## 4.4 Power, Taylor and Maclaurin Series

**Definition 4.4.1** *A series of the form*

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

*is called a* ***power series in powers of*** $x = c$ *or a* ***power series about*** $c$. *The constants* $a_0, a_1, a_2, \dots$ *are called the* ***coefficients*** *of the power series.*

The point $c$ is the center of **convergence**

**Theorem 4.4.2** *For any power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *one of the following alternatives must hold:*

*i) the series may converge only at* $x = c$,

*ii) the series may converges at every real number* $x$, *or*

*iii) there may exist a positive real number* $R$ *such that the series converges at every* $x$ *satisfying* $|x-c| > R$. *In this case the series may or may not converge at either of the two end points* $x = c - R$ *and* $x = c + R$.

*In each of these cases the convergence is* ***absolute*** *(i.e* $\sum_{n=0}^{\infty} |a_n(x-c)^n|$ *converges) except possibly at the end point* $x = c - R$ *and* $x = c + R$ *in case iii).*

**Definition 4.4.3** *Suppose that* $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ *exists or is* $\infty$. *Then the power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *has* ***radius of convergence*** $R = 1/L$. *If* $L = 0$ *then* $R = \infty$ *and if* $L = \infty$ *then* $R = 0$

**Theorem 4.4.4** *Let* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *be two series with radii of convergence* $R_a$ *and* $R_b$ *respectively and let* $c$ *be a constant. Then*

*i)* $\sum_{n=0}^{\infty} (ca_n)x^n$ *has radius of convergence* $R_a$, *and*

$$\sum_{n=0}^{\infty} (ca_n)x^n = c\sum_{n=0}^{\infty} a_n x^n$$

*wherever the series on the right converges.*

*ii)* $\sum_{n=0}^{\infty} (a_n + b_n)x^n$ *has radius of convergence* $R \geq \min\{R_a, R_b\}$ *and*

$$\sum_{n=0}^{\infty} (a_n + b_n)x^n = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

*wherever both series on the right converges.*

**Theorem 4.4.5** *If the series* $\sum_{n=0}^{\infty} a_n x^n$ *converges to the sum* $f(x)$ *on an interval* $(-R, R)$*, where* $R > 0$*, that is,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad (-R < x < R)$$

*then* $f$ *is differentiable on* $(-R, R)$*, and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots, \quad (-R < x < R).$$

*Also,* $f$ *is integrable over any closed sub-interval of* $(-R, R)$*, and if* $|x| < R$*, then*

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots.$$

**Theorem 4.4.6** *Suppose the series*

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

*converges to* $f(x)$ *for* $c - R < x < c + R$ *where* $R > 0$*. Then*

$$a_k = \frac{f^{(k)}(c)}{k!}, \quad k=0,1,2,3,\dots.$$

**Definition 4.4.7** *If* $f(x)$ *has derivatives of all orders at* $x = c$*, then the series*

$$\sum_{n=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}}{3!}(x-c)^3 + \dots$$

*is called the* ***Taylor series of*** $f$ ***about*** $c$*. If* $c = 0$*, the series is called the* ***Maclaurin series****.*

**Definition 4.4.8** *A function* $f$ *is* ***analytic*** *at* $c$ *if* $f$ *has a Taylor series at* $c$ *and that series converges to* $f(x)$ *in an open interval containing* $c$*. If* $f$ *is analytic at each point of the open interval, then we say it is analytic on that interval.*

**Example 4.4.9** *Find the Taylor series for* $e^x$ *about* $x = c$*. Where does the series converge to* $e^x$*? Where is* $e^x$ *analytic? What is the Maclaurin series for* $e^x$*.*

**Solution.** Since all the derivatives of $f(x) = e^x$ are $e^x$, we have that $f^{(n)}(c) = e^c$ for every $n \geq 0$. Thus the Taylor series for $e^x$ about $x = c$ is

$$\sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \dots$$

The radius of convergence $R$ is given by

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{e^c/(n+1)!}{e^c/n!} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Thus $R = \infty$ and the series converges for all $x$.

Suppose the sum is $g(x)$:

$$g(x) = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \dots$$

then

$$\begin{aligned} g'(x) &= 0 + e^c + \frac{e^c}{2!}2(x-c) + \frac{e^c}{3!}3(x-c)^2 + \dots \\ &= e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \dots \\ &= g(x) \end{aligned}$$

Also, $g(c) = e^c + 0 + 0 + \dots = e^c$. Since $g'(x) = g(x)$, we have $g(x) = Ce^x$. Substituting $x = c$ we get $e^c = Ce^c$ so that $C = 1$. Thus the Taylor series for $e^x$ in powers of $(x-c)$ converges to $e^x$ for every real number $x$; i.e,

$$e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \dots$$

for all $x$. In particular, $e^x$ is analytic on the whole real line $\mathbb{R}$.

Setting $c = 0$, we obtain the Maclaurin series for $e^x$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

for all $x$. ■

Here are some Maclaurin series:

i) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$ (for all $x$)

ii) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots,$ (for all $x$)

iii) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots, \quad (-1 < x < 1)$

iv) $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad -1 < x \leq 1$

v) $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, \quad -1 \leq x \leq 1$

Some Maclaurin series obtained from known series:

i) $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{n!} x^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots,$ for all $x$

ii) $\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$ for all $x$

iii) $\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$ for all $x$

iv) $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{(n-1)} = 1 + 2x + 3x^2 + 4x^3 + \ldots, \quad (-1 < x < 1)$