Chapter 5

Functions of several variables

5.1 Domain and Range

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there correspond a unique real number $f(x, y)$, then f is called a function of x and y. The set D is called the domain of f, and the corresponding set of values for $f(x, y)$ is the range of f.

Example 5.1.1 *i) $V = \pi r^2 h$ or $V(r, h)$ is a function of two variables.*

ii) $V = lbh$ or $V(l, b, h) = lbh$ is a function of three variables

Example 5.1.2 *Find the domain of*

a) $f(x, y) = \frac{\sqrt{x^2+y^2-9}}{x}$.

b) $g(x, y, z) = \frac{x}{\sqrt{9-x^2-y^2-z^2}}$.

Solution.

a) The domain consists of all point (x, y) such that $x \neq 0$ and $x^2+y^2 \geq 9$. In other words, the domain consists of all points outside and on the circle of radius 3 excluding the y-axis. See figure 5.1 below.

b) The domain consists of all point (x, y, z) such that $x^2 + y^2 + z^2 < 9$. In other words, the domain consists of all points inside a sphere of radius 3.

■

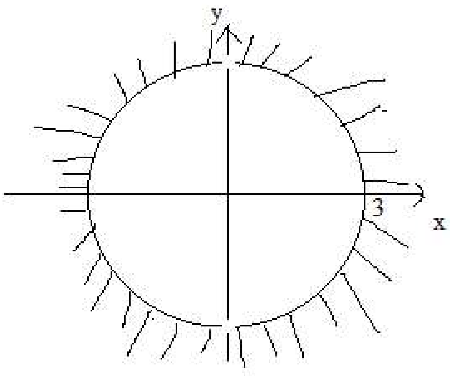

Figure 5.1: Domain of f(x,y)

5.2 Limits and Continuity

Definition 5.2.1 *Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disc centered at (x_0, y_0), and let L be a real number. Then*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for every $\epsilon > 0$ there corresponds a $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Theorem 5.2.2 *If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$. Then*

i) $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \pm g(x,y)) = L_1 \pm L_2$

ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)g(x,y) = L_1 L_2$

iii) $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL_1$ *for any real number k.*

iv) $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L_1}{L_2}$ *if $L_2 \neq 0$.*

Example 5.2.3 *By definition of the limit, show that* $\lim_{(x,y)\to(a,b)} x = a$

Solution. Let $f(x,y) = x$ and $L = a$. Let $\epsilon > 0$ be given. We need to show that there exists a $\delta > 0$ such that $|f(x,y) - L| = |x - a| < \epsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. So

$$|f(x,y) - L| = |x - a| = \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2 + (y-b)^2}.$$

Thus if we choose $\delta = \epsilon$, then the limit is verified. ■

Example 5.2.4 *Evaluate*

a) $\lim\limits_{(x,y)\to(3,-4)} (x^2 + y^2)$.

b) $\lim\limits_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$

Solution.

a) $\lim\limits_{(x,y)\to(3,-4)} (x^2 + y^2) = 3^2 + (-4)^2 = 9 + 16 = 25$.

b) $\lim\limits_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - 0(1) + 3}{(0^2)(1) + 5(0)(1) - (1)^3} = -3$.

■

Definition 5.2.5 *A function $f(x,y)$ is said to be continuous at the point (x_0, y_0) if*

i) *f is defined at (x_0, y_0)*

ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ *exist, and*

iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

Remark 5.2.6 *If $f(x,y)$ and $g(x,y)$ are continuous at a point then their sums, differences, products and there multiples are continuous. Their quotient is continuous whenever it is defined.*

Example 5.2.7 *Show that*

$$f(x,y) \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous everywhere except at the origin.

Solution. Note that a rational function is continuous at every point except where the denominator is zero. By definition we have

i) f is defined at every point including $(0,0)$.

ii) if $(x,y) \to (0,0)$ along $y = mx$ but $(x,y) \neq (0,0)$, then along $y = mx$ we have

$$f(x,y) = \frac{xy}{x^2+y^2} = \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

and

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{m}{1+m^2}.$$

But the this limit is not unique since it changes as m changes. Hence, it does not exist. Therefore, $f(x,y)$ cannot be continuous by definition.

■

Example 5.2.8 *Discuss the continuity of* $\frac{2}{y-x^2}$.

Solution. The function is continuous at every point except at $y = x^2$. Thus continuous at every point except at the points lying on the parabola $y = x^2$.

■

5.3 Partial derivatives

Definition 5.3.1 *If $z = f(x,y)$, then the first partial derivatives of f with respect to x and y are the functions f_x and f_y defined by*

$$f_x(x,y) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \quad \textit{and}$$

$$f_y(x,y) = \lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

provided the limits exist.

For $z = f(x,y)$, the partial derivatives f_x and f_y are also denoted by

$$\frac{\partial}{\partial x} f(x,y) \text{ or } z_x \text{ or } \frac{\partial z}{\partial x}, \text{ and}$$

$$\frac{\partial}{\partial y} f(x,y) \text{ or } z_y \text{ or } \frac{\partial z}{\partial y}$$

The first partials evaluated at the point (a, b) are denoted by

$$\frac{\partial z}{\partial y}\mid_{(a,b)}= f_x(a, b) \text{ and } \frac{\partial z}{\partial y}\mid_{(a,b)}= f_y(a, b)$$

Example 5.3.2 *By definition, find f_x and f_y for the function*

$$f(x, y) = x^2 + 3xy + y - 1$$

and evaluate $f_x(4, -5)$.

Solution. By definition we have

$$\begin{aligned}
f_x(x, y) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + 3(x + \Delta x)y + y - 1 - (x^2 + 3xy + y - 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2(\Delta x)x + (\Delta x)^2 + 3xy + 3(\Delta x)y + y - 1 - x^2 - 3xy - y + 1}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2 + 3(\Delta x)y}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2x + \Delta x + 3y \\
&= 2x + 3y.
\end{aligned}$$

and

$$\begin{aligned}
f_y(x, y) &= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{x^2 + 3x(y + \Delta y) + (y + \Delta y) - 1 - (x^2 + 3xy + y - 1)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{x^2 + 3xy + 3x(\Delta y) + y + \Delta y - 1 - x^2 - 3xy - y + 1}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{3x(\Delta y) + \Delta y}{\Delta y} \\
&= \lim_{\Delta y \to 0} 3x + 1 \\
&= 3x + 1.
\end{aligned}$$

At $(4, -5)$ we have $f_x(4, -5) = 2(4) + 3(-5) = -7$ ■

Example 5.3.3 *Find the first partial derivatives of*

$$f(x, y) = 3x - x^2y^2 + 2x^3y.$$

Solution. To find $f_x(x, y)$, we differentiate the function with respect to x and treat the variable y as a constant. Thus

$$f_x(x, y) = 3 - 2xy^2 + 6x^2y.$$

Similarly, treating x as a constant we have

$$f_y(x, y) = -2x^2y + 2x^3$$

■

We can also use partial derivatives to find the rate of change.

Example 5.3.4 *The area of a parallelogram with adjacent sides a and b and included angle θ is given by $A = ab\sin\theta$. When $a = 10,\ b = 20$ and $\theta = \pi/6$, find the rate of change of*

a) A with respect to a.

b) A with respect to θ.

Solution.

a) We need to find $A_a(a, b, \theta)$, thus

$$A_a(10, 20, \pi/6) = \frac{\partial A}{\partial a}|_{(10,20,\pi/6)} = b\sin\theta|_{(10,20,\pi/6)} = 20\sin\pi/6 = 10\,units$$

b) We need to find $A_\theta(a, b, \theta)$, thus

$$A_\theta(10, 20, \pi/6) = \frac{\partial A}{\partial \theta}|_{(10,20,\pi/6)} = ab\cos\theta|_{(10,20,\pi/6)} = 10(20)\cos\pi/6 = 100\sqrt{3}\,units$$

■

Given that $w = f(x, y, z)$, then

$$f_x(x, y, z) = \lim_{\Delta x\to 0}\frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$f_y(x, y, z) = \lim_{\Delta y\to 0}\frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}, \text{ and}$$

$$f_z(x, y, z) = \lim_{\Delta z\to 0}\frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

provided the limits exist.

In general , if $w = f(x_1, x_2, \ldots, x_n)$ there are n first partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), k = 1, 2, \ldots, n$$

5.4 Higher Order Partial Derivatives

The function $z = f(x, y)$ has the following second order partial derivatives:

i) $f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ (Differentiate twice with respect to x)

ii) $f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ (Differentiate twice with respect to y)

iii) $f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ (Differentiate first with respect to x and then with respect to y).

iv) $f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ (Differentiate first with respect to y and then with respect to x.)

The cases iii) and the iv) are called **mixed partial** derivatives.

Example 5.4.1 *Find the second partial derivatives of*

$$f(x, y) = 3xy^2 - 2y + 5x^2y^2$$

and determine the value of $f_{xy}(-1, 2)$.

Solution.

i) $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3y^2 + 10xy^2) = 10y^2$

ii) $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(6xy - 2 + 10x^2y) = 6x + 10x^2$

iii) $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3y^2 + 10xy^2) = 6y + 20xy.$

iv) $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(6xy - 2 + 10x^2y) = 6y + 20xy$

■

Example 5.4.2 *Show that* $f_{xz} = f_{zx}$ *and* $f_{xzz} = f_{zxz} = f_{zzx}$ *for the function* $f(x, y, z) = ye^x + x\ln z$.

Solution.

$$f_{xz}(ye^x + x\ln z) = f_z(ye^x + \ln z) = \frac{1}{z}, \text{ and}$$

$$f_{zx}(ye^x + x\ln z) = f_x\left(\frac{x}{z}\right) = \frac{1}{z};$$

thus $f_{xz} = f_{zx}$. Also,

$$f_{xzz} = f_{zz}(ye^x + \ln z) = f_z\left(\frac{1}{z}\right) = \frac{-1}{z^2},$$

$$f_{zxz} = f_{xz}\left(\frac{x}{z}\right) = f_z\left(\frac{1}{z}\right) = \frac{-1}{z^2}, \text{ and}$$

$$f_{zzx} = f_{zx}\left(\frac{x}{z}\right) = f_x\left(\frac{-x}{z^2}\right) = \frac{-1}{z^2}$$

showing that $f_{xzz} = f_{zxz} = f_{zzx}$ ■

Theorem 5.4.3 *If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.*

5.5 Chain rule

Theorem 5.5.1 *Let $w = f(x, y)$ be a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$ where g and h are differentiable functions of t, then w is a differentiable function of t and*

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dx} = w_x\frac{dx}{dt} + w_y\frac{dy}{dx}.$$

Example 5.5.2 *Use the chain rule to find the derivative of $f(x, y) = xy$ with respect to t along the path $x = \cos t$ and $y = \sin t$. What is the value of the derivative at $t = \pi/2$.*

Solution. By the chain rule we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Now $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$. substituting these in the formula above, we get

$$\begin{aligned}\frac{df}{dt} &= y(-\sin t) + x(\cos t)\\ &= \sin t(-\sin t) + \cos t(\cos t)\\ &= -\sin^2 t + \cos^2 t\\ &= \cos 2t.\end{aligned}$$

When $t = \pi/2$,

$$\frac{df}{dt}|_{\pi/2} = \cos 2(\pi/2) = \cos \pi - 1.$$

■

Example 5.5.3 *The radius of a right circular cone is increasing at a rate of 3cm/sec while its height is increasing at a rate of 5cm/sec. How fast is the volume increasing when $r = 15cm$ and $h = 25cm$?*

Solution. We need to find $\frac{dV}{dt}$ where $V = \frac{1}{3}\pi r^2 h$. By the chain rule, with $\frac{dr}{dt} = 3$ and $\frac{dh}{dt} = 5$, we have

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \\ &= \frac{2}{3}\pi rh(3) + \frac{1}{3}\pi r^2(5) \\ &= 2\pi rh + \frac{5}{3}\pi r^2 \\ &= \pi(2 \times 15 \times 25 + \frac{5}{3} \times (15)^2) \\ &= 1125\pi cm^3/sec.\end{aligned}$$

■

If $w = f(x, y, z)$ and its partial derivatives are continuous and $x = x(t), y = y(t)$ and $z = z(t)$ are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = w_x\frac{dx}{dt} + w_y\frac{dy}{dt} + w_z\frac{dz}{dt}.$$

Theorem 5.5.4 *Let f with its partial derivatives with respect to x, y and z be continuous and let x, y and z and their partial derivatives with respect to r and s be continuous. Then f has continuous partial derivatives with respect to r and s, and are given by*

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}, \text{ and}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

respectively.

Example 5.5.5 *Find* $\frac{\partial f}{\partial r}$ *and* $\frac{\partial f}{\partial s}$ *as functions of* r *and* s *if*

$$f(x,y,z) = x + 2y + z^2,\ x = \frac{r}{s},\ y = r^2 + \ln s \text{ and } z = 2r.$$

Solution. By Theorem 5.5.4, we have

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} = 1\left(\frac{1}{s}\right) + 2(2r) + 2z(2) = \frac{1}{s} + 12r, \text{ and}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} = 1\left(\frac{-r}{s^2}\right) + 2\left(\frac{1}{s}\right) - 2z(0) = -\frac{r}{s^2} + \frac{2}{s}$$

■

5.6 Application of Partial derivatives

Increments and Differentials

Let $z = f(x,y), \Delta x$ and Δy be increments of x and y respectively. Then the increment of z is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y).$$

Definition 5.6.1 *If* $z = f(x_1, x_2, \ldots, x_n)$ *and* $\Delta x_1, \Delta x_2, \ldots \Delta x_n$ *are increments in* $x_1, x_2, \ldots, x_n$ *respectively, then the differentials of the independent variables* x_i *are*

$$dx_1 = \Delta x_1,\ dx_2 = \Delta x_2 \ldots dx_n = \Delta x_n$$

and the total differential of the dependent variable z *is*

$$dz = \frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2 + \cdots + \frac{\partial z}{\partial x_n}dx_n$$

Note 5.6.2 *The increment* Δz *represents the change in the function value if* $(x_1, x_2, \ldots, x_n)$ *changes to* $(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n)$.

Example 5.6.3 *Find the total differential* dz *for* $z = 2x\sin y - 3x^2y^2$

Solution.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2\sin y - 6xy^2)dx + (2x\cos y - 6x^2y)dy$$

■

For small Δx and Δy, we can approximate Δz using dz, i.e, $\Delta z \approx dz$.

Example 5.6.4 *Use the differential* dz *to approximate the change in*

$$z = \sqrt{4 - x^2 - y^2}$$

as (x, y) *moves from* $(1, 1)$ *to* $(1.01, 0.97)$*. Compare this approximation with the actual change in* z*.*

Solution. Let $(x, y) = (1, 1)$ and $(x + \Delta x, y + \Delta y) = (1.01, 0.97)$, then $dx = \Delta x = 0.01$ and $dy = \Delta y = -0.03$. Thus

$$\begin{aligned}\Delta z \approx dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= \frac{-x}{\sqrt{4 - x^2 - y^2}}\Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}}\Delta y\end{aligned}$$

When $x = 1$ and $y = 1$ we have

$$\Delta z \approx \frac{-1}{\sqrt{2}}(0.01) - \frac{1}{\sqrt{2}}(-0.03) = \frac{0.02}{\sqrt{2}} = \sqrt{2}(0.01) = 0.0141.$$

The actual value is

$$\Delta z = f(1.01, 0.97) - f(1, 1) = \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2} \approx 0.0137.$$

■

Example 5.6.5 *Use the differential to estimate* $\sqrt{(2.98)^2 + (4.03)^2}$*.*

Solution. Let $f(x, y) = \sqrt{x^2 + y^2}$ and $(x, y) = (3, 4)$. Then $\Delta x = -0.02$ and $\Delta y = 0.03$. Now, $f(3, 4) = \sqrt{3^2 + 4^2} = 5$. Thus we need to find $f(3 - 0.02, 4 + 0.03)$.

$$\begin{aligned}df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \frac{x}{\sqrt{x^2 + y^2}}dx + \frac{y}{\sqrt{x^2 + y^2}}dy \\ &= \frac{3}{5}(-0.02) + \frac{4}{5}(0.03) \\ &= 0.012\end{aligned}$$

Hence,

$$\Delta z = f(3 - 0.02, 4 + 0.03) - f(3, 4) \approx df = 0.012$$

and so

$$\sqrt{(2.98)^2 + (4.03)^2} = f(2.98, 4.03) \approx f(3,4) + 0.012 = 5.012.$$

The actual value of $\sqrt{(2.98)^2 + (4.03)^2}$ is 5.012115, so $\Delta f = 0.012115$ ■

As you know every approximation has errors involved. We analyze such errors in the following example.

Example 5.6.6 *The possible error involved in measuring each dimension of a rectangular box is* $\pm 0.1mm$. *The dimensions of the box are* $x = 50cm$, $y = 20cm$ *and* $z = 15cm$. *Use* dV *to approximate the propagated error in the calculated volume of the box.*

Solution. $V = xyz$ and

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yzdx + xzdy + xydz.$$

Now $0.1mm = 0.01cm$ and $dx = dy = dz = \pm 0.01cm$. So the propagated error is approximately

$$\begin{aligned} dV &= (20)(15)(\pm 0.01) + (50)(15)(\pm 0.01) + (50)(20)(\pm 0.01) \\ &= 300(\pm 0.01) + 750(\pm 0.01) + 1000(\pm 0.01) \\ &= 2050(\pm 0.01) \\ &= \pm 20.5cm^3. \end{aligned}$$

The relative error is given by

$$R.E = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{20.5}{15000} = 0.0014.$$

Thus there is a relative error of approximately 0.14% ■