1. **ANALYTIC GEOMETRY**

1.1 **CONIC SECTIONS**

The intersection of a plane and a double-napped cone produces four (4) different types of conic sections (Conics). These are

1. Circle (plane perpendicular to cone axis)
2. Parabola (plane parallel to side of cone)
3. Ellipse (plane inclined at an angle)
4. Hyperbola (plane parallel to cone axis)

When the plane passes through the vertex, the resulting figure is a degenerate conic (see handout). Algebraically, conic sections are defined by the quadratic equation

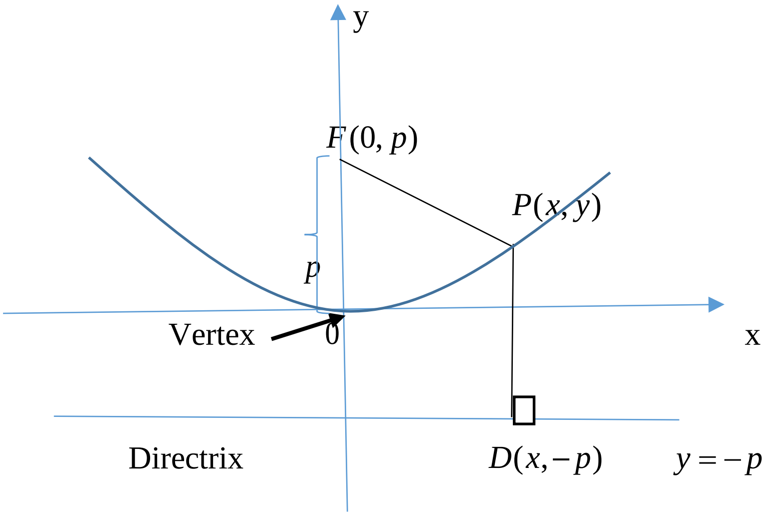
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \qquad \textbf{(1.1)}$$

In this chapter, we will consider the parabola, ellipse and hyperbola and be able to identify these from any given quadratic equation. Furthermore, the polar equations of these conics will be given. The circle will not be considered as it has been discussed extensively in the first year course.

1.1.1 **PARABOLA**

A parabola is the set of all points (x, y) that are equidistant from a fixed line called the **directrix** and a fixed point called the **focus** not on the line. The midpoint between the focus and the directrix is the **vertex** and the line passing through the focus and the vertex is the **axis**. We call the distance p along the axis from the focus to the vertex the **directed distance**.

Suppose that the vertex of a parabola is at the origin and y-axis is its axis



Then by definition

$$|FP| = |PD| \Rightarrow \sqrt{x^2 + (p-y)^2} = \sqrt{0^2 + (y+p)^2}$$
$$\Rightarrow x^2 + p^2 - 2py + y^2 = y^2 + 2py + p^2$$
$$\Rightarrow x^2 = 4py \qquad \textbf{(1.2)}$$

Similarly, the equation of a parabola with vertex at the origin, focus at $(0, -p)$ and directrix p units above the vertex is given by

$$\Rightarrow x^2 = -4py. \qquad \textbf{(1.3)}$$

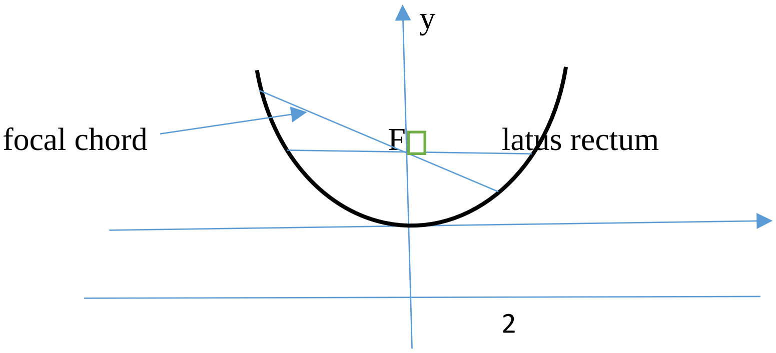
If the focus is at $(p, 0)$ and directrix is p units to the left of the vertex, then this parabola is given by

$$\Rightarrow y^2 = 4px \qquad \textbf{(1.4)}$$

And in a similar way the equation of a parabola with focus at $(-p, 0)$ and directrix p units to the right of the vertex is given by

$$\Rightarrow y^2 = -4px \qquad \textbf{(1.5)}$$

A line segment that passes through the focus of a parabola and has end points on the parabola is called <u>focal chord</u>. The specific focal chord perpendicular to the axis is called <u>latus rectum</u>.



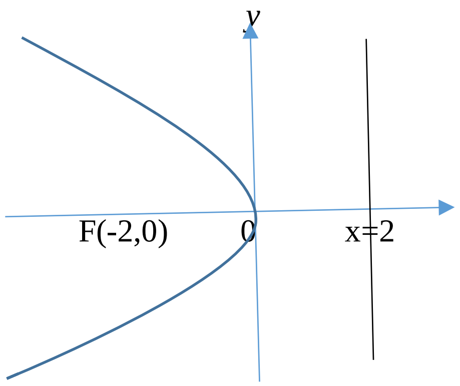
Example 1.1.1

1. Find the axis, vertex, focus and directrix of the parabolas

   (a) $y^2 = -8x$ (b) $x^2 = -2y$

2. Find the length of the latus rectum of the parabolas in question 1.

Solution



1. (a) $y^2 = -8x = -4(2)x$

   $\Rightarrow p = 2$

   $\therefore$ axis: x-axis or $y = 0$

   vertex: $(0, 0)$

   focus: $(-2, 0)$

   Directrix: $x = 2$

   (b) $x^2 = -2y \Rightarrow x^2 = -4\left(\frac{1}{2}\right)y$

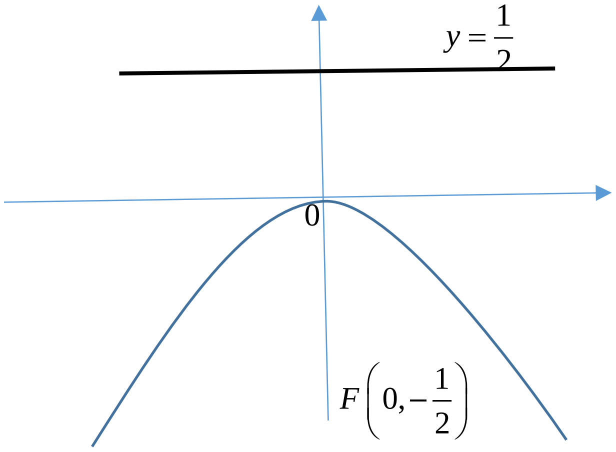
   $\Rightarrow p = \frac{1}{2}$

   $\therefore$ axis: y-axis or $x = 0$

   vertex: $(0, 0)$

   focus: $\left(0, -\frac{1}{2}\right)$

   Directrix: $x = \frac{1}{2}$



2. (a) When $x = -2$

   $y^2 = -8(-2)$

   $\Rightarrow y = \pm 4.$

   $\therefore$ Length of latus rectum is $8.$

   (b) When $y = -\frac{1}{2}$

   $x^2 = -2\left(-\frac{1}{2}\right)$

   $\Rightarrow x = \pm 1.$

   $\therefore$ Length of latus rectum is $2.$

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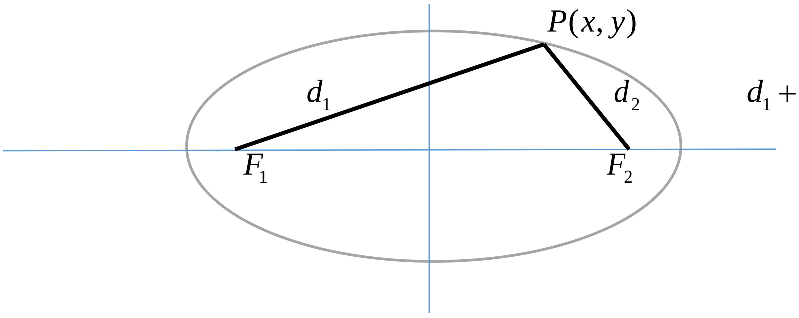
1.1.2 **ELLIPSE**

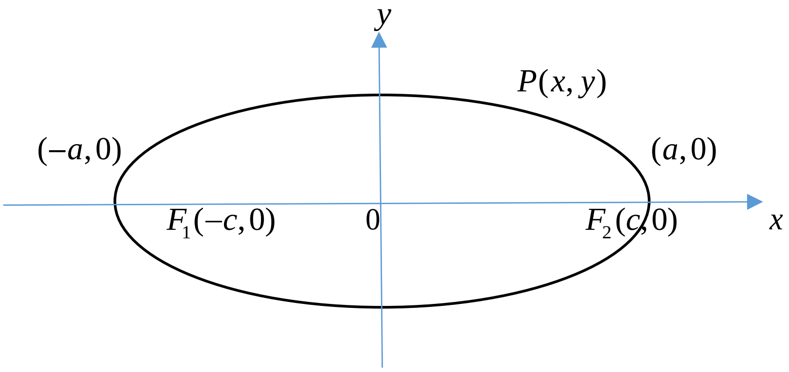
An ellipse is the set of all points (x, y) in the plane whose distances from two distinct fixed points called **foci** have a **constant sum.**



$d_1 + d_2$ is constant

- The line through the foci is the ellipse's **focal axis.**
- The focal axis intersects the ellipse at two points called the **vertices.**
- The chord joining the vertices is the **major axis**, and its midpoint is the **centre** of the ellipse.
- The chord perpendicular to the major axis at the centre is the **minor axis**.

Suppose that the ellipse is centred at the origin and x- and y-axis is the major and minor axis respectively. Also, suppose that the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ with vertices at $(-a, 0)$ and $(a, 0)$ meaning length of major axis is $2a$.



By definition,

$$|F_1P| + |PF_2| = 2a$$

$$\sqrt{(-c-x)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$
$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$
$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$
$$a\sqrt{(x-c)^2 + y^2} = a^2 - cx$$
$$a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2$$
$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1.$$

Since $a>c,\ a^2-c^2>0.$ Letting $b^2=a^2-c^2,$ we have that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \textbf{(1.6)}$$

where $2b$ is the length of minor axis.

Similarly, if the major axis is along the y-axis, then the equation of the ellipse is given by

$$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1. \qquad \textbf{(1.7)}$$

Example 1.1.2

Given the ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1,$$

Find the centre, foci and vertices and sketch it.

Solution

Clearly, the major axis is horizontal.

$a^2=16\Rightarrow a=4.$

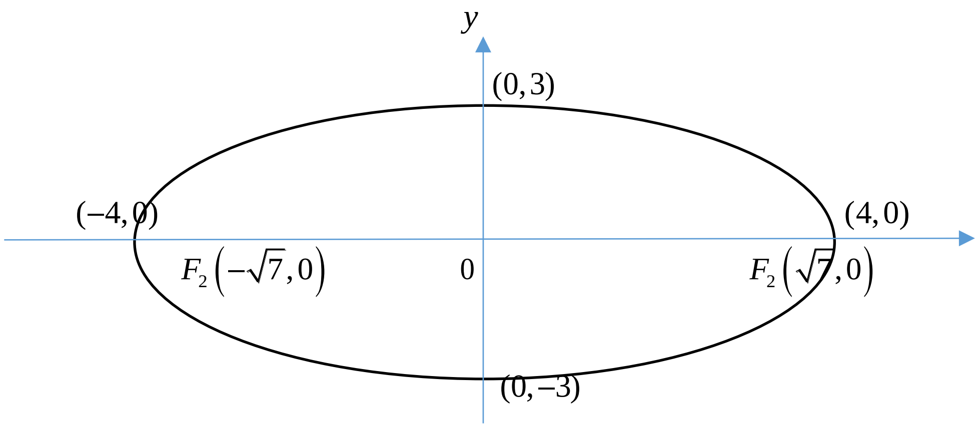
$b^2=9\Rightarrow b=3$

$c^2=a^2-b^2=7\Rightarrow c=\sqrt{7}.$

$\therefore$ centre: $(0,0)$

foci: $\left(-\sqrt{7},0\right)$ and $\left(-\sqrt{7},0\right)$

vertices: $(-4,0)$, $(4,0)$, $(0,-3)$ and $(0,3)$.



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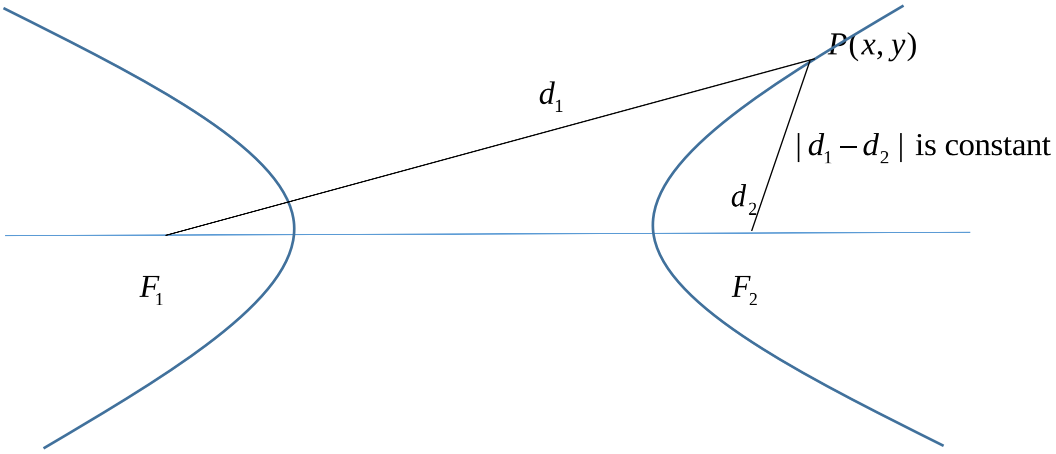
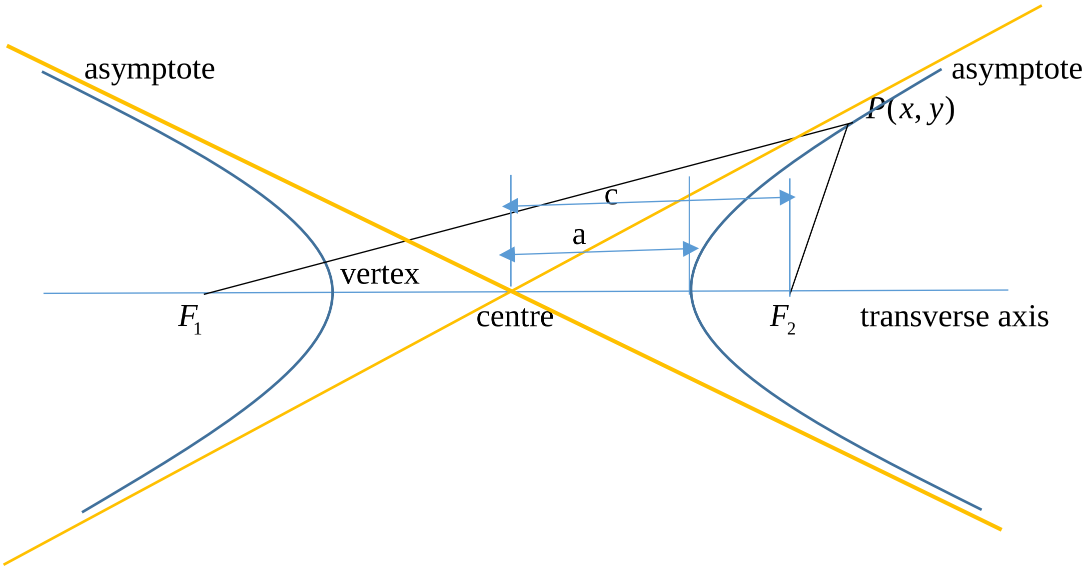
1.1.3 HYPERBOLA

A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two fixed points called **foci** is **constant**.



- The line through the foci intersects a hyperbola at two points called the **vertices**.
- The line segment connecting the vertices is called the **transverse axis** (focal axis) and its midpoint is known as the **centre** of the hyperbola.

A hyperbola has two asymptotes that intersect at the centre. They are an important aid in sketching its graph.



If the foci are at $F_1(-c,0)$ and $F_2(c,0)$ and the constant is $2a$, then by definition any point $P(x, y)$ lies on the hyperbola if and only if

$$\left(|F_1P| - |PF_2|\right) = \pm 2a.$$

Simplifying this equation, we get

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1.$$

Since $c>a,\ a^2-c^2<0,$ so we write this equation as

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$$

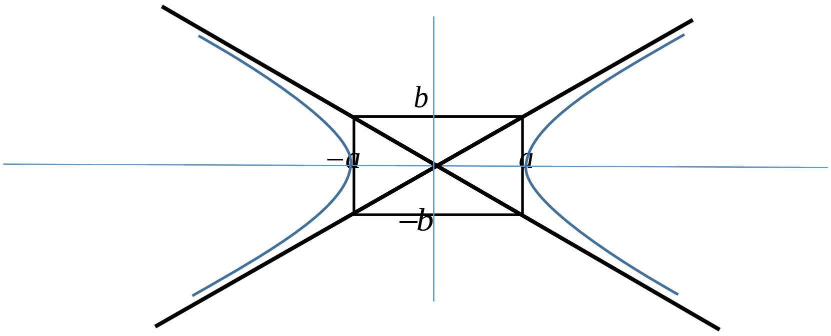
so that if we let $b^2=c^2-a^2,$ we have

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \qquad \textbf{(1.8)}$$

Similarly, if the transverse axis is vertical, then the equation of the hyperbola is given by

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1. \qquad \textbf{(1.9)}$$

If we draw a rectangle with horizontal line passing through b and $-b$ and vertical line passing through $-a$ and a (and vice versa for vertical transverse axis), then the asymptotes pass through the centre and the diagonals of this rectangle.



So for equation(1.8), the asymptotes are given by

$$y=\pm\frac{b}{a}x$$

and for equation (1.9), the asymptotes are given by

$$y=\pm\frac{a}{b}x.$$

<u>EXAMPLE 1.1.3</u>

A hyperbola is given by

$$4x^2-y^2=16.$$

Find the centre, foci, vertices and asymptotes and sketch it.

Solution

$$4x^2 - y^2 = 16 \Rightarrow \frac{x^2}{4} - \frac{y^2}{16} = 1$$

$\therefore$ transverse axis ia holizontal

$a^2 = 4 \Rightarrow a = 2$

$b^2 = 16 \Rightarrow b = 4$

$c^2 = a^2 + b^2 = 20 \Rightarrow c = \sqrt{20}.$

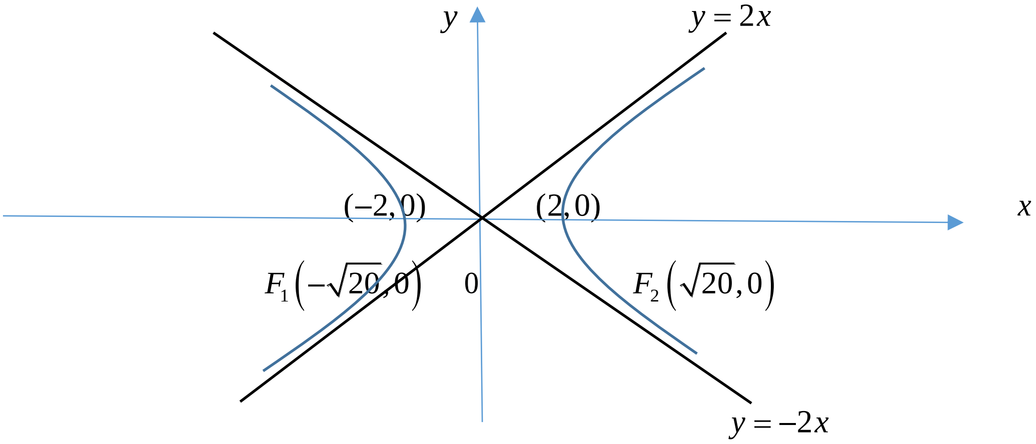
$\therefore$ cenre: $(0,0)$

foci: $(-\sqrt{20},0)$ and $(\sqrt{20},0)$

vertices: $(-2,0)$ and $(2,0)$

asymptotes: $y = \pm\frac{4}{2}x$

i.e $y = \pm 2x.$



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1.2 **CLASSIFYING CONICS BY ECCENTRICITY**

We consider the number called the conic section’s eccentricity and show how it is related to different conics we have discussed already. The eccentricity describes the conic section’s general proportions (in the case of ellipse and hyperbola) and is used to determine the type of a conic section. For example, given any ellipse, we can fix a and vary c over the interval $0 \le c \le a.$ We notice that when $c = 0,$ we get circles and as c increases, the ellipse flattens. If $c = a,$ the foci and vertices overlap and the ellipse degenerates into a line segment.

Definition 1.2.1

The eccentricity $e,$ of a parabola is $e = 1.$

Definition 1.2.2

The eccentricity $e,$ of an ellipse is given by the ratio

$$e = \frac{c}{a},$$

where $0 < c < a$ implying that $0 < e < 1.$

Definition 1.2.3

The eccentricity $e,$ of a hyperbola is given by the ratio

$$e = \frac{c}{a}.$$

Since $c > a$ for a hyperbola, $e > 1.$

In case of the ellipse, the eccentricity can be said to be the ratio of the distance of a point $P(x, y)$ from the foci to the distance of the same point from a fixed line called the directrix. That leads to the "focus-directrix" equation

$$|PF| = e.|PD|, \qquad \textbf{(1.10)}$$

where D is a point on the directrix such that the line segment PD is perpendicular to the directrix . This focus-directrix equation unites the parabola, ellipse and hyperbola in the following theorem:

Theorem 1.2.1

Suppose that the distance $|PF|$ of a point $P(x, y)$ from the focus F is a constant multiple of its distance from a directrix, i.e.

$$|PF| = e.|PD|,$$

where e is the constant of proportionality. Then, the path traced by P is

(a) A parabola

(b) An ellipse of eccentricity $e,$ if $e < 1,$ and

(c) A hyperbola of eccentricity $e,$ if $e > 1.$

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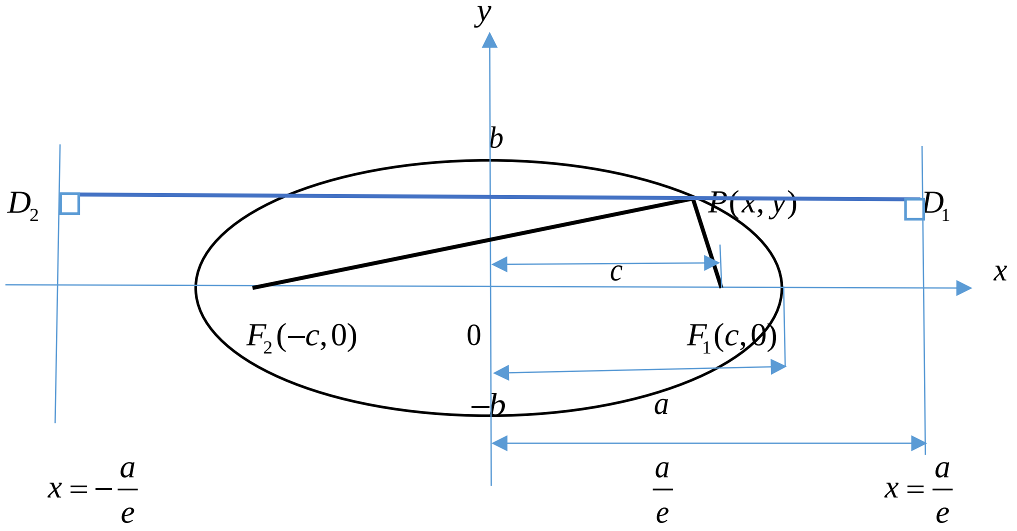
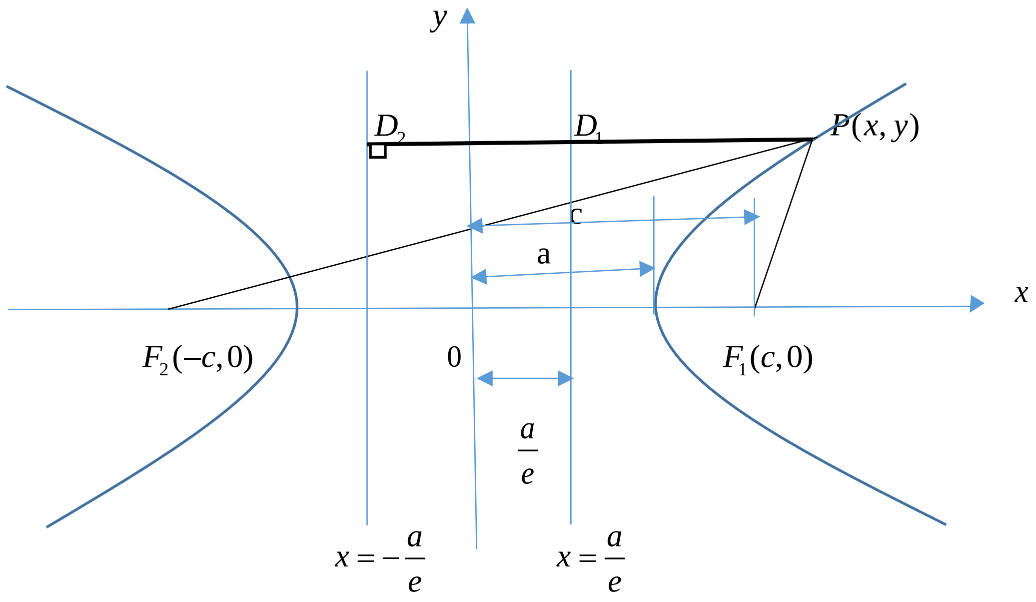


$$|PF_1| = e.|PD_1|$$
$$|PF_2| = e.|PD_2|$$



$$|PF_1| = e.|PD_1|$$
$$|PF_2| = e.|PD_2|$$

<u>EXAMPLE 1.2.1</u>

1. Identify the conic sections centred at the origin and given by the following values of a and b:
   (a) $a = 2,\ b = 5$
   (b) $a = 3,\ b = 1.$
2. A hyperbola of eccentricity $e = \frac{3}{2}$ has one focus at $(1, -3).$ The corresponding directrix is the line $y = 2.$ Find an equation for the hyperbola.

<u>Solution</u>

1. (a) $a = 2,\ b = 5 \Rightarrow c^2 = a^2 + b^2$ (only two possibilities but not $c^2 = a^2 - b^2$)

$$= 4 + 25$$
$$= 29$$
$$\Rightarrow c = \sqrt{29}$$

$$\therefore e = \frac{c}{a} = \frac{\sqrt{29}}{2} > 1$$

$\therefore$ The conic section is a hyperbola.

(b) $a=3,\ b=1.$ We have two possibilities:

(i) $c^2=a^2+b^2$

$=9+1$
$=10$

$\Rightarrow c=\sqrt{10}$ and $e=\dfrac{c}{a}=\dfrac{\sqrt{10}}{3}>1.$

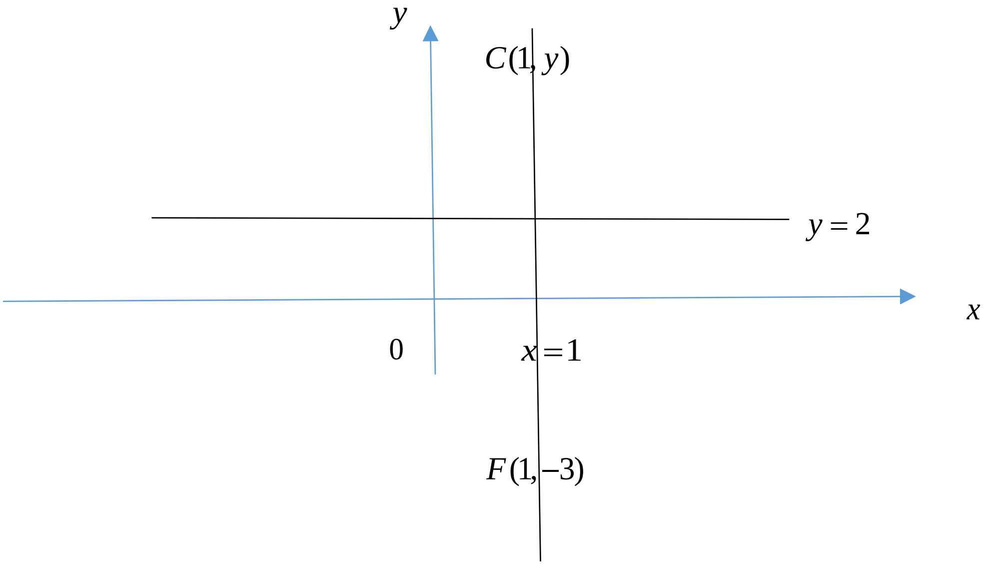
(ii) $c^2=a^2-b^2$

$=9-1$
$=8$

$\Rightarrow c=\sqrt{8}$ and $e=\dfrac{c}{a}=\dfrac{\sqrt{8}}{3}<1.$

$\therefore$ The conic section can a hyperbola or an ellipse.

2. It is indeed a hyperbola since $e=\dfrac{3}{2}.$



- $e=\dfrac{3}{2}=\dfrac{c}{a}\Rightarrow 3a=2c$..................................(i)
- The distance from the directrix to the focus is given by $c-\dfrac{a}{e}$, and in our case it is equal to $5.$ Thus,

  $c-\dfrac{a}{e}=5\Rightarrow\dfrac{3c}{2}-a=5\left(\dfrac{3}{2}\right)$

  $\Rightarrow 3c-2a=15$..............................(ii)

  Solving (i) and (ii) simultaneously, we get

  $3a-2c=0$

  $-2a+3c=15$

$$\Rightarrow 6a - 4c = 0$$
$$-6a + 9c = 45$$

$$\Rightarrow 5c = 45$$
$$\Rightarrow c = 9$$
$$\Rightarrow a = \left(\frac{2}{3}\right)(9) = 6.$$

$\therefore$ The centre is 9 units upward from the focus and the vertex is 6 units from the centre.

$\therefore$ Centre: $(1, 6)$

Vertex: $(1, 0)$

$b^2 = c^2 - a^2 = 81 - 36 = 45$

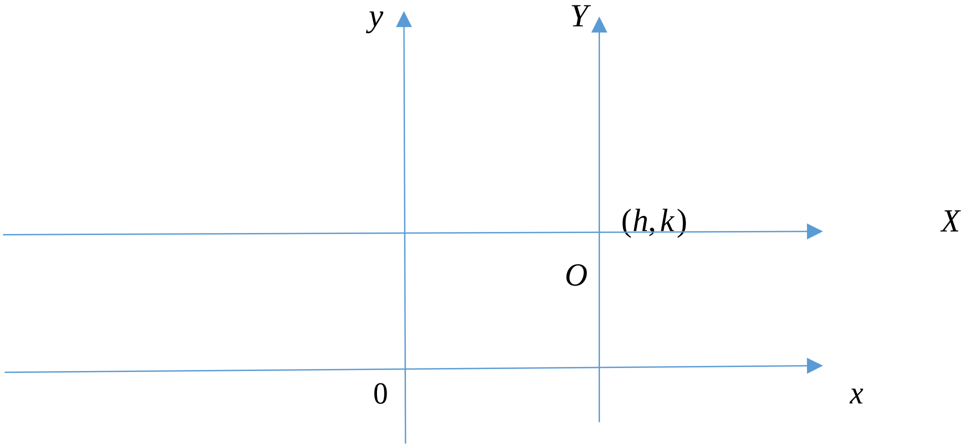
$\therefore$ The equation of the hyperbola is

$$\frac{(y-6)^2}{36} - \frac{(x-1)^2}{45} = 1.$$

△

1.3 TRANSLATION OF COORDINATES

In the previous section, we have considered conic sections centred at the origin. Suppose we shift the centre to the right to any point say $O(h, k)$. Then, we may assume that the axes passing through that point form another coordinate system we shall call the XY coordinate system.



If we consider this point O to be the origin for the XY coordinate system, then any point $P(x, y)$ in the original xy coordinate system will be translated to the point $P(X, Y)$ in such a way that

$$\left.\begin{aligned} X &= x - h \\ Y &= y - k \end{aligned}\right\} \qquad \textbf{(1.11)}$$

Similar translations can be done in the other three remaining quadrants. In each case, we obtain equations like (1.11) which relate the two coordinate systems xy and XY. We can then discuss conic sections in the XY coordinate system as well as in the original xy coordinate system.

<u>Example 1.3.1</u>

Discuss the conic sections given by

(a) $2x^2 - y^2 + 6y = 3$

(b) $4x^2 + y^2 - 8x + 4y - 8 = 0$

(c) $x^2 + 2y + 2x = 1.$

<u>Solution</u>

In each case, we complete the square.

(a) $2x^2 - y^2 + 6y = 3$

$$\begin{aligned} &\Rightarrow 2x^2 - (y^2 - 6y) = 3 \\ &\quad 2x^2 - [(y-3)^2 - 9] = 3 \\ &\quad 2x^2 - (y-3)^2 = -6 \\ &\Rightarrow \frac{(y-3)^2}{6} - \frac{x^2}{3} = 1. \end{aligned}$$

Letting $X = x$ and $Y = y - 3,$ we have

$$\frac{Y^2}{6} - \frac{X^2}{3} = 1,$$

which is a hyperbola with vertical transverse axis.

$$a^2 = 6 \Rightarrow a = \sqrt{6}$$
$$b^2 = 3 \Rightarrow b = \sqrt{3}$$
$$c^2 = a^2 + b^2 = 9 \Rightarrow c = 3.$$

In the XY coordinate system,

Centre: $(0, 0)$

Foci: $(0, -3)$ and $(0, 3)$

Vertices: $(0, -\sqrt{6})$ and $(0, \sqrt{6})$

Asymptotes: $Y = \pm \dfrac{\sqrt{6}}{\sqrt{3}}$ i.e. $Y = \pm\sqrt{2}X$

Directrices: $Y = \pm \dfrac{a^2}{c} = \pm 2.$

Using the equations $X = x$ and $Y = y - 3,$ we have that

for $Y = \pm 3,$ $y = 0$ and $y = 6,$

for $Y=\pm\sqrt{6}$, $y=3-\sqrt{6}$ and $y=3+\sqrt{6}$,

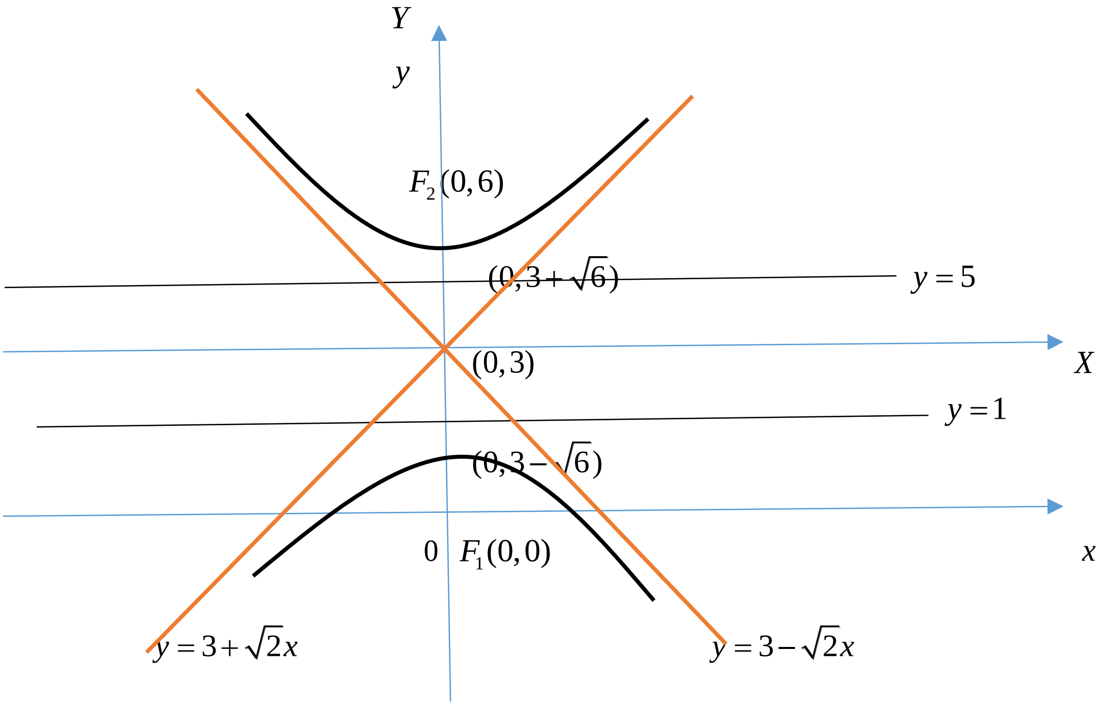
for $Y=\pm 2$, $y=5$ and $y=1$

and for the asymptotes $Y=\pm\sqrt{2}X$, we have that

$y-3=\pm\sqrt{2}x$, i.e. $y=3\pm\sqrt{2}x$.

Therefore, in the original xy coordinate system,

Centre: $(0,3)$
Foci: $(0,0)$ and $(0,6)$
Vertices: $(0,3-\sqrt{6})$ and $(0,3+\sqrt{6})$
Asymptotes: $y=3\pm\sqrt{2}x$
Directrices: $y=1$ and $y=5$.



(b) $4x^2+y^2-8x+4y-8=0$

$$4(x^2-2x)+y^2+4y-8=0$$
$$4(x-1)^2-4+(y+2)^2-4-8=0$$
$$4(x-1)^2+(y+2)^2=16$$
$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1.$$

Letting $X=x-1$ and $Y=y+2$, we get

$$\frac{X^2}{4}+\frac{Y^2}{16}=1,$$

which is an ellipse with vertical major axis.

$a^2 = 16 \Rightarrow a = 4$

$b^2 = 4 \Rightarrow b = 2$

$c^2 = 16 - 4 = 12 \Rightarrow c = \sqrt{12}.$

In the XY coordinate system

Centre: $(0,0)$

Foci: $\left(0, -\sqrt{12}\right)$ and $\left(0, \sqrt{12}\right)$

Vertices: $(0,-4)$, $(0,4)$, $(-2,0)$ and $(2,0)$

Directrices: $Y = \pm\frac{a^2}{c} = \pm\frac{16}{2\sqrt{3}} = \pm\frac{8}{\sqrt{3}}.$

When $X = 0$, $x = 1$

When $Y = 0$, $y = -2$

When $X = \pm 2$, $x = 3$ and $x = -1$

When $Y = \pm 4$, $y = 2$ and $y = -6$

When $Y = \pm\frac{8}{\sqrt{3}}$, $y = \pm\frac{8}{\sqrt{3}} - 2.$

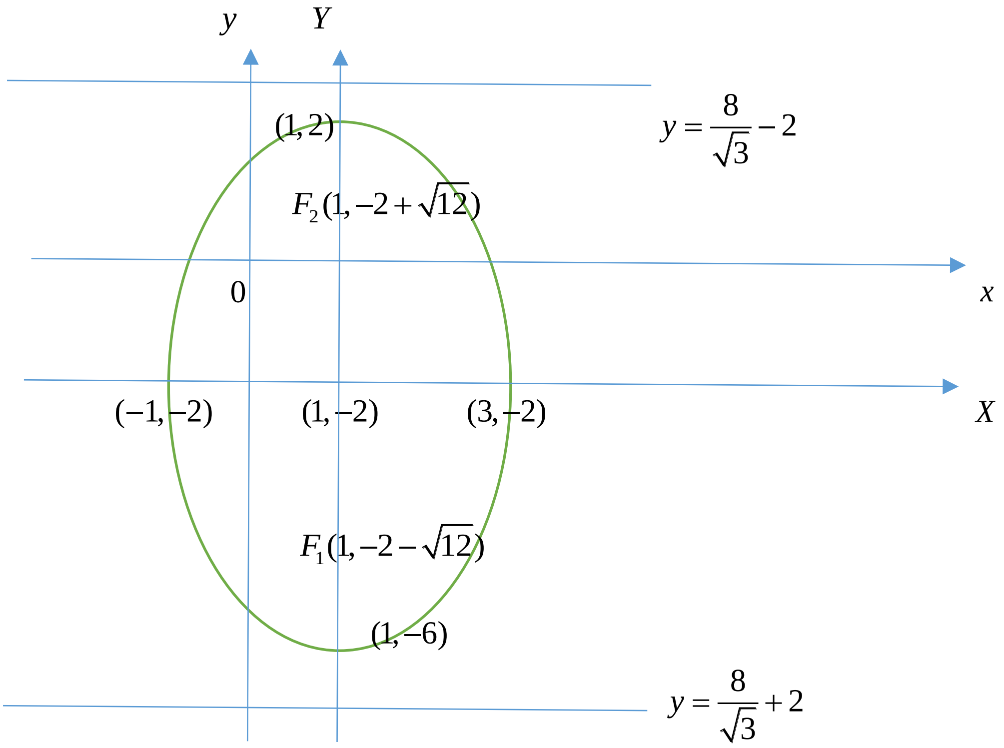
Therefore, in the xy coordinate system,

Centre: $(1,-2)$

Foci: $\left(1, -2-\sqrt{12}\right)$ and $\left(1, -2+\sqrt{12}\right)$

Vertices: $(1,-6)$, $(1,2)$, $(-1,-2)$ and $(3,-2)$

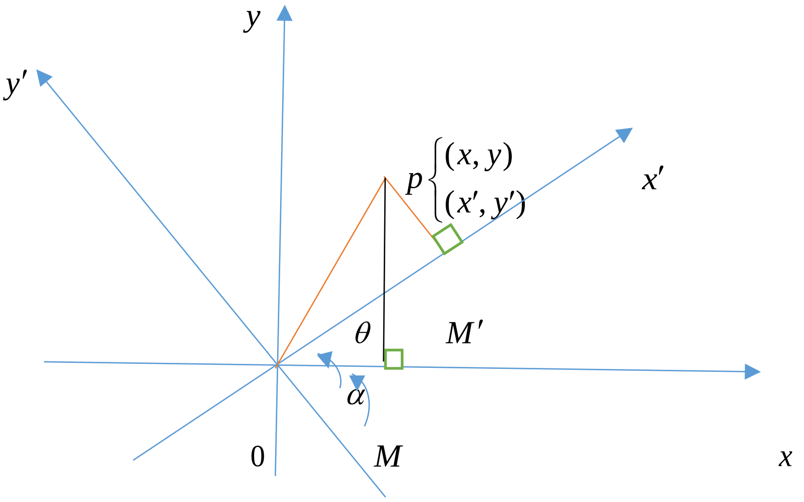
Directrices: $y = \pm\frac{8}{\sqrt{3}} - 2.$



(c) Exercise △

1.4 ROTATION OF AXES

It was stated at the beginning of this chapter that conic sections can be described algebraically by equation(1.1). You may have noticed that in the conic sections we have considered so far the Bxy- term did not appear because the axes were parallel to the coordinate axes. To eliminate this xy-term, we rotate the coordinate axes counter clockwise about the origin through an angle α.



$$\begin{aligned} x &= oM = (op)\cos(\theta+\alpha) \\ y &= Mp = (op)\sin(\theta+\alpha) \\ x' &= oM' = (op)\cos\theta \\ y' &= M'p = (op)\sin\theta \end{aligned}$$

$$\begin{aligned} \therefore\ x &= (op)\cos\theta\cos\alpha - (op)\sin\theta\sin\alpha \\ &= x'\cos\alpha - y'\sin\alpha \\ y &= (op)\sin\theta\cos\alpha + (op)\cos\theta\sin\alpha \\ &= y'\cos\alpha + x'\sin\alpha \end{aligned}$$

Equations for rotating coordinate axes:

$$\left.\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha \\ y &= x'\sin\alpha + y'\cos\alpha \end{aligned}\right\} \qquad \textbf{(1.12)}$$

If we solve (1.12) simultaneously for x' and y', we also get

$$\left.\begin{aligned} x' &= x\cos\alpha + y\sin\alpha \\ y' &= x\sin\alpha - y\cos\alpha \end{aligned}\right\} \qquad \textbf{(1.13)}$$

Example 1.4.1

The x- and y-axes are rotated through an angle $\frac{\pi}{4}$ radians about the origin. Find an equation for the hyperbola

$$2xy = 9$$

in the new coordinates.

Solution

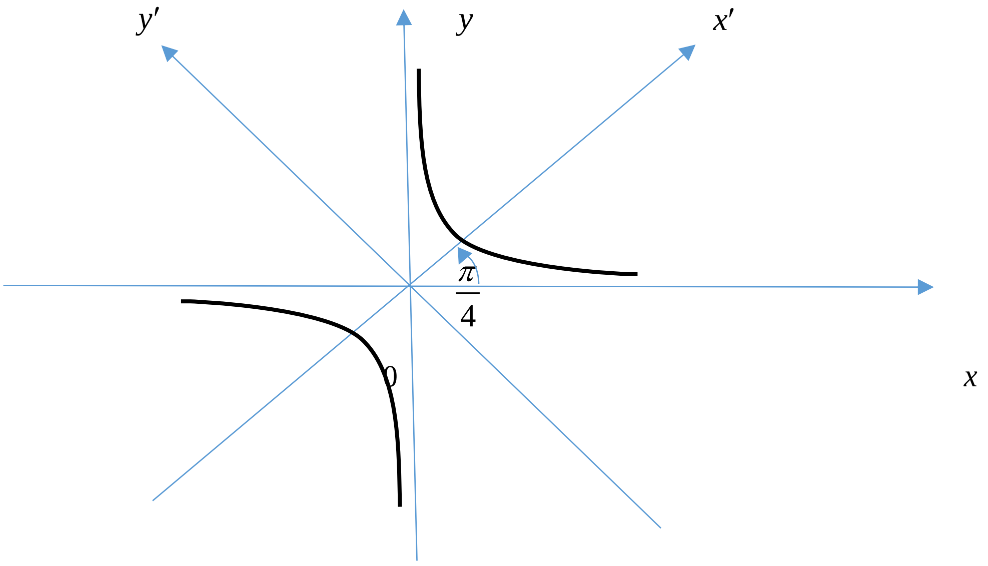
Here $\alpha = \frac{\pi}{4}$ so that

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{x'-y'}{\sqrt{2}}$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{x'+y'}{\sqrt{2}}.$$

Thus,

$$\begin{aligned} 2xy = 9 &\Rightarrow 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) = 9 \\ &\Rightarrow x'^2 - y'^2 = 9 \\ &\Rightarrow \frac{x'^2}{9} - \frac{y'^2}{9} = 1 \end{aligned}$$



△

The task at hand is determining the angle α. We notice that if we apply the equations (1.12) to the quadratic equation (1.1) with $B \neq 0$, we get a new equation with $B' = 0$, i.e.

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0,$$

where

$$\begin{aligned}
&A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha \\
&B' = B\cos 2\alpha + (C - A)\sin 2\alpha \\
&C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha \\
&D' = D\cos\alpha + E\sin\alpha \\
&E' = -D\sin\alpha + E\cos\alpha \\
&F' = F.
\end{aligned}$$

$$\cot 2\alpha = \frac{A-C}{B} \quad \text{or} \quad \tan 2\alpha = \frac{B}{A-C} \tag{1.14}$$

<u>Example 1.4.2</u>

For the equation

$$2x^2 + \sqrt{3}xy + y^2 - 10 = 0,$$

find the angle α and identify the conic section.

<u>Solution</u>

Here $A = 2,\ B = \sqrt{3},\ C = 1$

$$\begin{aligned}
&\therefore \tan 2\alpha = \frac{\sqrt{3}}{2-1} = \sqrt{3} \\
&\Rightarrow 2\alpha = \frac{\pi}{3} \\
&\Rightarrow \alpha = \frac{\pi}{6} \\
&\therefore x = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} = \frac{x'\sqrt{3}}{2} - \frac{y'}{2} \\
&\quad y = x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6} = \frac{x'}{2} + \frac{y'\sqrt{3}}{2} \\
&\therefore 2x^2 + \sqrt{3}xy + y^2 - 10 = 0 \\
&\Rightarrow 2\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^2 + \sqrt{3}\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) + \left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right)^2 - 10 = 0 \\
&\frac{3}{2}x'^2 - \sqrt{3}x'y' + \frac{1}{2}y'^2 + \frac{3}{2}x'^2 + \frac{3\sqrt{3}}{4}x'y - \frac{\sqrt{3}}{4}x'y' - \frac{3}{4}y'^2 + \frac{1}{4}x'^2 + \frac{\sqrt{3}}{2}x'y' + \frac{3}{4}y'^2 - 10 = 0 \\
&\Rightarrow \frac{5}{2}x'^2 + \frac{1}{2}y'^2 = 10 \quad \text{or} \quad \frac{x'^2}{4} + \frac{y'^2}{20} = 1.
\end{aligned}$$

$\therefore$ The conic section is an ellipse

$a^2 = 20 \Rightarrow a = \sqrt{20}$
$b^2 = 4 \Rightarrow b = 2$
$c^2 = 20 - 4 \Rightarrow c = 4.$

In the $x'y'$ coordinate system, we have

Centre: $(0,0)$

Foci: $(0,4)$ and $(0,-4)$

Vertices: $\left(0,2\sqrt{5}\right)$, $\left(0,-2\sqrt{5}\right)$, $(2,0)$ and $(-2,0)$.

Directrices: $y' = \pm\dfrac{8}{\sqrt{5}}$

For $(0,0)$, $x = 0$ and $y = 0$.

For $(0,4)$, $x = \dfrac{0-4}{2} = -2$ and $y = \dfrac{0+4\sqrt{3}}{2} = 2\sqrt{3}$

For $(0,-4)$, $x = \dfrac{0+4}{2} = 2$ and $y = \dfrac{0-4\sqrt{3}}{2} = -2\sqrt{3}$

For $\left(0,2\sqrt{5}\right)$, $x = \dfrac{0-2\sqrt{5}}{2} = -\sqrt{5}$ and $y = \dfrac{0+2\sqrt{5}\sqrt{3}}{2} = \sqrt{15}$

For $\left(0,-2\sqrt{5}\right)$, $x = \dfrac{0+2\sqrt{5}}{2} = \sqrt{5}$ and $y = \dfrac{0-2\sqrt{5}\sqrt{3}}{2} = -\sqrt{15}$

For $(2,0)$, $x = \sqrt{3}$ and $y = 1$

For $(-2,0)$, $x = -\sqrt{3}$ and $y = -1$.

Since $y' = \dfrac{x - y\sqrt{3}}{2}$, we have that $x - y\sqrt{3} = \pm\dfrac{16}{\sqrt{5}}$.

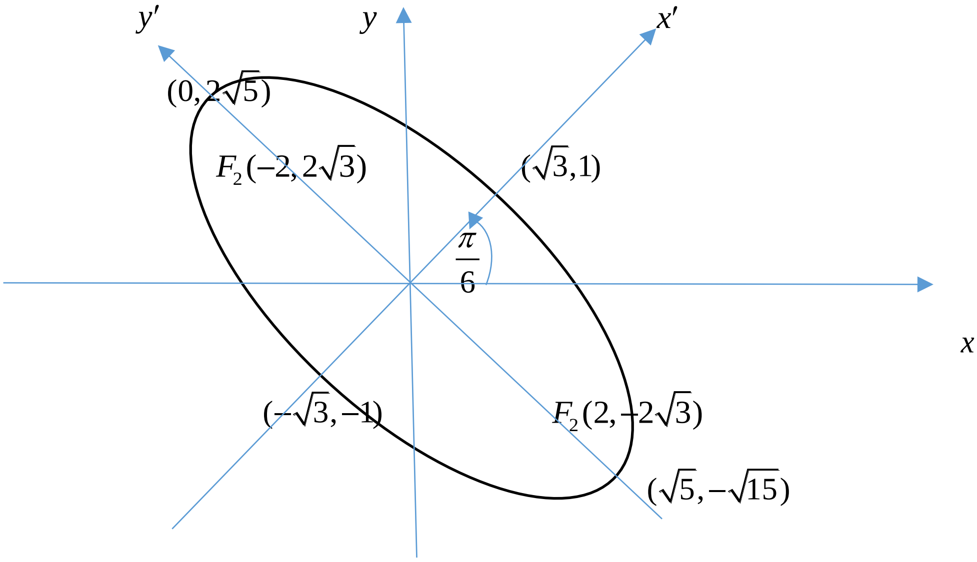
In the xy coordinate system,

Centre: $(0,0)$

Foci: $\left(-2,2\sqrt{3}\right)$ and $\left(2,-2\sqrt{3}\right)$

Vertices: $\left(-\sqrt{5},\sqrt{15}\right)$, $\left(\sqrt{5},-\sqrt{15}\right)$, $\left(\sqrt{3},1\right)$ and $\left(-\sqrt{3},-1\right)$

Directrices: $x - y\sqrt{3} = \pm\dfrac{16}{\sqrt{5}}$.



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1.4.1 THE DISCRIMINANT TEST

Sometimes, we do not need to eliminate the $xy-$term from the quadratic equation to be able to identify the conic section. We can easily do that by applying the discriminant test.

With the understanding that occasional degenerate cases may arise, the quadratic equation (1.1) is

(a) A Parabola if $B^2-4AC=0$
(b) An Ellipse if $B^2-4AC<0$ and
(c) A Hyperbola if $B^2-4AC>0$.

Example 1.4.3

(a) $3x^2-6xy+3y^2+2x-7=0$
represents a parabola because
$B^2-4AC=(-6)^2-4(3)(3)=0.$

(b) $x^2+xy+y^2-1=0$
represents an ellipse because
$B^2-4AC=(1)^2-4(1)(1)=-3<0.$

(c) $xy-y^2-5y+1=0$
Represents a hyperbola because
$B^2-4AC=(1)^2-4(-1)(0)=1>0.$

△

1.5 PARAMETRISATION OF PLANE CURVES

The graphs you have seen so far have been represented by a single equation involving two variables, for example

$$y = -\frac{x^2}{72} + x.$$

Although this equation can tell us where the object has been, it does not tell us when the object was at a given point (x, y). To determine the time, we introduce a third variable t called a parameter so that the object's coordinates are expressed as a function of t and then write x and y as

$$x = f(t) = 24\sqrt{2}t$$
$$y = g(t) = -16t^2 + 24\sqrt{2}t.$$

Definition 1.5.1

If x and y are given as continuous functions

$$x = f(t)$$
$$y = g(t),$$

over an interval of t-values, then the set of points $(x, y) = \left(f(t), g(t)\right)$ defined by these equations is a **curve** in the coordinate plane. The equations are **parametric equations** for the curve. The variable t is a **parameter** for the curve and its domain I is the **parameter interval**. If I is a closed interval, $a \le t \le b,$ then the point $\left(f(a), g(a)\right)$ is the **initial point** of the curve and $\left(f(b), g(b)\right)$ is the **terminal point** of the curve. When we give parametric equations and a parameter interval for a curve in the plane, we say that we have parametrised the curve. The equations and interval constitute a parametrisation of the curve.

Example 1.5.1

Sketch the curve represented by the following equations

(a) $x = t,\ y = 1 - t,\ -1 \le t \le 3.$

(b) $x = \dfrac{1}{\sqrt{t+1}},\ y = \dfrac{t}{t+1},\ t > -1.$

(c) $x = \cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi.$

(d) $x = 3\cos\theta,\ y = 4\sin\theta,\ 0 \le \theta \le 2\pi.$

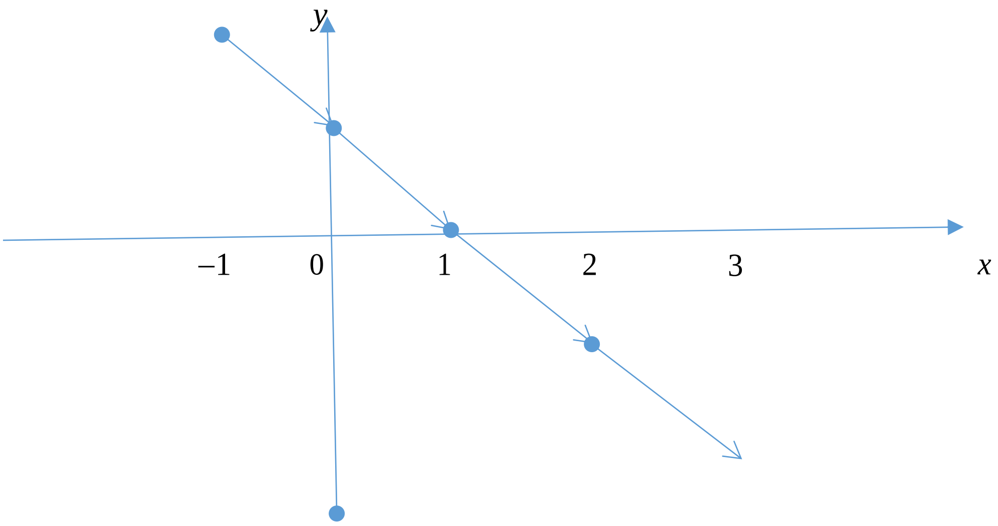
Solution

(a) $x = t,\ y = 1 - t = 1 - x$

t	-1	0	1	2	3
x	-1	0	1	2	3
y	2	1	0	-1	-2

Initial point: $(-1, 2)$

Terminal point: $(3, -2)$



(b) $y = \dfrac{t}{t+1} \Rightarrow yt + y = t$

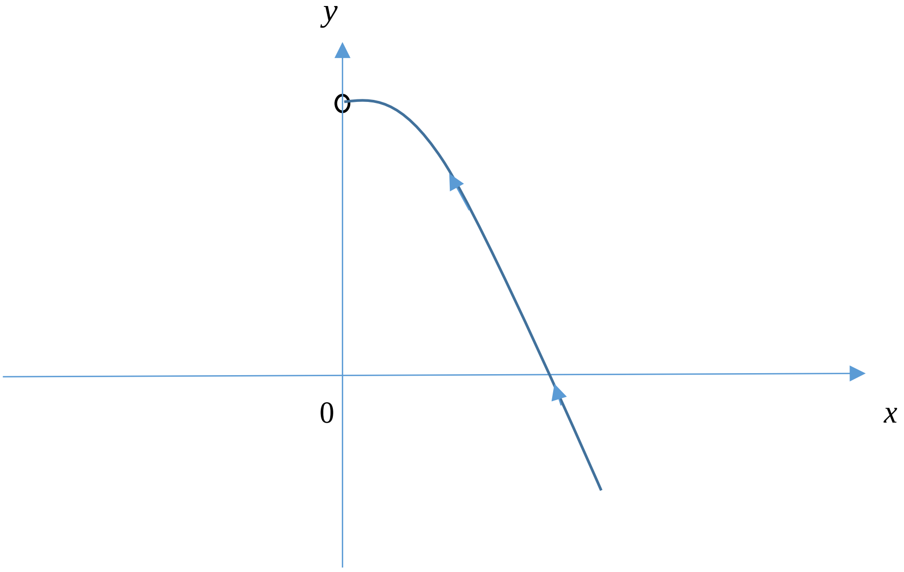
$$\Rightarrow t(y-1) = -y$$

$$\Rightarrow t = \frac{-y}{y-1}$$

$$\therefore x = \frac{1}{\sqrt{t+1}} = \frac{1}{\sqrt{\dfrac{-y}{y-1}+1}}$$

$$\Rightarrow x^2 = -(y-1).$$

Since $t > -1$ and $x = \dfrac{1}{\sqrt{t+1}}$, it follows that $x > 0$.

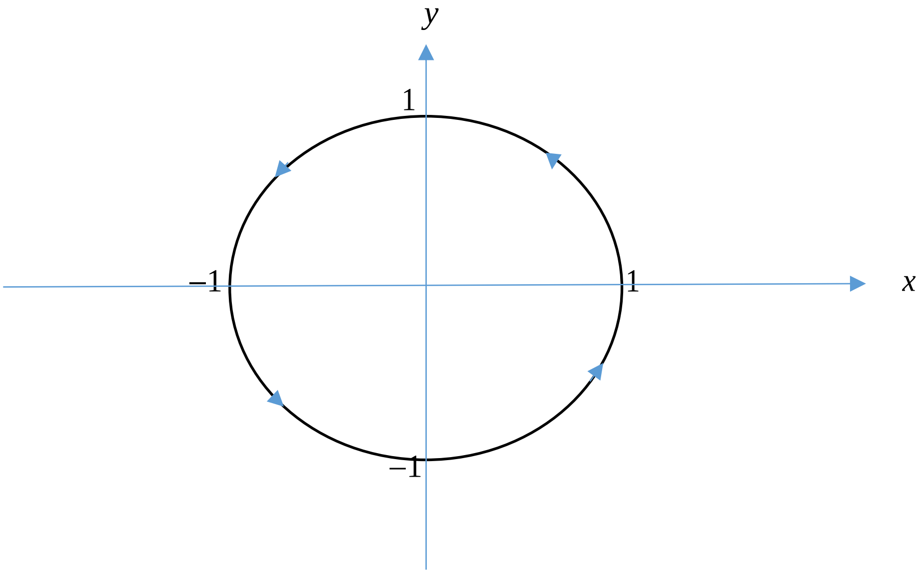


(c) $x = \cos\theta,\ \ y = \sin\theta$

Using the identity $\cos^2\theta + \sin^2\theta = 1,$ we have that

$$x^2 + y^2 = 1,$$

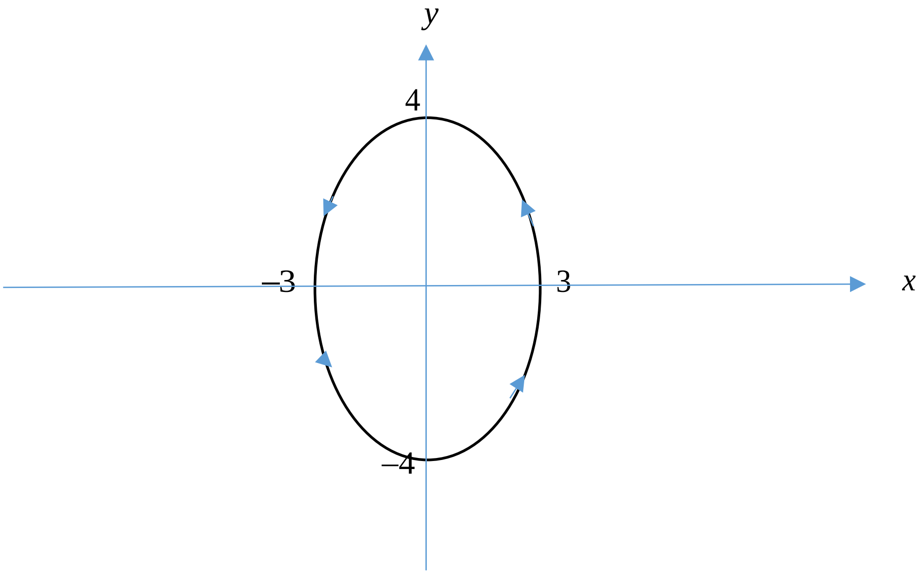
which is a circle centred at the origin with radius 1.



(d) $x = 3\cos\theta,\ \ y = 4\sin\theta\ \ \Rightarrow \cos\theta = \dfrac{x}{3}$ and $\sin\theta = \dfrac{y}{4}.$

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1,$$

which is the equation of an ellipse.



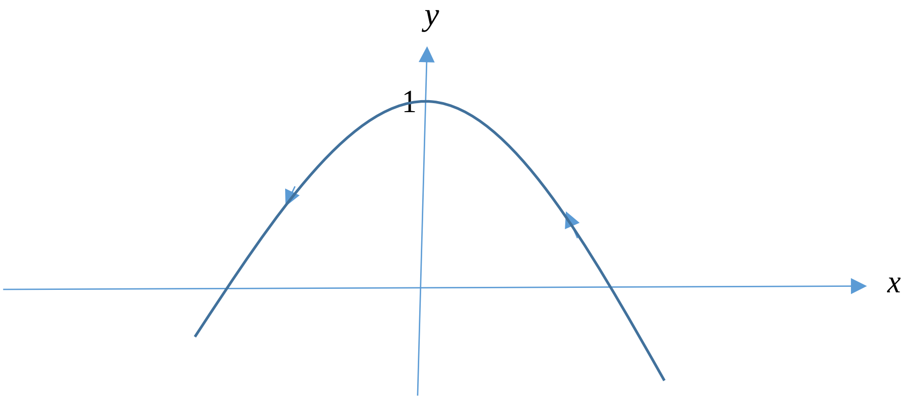
In the previous example, the parametric equations were given and all we wanted to do was to sketch the graph. Suppose we are given the equation and we want to parametrise it. The next example will illustrate just that.

<u>Example 1.5.2</u>

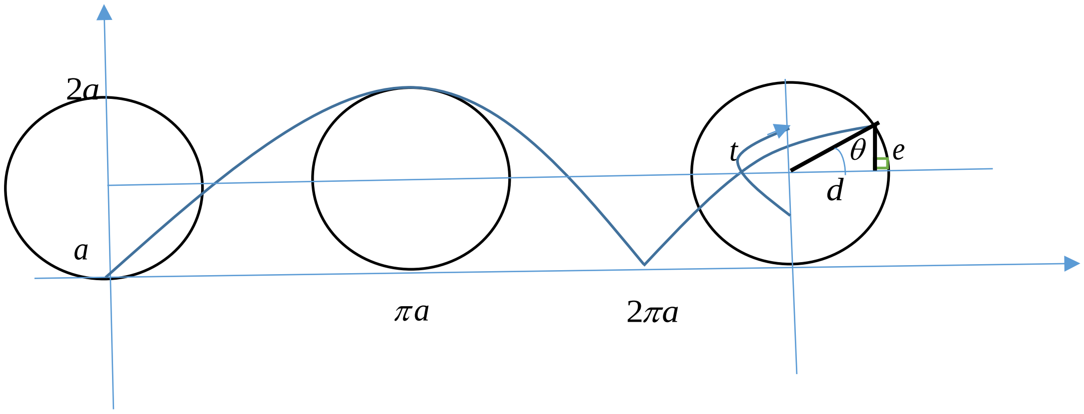
1. Find a set of parametric equations that represents the graph of $y = 1 - x^2$ using the slope $m = \frac{dy}{dx}$ at the point (x, y).
2. Find the parametric equations for the right-hand branch of the hyperbola
$$x^2 - y^2 = 1.$$
3. Find the parametric equations for the curve traced by a point p on the circumference of a circle of radius "a" rolling along a horizontal straight line in a plane. Such a curve is called a cycloid.

<u>Solution</u>

1. $y = 1 - x^2 \Rightarrow \frac{dy}{dx} = m = -2x$ and so $x = -\frac{m}{2}$. Thus,
$$y = 1 - \left(-\frac{m}{2}\right)^2 = 1 - \frac{m^2}{4}.$$
$$\therefore x = -\frac{m}{2},\ y = 1 - \frac{m^2}{4},\ -\infty < m < \infty$$



2. For the hyperbola $x^2 - y^2 = 1$, we think of one trigonometric identity that can help in parametrising. Since $\sec^2\theta - \tan^2\theta = 1$, letting
$$x = \sec\theta \text{ and } y = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
gives us the required parametric equations.
3. As a parameter, we will use the angle t through which the circle turns. When $t = 0$, p is at the origin. When $t = \pi$, p would have moved one-half of the circumference of the circle, i.e. πa and p is at $(\pi a, 2a)$. Suppose the base lies at "at" units from the origin.



$$\cos\theta = \frac{d}{a} \Rightarrow d = a\cos\theta \text{ and } \sin\theta = \frac{e}{a} \Rightarrow e = a\sin\theta$$

$$t + \theta = \frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{2} - t \text{ and so}$$

$$\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t$$

$$\sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t$$

$$\therefore x = at + d = at + a\cos\theta = at - a\sin t = a(t - \sin t)$$

$$y = a + e = a + a\sin\theta = a - a\cos t = a(1 - \cos t),$$

i.e. $x = a(t - \sin t),\ y = a(1 - \cos t),\ t \ge 0.$ △

1.5.1 STANDARD PARAMETRISATIONS

Circle: $x^2 + y^2 = a^2$

$$x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi.$$

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi.$$

Cycloid generated by a circle of radius a:

$$x = a(t - \sin t),\ y = a(1 - \cos t),\ t \ge 0.$$

1.6 POLAR COORDINATES

You have already seen how graphs can be represented by a collection of points (x, y) on the rectangular (Cartesian) coordinate system. We now want to study the coordinate system called polar coordinate system and see how it is related to the rectangular coordinate system. It is interesting to note that while a point in the plane has only one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates.

Definition 1.6.1

Fix an origin O (called the pole) and an initial ray from O. Then each point P can be located by assigning to it a polar coordinate pair (r,θ) in which r gives the directed distance from O to P, and θ gives the directed angle from the initial ray to OP.

As mentioned before, the point (r,θ) can have infinitely many pairs. This is because there are instances when r can be negative (hence the use of directed distance in the definition) and so (r,θ) and $(-r,\pi+\theta)$ represent same point. Similarly, (r,θ) and $(r,2\pi+\theta)$ represent same point. In general,

$$(r,\theta)=(r,2n\pi+\theta)$$

or

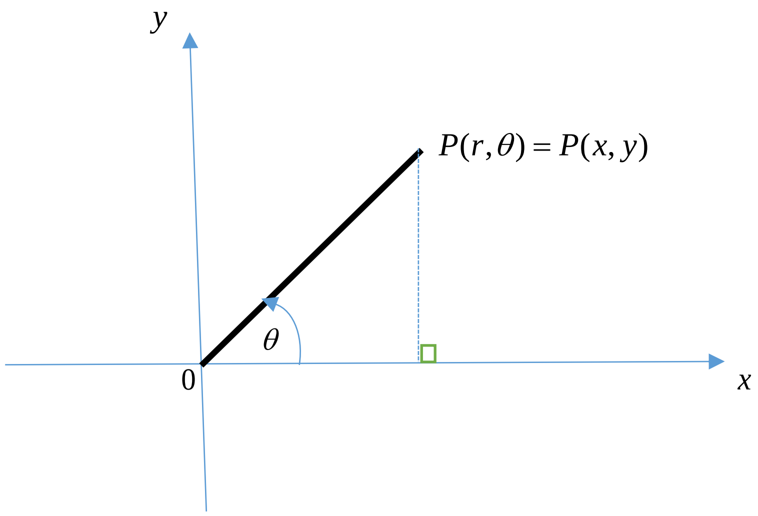
$$(r,\theta)=(-r,(2n+1)\pi+\theta),$$

where n is any integer. For example, the point $P\left(2,\frac{\pi}{6}\right)$ has the following polar coordinates:

$$\text{when } r=2,\ \theta=\left\{\frac{\pi}{6},\frac{\pi}{6}\pm 2\pi,\frac{\pi}{6}\pm 4\pi,\ldots\right.$$

$$\text{when } r=\text{-}2,\ \theta=\left\{-\frac{5\pi}{6},-\frac{5\pi}{6}\pm 2\pi,-\frac{5\pi}{6}\pm 4\pi,\ldots.\right.$$

Let us now see the relationship between polar and rectangular coordinates. We assume that the polar axis coincide with the positive x-axis and the pole with the origin.



Then $x=r\cos\theta,\ \ y=r\sin\theta$

which are the parametric equations for a circle centred at the origin with radius $r>0$ and so

$\tan\theta = \dfrac{y}{x}$. Thus, the equations relating polar and Cartesian coordinates are:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ x^2 + y^2 = r^2,\ \ \tan\theta = \frac{y}{x}.$$

__Example 1.6.1__

1. Convert the point $(-1,1)$ and $(0,2)$ to polar coordinates.
2. Convert the polar coordinates $(2,\pi)$ and $\left(\sqrt{3},\dfrac{\pi}{6}\right)$ to rectangular coordinates.

<u>Solution</u>

1. For $(-1,1)$, $\tan\theta = -1 \Rightarrow \theta = \dfrac{3\pi}{4}$

   We choose θ in the second quadrant where $r > 0$. So

   $r = \sqrt{x^2 + y^2} = \sqrt{2}$

   $\therefore$ One set of polar coordinates is $\left(\sqrt{2},\dfrac{3\pi}{4}\right)$

   For $(0,2)$, since x-coordinate is zero, we choose $\theta = \dfrac{\pi}{2}$ and automatically

   $r = \sqrt{0^2 + 2^2} = 2$

   $\therefore$ One set of polar coordinates is $\left(2,\dfrac{\pi}{2}\right)$.

2. For $(2,\pi)$, $x = r\cos\theta = 2\cos\pi = -2$

   $y = r\sin\theta = 2\sin\pi = 0$

   $\therefore$ Rectangular coordinate is $(-2,0)$.

   For $\left(\sqrt{3},\dfrac{\pi}{6}\right)$, $x = r\cos\theta = \sqrt{3}\cos\dfrac{\pi}{6} = \dfrac{3}{2}$

   $y = r\sin\theta = \sqrt{3}\sin\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

   $\therefore$ Rectangular coordinate is $\left(\dfrac{3}{2},\dfrac{\sqrt{3}}{2}\right)$.

△

__Example 1.6.2__

For each of the following polar equations, find their equivalent Cartesian equation and identify their graphs:

(a) $r\cos\theta = -4$
(b) $r^2 = 4r\cos\theta$

(c) $r=\dfrac{4}{2\cos\theta-\sin\theta}$.

Solution

(a) Since $x=r\cos\theta,$ we have that $x=-4$ which is a vertical line.

(b) Since $x^2+y^2=r^2$ and $y=r\sin\theta,$ we have that

$2x-y=4x,$ i.e. $(x-2)^2+y^2=4,$ which is a circle centred at $(2,0)$ with radius $2.$

(c) $r=\dfrac{4}{2\cos\theta-\sin\theta}\Rightarrow 2r\cos\theta-r\sin\theta=4.$

Since $x=r\cos\theta$ and $y=r\sin\theta,$ we have that

$2x-y=4,$ which is a straight line.

Example 1.6.3

Replace the Cartesian equations

(a) $y^2=4x$ (b) $x^2+xy+y^2=1$ and (c) $x^2+(y-2)^2=4$

by their equivalent polar equations.

Solution

(a) $y^2=4x\Rightarrow(r\sin\theta)^2=4(r\cos\theta)$

$$\Rightarrow r^2\sin^2\theta=4r\cos\theta$$

$$\Rightarrow r\sin^2\theta=4\cos\theta$$

(b) $x^2+xy+y^2=1\Rightarrow x^2+y^2+xy=1$

$$\Rightarrow r^2+r^2\sin\theta\cos\theta$$

$$\Rightarrow r^2(1+\sin\theta\cos\theta).$$

(c) $x^2+(y-2)^2=4\Rightarrow x^2+y^2-2y+4=4$

$$\Rightarrow r^2=2r\sin\theta$$

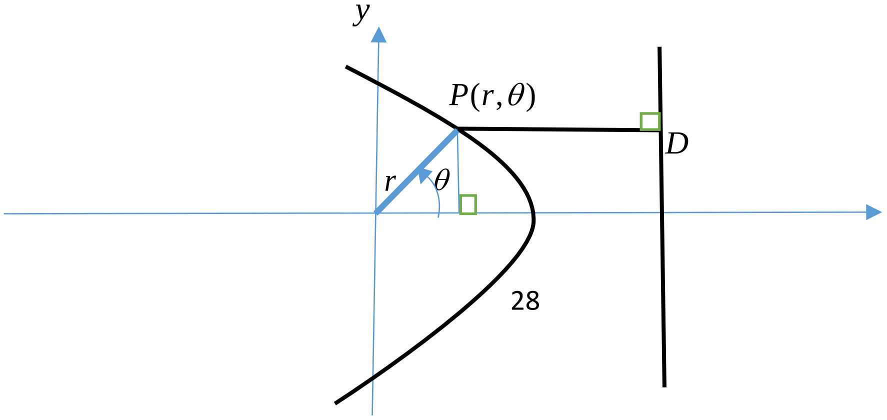
$$\Rightarrow r=2\sin\theta.$$

△

1.7 POLAR EQUATIONS FOR CONICS

We wish to use polar coordinates to describe conics with a single relatively simple coordinate equation.

Suppose one focus of a conic section is placed at the origin and the corresponding directrix is to the right of the origin along the vertical line $x=k.$



F B

$x = k$

Clearly, $|PF| = r$ and $|PD| = k - FB = k - x$

$$= k - r\cos\theta.$$

Then, using the 'focus-directrix' equation $|PF| = E.|PD|,$ we have that

$r = e(k - r\cos\theta).$

We can do the same for the other three (3) cases when the directrix is vertical to the left of the pole, and when it is above and below the pole.

<u>*Theorem 1.7.1*</u>

Let F be a focus placed at the origin, D be a point along the directrix. Let P be a point in the plane and e be eccentricity. Then, the polar equations of the conics are as given below depending on the position of the directrix:

1. Horizontal directrix above the pole: $r = \dfrac{ek}{1 + e\sin\theta}$
2. Horizontal directrix above the pole: $r = \dfrac{ek}{1 - e\sin\theta}$
3. Vertical directrix to the right of the pole: $r = \dfrac{ek}{1 + e\cos\theta}$
4. Vertical directrix to the left of the pole: $r = \dfrac{ek}{1 - e\cos\theta}.$ ☐

<u>***Example 1.7.1***</u>

Identify and sketch the graph of the polar equations

(a) $r = \dfrac{32}{3 + 5\sin\theta}$ (b) $r = \dfrac{15}{3 - 2\cos\theta}.$

<u>Solution</u>

(a) $r = \dfrac{32}{3 + 5\sin\theta} = \dfrac{32/3}{1 + 5/3\sin\theta}$

$$\Rightarrow e = \frac{5}{3} > 1.$$

$\therefore$ The conic section is a hyperbola with directrix above the pole.

$$ek = \frac{32}{3} \Rightarrow k = \frac{32}{3} \times \frac{3}{5} = \frac{32}{5}$$

For $\theta = 0,\ r = \frac{32}{3} \Rightarrow (r,\theta) = \left(\frac{32}{3}, 0\right) \to (x, y) = \left(\frac{32}{3}, 0\right)$

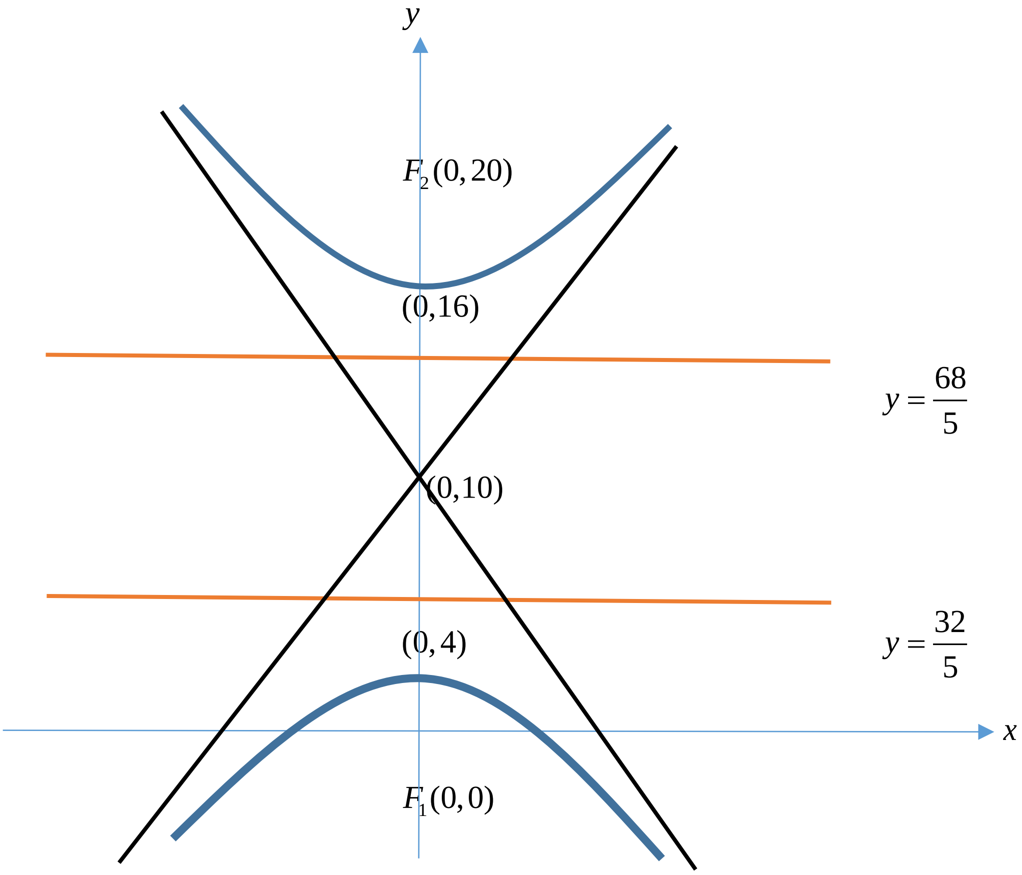
For $\theta = \frac{\pi}{2},\ r = 4 \Rightarrow (r,\theta) = \left(4, \frac{\pi}{2}\right) \to (x, y) = (0, 4)$

For $\theta = \pi,\ r = \frac{32}{3} \Rightarrow (r,\theta) = \left(\frac{32}{3}, \pi\right) \to (x, y) = \left(-\frac{32}{3}, 0\right)$

For $\theta = \frac{3\pi}{2},\ r = -16 \Rightarrow (r,\theta) = \left(-16, \frac{3\pi}{2}\right) \to (x, y) = (0, 16)$

Thus, length of vertical transverse axis is $16 - 4 = 12 \Rightarrow 2a = 12$
$\Rightarrow a = 6$

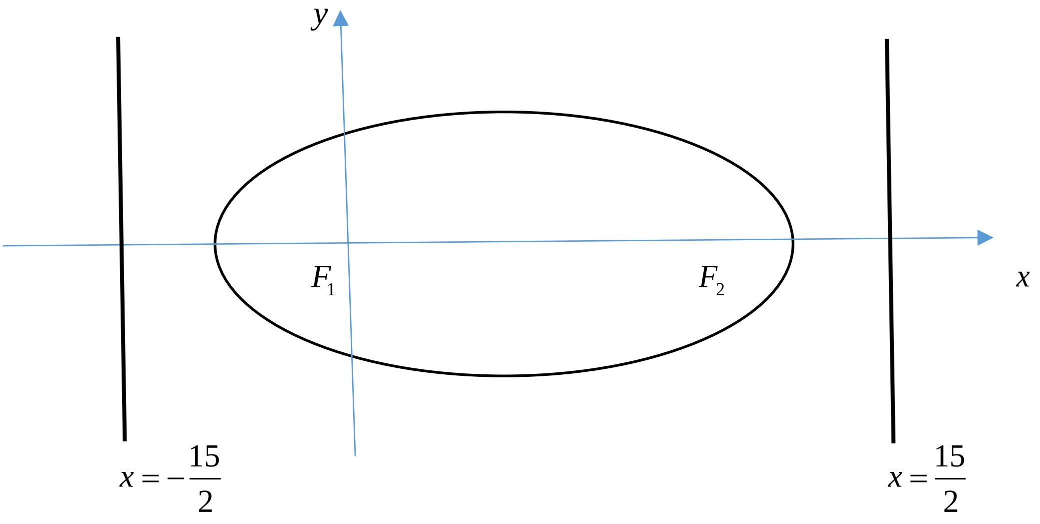
and centre is at $(0,10)$. It therefore follows that the hyperbola has been shifted upwards by 10 units implying that $c = 10$ and $b = 8$. Since one directrix is $y = \frac{32}{5}$, we should have another directrix given by $y = 16 - \left(\frac{32}{5} - 4\right) = \frac{68}{5}$ and the asymptotes are $y - 10 = \pm\frac{6}{8}x \Rightarrow y = 10 \pm \frac{3}{4}x$.



(b) $r = \frac{15}{3 - 2\cos\theta} = \frac{15/3}{1 - 2/3\cos\theta}$

$\Rightarrow e = \dfrac{2}{3} < 1$

$\therefore$ The conic section is an ellipse with vertical directrix to the left of the pole.



For $\theta = 0$ and $\theta = \pi$, $r = \dfrac{15}{3 - 2\cos 0} = 3$ and $r = \dfrac{15}{3 - 2\cos \pi} = 15$

$\therefore x = 3\cos\pi = -3 \qquad x = 15\cos 0 = 15$

$y = 3\sin\pi = 0 \qquad y = 15\sin 0 = 0$

$\therefore$ Foci: $(0,0)$ and $(12,0)$

Vertices: $(-3,0)$ and $(15,0)$.

Centre: $(6,0)$

$$ek = 5 \Rightarrow k = 5 \times \frac{3}{2} = \frac{15}{2}$$

Directrices: $x = -\dfrac{15}{2}$ and $x = \dfrac{39}{2}$.

△