to\infty} x^n = 0$.*

b) If x is any real number, then $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$.

Example 4.1.19 *Evaluate $\lim\limits_{n\to\infty} \frac{3^n + 4^n + 5^n}{5^n}$.*

Solution. $\lim\limits_{n\to\infty} \frac{3^n + 4^n + 5^n}{5^n} = \lim\limits_{n\to\infty} \left[\left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n + 1\right] = 0 + 0 + 1 = 1$ by Theorem 4.1.18 a) ■