Indeterminate forms and L'Hopital's Rule

If f and g are continuous at $x=a$ but $f(a)=g(a)=0$, then the limit

$$\lim_{x\to a}\frac{f(a)}{g(a)}$$

cannot be evaluated by substituting $x=a$ since this produces $\frac{0}{0}$, a meaningless expression known as an indeterminate form. L'Hopital's rule gives an explicit connection between derivatives and limits that leads to indeterminate forms.

Theorem 2.3.19 (First form of L'Hopital's Rule) *Suppose that* $f(a)=g(a)=0, f'(a)$ *and* $g'(a)$ *exist, and that* $g'(a)\neq 0$. *Then*

$$\lim_{x\to a}\frac{f(a)}{g(a)}=\frac{f'(a)}{g'(a)}|_{x=a}$$

Theorem 2.3.20 (Stronger form of L'Hopital's Rule) *Suppose that* $f(x_0)=g(x_0)=0$ *and that the functions* f *and* g *are both differentiable on an open interval* (a,b) *that contains the point* x_0, *suppose also that* $g'\neq 0$ *at every point in* (a,b) *except possibly at* x_0. *Then*

$$\lim_{x\to a}\frac{f(a)}{g(a)}=\lim_{x\to a}\frac{f'(a)}{g'(a)}$$

provided the limit on the right exists.

Example 2.3.21 *Using L'Hopital's Rule we have*

a) $\lim\limits_{x\to 0}\frac{3x-\sin x}{x}=\frac{3-\cos x}{1}|_{x=0}=2$

b) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{\frac{1}{2\sqrt{x+1}}}{1}|_{x=0}=\frac{1}{2}$

c) $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}=\frac{1-\cos x}{3x^2}$, *but this is still* $\frac{0}{0}$ *form. So we apply the stronger form of L'Hopital's Rule:*

$$\begin{aligned}\lim_{x\to 0}\frac{x-\sin x}{x^3}&=\lim_{x\to 0}\frac{1-\cos x}{3x^2}\ (\textit{still } \frac{0}{0}, \textit{ apply the rule again})\\&=\lim_{x\to 0}\frac{\sin x}{6x}\ (\textit{still } \frac{0}{0}, \textit{ apply the rule again})\\&=\lim_{x\to 0}\frac{\cos x}{6}\\&=\frac{1}{6}\end{aligned}$$

Another indeterminate form is $\frac{\infty}{\infty}$.

Example 2.3.22 *Using the stronger form of L'Hopital's Rule, we evaluate the following limits that lead to* $\frac{\infty}{\infty}$ *indeterminate forms:*

a) $\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{1+\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x} = 1$

b) $\lim_{x\to\infty} \frac{x-2x^2}{3x^2+5x} = \lim_{x\to\infty} \frac{1-4x}{6x+8}$*(still* $\frac{\infty}{\infty}$*, apply the rule again)* $= \lim_{x\to\infty} \frac{-4}{6} = \frac{-2}{3}$

If you get indeterminate forms $0\cdot\infty$ or $\infty\cdot 0$ or $\infty\pm\infty$, try to change them into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 2.3.23 *a)* $\lim_{x\to\infty} x\cdot\sin(\frac{1}{x})$ *leads to the form* $\infty\cdot 0$. *We can change to the form* $\frac{0}{0}$ *by writing* $x = \frac{1}{t}$ *and letting* $t \to 0^+$*:*

$$\begin{aligned}\lim_{x\to\infty} x\cdot\sin(\frac{1}{x}) &= \lim_{x\to 0^+} \frac{1}{t}\cdot\sin(t)\\ &= \lim_{x\to 0^+} \frac{\sin t}{t} \quad \left(\frac{0}{0}\, \textit{form}\right)\\ &= \frac{\cos 0}{1}\\ &= 1.\end{aligned}$$

b) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ *leads the* $\infty-\infty$ *form if* $x\to 0^+$*, and leads to* $-\infty+\infty$ *if* $x\to 0^-$ *which are both indeterminate. But we may also write*

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x-\sin x}{x\sin x}$$

which now leads to a $\frac{0}{0}$ *form and we can apply L'Hopital's Rule. Thus,*

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x\to 0} \frac{x-\sin x}{x\sin x} \quad \left(\frac{0}{0}\, \textit{form}\right)\\ &= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \quad \left(\frac{0}{0}\, \textit{form}\right)\\ &= \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x}\\ &= \frac{0}{2}\\ &= 0.\end{aligned}$$

Rates of Change

When two or more quantities that change with time are linked by an equation, that equation can be differentiated with respect to time to produce an equation linking the rates of change of the quantities. Any one of these rates may then be determined when the others, and the values of the quantities themselves, are known

Example 2.3.24 *An aircraft is flying horizontally at a speed of* 600 *km/h. How fast is the distance between the aircraft and the radio beacon increasing* 1 *min after the aircraft passes* 5 *km directly above the beacon?*

Solution

Let C be the point on the aircraft's path directly above the beacon B. Let A be the position of the aircraft t minutes after it is at C, and let x and s be the distances CA and BA respectively.

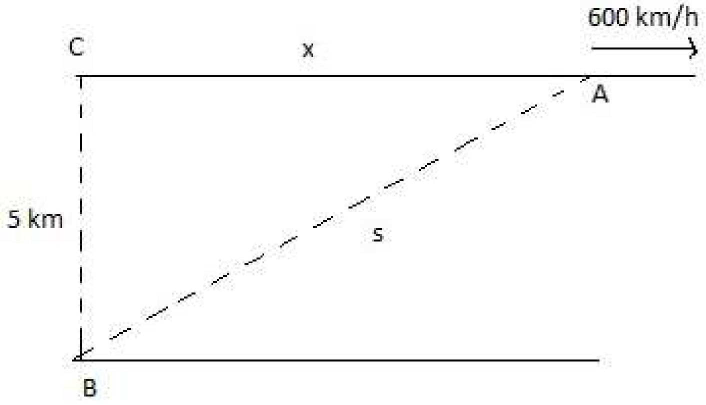

Figure 2.1:

From the right triangle BCA we have

$$s^2 = x^2 + 5^2.$$

Differentiating this equation implicitly with respect to time we obtain

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}.$$

We are given that

$$\frac{dx}{dt} = 600km/h = 10km/min;$$

hence, $x = 10km$ at time $t = 1$. At that time, $s = \sqrt{10^2 + 5^2} = 5\sqrt{5}$ and is increasing at the rate

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt} = \frac{10}{5\sqrt{5}}(600) = \frac{1200}{\sqrt{5}} \approx 536.7km/h.$$

Example 2.3.25 *How fast is the area of a rectangle changing if one side is* $10cm$ *long and is increasing at a rate of* $2cm/s$ *and the other side is* 8 *cm long and is decreasing at a rate of* $3cm/s$*.*

Solution

Let x cm and y cm be the lengths of the sides of the rectangle at time t, respectively. Then the area at time t is $xycm^2$. We want to know the value of $\frac{dA}{dt}$ when $x = 10$ and $y = 8$ given that $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = -3$ (the negative sign indicates that y is decreasing).

Differentiating $A = xy$ implicitly, we get

$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt} = 2(8) + 10(-3) = -14.$$

So at the time in question, the area of the rectangle is decreasing at a rate of $14cm^2/s$.

Example 2.3.26 *A lighthouse L is located on a small island* 2 *km from the nearest point A on a long, straight shoreline. If the lighthouse lamp rotates at* 3 *revolutions per minute, how fast is the illuminated spot P on the shoreline moving along the shoreline when it is* 4 *km from A?*

Solution

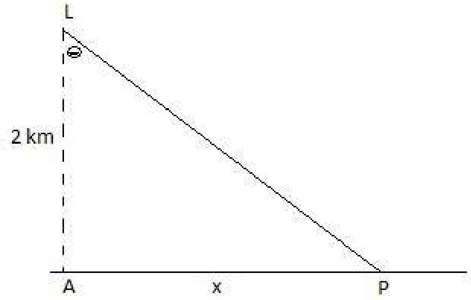

Figure 2.2:

Let x be the distance AP and let θ be the angle PLA. Then $x = 2\tan\theta$ and

$$\frac{dx}{dt} = 2\sec^2\theta\frac{d\theta}{dt}.$$

Now

$$\frac{d\theta}{dt} = 3rev/min \times 2\pi\ rad/rev = 6\pi\ rad/min.$$

Note that it is necessary to change revolutions per minute to radians per minute. When $x = 4$, we have $\tan\theta = 2$ and $\sec^2\theta = 1 + \tan^2\theta = 5$. Thus

$$\frac{dx}{dt} = 2 \times 5 \times 6\pi = 60\pi \approx 188.5.$$

The spot of light is moving along the shoreline at a rate of about 189 km/min when it is 4 km from A.

Example 2.3.27 *A leaky water tank is in the shape of an inverted right circular cone with depth* 5 *m and top radius* 2 *m. When the water in the tank is* 4 *m deep, it is leaking out at a rate of* $\frac{1}{12}$ m^2/min*. How fast is the water level in the tank dropping at that time?*

Solution

Let r and h denote the surface radius and depth of water in the tank at time t, respectively.

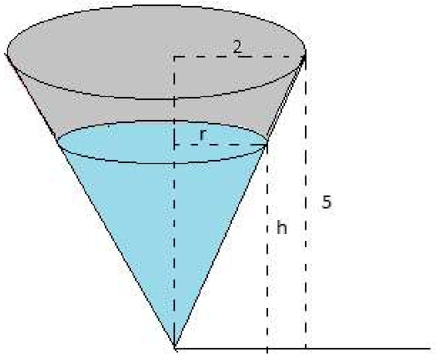

Figure 2.3:

Thus the volume V of water in the tank at time t is

$$V = \frac{1}{3}\pi r^2 h.$$

We find a relationship between r and h using similar triangles: $\frac{r}{h} = \frac{2}{5}$, so that $r = \frac{2h}{5}$ and

$$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h = \frac{4\pi}{75}h^3.$$

Implicitly differentiating this equation with respect to t we get

$$\frac{dV}{dt} = \frac{4\pi}{25}h^2\frac{dh}{dt}.$$

Since $dV/dt = -1/12$ when $h = 4$, we have

$$\frac{-1}{12} = \frac{4\pi}{25}(4)^2\frac{dh}{dt}, \quad \text{so that } \frac{dh}{dt} = \frac{-25}{768\pi}.$$

When the water in the tank is 4 m deep, its level is dropping at a rate of $\frac{25}{768\pi}$m/min or about 1.036cm/min.

Example 2.3.28 *At a certain instant an aircraft flying due east at* 400 *km/h passes directly over a car travelling due Southeast at* 100 *km/h on straight level road. If the aircraft is flying at an altitude of* 1 *km, how fast is the distance between the aircraft and the car increasing* 36 *seconds after the aircraft passes directly over the car?*

Solution

Let t be measured in hours from the time the aircraft was at position A directly above the car at position C. Let X and Y be the positions of the aircraft and car respectively at time t. Let x be the distance AX, y the distance CY, and s the distance XY, all measured in kilometres. Let Z be the point 1 km above Y. (See Figure 2.4)

Since $XAZ = 45°$, the Pythagoras theorem and the cosine law yield

$$s^2 = 1 + (ZX)^2 = 1 + x^2 + y^2 - 2xy\cos 45° = 1 + x^2 + y^2 - \sqrt{2}xy.$$

Thus

$$\begin{aligned} 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \sqrt{2}\frac{dx}{dt}y - \sqrt{2}x\frac{dy}{dt} \\ &= 400(2x - \sqrt{2}y) + 100(2y - \sqrt{2}x) \end{aligned}$$

When $t = \frac{1}{100}$ (i.e, 36 s after $t = 0$), we have $x = 4$ and $y = 1$. Hence,

$$s^2 = 1 + 16 + 1 - 4\sqrt{2} = 18 - 4\sqrt{2}$$

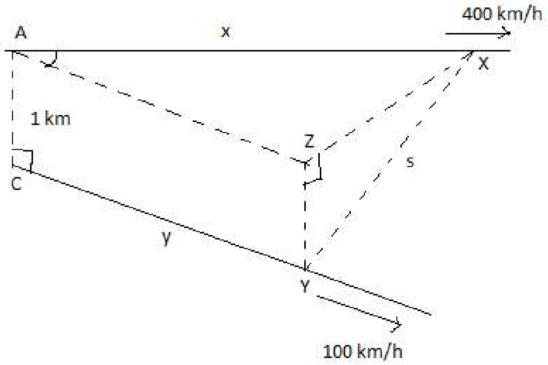

Figure 2.4:

so that $s \approx 3.5133$ and

$$\frac{ds}{dt} = \frac{1}{2s}(400(8-\sqrt{2}) + 100(2-4\sqrt{2})) \approx 322.86.$$

The aircraft and the car are separating at a rate of about 323km/h after 36s. Note that in our calculations, we converted 36 seconds to hours.

Optimization Problems

Here, we solve various word problems that, when translated into mathematical terms, require the finding of a maximum or minimum value of a function of one variable.

In solving this kind of problems, it is important to read the problem and make an appropriate diagram.

Example 2.3.29 *A rectangular animal enclosure is to be constructed having one side along an existing long wall and the other three sides fenced. If 100M of fence are available, what is the largest possible area for the enclosure.*

Solution

Let x m be the length and y m be the width of the enclosure and let A m^2 be its area. Thus $A = xy$.

Since the total length of the fence is 100 m, we must have

$$x + 2y = 100$$

or

$$x = 100 - 2y.$$

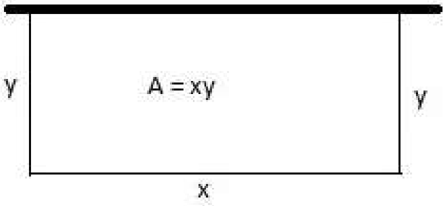

Figure 2.5:

So the area A is given by

$$A = xy = (100 - 2y)y = 100y - 2y^2.$$

Since $x \geq 0$, we have that $y \leq 50$, otherwise, x would be negative which wouldn't make sense. Thus we must maximize the function $A(y) = 100y - 2y^2$ on the interval $[0, 50]$. Observe that $A(0) = A(50) = 0$ and $A(y) > 0$ for $0 < y < 50$. Hence, $A(y)$ has maximum. This maximum must occur at a critical point. To find critical points, we solve

$$0 = A'(y) = 100 - 4y \Rightarrow y = 25.$$

Thus the maximum value occurs at $y = 25$. Thus the greatest possible area for the enclosure is therefore

$$A(25) = 100(25) - 2(25)^2 = 1250 \text{ m}^2.$$

Example 2.3.30 *A lighthouse L is located on a small island* 5 *km north of a point A on a straight east-west shoreline. A cable is to be laid from L to point B on the shoreline* 10 *km east of A. The cable will be laid through the water in a straight line from L to a point C on the shoreline between A and B, and from there to B along the shoreline.*

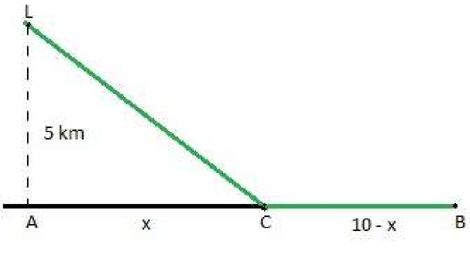

Figure 2.6:

The part of the cable lying in the water costs $K50,000/km$, and the part along the shoreline costs $K30,000/km$.

a) Where should C be chosen to minimize the total cost of the cable.

b) Where should C be chosen if B is only 3 km from A.

Solution

a) Let C be x km from A toward B. Thus $0 \leq x \leq 10$. The length of LC is $\sqrt{25+x^2}$ km and the length of CB is $(10-x)$ km (see Figure 2.6). Hence, the total cost of the cable in kwacha is T, where

$$T = T(x) = 50,000\sqrt{25+x^2} + 30,000(10-x).$$

Note that T is continuous on $[0,10]$, so it has a minimum value that can occur at one of the endpoints or a critical point in the interval $(0,10)$. For critical points we have

$$\begin{aligned}
0 = \frac{dT}{dx} &= \frac{50,000x}{\sqrt{25+x^2}} - 30,000 \\
&\Rightarrow 50,000x = 30,000\sqrt{25+x^2} \\
&\Rightarrow 5x = 3\sqrt{25+x^2} \quad \text{(squaring both sides)} \\
&\Rightarrow 25x^2 = 9(25+x^2) \\
&\Rightarrow 25x^2 = 225 + 9x^2 \\
&\Rightarrow 16x^2 = 225 \\
&\Rightarrow x^2 = \frac{225}{16} = \frac{15^2}{4^2}.
\end{aligned}$$

So the critical points are $x = \pm\frac{15}{4}$. Observe that only $x = \frac{15}{4}$ lies in the interval $(0,10)$. Since

$$T(0) = 50,000(5) - 30,000(10) = 550,000,$$

$$T\left(\frac{15}{4}\right) = 50,000\left(\sqrt{25+\frac{225}{16}}\right) - 30,000\left(100 - \frac{15}{4}\right) = 500,000$$

and

$$T(10) = 50,000(\sqrt{25+100}) + 0 = 559,016$$

the critical point determines the minimum value of $T(x)$. So for minimum cost, C should be $\frac{15}{4} = 3.75$ km from A.

b) If B is 3 km from A, we have

$$T(x) = 50,000\sqrt{25+x^2} + 30,000(3-x) \ (0 \leq x \leq 3)$$

which differs from T(x) in a) by an added constant. Hence, it has the same critical points $x = \pm\frac{15}{4}$ (verify this) neither of which lies in the interval $(0, 3)$. Now, $T(0) = 340,000$ and $T(3) = 291,548$. So to minimize the cost, we chose $x = 3$; that is, the cable should go straight from L to B.

Example 2.3.31 *Find the length of the shortest ladder that can extend from a vertical wall, over a fence* 2 *m high located* 1 *m away from the wall, to a point on the ground outside the fence.*

Solution

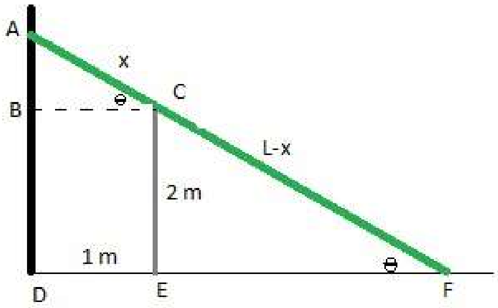

Figure 2.7:

Let θ be the angle of inclination of the ladder x be the length of AC as shown in Figure 2.7 and L be the length of the ladder AF. Using the two right-angled triangles ABC and CEF, we obtain the length L of the ladder as a function of θ:

$$x = \frac{1}{\sin\theta}, \quad L - x = \frac{2}{\sin\theta}$$

and solving for L we get

$$L = L(\theta) = \frac{1}{\cos\theta} + \frac{2}{\sin\theta}, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Since $\lim_{\theta\to(\pi/2)^-} L(\theta) = \infty$ and $\lim_{\theta\to(0)^+} L(\theta) = \infty$, $L(\theta)$ must have a minimum value on the interval $(0, \frac{\pi}{2})$ occurring at a critical point. To find critical points, we solve

$$0 = L'(\theta) = \frac{\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta} = \frac{\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta}.$$

So any critical point satisfies $\sin^3\theta = 2\cos^3\theta$ or equivalently, $\tan^3\theta = 2$. We don't need to solve this equation for θ since it is really the corresponding value of $L(\theta)$ that we want. Observe that

$$\frac{1}{\cos^2\theta} = \sec^2\theta = 1 + \tan^2\theta = 1 + 2^{\frac{2}{3}}.$$

It follows that $\cos\theta = \frac{1}{(1+2^{\frac{2}{3}})^{\frac{1}{2}}}$ and $\sin\theta = \tan\theta\cos\theta = \frac{2^{\frac{1}{3}}}{(1+2^{\frac{2}{3}})^{\frac{1}{2}}}$. Therefore, the minimal value for $L(\theta)$ is

$$\frac{1}{\cos\theta} + \frac{2}{\sin\theta} = (1+2^{\frac{2}{3}})^{\frac{1}{2}} + 2\frac{(1+2^{\frac{2}{3}})^{\frac{1}{2}}}{2^{\frac{1}{3}}} = (1+2^{\frac{2}{3}})^{\frac{3}{2}} \approx 4.16$$

Therefore, the length of the shortest ladder is 4.16 m long.