MAT2100 Tutorial Sheet 2

Handing-in date: 28 November 2014

1. Prove that if $A \neq 0$, the curve $y = Ax^2 + Bx + C$ is a parabola and give an equation for the focus

2. Identify the curve $16x^2 + y^2 = 32x - 4y - 16$ and characterise it

3. Determine the eccentricity, foci and directrices of the given conic section and sketch it.

(a) $4x(x-2) + 3y(y+2) = 41$

(b) $9y^2 + 96x = 16x^2 + 72y + 144$

4. Find the conic section with centre of symmetry at the origin and satisfying the given condition

(a) Focus at $(3,0)$ and $e = 1.5$

(b) Focus at $(4,0)$ and directrix at $x = -9$

(c) An ellipse passing through the points $(2,1)$ and $(1,3)$.

5. Find the angles between the curves $xy = 8$ and $y = 10 - x - x^2$ at the points of intersection

6.Prove that the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents a hyperbola if $B^2 - 4AC > 0$, a parabola if $B^2 - 4AC = 0$ and an ellipse if $B^2 - 4AC < 0$.

7. Show that the value of $B^2 - 4AC$ remains the same under a rotation, i.e., it is invariant.

8. Identify and characterise the given conic sections

(a) $4x^2 + 2xy + y^2 - 60x + 120y = 0$

(b) $2xy - 6y - 4x + 11 = 0$

(c) $x^2 + 2xy + y^2 + 8x - 8y = 0$

8. Prove that the distance between two points is invariant under a rotation.

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1. $y = Ax^2 + Bx + C$

$y - C = A\left(x^2 + \frac{B}{A}x\right)$

$= A\left(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}\right)$

$- \frac{B^2}{4A}$

$y + \frac{B^2}{4A} - C = A\left(x + \frac{B}{2A}\right)^2$

$y' = Ax'$

$y' = y + \frac{B^2}{A} - C$

$x' = x + \frac{B}{2A}$

$y' = Ax'^2$

When the focus is $(0, p)$ and the directrix is $y = -p$, the eqn is

$x^2 = 4py$

Here $x'^2 = \frac{1}{A}y'$

Hence $4p = \frac{1}{A}$ $p = 4A$

Hence the focus is at $(0, p) = (0, 4A)$ and the directrix is at $y = -p = -4A$

$y' = 4A \Rightarrow y + \frac{B^2}{A} - C = 4A$

$y = 4A + C - \frac{B^2}{C}$

Hence the focus is at $\left(0, y = 4A + C - \frac{B^2}{C}\right)$ and the directrix is at $y' = -p$ (1)

$\therefore y + \frac{B^2}{A} - C = -4A$

$y = C - 4A - \frac{B^2}{A}$

The axis is $x' = 0$. This is the same as

$x = -\frac{B}{2A}$ //

$\underline{\underline{2}}$ $16x^2+y^2=32x-4y-16$

$16x^2-32x+y^2+4y=-16$

$16(x^2-2x)+y^2+4y+4-4=-16$

$16(x^2-2x+1-1)+(y+2)^2=-12$

$16(x-1)^2-16+(y+2)^2=-12$

$16(x-1)^2+(y+2)^2=4$

$$(x-1)^2+\frac{(y+2)^2}{4^2}=1$$

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$$

Thus if $x'=x-1$, $y'=y+2$ then we have an ellipse in the $O'x'y'$ system. Its origin is at $(1,-2)$.

$a=1$, $b=4$

$c=\sqrt{b^2-a^2}=\sqrt{16-1}=\sqrt{15}$

The foci are at $y'=\pm\sqrt{15}$

~~The ecch~~

The eccentricity is $e=\frac{c}{B}$

$\therefore e=\frac{\sqrt{15}}{4}$.

The directrices are at

$y'=\pm d=\pm\frac{b}{c}=\pm\frac{4}{\sqrt{15}}$

Now $y=y'-2$ (2)

$x=x'+1$

Hence the foci are at

$(x'=0, y'=\pm\sqrt{15})=(1,\pm\sqrt{15}-2)$

The directrices are at

$y=\pm\frac{4}{\sqrt{15}}-2$

$\frac{144}{16}=9$

$9\overline{)144}$

9

54

63

$\frac{72}{8}=9$

3(a) $4x(x-2)+3y(y+2)=41$

$4x^2-8x+3y^2+6y=41$

$4(x^2-2x)+3(y^2+6y)=41$

$4(x^2-2x+1-1)+3(y^2+6y+9-9)=41$

$4(x-1)^2-4+3(y+3)^2-27=41$

$4(x-1)^2+3(y+3)^2=72$

$$\frac{(x-1)^2}{18}+\frac{(y+3)^2}{24}=1$$

$x'=x-1,\quad y'=y+3$

$x=x'+1 \qquad y=y'-3$

in $O'x'y'$, this is an ellipse with $a=\sqrt{18}$, $b=\sqrt{24}$

$c=\sqrt{b^2-a^2}=\sqrt{24-18}$

$c=\sqrt{6}$

The foci are at $(0,\pm\sqrt{6})$ in $O'x'y'$. The eccentricity is $e=\frac{c}{b}=\frac{\sqrt{6}}{\sqrt{24}}$

41
27
$\overline{}$
6

$=\sqrt{\frac{6}{24}}=\frac{1}{\sqrt{4}}=\frac{1}{2}$

The directrix is at

$y'=\pm d=\pm\frac{b}{c}=\frac{\sqrt{24}}{\sqrt{6}}$

$y'=\pm 2.$

In Oxy, the foci are at $(1,\pm\sqrt{6}-3)$

The directrices are at

$y=\pm 2-3$

These are

$y=-1,\ y=-5$

The centre of the ellipse is at $x'=0$ or $x=1$

$y'=0$ or $y=-3$

3(b)

$9y^2-72y-16x^2+96x=144$

$9(y^2-8y)-16(x^2-6x)=144$

$9(y^2-8y+16)-9\times16$

$-16(x^2-6x+9)+16\times19=144$

$9(y-4y)^2-16(x-3)^2=144$

$$\frac{(y-4)^2}{16}-\frac{(x-3)^2}{9}=1$$

$$\frac{y'^2}{a^2}-\frac{x'^2}{b^2}=4$$

Let $y'=y-4$, $y=y'+4$

$x'=x-3$ $\quad x=x'+3$

Then in $Ox'y'$, the curve is a hyperbola.

$c=\sqrt{a^2+b^2}$

$a=4,\ b=3$

$c=\sqrt{16+9}=5$

$e=\frac{c}{a}=\frac{5}{4}=1{,}25$

$d=\frac{a}{e}=\frac{4}{5/4}=\frac{16}{5}$

The foci are at $(x'=\pm c, y'=0)$

These are at $(\pm5,0)$

~~In Oxy,~~

The directrices are at

$x'=\pm\frac{16}{5}$,

Hence in Oxy, the foci are at $(3\pm5,4)$ 3

The directrices are at

$x'=x-3=\pm\frac{16}{3}$

$\therefore x=3\pm\frac{16}{3}$

4 (a)

Focus at $(3,0)$, $e = 1.5$

This is a hyperbola. Hence the other focus is at $(-3,0)$

Here $c = 3$, $e = \frac{c}{a}$

$$b = \frac{c}{e} = \frac{3}{1.5} = 2$$

$$c^2 = a^2 + b^2$$

$$a = \sqrt{c^2 - b^2}$$

$$= \sqrt{9-4} = \sqrt{5}$$

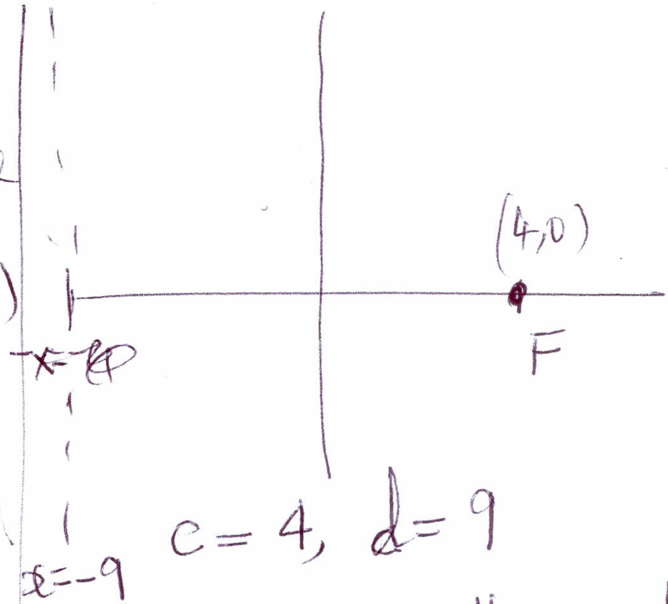
~~This gives~~ $\cancel{\frac{y^2}{4} - x^2}$

Thus

$$\frac{x^2}{5} - \frac{y^2}{4} = 1 \;//$$

(b) Focus at $(4,0)$, directrix at $x = -9$

$\therefore d = -9$, $c = 4$.



$c = 4$, $d = 9$

This must be an ellipse. Hence ~~a = 4~~ Another focus is $(-4,0)$. $c = 4$. Also, $d = 9$

$\frac{c}{a} = e$. $d = \frac{a}{e} \Rightarrow e = \frac{a}{d}$

$$\frac{c}{a} = \frac{a}{d} \qquad a^2 = cd$$

$$a^2 = 4 \times 9 = 36$$

$$a = 6.$$

$$c^2 = a^2 - b^2$$

$$b^2 = a^2 - c^2 = 6^2 - 4^2$$

$$= 36 - 16 = 20$$

$b = \sqrt{20}$, $e = \frac{6}{9} = \frac{2}{3}$

$$\therefore \frac{x^2}{36} + \frac{y^2}{20} = 1 \;//$$

4(c)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{4}{a^2}+\frac{1}{b^2}=1$$

$$\frac{1}{a^2}+\frac{9}{b^2}=1$$

$$\frac{1}{a^2}=\frac{\begin{vmatrix}1 & 1\\ 1 & 9\end{vmatrix}}{\begin{vmatrix}4 & 1\\ 1 & 9\end{vmatrix}}$$

$$=\frac{9-1}{36-1}=\frac{8}{35}$$

$$a=\sqrt{\frac{35}{8}}$$

$$\frac{1}{b^2}=\frac{\begin{vmatrix}4 & 1\\ 1 & 1\end{vmatrix}}{35}=\frac{4-1}{35}$$

$$b^2=\frac{35}{3},\ b=\sqrt{\frac{35}{3}}$$

$\therefore$ The ellipse is

$$\frac{8x^2}{35}+\frac{3y^2}{35}=1$$

$$8x^2+3y^2=35$$

5.

$$\tan\varphi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$y = 10 - x - x^2$$

$$y = \frac{8}{x}$$

At the intersection,

$$\therefore \frac{8}{x} = 10 - x - x^2$$

$$10x - x^2 - x^3 - 8 = 0$$

Let $x = 1$,

$$10 - 1 - 1 - 8 = 0$$

$$\therefore y = \frac{8}{x} = 8$$

Intersection is at $(1, 8)$.

$$\frac{dy}{dx} = -\frac{8}{x^2} \text{ for } y = \frac{8}{x}$$

When $x = 1$, $\frac{dy}{dx} = m_1 = -8$

For $y = 10 - x - x^2$,

$$\frac{dy}{dx} = -1 - 2x$$

$$\left.\frac{dy}{dx}\right|_{x=1} = -1 - 2 = -3 = m_2$$

The angle is

$$\tan\varphi = \frac{-3 + 8}{1 + (-3) \times (-8)}$$

$$= \frac{5}{25} = \frac{1}{4}$$

$$\varphi = \arctan\left(\frac{1}{4}\right).$$

6. Under the appropriate rotation this becomes

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

with

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$$

$$B' = B(\cos^2\alpha - \sin^2\alpha) + 2(C-A)\sin\alpha\cos\alpha$$

$$C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha$$

$$D' = D\cos\alpha + E\sin\alpha$$

$$E' = -D\sin\alpha + E\cos\alpha$$

$$F' = F$$

The quantity $B^2 - 4AC$ is invariant

$$\therefore B^2 - 4AC = B'^2 - 4A'C'$$

Since the purpose of the rotation was to remove the cross terms, $B' = 0$

$$B^2 - 4AC = -4A'C'$$

where

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

If both A' and C' have the same sign, the curve is an ellipse. But in that case

$$-4A'C' < 0$$

$\therefore B^2 - 4AC < 0$ for an ellipse.

If A' or $D' = 0$, $-4A'C' = 0$

Hence $B^2 - 4AC = 0$. But this means the eqn represents a parabola.

If A' and C' have opposite signs $-4A'C' > 0$

$\therefore B^2 - 4AC > 0$.

This means we have a hyperbola.

7. If B^2-4AC is invariant, then $B'^2-4A'C'=B^2-4AC$

$$B' = B\cos 2\alpha + (C-A)\sin 2\alpha$$

$$B'^2 = B^2\cos^2 2\alpha + (C-A)^2\sin^2 2\alpha + 2B(C-A)\cos 2\alpha \sin 2\alpha$$

$$A'C' = (A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha)(A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha)$$

$$= A^2\cos^2\alpha\sin^2\alpha - AB\sin\alpha\cos^3\alpha + \cancel{AC\cos^4\alpha}$$
$$+ AB\cos\alpha\sin^3\alpha - B^2\sin^2\alpha\cos^2\alpha + BC\cos^3\alpha\sin\alpha$$
$$+ \cancel{AC\sin^4\alpha} - BC\sin^3\alpha\cos\alpha + C^2\cos^2\alpha\sin^2\alpha$$

$$= A^2\cos^2\alpha\sin^2\alpha + AB\sin\alpha\cos\alpha(\sin^2\alpha - \cos^2\alpha)$$
$$+ AC(\cos^4\alpha + \sin^4\alpha) - BC\sin\alpha\cos\alpha(\cos^2\alpha - \sin^2\alpha)$$
$$+ C^2\cos^2\alpha\sin^2\alpha - B^2\sin^2\alpha\cos^2\alpha$$

$=$

$$B'^2 - 4A'C' = B^2\cos^2 2\alpha + C^2\sin^2 2\alpha + A^2\sin^2 2\alpha - 2AC\sin^2 2\alpha$$
$$+ 2BC\cos 2\alpha\sin 2\alpha - 2BA\cos 2\alpha\sin 2\alpha$$
$$- 4A^2\cos^2\alpha\sin^2\alpha + 4AB\sin\alpha\cos\alpha\cos 2\alpha$$
$$- 4AC(\cos^4\alpha + \sin^4\alpha) - 4BC\sin\alpha\cos\alpha\cos 2\alpha$$
$$- 4C^2\cos^2\alpha\sin^2\alpha + B^2\sin^2 2\alpha$$

$$= B^2(\cos^2 2\alpha + \sin^2 2\alpha) + C^2\cancel{[\sin^2 2\alpha - \sin^2 2\alpha]}$$

$$+ A^2\cancel{[\sin^2 2\alpha - \sin^2 2\alpha]}$$

$$+ BC[2\cos 2\alpha \sin 2\alpha + 4\cos^3\alpha\sin\alpha - 4\sin^3\alpha\cos\alpha]$$

$$+ AB[4\cos\alpha\sin^3\alpha - 4\sin\alpha\cos^3\alpha - 2\cos 2\alpha\sin 2\alpha]$$

$$+ AC[-4\cos^4\alpha - 4\sin^4\alpha - 2\sin^2 2\alpha]$$

$$= B^2 - 4(\cos^4\alpha + \sin^4\alpha + 2\sin^2\alpha\sin^2\alpha)AC$$

$$+ BC[2\cos 2\alpha\sin 2\alpha - 4\cos\alpha\sin\alpha(\cos^2\alpha - \sin^2\alpha)]$$

$$+ AB[4\cos\alpha\sin\alpha(-\sin^2\alpha + \cos^2\alpha) - 2\cos 2\alpha\sin 2\alpha]$$

$$= B^2 - 4(\cos^2\alpha + \sin^2\alpha)^2 AC$$

$$+ BC[2\cos 2\alpha\sin 2\alpha - 2\sin 2\alpha\cos 2\alpha]$$

$$+ AB[2\sin 2\alpha\cos 2\alpha - 2\cos 2\alpha\sin 2\alpha]$$

$$= B^2 - 4AC$$

$$\therefore B'^2 - 4A'C' = B^2 - 4AC \;//$$

8(a) $4x^2+4xy+y^2-60x+120y=0$

$A=4,\ B=4,\ C=1\ D=-60\ E=120$

Under the rotation, this becomes

$A'x'^2+B'x'y'+C'y'^2+D'x'+E'y'+F'=0$

$A'=A\cos^2\alpha+B\cos\alpha\sin\alpha+C\sin^2\alpha$

$B'=B\cos2\alpha+(C-A)\sin2\alpha$

$C'=A\sin^2\alpha-B\sin\alpha\cos\alpha+C\cos^2\alpha$

$D'=D\cos\alpha+E\sin\alpha$

$E'=-D\sin\alpha+E\cos\alpha$

$F'=F$

We want $B'=0$, and so

$\tan\alpha=\frac{C-A}{B}+\sqrt{\left(\frac{C-A}{B}\right)^2+1}$

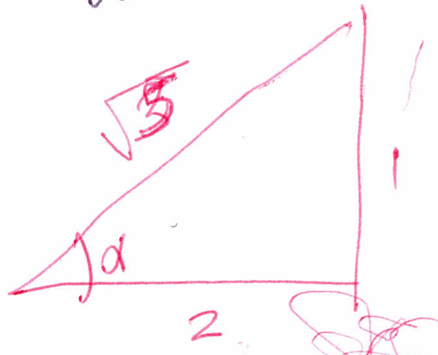
$=\frac{1-4}{4}+\sqrt{\left(\frac{1-4}{4}\right)^2+1}$

$=-\frac{3}{4}+\sqrt{\frac{9}{16}+1}$

$=-\frac{3}{4}+\sqrt{\frac{9+16}{16}}$

$=-\frac{3}{4}+\frac{5}{4}=\frac{2}{4}=\frac{1}{2}$

8a



$\cos\alpha=\frac{2}{\sqrt5},\ \sin\alpha=\frac{1}{\sqrt5}$

$A'=4\cdot\frac{4}{5}+4\times\frac{2}{5}\times1+1\times\frac{1}{5}$

$=\frac{16+8+1}{5}=\frac{25}{5}=5$

$C'=4\times\frac{1}{5}-4\times\frac{1\times2}{5}+\frac{1\times4}{5}$

$=\frac{4-8+4}{5}=0$

$D'=-\frac{60\times2}{\sqrt5}+120\times\frac{1}{\sqrt5}=0$

$E'=+60\times\frac{1}{\sqrt5}+\frac{120\times2}{\sqrt5}$

$=\frac{300}{\sqrt5}$

$\Rightarrow$

$5x'^2+\frac{300}{\sqrt5}y'=0$

This is a parabola:

$x'^2=-\frac{60}{\sqrt5}y'$

Comparing with

$$x^2 = 4py,$$

we see that

$$4p = -\frac{60}{\sqrt{5}}$$

$$p = -\frac{15}{\sqrt{5}} = -\frac{15\sqrt{5}}{5}$$

$$= -3\sqrt{5}$$

Hence the focus is at

$$y' = (0, -3\sqrt{5})$$

and the directrix is

$$y' = 3\sqrt{5}$$

But $y' = x\cos\alpha + y\sin\alpha$

Hence in Oxy, the directrix is the line

$$3\sqrt{5} = \frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y$$

$$3\times 5 = 2x + y$$

$$2x + y = 15$$

The focus is at $(x'=0, y'=3\sqrt{5})$

~~since $x' = (x\cos\alpha +$~~

But $x = x'\cos\alpha - y'\sin\alpha$

$y = x'\sin\alpha + y'\cos\alpha$

$\Rightarrow$ The focus is at

$$x = 0 + 3\sqrt{5}\times\frac{1}{\sqrt{5}} = +3$$

$$y = 0 + (-3\sqrt{5})\times\frac{2}{\sqrt{5}}$$

$$= -6$$

Hence focus is at

$$(x, y) = (3, -6)$$

and the directrix is the line

$$y = -2x + 15$$

The eccentricity is

$e = 1$, of course.

The vertex is at $y' = 0$

$x' = 0$

$x = 0, y = 0$

Vertex is at $(0, 0)$.

(b) $2xy - 6y - 4x + 11 = 0$

$A = 0,\ B = 2,\ C = 0,\ D = -4,\ E = -6$

$F = 11$

$A' = \cancel{0}$

$$\tan\alpha = \frac{0-0}{2} + \sqrt{\frac{0}{2} + 1}$$

$\tan\alpha = 1$, $\cos\alpha = \frac{1}{\sqrt{2}}$, $\sin\alpha = \frac{1}{\sqrt{2}}$

$$A' = 0 + 2 \times \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 = 1$$

$B' = 0$

$$C' = 0 - 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 = -1$$

$$D' = -4 \times \frac{1}{\sqrt{2}} - 6 \times \frac{1}{\sqrt{2}} = -\frac{10}{\sqrt{2}}$$

$$E' = 4 \times \frac{1}{\sqrt{2}} - 6 \times \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$$

$$\Rightarrow x^2 - y^2 - \frac{10}{\sqrt{2}}x - \frac{2}{\sqrt{2}}y + 11 = 0$$

$$\cancel{\left(x^2 - \frac{10}{\sqrt{2}}x\right)}$$

This is a hyperbola

$$x^2 - \frac{10}{\sqrt{2}}x - \left(y^2 + \frac{2}{\sqrt{2}}y\right) + 11 = 0$$

$$\left(x^2 - \frac{10x}{\sqrt{2}} + \frac{25}{2} - \frac{25}{2}\right)$$

$$- \left(y^2 + \frac{2}{\sqrt{2}}y + \frac{1}{2} - \frac{1}{2}\right) + 11 = 0$$

$$\left(x - \frac{5}{\sqrt{2}}\right)^2 - \frac{25}{2} - \left(y + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} + 11 = 0$$

$$\cancel{\left(x - \frac{5}{\sqrt{2}}\right)^2 - 13 + 11 - \left(y + \frac{1}{\sqrt{2}}\right)}$$

$$\left(x - \frac{5}{\sqrt{2}}\right)^2 - \left(y + \frac{1}{\sqrt{2}}\right)^2 - \frac{25}{2} + \frac{1}{2} + 11 = 0$$

$$\left(x - \frac{5}{\sqrt{2}}\right)^2 - \left(y + \frac{1}{\sqrt{2}}\right)^2 = 1$$

$x' = x - \frac{5}{\sqrt{2}}$

$x'^2 - y'^2 = 1$ $\quad y' = y + \frac{1}{\sqrt{2}}$

This is a hyperbola with

$a = 1,\ b = 1;$

$$\frac{x'^2}{1} - \frac{y'^2}{1} = 1$$

$$c = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Hence the ~~foci~~ foci are at

$F = (\pm\sqrt{2}, 0) = (x', y')$

The symmetry axes are

$x' = 0,\ y' = 0$

The eccentricity is

$e = \frac{c}{a} = \frac{\sqrt{2}}{1} = \sqrt{2}$

The directrices are the lines $x' = \pm d = \pm \frac{a}{c}$,

$\therefore x' = \pm \frac{1}{\sqrt{2}}$

$x = x'\cos\alpha - y'\sin\alpha$

$y = x'\sin\alpha + y'\cos\alpha$

Hence the ~~focus~~ foci have the coordinates

$x = x'\cos\alpha - y'\sin\alpha$

$= \pm\sqrt{2} \cdot \frac{1}{\sqrt{2}} - 0 \times \frac{1}{\sqrt{2}}$

$= \pm 1$

$y = \pm\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 0$

$= \pm 1$

When $(x', y') = (\sqrt{2}, 0)$

then $(x, y) = (1, 1)$

When $(x', y') = (-\sqrt{2}, 0)$,

$(x, y) = (-1, -1)$

These are the foci

$x' = \pm \frac{1}{\sqrt{2}}$ gives

$x\cos\alpha + y\sin\alpha = \pm \frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = \pm \frac{1}{\sqrt{2}}$

$x + y = \pm 1$

$y = -x \pm 1$

are the directrices

The symmetry axes are $x' = 0$ and $y' = 0$

These gives

$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = 0$

$\Rightarrow x + y = 0$

and $-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 0$

$-x + y = 0$

The vertices are $x' = \pm 1, y' = 0$

$x = \pm 1 \frac{1}{\sqrt{2}} \times 0 = \pm \frac{1}{\sqrt{2}}$

$y = \pm \frac{1}{\sqrt{2}} = \pm \frac{1}{\sqrt{2}}$

$\Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(c) $x^2+2xy+y^2+8x-8y=0$

$A=1, B=2, C=1, D=8, E=-8$

$\tan\alpha = \frac{1-1}{2} + \sqrt{\left(\frac{1-1}{2}\right)^2+1}$

$\tan\alpha = 1$

$\cos\alpha = \sin\alpha = \frac{1}{\sqrt{2}}$

~~We have an ellipse~~.

$A' = 1\times\frac{1}{2} + 2\times\frac{1}{2} + \cancel{8}\,1\times\frac{1}{2}$

$= \frac{1}{2}+\frac{1}{2}+1 = 2$

$B' = 0$

$C' = 1\times\frac{1}{2} - 2\times\frac{1}{2} + 1\times\frac{1}{2}$

$= 1-1 = 0$

$D' = \frac{8}{\sqrt{2}} \cancel{+} - \frac{8}{\sqrt{2}} = 0$

$E' = -8\times\frac{1}{\sqrt{2}} - 8\times\frac{1}{\sqrt{2}} = -\frac{16}{\sqrt{2}}$

$= \frac{-16\sqrt{2}}{2} = -8\sqrt{2}$

$\cancel{F0}$

$\Rightarrow 2x'^2 - 8\sqrt{2}y' = 0$

$x'^2 = 4\sqrt{2}y'$

$x'^2 = 4py'$

gives $p=\sqrt{2}$.

The focus is $(0,\sqrt{2})$ and the directrix is $y' = -\sqrt{2}$

The lines of symmetry are $x'=0$, $y'=0$.

In Oxy, we have

$x = x'\cos\alpha - y'\sin\alpha$

$y = x'\sin\alpha + y'\cos\alpha$

$x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$

$y = \frac{1}{\sqrt{2}}x^2 + \frac{1}{\sqrt{2}}y'$

~~The line x'=0 gi~~

Also,

$x' = x\cos\alpha + y\sin\alpha$

$y' = -x\sin\alpha + y\cos\alpha$

$x'=0$ is the line

$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = 0$

$x+y=0$

$y'=0$ is the line

$0 = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$

$x-y=0$

The directrix is the line

$y' = -\sqrt{2}$

$\therefore\ -x\frac{1}{\sqrt{2}} + y\frac{1}{\sqrt{2}} = -\sqrt{2}$

$-x + y = -2$

$y = x - 2$

This is the directrix.

The ~~verticea~~ vertex is $(0,0)$ in both Oxy and $Ox'y'$

9. $d' = \sqrt{(x_1'-x_2')^2+(y_1'-y_2')^2}$

$\cancel{x = (x' \cos}$

$d^2 = x_1'^2 + x_2'^2 - 2x_1'x_2'$
$\quad + y_1'^2 + y_2'^2 - 2y_1'y_2'$

If d is invariant,

$d^2 = d'^2$

$d'^2 = (x_1\cos\alpha + y_1\sin\alpha)^2 + (x_2\cos\alpha + y_2\sin\alpha)^2$
$- 2(x_1\cos\alpha + y_1\sin\alpha)(x_2\cos\alpha + y_2\sin\alpha)$
$+ (-x_1\sin\alpha + y_1\cos\alpha)^2 + (-x_2\sin\alpha + y_2\cos\alpha)^2$
$- 2(-x_1\sin\alpha + y_1\cos\alpha)(-x_2\sin\alpha + y_2\cos\alpha)$

$\Rightarrow x_1^2\cancel{\cos^2\alpha} + y_1^2\cancel{\sin^2\alpha} + \cancel{2x_1y_1\cos\alpha\sin\alpha}$
$+ x_2^2\cancel{\cos^2\alpha} + y_2^2\cancel{\sin^2\alpha} + \cancel{2x_2y_2\cos\alpha\sin\alpha}$
$- 2x_1x_2\cancel{\cos^2\alpha} - \cancel{2x_1y_2\cos\alpha\sin\alpha}$
$- \cancel{2y_1x_2\sin\alpha\cos\alpha} - 2y_1y_2\sin^2\alpha$
$+ x_1^2\cancel{\sin^2\alpha} + y_1^2\cancel{\cos^2\alpha} - \cancel{2x_1y_1\sin\alpha\cos\alpha}$
$+ x_2^2\cancel{\sin^2\alpha} + y_2^2\cancel{\cos^2\alpha} - \cancel{2x_2y_2\sin\alpha\cos\alpha}$
$- 2x_1x_2\cancel{\sin^2\alpha} + \cancel{2x_1y_2\sin\alpha\cos\alpha}$
$+ \cancel{2y_1x_2\cos\alpha\sin\alpha} - 2y_1y_2\cos^2\alpha$

$= x_1^2 + y_1^2 + x_2^2 + y_2^2$
$- 2x_1x_2 - 2y_1y_2$

$= (x_1^2 + y_1^2 - 2y_1y_2)$
$\cancel{+ (x_2^2 + y_2^2 - 2}$

$= (x_1^2 + x_2^2 - 2x_1x_2) + (y_1^2 + y_2^2 - 2y_1y_2)$

$= (x_1 - x_2)^2 + (y_1 - y_2)^2$

$= d^2$

$\therefore d'^2 = d^2.$