Extrema of Functions of two variables.

Consider the continuous function f of two variables defined on a closed bounded region R. The values $f(a,b)$ and $f(c,d)$ such that

$$f(a,b) \leq f(x,y) \leq f(c,d)$$

for all (x,y) in R are called the minimum and the maximum of f in the region R.

Theorem 5.6.7 *Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.*

i) There is at least one point in R where f takes a minimum value.

ii) There is at least one point in R where f takes a maximum value.

A maximum is also called an absolute maximum, and the minimum the absolute minimum.

Definition 5.6.8 *Let f be a function defined on a region R containing (x_0, y_0).*

i) The function f has a relative minimum at (x_0, y_0) if $f(x,y) \geq f(x_0, y_0)$ for all (x,y) in an open disc containing (x_0, y_0).

ii) The function f has a relative maximum at (x_0, y_0) if $f(x,y) \leq f(x_0, y_0)$ for all (x,y) in an open disc containing (x_0, y_0).

To locate the relative extrema of f, we find the points at which the gradient of f is equal to zero or is undefined. Such points are called critical points of f.

Definition 5.6.9 *Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a critical point of f if one of the following is true*

i) $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$,

ii) $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Example 5.6.10 *Find the absolute maximum and minimum values of*

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular plate in the first quadrant bounded by the lines $x = 0, y = 0$ and $y = 9 - x$.

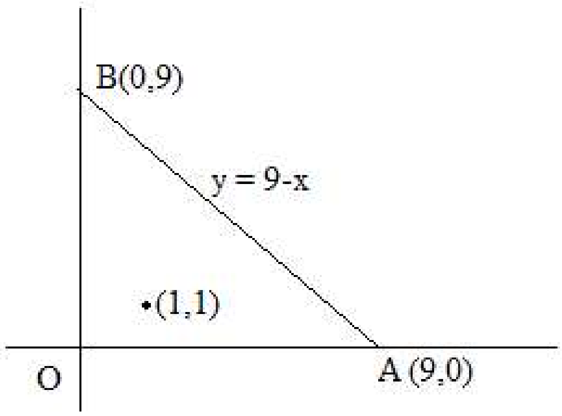

Figure 5.2: Triangular plate

Solution. The absolute maximum and minimum can occur at the critical points, end points or boundary points.

Critical points : $f_x = 2 - 2x, f_y = 2 - 2y$ and $f_x = 0$ and $f_y = 0$ implies $2 - 2x = 0$ so that $x = 1$, and $2 - 2y =$ so that $y = 1$. Thus the critical point is $(1, 1)$ and $f(1, 1) = 2 + 2 + 2 - 1 - 1 = 4$.

End points : The end points are $(0, 0)$, $(9, 0)$ and $(0, 9)$. So we have

$$f(0, 0) = 2,\ f(9, 0) = 2 + 18 - 81 = -61 \text{ and } f(0, 9) = -61$$

Boundary points :

i) Along $y = 0$ we have $f(x, 0) = 2 + 2x - x^2$, $f'(x) = 2 - 2x$ and $2 - 2x = 0$ implies $x = 1$. So $f(1, 0) = 3$.

ii) Along $x = 0$ we have $f(0, y) = 2 + 2y - y^2$, $f'(y) = 2 - 2y$ and $2 - 2y = 0$ implies $y = 1$. So $f(0, 1) = 3$.

iii) Along $y = 9 - x$ we have

$$f(x, y) = f(x, 9-x) = 2+2x+2(9-x)-x^2-(9-x)^2 = -61+18x-2x^2.$$

So

$$0 = f'(x, 9 - x) = 18 - 4x$$

so that $x = \frac{18}{4} = \frac{9}{2}$. Now, when $x = \frac{9}{2}$, we have $y = 9 - \frac{9}{2} = \frac{9}{2}$. Thus

$$f(x, y) = f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}$$

So the values of f at the extreme points are $4, 2, -61, 3$, and $-\frac{41}{2}$. Hence the absolute maximum is 4 and the absolute minimum is -61. ■

Example 5.6.11 *Determine the relative extrema of*

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$$

Solution. The critical points: $f_x(x, y) = 4x + 8$ and $f_y(x, y) = 2y - 6$. Solving for x and y, we get $x = -2$ and $y = 3$. Thus the critical point is $(-2, 3)$.

To determine if this critical point gives a minimum or a maximum value, we complete the square of $f(x, y)$:

$$\begin{aligned} 2x^2 + y^2 + 8x - 6y + 20 &= 2x^2 + 8x + y^2 - 6y + 20 \\ &= 2(x^2 + 4x) + (y^2 - 6y) + 20 \\ &= 2(x^2 + 4x + 4 - 4) + (y^2 - 6y + 9 - 9) + 20 \\ &= 2(x + 2)^2 - 8 + (y - 3)^2 - 9 + 20 \\ &= 2(x + 2)^2 + (y - 3)^2 + 3 \end{aligned}$$

Now, $f(-2, 3) = 2(-2 + 2)^2 + (3 - 3)^2 + 3 = 3$. So we see that

$$f(x, y) \geq 3 = f(-2, 3)$$

for all (x, y), hence by definition, f has a relative minimum at $(-2, 3)$. ■

Definition 5.6.12 *A differentiable function has a saddle point at a critical point (a, b) if every open disc centred at (a, b) there are domain points (x, y) where $f(x, y) \geq f(a, b)$ and domain points (x, y) such that $f(x, y) \leq f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface.*

Example 5.6.13 *Find the extrema of $f(x, y) = y^2 - x^2$*

Solution. For the critical points, we have $0 = f_x(x, y) = -2x$ so that $x = 0$, and $f_y(x, y) = 2y$ so that $y = 0$. Thus $(0, 0)$ is the only critical point of f and $f(0, 0) = 0$.

However, $f(0, y^2) = y^2 > 0$, and $f(x, 0) = -x^2 < 0$. Thus we can find points in the xy-plane where $f(x, y) \geq 0 = f(0, 0)$ and where $f(x, y) \leq 0 = f(0, 0)$. Hence, $(0, 0, 0)$ is a saddle point. ■

Second partial derivative test for extrema

Let f be a function of two variables that has continuous second partial derivatives on an open region. Consider the quantity

$$g(x,y) = \det \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y). \end{pmatrix} = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2.$$

If (a,b) is in R and $f_x(a,b) = 0 = f_y(a,b)$ then

i) $f(a,b)$ is a relative minimum of f if $g(a,b) > 0$ and $f_{xx}(a,b) > 0$,

ii) $f(a,b)$ is a relative maximum of f if $g(a,b) > 0$ and $f_{xx}(a,b) < 0$,

iii) $f(a,b)$ is a saddle point of f if $g(a,b) < 0$.

Note that f_{xx} can be replaced with f_{yy} in i) and ii) since $g(a,b) > 0$ implies that f_{xx} and f_{yy} have the same sign. Further, if $g(a,b) = 0$, the test gives no information.

Example 5.6.14 *Find the relative extrema of* $f(x,y) = -x^3 + 4xy - 2y^2 + 1$.

Solution. Critical points:

$$0 = f_x(x,y) = -3x^2 + 4y$$

$$0 = f_y(x,y) = 4x - 4y$$

From the second equation we have $x = y$ and substituting this in the first equation we get

$$0 = -3x^2 + 4x = x(-3x + 4)$$

so that $x = 0$ or $x = \frac{4}{3}$. Thus the critical points are $(0,0)$ and $(4/3, 4/3)$. Now

$$f_{xx} = -6x, \ f_{yy} = -4 \text{ and } f_{xy} = 4$$

At $(0,0)$, we have

$$g = \det \begin{pmatrix} 0 & 4 \\ 4 & -4 \end{pmatrix} = -16 < 0$$

Hence $(0,0,1)$ is a saddle point.

At $(4/3, 4/3)$, we have

$$g = \det \begin{pmatrix} -8 & 4 \\ 4 & -4 \end{pmatrix} = 32 - 16 = 16 > 0$$

and $f_{xx}(4/3, 4/3) = -8 < 0$. Hence f has a relative maximum at $(4/3, 4/3)$. ■

Lagrange Multipliers

Sometimes there is need to find the maximum and minimum values of functions whose domains are confined to lie within some particular subset of the plane. For example, suppose that one wants ro find the rectangle of maximum area that can be inscribed in the ellipse given by

$$\frac{x^2}{3^2}+\frac{y^2}{4^2}=1.$$

This can easily be done by the method of Lagrange multipliers.

Theorem 5.6.15 *Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on a smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq$, then there is a real number λ such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0 . y_0).$$

Here $\nabla = f_x i + f_y j$ and is called the gradient equation of f.

Definition 5.6.16 *The scalar λ is called a Lagrange multiplier.*

To find the minimum or maximum of f, use the following steps:

i) solve the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ simultaneously by solving the following system of equations

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y) \\ f_y(x,y) &= \lambda g_y(x,y) \\ g(x,y) &= c \end{aligned}$$

ii) Evaluate f at each solution point obtained in i). The largest value is the maximum of f subject to the constraint $g(x, y) = c$, and the smallest value is a minimum of f subject to the constraint $g(x, y) = c$.

Example 5.6.17 *Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse*

$$\frac{x^2}{8}+\frac{y^2}{2}=1.$$

Solution. Let $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$, we need to find $((x, y)$ and λ such that

$$\nabla f = \lambda \nabla g \text{ and } g(x, y) = 0.$$

Now

$$\nabla f = yi + xj \text{ and } \nabla g(x,y) = \frac{x}{4}i + yj$$

so $yi + xj = \lambda(\frac{x}{4}i + yj)$ implies $y = \frac{\lambda}{4}x$ and $x = \lambda y$. Solving these equations simultaneously we get

$$y = \frac{\lambda}{4}(\lambda y) \Rightarrow 4y = \lambda^2 y \Rightarrow y(4 - \lambda^2) = 0.$$

From this we get $y = 0$ and $\lambda = \pm 2$.

case 1: If $y = 0$, then $x = y = 0$. But $(0,0)$ is not on the ellipse. Hence $y \neq 0$

case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in $g(x,y) = 0$ gives

$$\begin{aligned} &\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1 \\ &\Rightarrow 4y^2 + 4y^2 = 8 \\ &\Rightarrow y = \pm 1 \end{aligned}$$

Thus the extreme values occur at four points $(-2,1)$, $(2,1)$, $(2,-1)$ and $(-2,-1)$ where $xy = 2$ is the maximum and $xy = -2$ is the minimum.

■

Example 5.6.18 *Find the minimum value of* $f(x,y,z) = 2x^2 + y^2 + 3z^2$ *subject to the constraint* $2x - 3y - 4z = 49$.

Solution. Note that $f(x,y,z)$ has no maximum. Let $g(x,y,z) = 2x - 3y - 4z = 49$. Then

$$\nabla f(x,y,z) = 4xi + 2yj + 6zk \text{ and } \lambda\nabla g(x,y,z) = 2\lambda i - 3\lambda j - 4\lambda k.$$

Now $\nabla f = \lambda\nabla g$ implies $4x = 2\lambda$, $2y = -3\lambda$ and $6z = -4\lambda$ from which we get $x = \frac{1}{2}\lambda$, $y = \frac{-3}{2}\lambda$ and $\frac{-2}{3}\lambda$. Substituting these in $g(x,y,z)$ we get

$$\begin{aligned} 49 &= 2\left(\frac{1}{2}\lambda\right) - 3\left(\frac{-3}{2}\lambda\right) - 4\left(\frac{-2}{3}\lambda\right) \\ &= \lambda + \frac{9}{2}\lambda + \frac{8}{3}\lambda \\ &= \frac{6\lambda + 27\lambda + 16\lambda}{6} = \frac{49\lambda}{6} \end{aligned}$$

Solving for λ we get $\lambda = 6$ and so $x = 3,\ y = -9$ and $z = -4$. Therefore, the minimum value of f is $f(3, -9, -4) = 2(3)^2 + (-9)^2 + 3(-4)^2 = 147$. ■
When there are two constraints on the variables of $f(x, y, z)$, say $g(x, y, z) = c$ and $h(x, y, z) = d$, we introduce a second Lagrange multiplier μ and solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

Example 5.6.19 *Let* $T(x, y, z) = 20+2x+2y+z^2$ *represent the temperature at each point on the sphere* $x^2+y^2+z^2 = 11$. *Find the extreme temperatures on the curve formed by the intersection of the plane* $x + y + z = 3$ *and the sphere.*

Solution. Let $g(x, y, z) = x^2 + y^2 + z^2 = 11$ and $h(x, y, z) = x + y + z = 3$. Then

$$\begin{aligned}
\nabla T(x, y, z) &= 2\mathbf{i} + 2\mathbf{j} + 2z\mathbf{k} \\
\lambda \nabla g(x, y, z) &= 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j} + 2\lambda z\mathbf{k} \\
\mu \nabla h(x, y, z) &= 2\mu\mathbf{i} + \mu\mathbf{j} + \mu\mathbf{k}
\end{aligned}$$

From these we get the system of equations

$$\begin{cases}
\quad 2 = 2\lambda x + \mu \\
\quad 2 = 2\lambda y + \mu \\
2z \quad = 2\lambda z + \mu \\
\quad 11 = x^2 + y^2 + z^2 \\
\quad 3 = x + y + z
\end{cases}$$

We need to solve this system simultaneously. Subtracting the second equation from the first we get

$$\lambda(x - y) = 0$$

so that $\lambda = 0$ or $y = x$. We now have two cases to consider.

Case 1: If λ=0, then from the first equation, we get $\mu = 2$. Substituting these in the third equation, we get $2z = 0+2$ so that $z = 1$. Substituting $z = 1$ in the fifth equation we get $x + y + 1 = 3$ so that $y = 2 = x$. We now substitute $y = 2 - x$ and $z = 1$ in the fourth equation to get

$$11 = x^2 + (2 - x)^{+}1 = x^2 + 4 - 4x + x^2 + 1 = 2x^2 - 4x + 5.$$

Thus we

$$2x^2 - 4x - 6 = 0 \text{ or } x^2 - 2x - 3 = 0$$

Solving for x we get

$$0 = x^2 - 2x - 3 = (x+1)(x-3)$$

so that $x = -1$ or $x = 3$. Substituting in $y = 2 - x$ we get the following critical points $A = (-1, 3, 1)$ and $B = (3, -1, 1)$.

Case 2: If $x = y$, then from the fifth equation we get

$$x + x + z = 3 \text{ or } z = 3 - 2x.$$

Sub substituting this in the fourth equation, we get

$$11 = x^2 + x^2 + (3-2x)^2 = 2x^2 + 9 - 12x + 4x^2 = 6x^2 - 12x - 9$$

From this we get $6x^2 - 12x - 2 = 0$ or $3x^2 - 6x - 1 = 0$. Solving for x we have

$$\begin{aligned} x &= \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)} \\ &= \frac{6 \pm \sqrt{36+12}}{6} \\ &= \frac{6 \pm \sqrt{48}}{6} \\ &= \frac{6 \pm 4\sqrt{3}}{6} \\ &= \frac{3 \pm 2\sqrt{3}}{3} \end{aligned}$$

Substituting this in $z = 3 - 2x$ we get

$$z = 3 - 2\left(\frac{3 \pm 2\sqrt{3}}{3}\right) = \frac{9 - 6 \pm 4\sqrt{3}}{3} = \frac{3 \pm 4\sqrt{3}}{3}$$

So the critical points are

$$C = \left(\frac{3+2\sqrt{3}}{3}, \frac{3+2\sqrt{3}}{3}, \frac{3-4\sqrt{3}}{3}\right) \text{ and } D = \left(\frac{3-2\sqrt{3}}{3}, \frac{3-2\sqrt{3}}{3}, \frac{3+4\sqrt{3}}{3}\right).$$

At the critical points, we have

$$T(A) = T(3, -1, 1) = 20 + 6 - 2 + 1 = 25 = T(3, -1, 1) = T(B),$$

$$\begin{aligned}
T(C) &= T\left(\frac{3+2\sqrt{3}}{3}, \frac{3+2\sqrt{3}}{3}, \frac{3-4\sqrt{3}}{3}\right) \\
&= 20 + 2\left(\frac{3+2\sqrt{3}}{3}\right) + 2\left(\frac{3+2\sqrt{3}}{3}\right) + \left(\frac{3-4\sqrt{3}}{3}\right)^2 \\
&= \frac{91}{3} \\
&= T\left(\frac{3-2\sqrt{3}}{3}, \frac{3-2\sqrt{3}}{3}, \frac{3+4\sqrt{3}}{3}\right) \\
&= T(D).
\end{aligned}$$

Thus $T = 25$ is the minimum temperature and $t = 91/3$ is the maximum temperature.

■