MAT2110 Test 2
25 April 2015
Time: Two Hours
Answer: Any Three Questions

1. (a) Explain why the conic section

$$r = \frac{5}{2 - 3\cos\theta}$$ 2 +4

is a hyperbola and determine its centre in Cartesian coordinates.

(b) Prove that the differential arc length can be written as $ds = \sqrt{1 + (dx/dy)^2}dy$ and hence calculate the length of the ~~parabola~~ line $y = 2x + 7$ from the point (0,7) to the point ((3,13) 2 + 3

(c) Determine the equation of the plane which contains the point $P(1, 2, -1)$ and the line 3 + 3 + 3

$$\frac{x-2}{2} = \frac{y+2}{-3} = z$$

2. (a) (i) Show that the parametric equations $x = sin\alpha$, $y = cos2\alpha$ describe a parabola. 2

(ii) Determine the curvature of the parabola at the point $\alpha = 0$. 4

(b) Determine the equation of the line tangent to the curve in **2**. (**a**) (**i**) at the point $\alpha = \pi/4$. 10

(c) Determine the rational fraction in lowest terms that is equal to 8.32323232...... 4

3 (a) Determine the Maclaurin expansion of $\ln(1 + x)$. 6

(b) (i) Find the equation of the line that passes through the point $(1, 1, 1)$ and is parallel to both the planes $3x - 4y + z = 10$ and $2x + y + z = 12$. 7

(ii) Determine the distance of the line from each of the planes.

(c) Determine the unit normal of the curve whose parametric equations are $x = t^3 - 3t$ and $y = 4 - t^2$. 3

4. (a) Determine the lengths of the sides and the interior angles for the triangle whose vertices are (1.2,3), (2,3,1) and (3,2,1). 6

(b) (i) Find the polar and Cartesian equations of a circle of radius 4 centred on $(-1, 3)$ 4

(ii) Determine the intersection of the circle and the line $y = x$, expressing your answer in both Cartesian and polar coordinates. 10

(c) The position vector of a particle of mass $m = 2$ is given by $\mathbf{R} = t^3\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$, where t is the time. For $t = 3$ s, determine

(i) the position,

(ii) the velocity

(iii) the force acting on the particle 4

(iv) the instantaneous direction of motion

MAT 2110 Test 2 Solns

(a) $r = \dfrac{5}{2-3\cos\theta}$

$= \dfrac{5/2}{1-\frac{3}{2}\cos\theta}$

2

From $r = \dfrac{ep}{1-e\cos\theta}$

we see that $e > \frac{3}{2}$ and so the curve is a hyperbola.

$r\left(1-\frac{3}{2}\cos\theta\right) = 5/2$

$r - \frac{3}{2}r\cos\theta = 5/2$

$r = \frac{3}{2}x + \frac{5}{2}$

$2r = 3x + 5$

$4r^2 = 9x^2 + 30x + 25$

$-5x^2 + 4y^2 - 30x = 25$

$-5(x^2+6x) + 4y^2 = 25$

$-5(x^2+6x+9-9) + 4y^2 = 25$

$-5(x+3)^2 + 45 + 4y^2 = 25$

$4y^2 - 5(x+3)^2 = 20$

$5(x+3)^2 - 4y^2 = 20$

$\dfrac{(x+3)^2}{4} - \dfrac{y^2}{5} = 1$

This hyperbola has centre $(-3,0)$ //

4

1(b) $ds = \sqrt{dx^2 + dy^2}$

$= \left[dy^2\left(1 + \frac{dx^2}{dy^2}\right)\right]^{1/2}$ $1\frac{1}{2}$

$= dy\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2}$ 2 3 //

$$s = \int_{P_1}^{P_2} ds$$

Here $x = \frac{y-7}{2}$

$$\frac{dx}{dy} = \frac{1}{2}$$

2 $\frac{1}{2}$ 3

$$s = \int_{P_1}^{P_2} \left(1 + \left(\frac{1}{2}\right)^2\right)^{1/2} dy$$

$= \left(1+\frac{1}{4}\right)^{1/2} y\Big|_7^{13}$

$= \left(\frac{5}{4}\right)^{1/2}\left[13 - 7\right]$

$= \frac{6}{2}\sqrt{5} = 3\sqrt{5}$ //

2

1(c): The line

$$\frac{x-2}{2} = \frac{y+2}{-3} = \frac{z}{1}$$

~~contain is to~~ ~~is~~ coincides with the vector

$$\vec{V}_1 = (2, -3, 1)$$

A vector in the same plane as the line is

$$\vec{V}_2 = (1,2,-1) - (2,-2,0)$$

$$= (-1, 4, -1)$$ 3

since the point $(2,-2,0)$ ~~also~~ lies on the line

The normal vector of the plane is 3

$$\vec{N} = \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 1 \\ -1 & 4 & -1 \end{vmatrix}$$

$$= \hat{i}(3-4) - \hat{j}(-2+1) + \hat{k}(8-3)$$

$$= (-1, 1, 5)$$

Thus, using the point $(2,-2,0)$, we get

$$(x-2, y+2, z) \cdot \vec{N} = 0$$

$$(x-2, y+2, z) \cdot (-1, 1, 5) = 0$$

$$\therefore -x + y + 5z = -2-2$$

$$-x + y + 5z = -4 \;//$$

5 3

2(a) $x = \sin\alpha$

$y = \cos 2\alpha$

$y = \cos^2\alpha - \sin^2\alpha$

$= 1 - \sin^2\alpha - \sin^2\alpha$

$= 1 - 2\sin^2\alpha$

Hence $y = 1 - 2x^2$ 2

This is a parabola.

(b) $$K = \frac{f''(x)}{[1 + [f'(x)]^2]^{3/2}}$$

Where $y = f(x)$.

Here $f(x) = 1 - 2x^2$

$f'(x) = -4x$

$f''(x) = -4$

$$K = \frac{-4}{[1 + [(-4x)]^2]^{3/2}}$$

$$= \frac{-4}{[1 + 16x^2]^{3/2}}$$

When $\alpha = 0$, $x = \sin\alpha = 0$

Hence

$K = -4$ 4

Q2a 4

Q2

2(b). The required line has the slope of $\frac{d\vec{R}}{d\alpha}$

Here $\vec{R} = (\sin\alpha, \cos 2\alpha)$

$$\frac{d\vec{R}}{d\alpha} = (\cos\alpha, -2\sin 2\alpha)$$

When $\alpha = \frac{\pi}{4}$, $\cos\alpha = \frac{1}{\sqrt{2}}$

$\sin 2\alpha = \sin\frac{\pi}{2} = 1$ 3

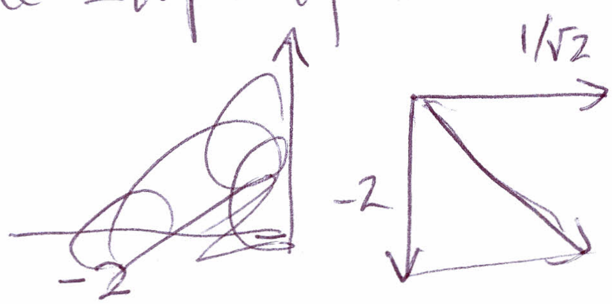
Hence $\left.\frac{d\vec{R}}{d\alpha}\right|_{\alpha=\frac{\pi}{2}} = \left(\frac{1}{\sqrt{2}}, -2\right)$

This vector is parallel to the required line, which passes through the point $\alpha = \pi/4$.

For this point $x = \frac{1}{\sqrt{2}}$, $y = \cos\frac{\pi}{2} = 0$

Hence the point is $\left(\frac{1}{\sqrt{2}}, 0\right)$. 2

The slope of the vector is



$s = \cancel{\frac{1/\sqrt{2}}{-2}} = -\cancel{\frac{1}{2\sqrt{2}}}$ $-\frac{2}{1/\sqrt{2}}$

Now $y = mx + C$

$y = -\frac{1}{2\sqrt{2}}x + C$

When $x = \frac{1}{\sqrt{2}}$, $y = 0$

$0 = -\frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + C$ 2

$C = \frac{1}{4}$

$y = -\frac{1}{\ }$

$s = -2\sqrt{2}$

$y = mx + C$

$y = -2\sqrt{2}x + C$

When $x = \frac{1}{\sqrt{2}}$, $y = 0$

$0 = -2\sqrt{2} \cdot \frac{1}{\sqrt{2}} + C$ 3

$C = 2$

$y = -2\sqrt{2}x + 2$

<u>Or</u>

$$\frac{x - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{y}{-2}$$

$$x - \frac{1}{\sqrt{2}} = -\frac{y}{2\sqrt{2}}$$ //

2(c)

$8.3232 32 \ldots$

$= 8 + \frac{32}{100} + \frac{32}{10000} + \frac{32}{1000000} + \ldots$

$= 8 + \frac{32}{100}\left(1 + \frac{1}{100} + \frac{1}{10000} + \ldots\right)$

$= 8 + \frac{32}{100}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \ldots\right)$

But $1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots$ is

a geometric series with

$a = 1, \ r = 0.01$

The sum is

$$S = \frac{a}{1-r} = \frac{1}{1-0.01}$$

$$\begin{array}{r} 1.00 \\ .01 \\ \hline 0.99 \end{array}$$

$$S = \frac{1}{0.99} = \frac{100}{99}$$

$\therefore 8.\overline{32} = 8 + \frac{32 \times \cancel{100}}{99 \times \cancel{100}}$

$= \frac{8 \times 99 + 32}{99}$

$=$

$= \frac{792 + 32}{99}$

$$\begin{array}{r} 792 \\ 32 \\ \hline 824 \end{array}$$

$= \frac{824}{99}$ ///

P2

4

3(a): $f(x) = \ln(1+x)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x=0)}{n!} x^2$$

$f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{1+x}$, $f'(0) = 1$

$f'(x) = (1+x)^{-1}$

$f''(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^{-2}}$

$f''(0) = -1$

$f^{(3)}(x) = 2(1+x)^{-3}$

$f^{(3)}(0) = 2$

$f^{(4)}(x) = -2 \cdot 3(1+x)^{-4}$

$f^{(4)}(0) = -6$

$f^{(5)}(x) = -2 \times 3 \times 4 (1+x)^{-5}$

$f^{(5)}(0) = +4! \times 1$

Hence

$f^{(n)}(0) = (-1)^{n+1}(n-1)!$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} x^n$$

$$= -\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} //$$

4

P3

$\ln(1+x) = \cancel{x} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

$+ \ldots$

3(b)(i) The direction vector of this line at right angles to the normal ~~plane~~ vectors of the plane. A vector that is simultaneously normal to both normal vectors is the cross product of the normal vectors of the planes.

The planes are

$$3x - 4y + z = 10 \quad 2$$

and $2x + y + z = 12$

The normal vectors are

$\vec{N_1} = (3, -4, 1)$

and $\vec{N_2} = (2, 1, 1)$ 2

The vector normal to both is

$$\vec{V} = \vec{N_1} \times \vec{N_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -4 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \vec{i}(-4-1) - \vec{j}(3-2) + \vec{k}(3+8)$$

$$= (-5, -1, 11)$$

Hence the required line is Q3b 6 2

$$\frac{x-1}{-5} = \frac{y+1}{-1} = \frac{z-1}{11} \quad //$$

(ii) This is given by

$$d = \frac{|\vec{PQ} \cdot \vec{N}|}{|\vec{N}|}$$

Here Q is (1,1,1) and P is any point on the plane. 4

For $3x - 4y + z = 10$, ~~a pt~~ we have $x = \frac{10}{3}$ if $y = 0, z = 0$

Hence $(\frac{10}{3}, 0, 0)$ is on the plane.

$\therefore P = P(\frac{10}{3}, 0, 0)$

We are using Q because it lies on the line

$\overrightarrow{PQ} = (1,1,1) - (\frac{10}{3},0,0)$

$= (1 - 3.333, 0, 0)$

$= \cancel{-2.333} = -7/3$

$\overrightarrow{PQ} = (-\frac{7}{3},1,1)$

$\vec{N} = (3,-4,1)$

$\overrightarrow{PQ} \cdot \vec{N} = (-\frac{7}{3},1,1) \cdot (3,-4,1)$

$= -7 - 4 + 1 = -10$

$|\vec{N}| = [9+16+1] = \sqrt{26}$

$d = \frac{10}{\sqrt{26}}$ //

For $2x+y+z=12$,

$\vec{N} = (2,1,1)$.

When $x=y=0$, $z=12$.

Hence a point in this plane is $P(0,0,12)$.

$\overrightarrow{PQ} = (0,0,12) - (2,1,1)$

$= (-2,-1,12)$

$-4+1$

$\overrightarrow{PQ} \cdot \vec{N} = (-2,-1,12) \cdot (2,1,1)$

$= -4 - 1 + 11 = 12 - 5 = 7$

$|\vec{N}| = \sqrt{4+1+1}$

Hence $d = \frac{7}{\sqrt{6}} = \sqrt{6}$

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7

6

3(c) If $\vec{R} = f(t)\vec{i} + g(t)\vec{j}$,

then

$$\vec{N} = -\frac{g'(t)\vec{i}}{\sqrt{f'^2+g'^2}} + \frac{f'(t)\vec{j}}{\sqrt{f'^2+g'^2}}$$

Here $f = t^3 - 3t$

$f' = 3t^2 - 3$

$g = 4 - t^2$

3

$g' = -2t$

$$f'^2 + g'^2 = (3t^2-3)^2 + 4t^2$$

$$= 9t^4 + 9 - 18t^2 + 4t^2$$

$$= 9t^4 - 1\cancel{4}t^2 + 9$$

$$\vec{N} = \frac{2t\vec{i} + (3t^2-3)\vec{j}}{[9t^4 - 18t^2 + 9]^{1/2}}$$

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4(a). Let the vertices be A(1,2,3), B(2,3,1) and C(3,2,1).

$\vec{AB} = (2,3,1)-(1,2,3)$

$= (1,1,-2)$

Q4

$|\vec{AB}| = \sqrt{1+1+4} = \sqrt{6}$

$\vec{AC} = (3,2,1)-(1,2,3)$

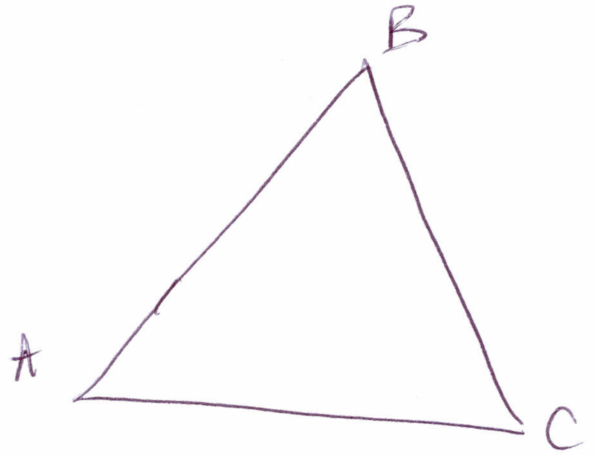
$= (2,0,-2)$

$|\vec{AC}| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\vec{BC} = (3,2,1)-(2,3,1)$

$= (1,-1,0)$

$\vec{BC} = \sqrt{1+1} = \sqrt{2}$



For the angle ~~ABC~~ $B\hat{A}C$

$\vec{AB}\cdot\vec{AC} = |AB|\,|AC|\cos\theta$

$\cos\theta = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}||\vec{AC}|}$

$= \frac{(1,1,-2)\cdot(2,0,-2)}{\sqrt{6}\cdot 2\sqrt{2}}$

$= \frac{2+4}{2\sqrt{2\times6}} = \frac{6}{2\sqrt{12}}$

$= \frac{3}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$

$= \frac{3\sqrt{3}}{2\sqrt{3}\sqrt{3}} = \frac{3\sqrt{3}}{2\times 3} = \frac{\sqrt{3}}{2}$

$\theta = 30^\circ$

For angle $A\hat{B}C$

$\vec{AB}\cdot\vec{BC} = |\vec{AB}|\,|\vec{BC}|\cos\theta$

$\cos\theta = \frac{\vec{AB}\cdot\vec{BC}}{|\vec{AB}||\vec{BC}|}$

$= \frac{(1,1,-2)\cdot(1,-1,0)}{\sqrt{6}\cdot\sqrt{2}}$

$$= \frac{1-1}{\sqrt{12}} = 0$$

Hence $\theta = 90^\circ$

For angle $B\hat{C}A$,

~~AB $\vec{AC} \cdot \vec{CA} \cdot \vec{CB}$~~

QP4

$$\vec{CA} \cdot \vec{CB} = |\vec{CA}||\vec{CB}| \cos\theta$$

6

$$\cos\theta = \frac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}||\vec{CB}|}$$

8

$$= \frac{(-2,0,2) \cdot (-1,1,0)}{2\sqrt{2} \cdot \sqrt{2}}$$

$$= \frac{2+0+0}{2 \times 2} = \frac{1}{2}$$

$\theta = 60^\circ$

~~For the area, this is~~

AE

4 (b). In Cartesian coordinates we have

$(x+1)^2+(y-3)^2=16$

$x^2+2x+1+y^2+9-6y=16$

$x^2+2x+y^2-6y=6$

$x^2+y^2+2x-6y=6$ 2

$r^2+2r\cos\theta-6r\sin\theta=6$

For the intersection $y=x$

$\therefore 2x^2-4x=6$ 4

$x^2-2x=3$

$x^2-2x-3=0$

$x=\dfrac{2\pm\sqrt{4+12}}{2}$

$=\dfrac{2\pm\sqrt{16}}{2}$ 2

$=\dfrac{2\pm4}{2}=1\pm2$

$x=3$ or -1

Hence the intersection is $(-1,-1)$ or $(3,3)$ in Cartesian coordinates. ~~4~~ 2

In polar coordinates:

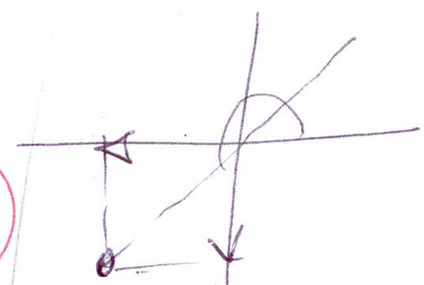
For $(x,y)=(-1,-1)$, we have

$-1=r\cos\theta$

$-1=r\sin\theta$

$r^2=2,\ r=\sqrt{2}$

$\tan\theta=\left(\dfrac{-1}{-1}\right)=1$ 2



$\theta=\dfrac{\pi}{2}+\dfrac{\pi}{4}$

$=\dfrac{2\pi}{4}+\dfrac{\pi}{4}=\dfrac{3\pi}{4}$

Hence $(r,\theta)=\left(\sqrt{2},\dfrac{3\pi}{4}\right)$

For $(x,y)=(3,3)$,

$x=3=r\cos\theta$

$y=3=r\sin\theta$ 2

$r^2=18,\ r=3\sqrt{2}$

$\tan\theta=1,\ \theta=\dfrac{\pi}{4}$,

$(r,\theta)=\left(3\sqrt{2},\dfrac{\pi}{4}\right)$

(c) $\vec{R} = t^3\hat{i} + t^2\hat{j}$

(i) When $t = 3s$,

$\vec{R} = 27\hat{i} + 9\hat{j}$ 1/2

(ii) $\vec{v} = \frac{d\vec{R}}{dt} = 3t^2\hat{i} + 2t\hat{j}$

When $t = 3s$,

$\vec{v} = 3\times 9\hat{i} + 2\times 3\hat{j}$

$= 27\hat{i} + 6\hat{j}$ 1/2

(iii) $\vec{a} = \frac{d\vec{v}}{dt} = 6t\hat{i} + 2\hat{j}$

When $t = 3$,

$\vec{a} = 18\hat{i} + 2\hat{j}$ 1

$\vec{F} = m\vec{a} = 36\hat{i} + 4\hat{j}$

(iv) When $t = 3s$,

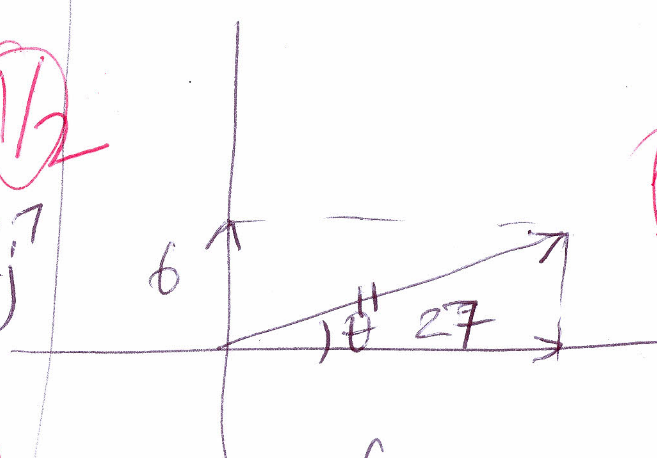
the velocity is

$\vec{v} = 27\hat{i} + 6\hat{j}$ ~~1~~

The velocity vector is tangent to the direction of motion. It ~~is~~ ~~therefore~~ thus gives the direction of motion.

For $t = 3s$,

$\vec{v} = 27\hat{i} + 6\hat{j}$



2

$\tan\theta = \frac{6}{27}$; $\theta = 12.5°$

~~The directio~~

9+