Chapter 3

Integration

In first year, you were introduced to some integration techniques. Here we review some and introduce new ones. We first give the table of some elementary integrals.

1. $1\int 1dx = x + C$	2. $\int xdx = \frac{x^2}{2} + C$
3. $\int x^2dx = \frac{x^3}{3} + C$	4. $\int \frac{1}{x^2} = \frac{-1}{x} + C$
5. $\int \sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}} + C$	6. $\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x} + C$
7. $\int x^r dx = \frac{1}{r+1}x^{r+1} + C\ (r \neq -1)$	8. $\int \frac{1}{x}dx = \ln\|x\| + C$
9. $\int \cos axdx = \frac{1}{a}\sin ax + C$	10. $\int \sin axdx = -\frac{1}{a}\cos ax + C$
11. $\int \sec^2 axdx = \frac{1}{a}\tan ax + C$	12. $\int \csc axdx = -\frac{1}{a}\cot ax + C$
13. $\int \sec ax \tan axdx = \frac{1}{a}\sec ax + C$	14. $\int \csc ax \cot axdx = -\frac{1}{a}\cot ax + C$
15. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C\ (a > 0)$	16. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\ (a > 0)$
17. $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$	18. $\int b^{ax}dx = \frac{1}{a\ln b}b^{ax} + C$
19. $\int \cosh axdx = \frac{1}{a}\sinh ax + C$	20. $\int \sinh axdx = \frac{1}{a}\cosh ax + C$

The linearity formula

$$\int (Af(x) + Bg(x))dx = A\int f(x)dx + B\int g(x)dx$$

makes it possible to integrate sums and constant multiples of functions.

3.1 Substitution

The method of substitution is the most important technique of integration. It is the integral version of the chain rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

and in integral form we get

$$\int f'(g(x))g'(x)dx = f(g(x)) + C.$$

To obtain this, let $u = g(x)$, then $\frac{du}{dx} = g'(x)$, or in differential form, $du = g'(x)dx$. Thus

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(u) + C = f(g(x)) + C.$$

Example 3.1.1 *Evaluate the indefinite integrals a)* $\int \frac{x}{x^2+1}dx$ *b)* $\int \frac{\sin(3\ln x)}{x}dx$ *c)* $\int e^x\sqrt{1+e^x}dx$.

Solution

a) Let $u = x^2 + 1$, then $du = 2xdx$ so that $xdx = \frac{1}{2}du$. Substituting this in the given integral we have

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C.$$

b) Let $u = 3\ln x$, then $du = \frac{3}{x}dx$ so that $\frac{dx}{x} = \frac{1}{3}du$. Substituting this in the given integral we have

$$\int \frac{\sin(3\ln x)}{x}dx = \frac{1}{3}\int \sin u du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(3\ln x) + C.$$

c) Let $v = 1 + e^x$, then $dv = e^x dx$. Substituting this in the given integral we have

$$\int e^x \sqrt{1+e^x}dx = \int \sqrt{v}dv = \int v^{\frac{1}{2}}dv = \frac{2}{3}v^{\frac{3}{2}} + C = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + C.$$

Sometimes appropriate substitutions are not obvious. So there is need to manipulate the integrand algebraically to put it into a better form for substitution as we do in the next example.

Example 3.1.2 *Evaluate* $\int \frac{1}{x^2+4x+5}dx.$

Solution

We first complete the square of the denominator in the integrand

$$\begin{aligned} x^2 + 4x + 5 &= (x^2 + 4x + 2^2) - 2^2 + 5 \\ &= (x^2 + 4x + 4) - 4 + 5 \\ &= (x+2)^2 + 1 \end{aligned}$$

Now we can do the substitution; let $t = x + 2$, then $dt = dx$ and so

$$\begin{aligned} \int \frac{1}{x^2+4x+5}dx &= \int \frac{dx}{(x+2)^2+1} \\ &= \int \frac{dt}{t^2+1} \\ &= \tan^{-1} t + C \\ &= \tan^{-1}(x+2) + C. \end{aligned}$$

We now consider integrals of the form

$$\int \sin^m x \cos^n x dx.$$

If either m or n is odd, positive integer, the integrand can be done easily by substitution. If say $n = 2k + 1$ where k is an integer, then we can use the identity $\sin^2 x + \cos^2 x = 1$ to rewrite the integrand as follows

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x \cos^{2k} x \cos x dx \\ &= \int \sin^m x (\cos^2 x)^k \cos x dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x dx \end{aligned}$$

which can be integrated using the substitution $u = \sin x$. Similarly, $u = \cos x$ can be used if m is odd.

Example 3.1.3 *Evaluate a)* $\int \sin^3 x \cos^8 x dx$, *b)* $\int \cos^5(ax)dx$.

Solution

a) Setting $I = \int \sin^3 x \cos^8 x dx$, we have

$$I = \int \sin^2 x \cos^8 x \sin x dx = \int (1 - \cos^2 x) \cos^8 x \sin x dx.$$

Now, let $u = \cos x$, then $du = -\sin x dx$. Thus

$$\begin{aligned} I &= -\int (1 - u^2) u^8 du \\ &= \int (u^{10} - u^8) du \\ &= \frac{u^{11}}{11} - \frac{u^9}{9} + C \\ &= \frac{1}{11} \cos^{11} x - \frac{1}{9} \cos^9 x + C \end{aligned}$$

b) Setting $J = \int \cos^5(ax) dx$, we have

$$J = \int \cos^4(ax) \cos(ax) dx = \int (1 - \sin^2(ax))^2 \cos(ax) dx.$$

Let $u = \sin(ax)$, then $du = a \cos(ax) dx$. Thus

$$\begin{aligned} J &= \frac{1}{a} \int (1 - u^2)^2 du \\ &= \frac{1}{a} \int (1 - 2u^2 + u^4) du \\ &= \frac{1}{a} (u - \frac{2}{3} u^3 + \frac{1}{5} u^5) + C \\ &= \frac{1}{a} (\sin(ax) - \frac{2}{3} \sin^3(ax) + \frac{1}{5} \sin^5(ax)) + C. \end{aligned}$$

If both m and n are even, we can use the double angle formulas:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \text{ and } \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

Example 3.1.4 *Evaluate* $\int \sin^4 x dx$.

Solution

We apply the double angle formula twice.

$$\begin{aligned}
\int \sin^4 x dx &= \int \sin^2 x)^2 dx \\
&= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx \\
&= \frac{1}{4}\int (1-\cos 2x)^2 \\
&= \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x) dx \\
&= \frac{1}{4}\int 1 dx - \frac{2}{4}\int \cos 2x dx + \frac{1}{4}\int \cos^2 2x dx \text{ (apply the formula again)} \\
&= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{4}\int \left(\frac{1}{2}(1+\cos 4x)\right) dx \\
&= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\, dx \\
&= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.
\end{aligned}$$

Similarly, integrals of the form

$$\int \sec^m x \tan^n x dx \text{ and } \int \csc^m x \cot^n x dx$$

for both m and n even positive numbers can be respectively evaluated by using the identities

$$\sec^2 x = 1 + \tan^2 x \text{ and } \csc^2 x = 1 + \cot^2 x.$$

If either m or n is odd, the integrals cannot be handled by substitution.

3.2 Integration by parts

Suppose that $U(x)$ and $V(x)$ are two differentiable functions. According to the product rule,

$$\frac{d}{dx}(U(x)V(x)) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}.$$

Integrating both sides, we get

$$\int \frac{d}{dx}(U(x)V(x))dx = \int U(x)\frac{dV}{dx}dx + \int V(x)\frac{dU}{dx}dx$$

which implies that

$$\int U(x)\frac{dV}{dx}dx = U(x)V(x) - \int V(x)\frac{dU}{dx}dx$$

or simply

$$\int UdV = UV - \int VdU.$$

This is the formula for carrying out integration by parts. We now use the formula to solve the following examples:

Example 3.2.1 *Evaluate a)* $\int xe^x dx$, *b)* $\int \ln x$, *c)* $\int x^2 \sin x dx$ *and d)* $x \tan^{-1} x dx$

Solution

a) Let $U = x$ and $dV = e^x dx$, then $dU = dx$ and

$$V = \int dV = \int e^x dx = e^x + K.$$

Thus

$$\begin{aligned}\int UdV &= UV - \int VdU \\ &= x(e^x + K) - \int (e^x + K)dx \\ &= xe^x + Kx - e^x - Kx + C \\ &= xe^x - ex + C.\end{aligned}$$

Notice that the constant of integration K cancels out at the end. So we usually leave it out as we shall do from now on.

b) Let $U = \ln x$ and $dV = dx$, then $dU = \frac{1}{x}dx$ and $V = x$. By the integration by parts formula we have

$$\int \ln x = UV - \int VdU = x \ln x - \int x\frac{1}{x}dx = x \ln x - x + C.$$

c) We do integration by parts twice: Let $U = x^2$ and $dV = \sin x dx$, then $dU = 2xdx$ and $V = -\cos x$. Thus

$$\int x^2 \sin x dx = -x^2 \cos x + 2 \int x \cos x dx.$$

We again apply the formula on the second term on the right hand side. Let $U = x$ and $dV = \cos x$, then $dU = dx$ and $V = \sin x$. So

$$\begin{aligned}\int x^2 \sin x dx &= -x^2 \cos x + 2(x \sin x - \int \sin x dx) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C \\ &= (2 - x^2) \cos x + 2x \sin x + C.\end{aligned}$$

d) Let $U = tan^{-1}x$ and $dV = xdx$, then $dU = \frac{dx}{1+x^2}$ and $V = \frac{1}{2}x^2$. So

$$\begin{aligned}\int x \tan^{-1} dx &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2} \tan^{-1} x + C \\ &= \frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{1}{2}x + C.\end{aligned}$$

The following are useful rules for choosing U and dV:

i) If the integrand involves a polynomial multiplied by an exponential, a sine or a cosine, or some other readily integrable function, try U equals the polynomial and dV equals the rest.

ii) If the integrand involves a logarithm, an inverse trigonometric function or some other function which is not readily integrable but whose derivative is readily calculated, try that function for U and let dV equals the rest.

3.3 Reduction Formula

Consider the problem of finding $\int x^4 e^{-x} dx$. We can of course use integration by parts, but it is repetitive and tedious. We can use the following approach: For $n \geq 0$, let

$$I_n = \int x^n e^{-1} dx.$$

We want to find I_4. First we integrate I_n by parts to get the general formula: Let $U = x^n$ and $dV = e^{-x}$, then $dU = nx^{n-1}dx$ and $V = -e^{-x}$. Thus

$$I_n = -xe^{-x} + n\int x^{n-1}e^{-x}dx = -x^n e^{-1} + nI_{n-1}.$$

The formula

$$I_n = -x^n e^{-1} + nI_{n-1}$$

is called a reduction formula because it gives the value of the integral I_n in terms of I_{n-1}, an integral corresponding to a reduced value of the exponent n.

We now evaluate $\int x^4 e^{-x}dx$ by finding I_4 using the reduction formula. Starting with

$$I_0 = \int x^0 e^{-x}dx = \int e^{-x} = -e^{-x} + C$$

we apply the reduction formula four times to get

$I_1 = -xe^{-x} + I_0 = -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C_1$
$I_2 = -x^2e^{-x} + 2I_1 = -x^2e^{-x} + 2(e^{-x}(x+1) + C) = -e^{-x}(x^2+2x+2) + C_2$
$I_3 = -x^3e^{-x} + 3I_2 = -e^{-x}(x^3 + 3x^2 + 6x + 6) + C_3$
$I_4 = -x^4e^{-x} + 4I_3 = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + C_4$

Example 3.3.1 *Obtain and use a reduction formula to evaluate*

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx \quad (n = 0, 1, 2, 3, \dots)$$

Solution

Observe first that $I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$ and $I_1 = \int_0^{\frac{\pi}{2}} \cos x dx = \sin x \mid_0^{\frac{\pi}{2}} = 1$. Now, let $n \geq 2$:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x dx$$

We apply integration by parts: Let $U = \cos^{n-1} x$ and $dV = \cos x dx$, then $dU = -(n-1)\cos^{n-2} x \sin x dx$ and $V = \sin x$. Thus

$$\begin{aligned} I_n &= \cos^{n-1} x \sin x|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx \\ &= 0 - 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x(1 - \cos^2 x)dx \\ &= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x dx \\ &= (n-1)I_{n-2} - (n-1)I_n. \end{aligned}$$

Solving for I_n, we obtain $nI_n = (n-1)I_{n-2}$ or

$$I_n = \frac{n-1}{n} I_{n-2}$$

which is the required reduction formula. This formula is valid for $n \geq 2$, which was need to ensure that $\cos^{n-1}(\frac{\pi}{2}) = 0$.

If $n \geq 2$ is an even integer, we have

$$\begin{aligned} I_n &= \frac{n-1}{n} I_{n-2} = \frac{n-1}{n} \cdot \frac{n-3}{n-2} I_{n-4} \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \end{aligned}$$

For instance,

$$I_4 = \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}.$$

If $n \geq 3$ is an odd integer, we have

$$\begin{aligned} I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \end{aligned}$$

For example

$$I_7 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{48}{105}.$$

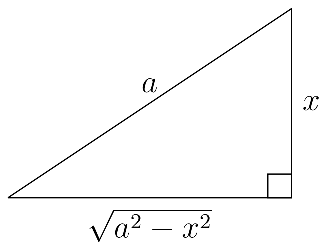
3.4 Inverse trigonometric substitution

The substitution we have seen so far is where we replace an expression in the integrand with a single variable. In this section, we consider the reverse approach. We replace a variable of integration with a function of a new variable. Such substitutions, called inverse substitutions may appear on the surface to make the integral more complicated. However, as we will see, such substitutions can actually simplify and transform the integral into one that can be evaluated by inspection or to which other techniques can readily be applied.

We now outline some very useful inverse substitutions:

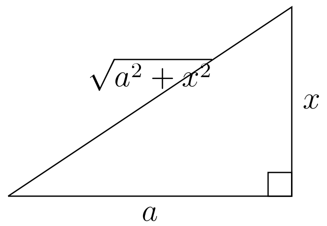
1. Integrals involving $\sqrt{a^2 - x^2}$ (where $a > 0$) can frequently be reduced to a simpler form by means of the substitution

$$x = a \sin\theta \text{ or equivalently } \theta = \sin^{-1}\frac{x}{a}$$



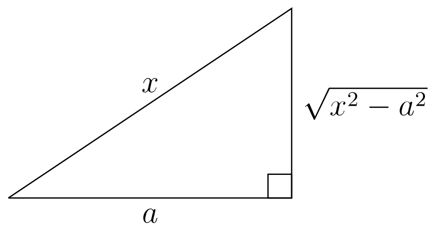
2. For integrals involving $\sqrt{a^2 + x^2}$ or $\frac{1}{x^2+a^2}$ (where $a > 0$) substitute

$$x = a \tan \theta \text{ or equivalently } \theta = \tan^{-1} \frac{x}{a}$$



3. For integrals involving $\sqrt{x^2 - a^2}$, $a > 0$, use the substitution

$$x = a \sec \theta \text{ or equivalently } \theta = \sec^{-1} \frac{x}{a}.$$



These substitutions are valid for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

4. For integrals involving $\sqrt{ax + b}$ use the substitution $ax + b = u^2$ and integrals involving $\sqrt[n]{ax + b}$ use the substitution $ax + b = u^n$.

Example 3.4.1 *Evaluate* $\int \frac{1}{(5-x^2)^{\frac{3}{2}}} dx$.

Solution

Observe that the integral is in the form described in 1. with $a = \sqrt{5}$. Let $x = \sqrt{5}\sin\theta$. Then $dx = \sqrt{5}\cos\theta d\theta$ and

$$\begin{aligned}(5-x^2)^{\frac{3}{2}} &= (5-(\sqrt{5}\sin\theta)^2)^{\frac{3}{2}} \\ &= (5-5\sin^2\theta)^{\frac{3}{2}} \\ &= (5(1-\sin^2\theta))^{\frac{3}{2}} \\ &= (5\cos^2\theta)^{\frac{3}{2}} \\ &= 5^{\frac{3}{2}}\cos^3\theta.\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{1}{(5-x^2)^{\frac{3}{2}}}dx &= \int \frac{\sqrt{5}\cos\theta d\theta}{5^{\frac{3}{2}}\cos^3\theta} \\ &= \frac{1}{5}\int \sec^2\theta d\theta \\ &= \frac{1}{5}\tan\theta + C \\ &= \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C. \text{ (Refer to the triangle in 1.)}\end{aligned}$$

Example 3.4.2 *Evaluate* $\int \frac{1}{\sqrt{4+x^2}}dx$.

Solution

The integral is in the form described in 2. with $a = 2$. Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta d\theta$ and

$$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = \sqrt{4(1+\tan^2\theta)} = \sqrt{4\sec^2\theta} = 2\sec\theta.$$

Thus

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}dx &= \int \frac{2\sec^2\theta d\theta}{2\sec\theta} \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C \text{ (Refer to the triangle in 2.)} \\ &= \ln|\sqrt{4+x^2} + x| - \ln 2 + C \\ &= \ln|\sqrt{4+x^2} + x| + C_1\end{aligned}$$

where $C_1 = C - \ln 2$.

Example 3.4.3 *Evaluate* $I = \int \frac{dx}{\sqrt{x^2-a^2}}$, *where* $a > 0$.

Solution

For the moment, assume that $x \geq a$. If $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta d\theta$ and $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = a\tan\theta$. Thus

$$\begin{aligned}
I &= \int \frac{a\sec\theta\tan\theta d\theta}{a\tan\theta} \\
&= \int \sec\theta d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C \text{ (Refer to the triangle in 3.)} \\
&= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C \\
&= \ln|x+\sqrt{x^2-a^2}| - \ln a + C \\
&= \ln|x+\sqrt{x^2-a^2}| + C_1
\end{aligned}$$

where $C_1 = C - \ln a$.

If $x \leq a$, let $u = -x$ so that $u \geq a$ and $du = -dx$. Thus we have

$$\begin{aligned}
I &= -\int \frac{du}{\sqrt{u^2-a^2}} \\
&= -\ln|u+\sqrt{u^2-a^2}| + C_1. \text{ (From the } x \geq a \text{ case above)} \\
&= -\ln|-x+\sqrt{(-x)^2-a^2}| + C_1 \\
&= \ln\left|\frac{1}{-x+\sqrt{x^2-a^2}}\right| + C_1 \\
&= \ln\left|\frac{1}{-x+\sqrt{x^2-a^2}} \cdot \frac{x+\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}\right| + C_1 \\
&= \ln\left|\frac{x+\sqrt{x^2-a^2}}{-a^2}\right| + C_1 \\
&= \ln|x+\sqrt{x^2-a^2}| - \ln|-a^2| + C_1 \\
&= \ln|x+\sqrt{x^2-a^2}| + C_2
\end{aligned}$$

where $C_2 = C_1 - 2\ln a$.

Thus in either case, we have seen that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}| + C.$$

Example 3.4.4 *Evaluate* $\int \frac{1}{1+\sqrt{2x}} dx$.

Solution

We use the substitution in 4. Let $2x = u^2$, then $2dx = 2udu$ or $dx = udu$. Thus

$$\begin{aligned}
\int \frac{1}{1+\sqrt{2x}} dx &= \int \frac{u}{1+u} du \\
&= \int \frac{1+u-1}{1+u} \\
&= \int \left(1 - \frac{1}{1+u}\right) du \text{ (Let } v = 1+u,\ dv = du) \\
&= u - \int \frac{dv}{v} \\
&= u - \ln|v| + C \\
&= u - \ln|1+u| + C \\
&= \sqrt{2x} - \ln(1+\sqrt{2x}) + C.
\end{aligned}$$