

Chapter 2

Differential Calculus

2.1 Higher order derivatives

If the derivative $y' = f'(x)$ of a function $y = f(x)$ is itself differentiable at x , we can calculate its derivative, which we call the **second derivative** of f and denoted by $y'' = f''(x)$. Other notations used include

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx} \frac{d}{dx} f(x) = \frac{d^2}{dx^2} f(x) = D_x^2 y = D_x^2 f(x).$$

In general, the n^{th} derivative of $y = f(x)$ is

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} f(x) = D_x^n y = D_x^n f(x)$$

and is defined to be the derivative of the $(n - 1)^{\text{st}}$ derivative. For $n = 1, 2, 3$, primes are still normally used: $f^{(2)}(x) = f''(x)$, $f^{(3)}(x) = f'''(x)$. Sometimes we denote $f^{(0)}(x) = f(x)$.

Example 2.1.1 *The velocity of a moving object is the rate of change of the position of the object with respect to time. If the object moves along the x -axis and is at position $x = f(t)$ at time t , then its velocity at that time is given by*

$$v = \frac{dx}{dt} = f'(t).$$

Similarly, the acceleration of the object is the rate of change of velocity. Thus, the acceleration is the second derivative of the position:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = f''(t).$$

Example 2.1.2 If $y = x^3$, then $y' = 3x^2$, $y'' = 6x$, $y''' = 6$, $y^{(4)} = 0$ and all higher derivatives are zero. In general, if $y = x^n$, then

$$\begin{aligned} y^{(k)}(x) &= n(n-1)(n-2)\dots(n-(k-1))x^{n-k} \\ &= \begin{cases} \frac{n!}{(n-k)!}x^{n-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases} \end{aligned}$$

where $n!$ (called n **factorial**) is defined by

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 1! \times 2 = 1 \times 2 = 2 \\ 3! &= 2! \times 3 = 2 \times 3 = 6 \\ 4! &= 3! \times 4 = 6 \times 4 = 24 \\ &\vdots \\ n! &= (n-1)!n = 1 \times 2 \times 3 \times \dots \times (n-1) \times n. \end{aligned}$$

It follows that if P is a polynomial of degree n ,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where $a_n, a_{n-1}, \dots, a_1, a_0$ are constants, then $P^{(k)}(x) = 0$ for $k > n$. For $n \leq k$, $P^{(k)}$ is a polynomial of degree $n - k$; in particular, $P^{(n)}(x) = n!a_n$, a constant function.

Example 2.1.3 Show that, if A, B and κ are constants, then the function $y = A \cos(\kappa t) + B \sin(\kappa t)$ is a solution of a second-order differential equation of simple harmonic motion

$$\frac{d^2 y}{dt^2} + \kappa^2 y = 0.$$

Solution

To be a solution, the function $y(t)$ must satisfy the equation identically, i.e.,

$$\frac{d^2 y(t)}{dt^2} + \kappa^2 y(t) = 0. \quad (2.1)$$

must hold for every real number t . We calculate the first and second derivatives, i.e.,

$$\frac{dy}{dt} = -A\kappa \sin(\kappa t) + B\kappa \cos(\kappa t)$$

and

$$\frac{d^2y}{dt^2} = -A\kappa^2 \cos(\kappa t) - B\kappa^2 \sin(\kappa t) = -\kappa^2 y(t)$$

Substituting this in equation (2.1), we get

$$-A\kappa^2 \cos(\kappa t) - B\kappa^2 \sin(\kappa t) + \kappa^2(A \cos(\kappa t) + B \sin(\kappa t)) = 0.$$

Thus, $y(t)$ is a solution to the given equation.

Example 2.1.4 Find the n^{th} derivative of $y = \frac{1}{1+x}$.

Solution

First note that

$$\frac{1}{1+x} = (1+x)^{-1}.$$

We begin by calculation a few derivatives:

$$\begin{aligned}y' &= (-1)(1+x)^{-2} \\y'' &= (-1)(-2)(1+x)^{-3} = 2(1+x)^{-3} \\y''' &= 2(-3)(1+x)^{-4} = -3!(1+x)^{-4} \\y^{(4)} &= -3!(-4)(1+x)^{-5} = 4!(1+x)^{-5}\end{aligned}$$

We are now able to predict the n^{th} derivative from the above pattern, which is

$$y^{(n)} = (-1)^n n!(1+x)^{-n-1}.$$

However, this does not prove that this formula is true for every n . So to complete the proof, we use mathematical induction on n . We see from the first derivative above that the formula is true for $n = 1$. Suppose that the formula is valid for $n = k$, where k is some positive integer. Consider y^{k+1} :

$$\begin{aligned}y^{k+1} &= \frac{d}{dx} y^{(k)} = \frac{d}{dx} ((-1)^k k!(1+x)^{-k-1}) \\&= (-1)^k k!(-k-1)(1+x)^{-k-2} \\&= (-1)^{k+1} (k+1)!(1+x)^{-(k+1)-1}.\end{aligned}$$

This is what the formula predicted for the $(k+1)^{\text{st}}$ derivative. Thus, if the formula is true for $n = k$, then it is true for $n = k+1$. Therefore, by induction, the formula is true for all $n \geq 1$.

Example 2.1.5 Find a formula for $f^{(n)}(x)$, given that $f(x) = \sin(ax + b)$.

Solution

We again begin by calculating a few derivatives and see if we can predict the n^{th} derivative.

$$\begin{aligned}f'(x) &= a \cos(ax + b) \\f''(x) &= -a^2 \sin(ax + b) = -a^2 f'(x) \\f'''(x) &= -a^3 \cos(ax + b) = -a^2 f''(x) \\f^{(4)}(x) &= a^4 \sin(ax + b) = a^4 f'(x) \\f^{(5)}(x) &= a^5 \cos(ax + b) = a^4 f''(x)\end{aligned}$$

We can see from the pattern above that each derivative is $-a^2$ times the second previous one. Therefore, a formula that gives all derivatives is

$$f^{(n)}(x) = \begin{cases} (-1)^k a^n \sin(ax + b) & \text{if } n = 2k \\ (-1)^k a^n \cos(ax + b) & \text{if } n = 2k + 1 \end{cases} \quad (k = 0, 1, 2, \dots)$$

Example 2.1.6 Calculate f' , f'' , and f''' for $f(x) = \sqrt{x^2 + 1}$. Can you predict $f^{(4)}$ from the pattern?

Solution

Since $f(x) = (x^2 + 1)^{\frac{1}{2}}$ we have

$$\begin{aligned}f'(x) &= \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}(2x) = x(x^2 + 1)^{-\frac{1}{2}}, \\f''(x) &= (x^2 + 1)^{-\frac{1}{2}} + x\left(-\frac{1}{2}\right)(x^2 + 1)^{-\frac{3}{2}}(2x) = (x^2 + 1)^{-\frac{3}{2}}, \\f'''(x) &= -\frac{3}{2}(x^2 + 1)^{-\frac{5}{2}}(2x) = -3x(x^2 + 1)^{-\frac{5}{2}}.\end{aligned}$$

The pattern is not obvious to predict $f^{(4)}$ without calculating it. In fact,

$$f^{(4)} = 3(4x^2 - 1)(x^2 + 1)^{-\frac{7}{2}}.$$

So the pattern doesn't become clearer even at this stage.

2.2 Implicit Differentiation

Curves are generally the graphs of equations in two variables. Such equations can be written in the form

$$F(x, y) = 0$$

where $F(x, y)$ denotes an expression involving the two variables x and y . For example, a circle of radius 5 and centre at the origin has equation

$$F(x, y) = x^2 + y^2 - 5.$$

To find the derivative of F , we use a technique called **implicit differentiation**. In this technique, we differentiate F with respect to x while treating y as a function of x having derivative $\frac{dy}{dx}$ or y' .

Example 2.2.1 Find $\frac{dy}{dx}$ if $y^2 = x$.

Solution

We differentiate both sides of the equation with respect to x using the chain rule:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x) \Rightarrow 2y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y} (= \frac{1}{2\sqrt{x}} \text{ or } -\frac{1}{2\sqrt{x}})$$

Note that $y^2 = x$ implies $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$. So

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \text{ and } \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}$$

which agrees with the above derivative.

Example 2.2.2 Find the slope of the circle $x^2 + y^2 = 25$ at $(3, -4)$.

Solution

This equation of the circle combines two graphs namely $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$. Observe that the point $(3, -4)$ lies on the graph of y_2 . So we can find the slope by calculating explicitly

$$\frac{dy_2}{dx} \Big|_{x=3} = -\frac{-2x}{2\sqrt{25 - x^2}} \Big|_{x=3} = \frac{6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

But we can also differentiate implicitly with respect to x :

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25) \Rightarrow 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}.$$

Therefore, the slope at $(3, -4)$ is

$$\frac{-x}{y} \Big|_{(3, -4)} = -\frac{-3}{-4} = \frac{3}{4}.$$

So why implicit differentiation if we can explicitly differentiate with respect to x ? Well, it is not all functions that we can easily express y as a function of x as the following example shows.

Example 2.2.3 Find $\frac{dy}{dx}$ if $y \sin x = x^3 + \cos y$.

Solution

Here, we can only use implicit differentiation. We have

$$\begin{aligned}\frac{d}{dx}(y \sin x) &= \frac{d}{dx}(x^3) + \frac{d}{dx}(\cos y) \\ \Rightarrow \sin x \frac{dy}{dx} + y \cos x &= 3x^2 - \sin y \frac{dy}{dx} \\ \Rightarrow (\sin x + \sin y) \frac{dy}{dx} &= 3x^2 - y \cos x \\ \Rightarrow \frac{dy}{dx} &= \frac{3x^2 - y \cos x}{\sin x + \sin y}.\end{aligned}$$

Example 2.2.4 Find an equation of the tangent to $x^2 + xy + 2y^3 = 4$ at $(-2, 1)$.

Solution

Note that $(-2, 1)$ lies on the curve. To find the slope of the tangent, we differentiate implicitly with respect to x to get:

$$2x + y + xy' + 6y^2y' = 0.$$

Substituting $x = -2$, $y = 1$ and solving for y' we get

$$-4 + 1 - 2y' + 6y' = 0 \Rightarrow y' = \frac{3}{4}.$$

Thus, the slope of the tangent at $(-2, 1)$ is $\frac{3}{4}$, and its equation is

$$y = \frac{3}{4}(x + 2) + 1 \text{ or } 3x - 4y = -10$$

Example 2.2.5 Find $\frac{d^2y}{dx^2}$ if $xy + y^2 = 2x$.

Solution

We implicitly differentiate twice both sides of the equation. Thus

$$y + xy' + 2yy' = 2 \tag{2.2}$$

and solving (2.2) for y' we get

$$y' = \frac{2 - y}{x + 2y}. \tag{2.3}$$

Implicitly differentiating (2.2) again with respect to x we get

$$y' + y' + xy'' + 2(y')^2 + 2yy'' = 0 \Rightarrow (x + 2y)y'' = -2y' - 2(y')^2$$

so that

$$y'' = \frac{-2y' - 2(y')^2}{x + 2y}. \quad (2.4)$$

Substituting (2.3) into (2.4) we get

$$\begin{aligned} y'' &= \frac{-2\left(\frac{2-y}{x+2y}\right) - 2\left(\frac{2-y}{x+2y}\right)^2}{x + 2y} \\ &= -2 \left(\frac{2-y}{(x+2y)^2} \right) - 2 \left(\frac{(2-y)^2}{(x+2y)^3} \right) \\ &= \frac{-2(2-y)(x+2y+2-y)}{(x+2y)^3} \\ &= \frac{-2(2-y)(x+y+2)}{(x+2y)^3} \\ &= \frac{-2(2x-xy+2y-y^2+4-2y)}{(x+2y)^3} \\ &= \frac{-2(2x-xy-y^2+4)}{(x+2y)^3} \\ &= \frac{-8}{(x+2y)^3} \quad (\text{since } 2x-xy-y^2=0) \end{aligned}$$

(We have used the given equation (in the question) to simplify the numerator.)