2. **DIFFERENTIAL CALCULUS**

Differential calculus, being a subfield of calculus, is concerned with the study of the rates at which quantities change, primarily the derivatives of functions. You have already learnt about derivatives of elementary real-valued functions of single variables as well as different rules of differentiation. In this chapter, we will focus mainly on the application of derivatives.

2.1 **HIGHER ORDER DERIVATIVES**

If $y = f(x)$ is a differentiable function in a given interval, then its derivative is given by

$$y' = \frac{dy}{dx}.$$

This derivative is called the first (first order) derivative of y with respect to x. If y' is also a differentiable function of x in the same interval, then its derivative

$$y'' = \frac{d}{dx} y' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2}$$

is called the second (second order) derivative of y with respect to x. Similarly, the n^{th} derivative of y, if it exists , is given by

$$y^{(n)} = \frac{d}{dx}\left(y^{(n-1)}\right),$$

for any positive integer n. Therefore, computation of higher order derivatives follows by repeated application of the differentiation rules you already know.

Example 2.1.1

If the gas in a closed container is maintained at a constant temperature T, the pressure P is related to the volume by a formula

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

where a, b, n and R are constants. Find (a) $\frac{dP}{dV}$ (b) $\frac{d^3P}{dV^3}$.

Solution

(a) $P = (nRT)(V - nb)^{-1} - (an^2)V^{-2}$

$$\frac{dP}{dV} = -(nRT)(V - nb)^{-2} + 2an^2V^{-3}$$

$$\Rightarrow \frac{dP}{dV} = -\frac{(nRT)}{(V - nb)^2} + \frac{2an^2}{V^3}.$$

(b) Since $-(nRT)(V - nb)^{-2} + 2an^2V^{-3}$, we have that

$$\frac{d^2P}{dV^2} = 2nRT(V - nb)^{-3} - 6an^2V^{-4} \quad \text{so that}$$

$$\frac{d^3P}{dV^3} = -\frac{6nRT}{(V - nb)^4} + \frac{24an^2}{V^5}$$

$\triangle$

Example 2.1.2

1. Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$.

2. Given that $y = \cosh x - x^n$, where $n \geq 1$ is an integer, find an expression for $y^{(n)}$ and use it to find $y^{(8)}(0)$.

Solution

1. We differentiate implicitly with respect to x.

$$\frac{d}{dx}\left(2x^3\right) - \frac{d}{dx}\left(3y^2\right) = \frac{d}{dx}(7)$$

$$6x^2 - 6y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x^2}{y}.$$

Using quotient rule, we get

$$\frac{d^2y}{dx^2} = \frac{y\frac{d}{dx}\left(x^2\right) - x^2\frac{d}{y}\left(y\right)}{y^2} = \left(\frac{2xy - x^2\frac{dy}{dx}}{y^2}\right).$$

Since $\frac{dy}{dx} = \frac{x^2}{y}$, we have that

$$\frac{d^2y}{dx^2} = \frac{2xy - x^2\left(\frac{x^2}{y}\right)}{y^2} = \frac{2xy^2 - x^4}{y^3}.$$

2. $y = \cosh x - x^n \Rightarrow y' = \sinh x - nx^{n-1}$

$$\begin{aligned} y'' &= \cosh x - n(n-1)x^{n-2} \\ y''' &= \sinh x - n(n-1)(n-2)x^{n-3} \\ &\vdots \\ y^{(n)} &= \begin{cases} \sinh x - n!, & n \text{ is odd} \\ \cosh x - n!, & n \text{ is even} \end{cases}. \end{aligned}$$

Thus, $y^{(8)} = \cosh x - 8! \Rightarrow y^{(8)}(0) = 1 - 8! = -40319.$ △

2.2 TANGENTS AND NORMALS TO GENERAL PLANE CURVES

Recall that for a function $y = f(x)$, the first derivative

$$y' = f'(x) = \frac{dy}{dx}$$

is called the gradient function and so y' gives the gradient of the curve at every point in a given interval.

Definition 2.2.1

The gradient of the curve $y = f(x)$ at a point $x = x_0$ is given by

$$y'(x_0) = f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0},$$

and the tangent line to this curve at $x = x_0$ is the line through $(x_0, f(x_0))$ with the gradient $y'(x_0)$, i.e.

$$y - f(x_0) = y'(x_0)(x - x_0)$$

is the tangent line to this curve at $x = x_0$.

<u>Definition 2.2.2</u>

A line is normal to a curve at a point if it is perpendicular to the curve's tangent line there. The line is called the normal line to the curve at that point.

Recall that if M_1 is the gradient of the tangent line at a point and M_2 is the gradient of the normal line at the same point, then

$$M_1.M_2 = -1.$$

<u>Example 2.2.1</u>

Find the tangent and normal lines to the curve $x^2 - xy + y^2 = 7$ at the point $(-1,2)$.

<u>Solution</u>

Recall that this curve is an ellipse rotated at an angle $\alpha = \frac{\pi}{4}$.

Then, using implicit differentiation, we get

$$2x - x\frac{dy}{dx} - y\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{y-2x}{2y-x}$$
$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x,y)=(-1,2)} = \frac{2+2}{2(2)+1} = \frac{4}{5}$$

$\therefore$ equation of tangent line at $(-1,2)$ is

$$y - 2 = \frac{4}{5}\left(x+1\right), \text{ i.e. } y = \frac{4x+14}{5}.$$

The gradient of the normal is

$$M_2 = \frac{-1}{\frac{4}{5}} = -\frac{5}{4}$$

$\therefore$ equation of the normal at $(-1,2)$ is

$$y - 2 = -\frac{5}{4}(x+1), \text{ i.e. } y = \frac{3-5x}{4}.$$

△

Recall that derivatives can be used to find extreme values of a function, points of inflection and to predict and analyse the shapes of graphs. Let us consider the application of derivatives to the mean value theorems, Newton's method and L'Hospital's rule.

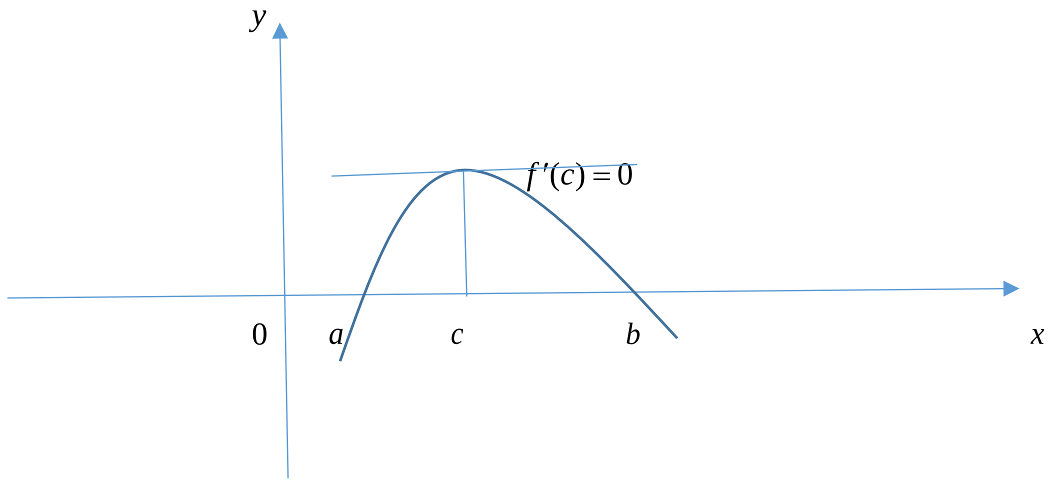
Theorem 2.2.1 (Rolle's Theorem)

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable at every point of its interior (a,b). If

$$f(a) = f(b) = 0,$$

Then there is at least one number c in (a,b) at which

$$f'(c) = 0.$$



Example 2.2.2

Verify the Rolle's Theorem for the polynomial function

$$f(x) = \frac{x^3}{3} - 3x$$

in the interval $[-3,3]$.

Solution

Clearly, f is continuous on $\mathbb{R}$ and differential hence continuous on every point of $[-3,3]$ and differentiable at every interior point of $(-3,3)$. Also

$$f(-3)\frac{(-3)^3}{3}-3(-3)=0=f(3)=\frac{3^3}{3}-3(3).$$

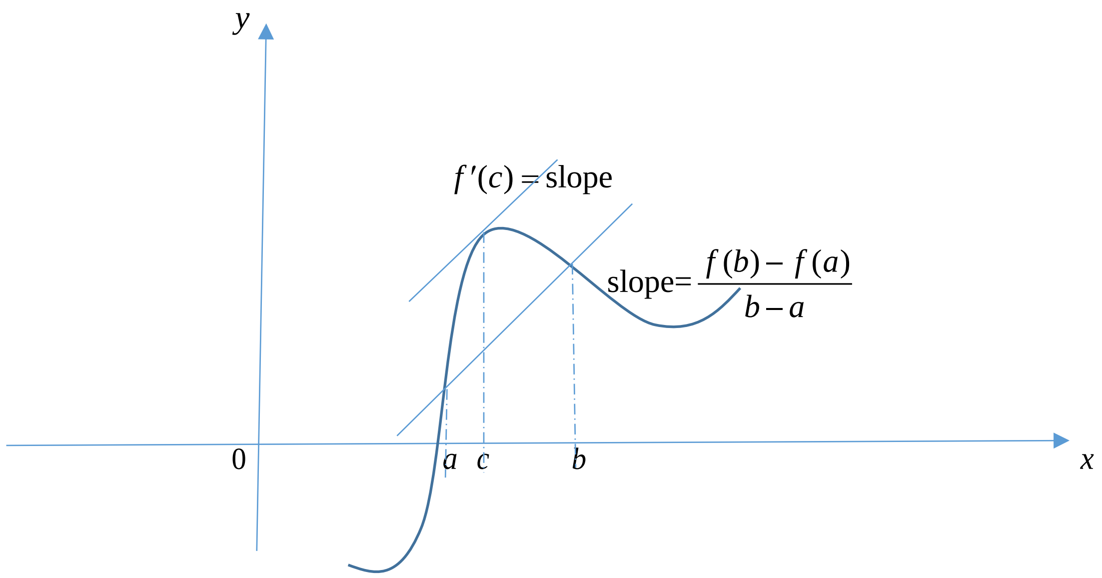
$$f'(x)=x^2-3. \quad f'(x)=0 \Rightarrow x=\pm\sqrt{3}.$$

Therefore, $f'(x)$ is zero at two points $c_1=-\sqrt{3}$ and $c_2=\sqrt{3}$ and that $c_1, c_2 \in (-3,3)$. Hence verified! △

<u>Theorem 2.2.2</u> (Mean Value Theorem)

Suppose that $y=f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the interval's interior (a,b). Then, there is at least one point c in (a,b) at which

$$\frac{f(b)-f(a)}{b-a}=f'(c) \tag{2.1}$$



<u>Theorem 2.2.3</u> (Cauchy's Generalised Mean Value Theorem)

Suppose that $f(x)$ and $g(x)$ are continuous functions in $[a,b]$ and differentiable in (a,b). Then, there is at least one point c in (a,b) at which

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}, \tag{2.2}$$

provided $g(b)\neq g(a)$ and $f'(x)$, $g'(x)$ are not simultaneously zero.

<u>NOTE</u>: (2.1) is a special case of (2.2) with $g(x)=x$.

Example 2.2.3

Given that

$$f(x) = 2x^2 - 7x + 10,$$

satisfies the Mean Value Theorem in the interval $[2,5]$, find c.

Solution

$$\frac{f(5) - f(2)}{5-2} = \frac{25-4}{3} = 7.$$
$$\therefore\ f'(x) = 4x - 7 = 7 \Rightarrow x = \frac{7}{2}$$
$$\therefore\ c = \frac{7}{2} \in (2,5).$$

Corollary 2.2.1

If $f'(x) = 0$ at each point of an interval I, then $f(x) = C$ for all x in I, where C is a constant.

Corollary 2.2.2

If $f'(x) = g'(x)$ at each point of an interval I, then there exists a constant C such that

$$f(x) = g(x) + C, \text{ for all } x \in I.$$

Example 2.2.4

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through $(0,2)$.

Solution

Suppose there is a function $g(x)$ such that $f'(x) = g'(x)$. Then, $g'(x) = \sin x$ implying $g(x) = -\cos x$ so that by Corollary 2.2.2

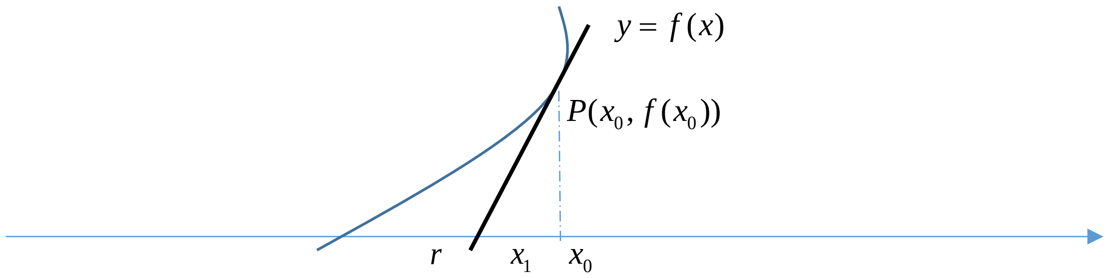
$$f(x) = -\cos x + C.$$

Since $f(x)$ passes through $(0,2)$, we have that

$$2 = -\cos 0 + C \Rightarrow C = 3$$
$$\therefore\ f(x) = -\cos x + 3.$$

2.3 **NEWTON'S METHOD**

Given the graph $y = f(x)$, we wish to find roots of the equation $f(x) = 0$. By graphing or by trial and error, consecutive integers can be found between which a root lies. Newton's method is a systematic way of using tangents to obtain a better approximation of a specific root.



Let r be a root and x_0 be an integer proceeding r. Then, the tangent at P will be

$$y - f(x_0) = f'(x_0)(x - x_0)$$
$$\Rightarrow 0 - f(x_0) = f'(x_0)(x - x_0)$$

so that

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

For one particular value of x, say $x = x_1$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, a tangent at $P_1(x_1, f(x_1))$ gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f(x_1)}{f'(x_1)}$$

and in general,

$$x_n = x_0 - \sum_{k=0}^{n} \left[\frac{f(x_k)}{f'(x_k)} \right].$$

NOTE: Success with Newton's method depends on the shape of the function's graph in the neighbourhood of the root.

Example 2.3.1

Determine the value of $\sqrt{3}$ to three decimal points.

Solution

Note that $\sqrt{3}$ is a solution of the equation $x^2 - 3 = 0$ and it lies between 1 and 2. Letting $f(x) = x^2 - 3,$ we have that $x_0 = 2$ and $f'(x_0) = f'(2) = 2(2) = 4$ so that by Newton's method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{1}{4} = 1.75.$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.75 - \frac{(3.0625 - 3)}{3.50} = 1.732.$$

Clearly, $(1.732)^2 = 2.999824.$ △

2.4 L'HOSPITAL'S RULE (OR L'HOPITAL'S RULE)

If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B,$ where A and B are either both zero or infinite, then

$$\lim_{x \to x_0} \left[\frac{f(x)}{g(x)} \right]$$

is often called an indeterminate of the form $\frac{0}{0}$ or $\frac{\infty}{\infty},$ respectively. L'Hospital's rule uses derivatives to evaluate such limits and it states that

(1) If $f(x)$ and $g(x)$ are differentiable in the interval (a, b) except possibly at a point x_0 in this interval, and if $g'(x) \neq 0,$ for $x \neq x_0,$ then

$$\lim_{x \to x_0} \left[\frac{f(x)}{g(x)} \right] = \lim_{x \to x_0} \left[\frac{f'(x)}{g'(x)} \right], \tag{2.3}$$

whenever the limit on the right can be found. In the case where $f'(x)$ and $g'(x)$ satisfy the same conditions as $f(x)$ and $g(x)$ given above, the process can be repeated.

(2) If $\lim_{x \to x_0} f(x) = \infty$ and $\lim_{x \to x_0} g(x) = \infty,$ the result (2.3) is also valid.

These can be extended to cases where $x \to \infty$ or $-\infty,$ and to cases where $x_0 = a$ or $x_0 = b$ in which only one sided limit, such as $x \to a^+$ or $x \to b^-,$ are involved. Other so-called indeterminate forms are $0.\infty,\ \infty^0,\ 0^0,\ 1^\infty$ and $\infty - \infty,$ and can be evaluated on replacing them by equivalent limits for which the above rules are applicable.

Example 2.4.1

Evaluate the following limits:

(a) $\lim_{x\to 1}\left(\dfrac{1+\cos \pi x}{x^2 - 2x + 1}\right)$ (b) $\lim_{x\to 0^+}(x^2 \ln x)$ (c) $\lim_{x\to 0}\left[(\cos x)^{1/x^2}\right].$

Solution

(a) The limit has indeterminate form $\dfrac{0}{0},$ so applying L'Hospital's rule, we get

$$\lim_{x\to 1}\left(\frac{1+\cos \pi x}{x^2 - 2x + 1}\right) = \lim_{x\to 1}\left(\frac{-\pi \sin \pi x}{2x - 2}\right) = \lim_{x\to 1}\left(\frac{-\pi^2 \cos \pi x}{2}\right) = \frac{\pi^2}{2}.$$

(b) The limit has the indeterminate form $0.\infty,$ so rewriting it and applying L'Hospital's rule, we get

$$\lim_{x\to 0^+}(x^2 \ln x) = \lim_{x\to 0^+}\left(\frac{\ln x}{1/x^2}\right) = \lim_{x\to 0^+}\left(\frac{1/x}{-2/x^3}\right) = \lim_{x\to 0^+}\left(-\frac{x^2}{2}\right) = 0.$$

(c) Since $\lim_{x\to 0}(\cos x) = 1$ and $\lim_{x\to 0}\left(\dfrac{1}{x^2}\right) = \infty,$ the limit is of the indeterminate form $1^\infty.$ Let $F(x) = (\cos x)^{1/x^2}.$ Then, $\ln\left[F(x)\right] = \dfrac{1}{x^2}\ln(\cos x).$ Thus,

$$\begin{aligned}\lim_{x\to 0}\left[\ln\left(F(x)\right)\right] &= \lim_{x\to 0}\left[\frac{\ln(\cos x)}{x^2}\right] \\ &= \lim_{x\to 0}\left(\frac{-\sin x}{2x \cos x}\right) \\ &= \lim_{x\to 0}\left(\frac{-\cos x}{2[-x\sin x + \cos x]}\right) \\ &= -\frac{1}{2}.\end{aligned}$$

Since the logarithm function is continuous,

$$\lim_{x\to 0}\left(\ln\left(F(x)\right)\right)=\ln\left(\lim_{x\to 0}\left(\cos x\right)^{1/x^2}\right)=-\frac{1}{2}$$
$$\Rightarrow \lim_{x\to 0}\left(\cos x\right)^{1/x^2}=e^{-1/2}.$$

△

2.5 **LINEARIZATION AND DIFFERENTIALS**

In this section, we want to use tangent lines to approximate functions. For the Leibniz notation $\frac{dy}{dx}$, we will use dy to estimate error in measurement and sensitivity to change.

Definition 2.5.1

If f is differentiable at $x=a$, then the approximating function

$$L(x)=f(a)+f'(a)(x-a)$$

is the linearization of f at a. The approximation

$$f(x)\approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x=a$ is the centre of the approximation.

NOTE: L is a tangent line at $x=a$ and it should be close to the graph for it to be a good approximation of f. Linear approximation loses accuracy away from its centre.

Example 2.5.1

1. Find the linearization of

$$f(x)=\sqrt{1+x}$$

   at $x=0$.

2. Use the linearization in (1) to find $\sqrt{4.2}$ and compare the result with the one gotten by using the linearization at $x=3$.

Solution

1. $f(x)=\sqrt{1+x}\Rightarrow f'(x)=\frac{1}{2}(1+x)^{-1/2}$

$$L(x) = \sqrt{1+0} + \frac{1}{2}(1+0)^{-1/2}(x-0) = 1 + \frac{x}{2}.$$

$$\therefore \sqrt{1+x} \approx 1 + \frac{x}{2}.$$

2. $\sqrt{4.2} = \sqrt{1+3.2} \approx 1 + \frac{3.2}{2} = 2.6$

At $x = 3,$

$$L(x) = \sqrt{1+3} + \frac{1}{2}(1+3)(x-3) = \frac{5}{4} + \frac{x}{4}$$

$$\therefore \sqrt{4.2} = \sqrt{1+3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 2.05$$

It can be seen that the value is more accurate when the linearization is closer to the centre.

Definition 2.5.2 △

Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is

$$dy = f'(x)dx,$$

which can also be written as

$$df = f'(x)dx.$$

If x changes from x_0 to $x_0 + dx,$ then the differential estimate of change is

$$df = f'(x_0)dx.$$

If $\Delta x = dx$ is an increment of x, then the true change of f as x changes from x_0 to $x_0 + dx$ is

$$\Delta f = f(x_0 + \Delta x) - f(x_0)$$

and the differential estimate is

$$df = f'(x_0)\Delta x$$

so that the approximate error is

$$|\Delta f - df|.$$

Example 2.5.2

1. Find dy if (a) $y = \sin 3x$ (b) $y = 2^x \ln x.$
2. The function

$$f(x) = \frac{1}{x}$$

changes value when x changes from x_0 to $x_0 + dx,$ where $x_0 = 0.5$ and $dx = 0.1.$ Find

(a) the true change of f

(b) the differential estimate of f

(c) the approximation error.

Solution

1. (a) $y = \sin 3x \Rightarrow dy = (3\cos 3x)dx$ (b) $y = 2^x \ln x \Rightarrow dy = \left(2^x.\frac{1}{x} + 2^x \ln 2 \ln x\right)dx.$

2. (a) $f(x) = \frac{1}{x} \Rightarrow \Delta f = f(x_0 + \Delta x) - f(x_0)$

$$= f(0.6) - f(0.5)$$

$$= \frac{5}{3} - 2$$

$$= -\frac{1}{3}.$$

(b) $f'(x) = -\frac{1}{x^2}$

$$\therefore df = f'(x_0)\Delta x = -\frac{1}{(0.5)^2}.(0.1) = (-4)(0.1) = -\frac{2}{5}.$$

(c) Approximation error $= |\Delta f - df| = \left|-\frac{1}{3} - \left(-\frac{2}{5}\right)\right| = \frac{1}{15}.$ △

2.6 **OPTIMISATION**

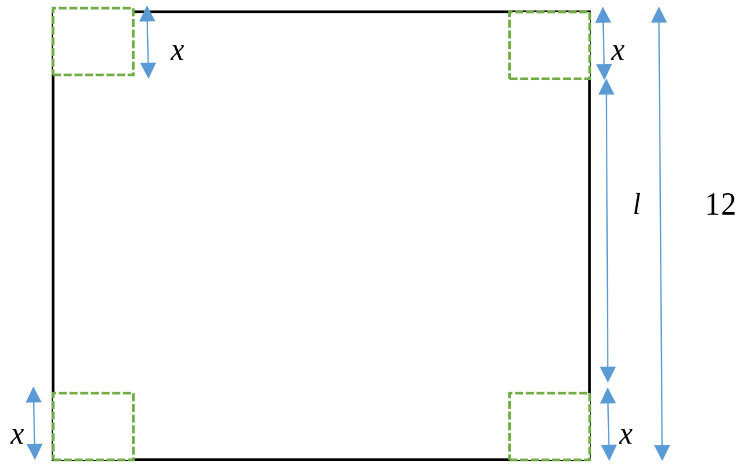
To optimise something means to maximise or minimise some aspect of it. Thus, in optimisation problems we look for the largest or smallest values that a function can take. Recall that the only domain points where a function can assume extreme values are critical points and endpoints.

Example 2.6.1

1. An open-top box is to be made by cutting small congruent squares from the corners of a $12cm \times 12cm$ sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much volume as possible?
2. An $1000cm^3$ oil can shaped like a right circular cylinder is to be made. What dimensions will use the least material?

Solution

1.



Since $l = 12 - 2x$, we should have that $l \geq 0 \Rightarrow 0 \leq x \leq 6$. Then, the volume will be given by

$$V = (12 - 2x)^2 x$$

$$\begin{aligned}\frac{dV}{dx} &= -2x(2)(12-2x) + (12-2x)^2 \\ &= (12-2x)(-4x+12-2x) \\ &= (12-2x)(12-6x) \\ &= 12(6-x)(2-x)\end{aligned}$$

$\therefore$ critical values are $x=6$ and $x=2$. But $x=6$ is an end point, so we only take $x=2$ as the critical point which lies in the interior of $[0,6]$.

At critical point, $V(2)=128$

At end point, $V(0)=0,\ V(6)=0.$

Therefore, the cut-out squares should have sides of length $2cm,$ for it to hold maximum volume.

2. The phrase 'least material' refers to the total area of the material to be used. We ignore thickness and the waste in manufacturing. Then

$$A=2\pi r^2+2\pi rh,$$

where r is the radius of the base-circle and h is the height of the right circular cylinder.

$$V=1000=\pi r^2 h \Rightarrow h=\frac{1000}{\pi r^2}$$
$$\Rightarrow A=2\pi r^2+2\pi r\left(\frac{1000}{\pi r^2}\right)=2\pi r^2+\frac{2000}{r}.$$

Thus, $0<r<\infty$ meaning that we have no end point, so we consider only the critical value.

$$\frac{dA}{dr}=4\pi r-\frac{2000}{r^2}.\ \frac{dA}{dx}=0, \Rightarrow 4\pi r^3=2000$$
$$\Rightarrow r=\left(\frac{500}{\pi}\right)^{1/3}.$$

Using second derivative test, we get

$\left.\frac{d^2A}{dx^2}\right|_{r=\left(\frac{500}{\pi}\right)^{1/3}}=\left.\left(4\pi+\frac{4000}{r^3}\right)\right|_{r=\left(\frac{500}{\pi}\right)^{1/3}}=12\pi>0.$ Therefore, the value of A at the critical point is an absolute minimum. This value of r gives $h=\frac{1000}{\pi r^2}=2\left(\frac{500}{\pi}\right)^{1/3}.$

△