MAT2110 Test 2

3 May 2015

Time: Two Hours

Answer: Any Three Questions

1. (a) Determine the eccentricity and the major and minor axes of the ellipse $r = 5/(4 - 2\cos\theta)$. 9

(b) Use the differential arc length ds to prove that the length of a quarter of the circumference of the circle $x^2 + y^2 = 16$ is 2π. 6

(c) (i) Determine the equation of the line that passes through the points $P_1(1, 1, 1)$ and $P_2(2, 0, 3)$. 2

(ii) Determine the equation of the plane perpendicular to this line and passing through the point $P_2(2, 0, 3)$. 3

2. (a) One of the parametric equations of the circle $x^2 + y^2 = 4$ is $x = \sqrt{1 - t^2}$. 4

(i) Determine what the other parametric equation must be.

(ii) Determine the curvature of the circle at the points $t = 0.5$ and $t = 0.25$ and comment. 5

(iii) Determine the equation of the line tangent to the circle at the point $t = \sqrt{5/9}$. 5

(b) Find the sum of the series $S = \sum_{k=1}^{\infty} \frac{1}{5^k}$. 2

(c) Obtain the Maclaurin series of $\sin x$. 3

3 (a) Determine the power series expansion of $x/(1 - x^2)$. 5

(b) (i) Find the line of intersection of the planes $3x - 4y + z = 10$ and $2x + y + z = 12$. 4

(ii) Determine the angle between the planes. 3

(c) The position vector of a certain particle is given by $\mathbf{R} = (a + mt)\,\hat{\mathbf{i}} + (b + nt)\,\hat{\mathbf{j}}$.

(i) Show that the particle moves in a straight line 3

(ii) Change the parameter to the arc length s measured from the point $P_0(a, b)$ and verify that your calculation is correct. 5

4. (a) The work done on a particle by the force $\mathbf{F}$ between the points $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by $W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}$ where $d\mathbf{r} = (dx, dy, dz)$. The force acting on one such particle is $\mathbf{F} = (2x, 3e^{-y}, -2z^2)$ in Newtons.

(i) Determine the work done on the particle between the origin $\mathbf{r}_1$ and the point $\mathbf{r}_2 = (1, 1, 1)$ 3

(ii) If the mass of the particle is 3 kg, find the acceleration of the particle at the point $(0.5, 1.5, 1)$. 3

(b) (i) Show that the polar equation of the line $y = x$ is $\theta = \pi/4$. 2

(ii) Express the intersection of this line with the hyperbola $r = 10/(3 - 4\cos\theta)$ in Cartesian coordinates. 6

(c) Determine the area of the triangle with the vertices $P_1(0, 0, 0)$, $P_2(1, 2, 3)$ and $P_3(1, 1, 1,)$. 6

MAT 2110 Make-up Test Solns
7/6/2015

Q1. $r = \frac{5}{4-2\cos\theta}$

$= \frac{5/4}{1-\frac{1}{2}\cos\theta}$

Comparing with

$r = \frac{eP}{1-e\cos\theta}$

we see that $e = \frac{1}{2}$.

$4r - 2r\cos\theta = 5$

$4r = 5 + 2x$

$16r^2 = 25 + 20x + 4x^2$

$16x^2 + 16y^2 - 20x - 4x^2 = 25$

$12x^2 - 20x + 16y^2 = 25$

$12\left(x^2 - \frac{20}{12}x\right) + 16y^2 = 25$

$12\left(x^2 - \frac{5}{3}x\right) + 16y^2 = 25$

$12\left(x^2 - \frac{5}{3}x + \frac{25}{36} - \frac{25}{36}\right) + 16y^2 = 25$

$12\left(x - \frac{5}{6}\right)^2 - \frac{25}{3} + 16y^2 = 25$

$36\left(x - \frac{5}{6}\right)^2 + 48y^2 = 75 + 25$

$\frac{\left(x-\frac{5}{6}\right)^2}{100/36} + \frac{y^2}{100/48} = 1$

$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$

$a = \sqrt{\frac{100}{36}} = \frac{10}{6} = \frac{5}{3}$

$b = \sqrt{\frac{100}{48}} = \frac{10}{4\sqrt{3}} = \frac{5}{2\sqrt{3}}$

Major axis is $2a = \frac{10}{3}$

Minor axis $2b = \frac{5}{\sqrt{3}}$

(b) $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$s = \int_{x_0}^{x_1} \left[1 + [y']^2\right]^{1/2} dx$

~~$y = (16 - x^2)^{1/2}$~~

$y = [16 - x^2]^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}\left[16 - x^2\right]^{-1/2}[-2x]$

$= \frac{-x}{(16-x^2)^{1/2}}$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{16-x^2}$$

$$= \frac{16-x^2+x^2}{16-x^2} = \frac{16}{16-x^2}$$

$$s = \int_0^4 \sqrt{\frac{16}{16-x^2}}\,dx$$

$$= 4\int_0^4 \frac{1}{(16-x^2)^{1/2}}\,dx$$

let $x = 4\cos\theta$

$dx = -4\sin\theta\, d\theta$

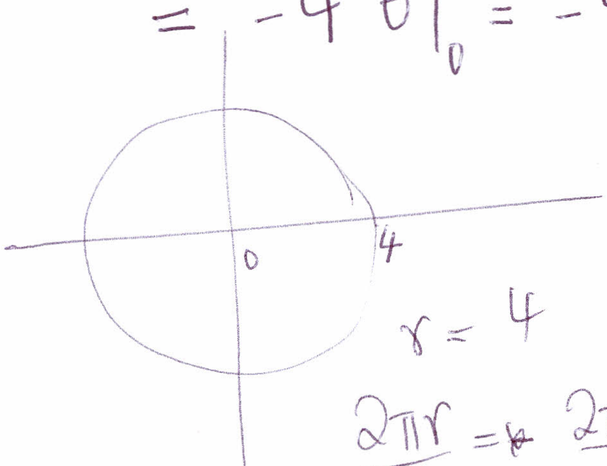
$16-x^2 = 16-16\cos^2\theta$

$= 16\sin^2\theta$

$$s = -4\int_0^{\pi/2} \frac{4\sin\theta\, d\theta}{4\sin\theta}$$

$$= -4\int_0^{\pi/2} d\theta$$

$$= -4\,\theta\Big|_0^{\pi/2} = -2\pi$$



$r = 4$

$$\frac{2\pi r}{4} = \frac{2\pi\times 4}{4}$$

(c) i) $\vec{v} = (2,0,3) - (1,1,1)$

$= (1,-1,2)$

∴ Using the point $P(2,0,3)$ the eqn is

$$\frac{x-2}{1} = \frac{y}{-1} = \frac{z-3}{2}$$ (2)

(ii) The normal vector is $\vec{N} = \vec{v} = (1,-1,2)$

A point in the plane is $P_2(2,0,3)$

∴ $\vec{N}\cdot(x-2, y, z-3) = 0$

∴ $x - y + 2z = 2+0+6$

$x - y + 2z = 8$ //

2(a)(i) $x = \sqrt{1-t^2}$

$y = f(t)$

But $x^2 + y^2 = 4$

$x^2 + [f(t)]^2 = 4$

$1 - t^2 + [f(t)]^2 = 4$

$[f(t)]^2 = 4 - 1 + t^2$

$[f(t)]^2 = 3 + t^2$

$y(t) = \sqrt{3+t^2}$

(ii) The curvature is

$$K = \frac{f''(x)}{[1 + [f'(x)]^2]^{3/2}}$$

$f(x) = y = \sqrt{4-x^2}$

$f'(x) = \frac{1}{2}(4-x^2)^{-1/2}(-2x)$

$= -\frac{x}{\sqrt{4-x^2}} = -\frac{x}{y}$

$f''(x) = -\frac{1}{y} + \frac{x}{y^2} y'$

$= -\frac{1}{y} + \frac{x}{y^2} \cdot \left(-\frac{x}{y}\right)$

$= -\frac{1}{y} - \frac{x^2}{y^3}$ (3)

$= -\frac{1}{y}\left(1 + \frac{x^2}{y^2}\right)$

$= -\frac{1}{y}\left(\frac{x^2+y^2}{y^2}\right)$

$= -\frac{1}{y^3} r^2 = -\frac{4}{y^3}$

$$\therefore K = \frac{-4/y^3}{\left[1 + \left(\frac{-x}{y}\right)^2\right]^{3/2}}$$

$$= \frac{-4/y^3}{\left[\frac{y^2+x^2}{y^2}\right]^{3/2}}$$

$$= \frac{-4}{y^3 \frac{[y^2+x^2]^{3/2}}{y^3}} = \frac{-4}{(x^2+y^2)^{3/2}}$$

$$= -\frac{4}{4^{3/2}} = -\frac{4}{8} = -\frac{1}{2}$$

$K \neq K(x,y). = -\frac{1}{r}$ //

The slope is $m = y' = -\frac{x}{y}$

At $t = \sqrt{5/9}$

$x = \sqrt{1-t^2} = \sqrt{1 - \frac{5}{9}}$

$= \sqrt{\frac{9-5}{9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$

$y^2 = 4 - x^2 = 4 - \frac{4}{9}$

$y^2 = \frac{36-4}{9} = \frac{32}{9}$

$y = \sqrt{\frac{32}{9}} = \sqrt{\frac{2 \times 16}{9}}$

$= \frac{4}{3}\sqrt{2}$

$m = \frac{-2/3}{4/3\sqrt{2}} = -\frac{1}{2\sqrt{2}}$

$y = -\frac{1}{2\sqrt{2}}x + C$

Using $(x, y) = \left(\frac{2}{3}, \frac{4}{3}\sqrt{2}\right)$;

$\frac{4}{3}\sqrt{2} = -\frac{1}{2\sqrt{2}} \cdot \frac{2}{3} + C$

$C = \frac{4\sqrt{2}}{3} + \frac{1}{3\sqrt{2}}$

$= \frac{1}{3}\left(4\sqrt{2} + \frac{1}{\sqrt{2}}\right)$

$= \frac{1}{3}\left(\frac{4\sqrt{2}\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$

$= \frac{1}{3\sqrt{2}}(8+1) = \frac{3}{\sqrt{2}}$

$y = -\frac{1}{2\sqrt{2}}x + \frac{3}{\sqrt{2}}$ //

(b) $S = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$

$= \frac{1}{5}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots\right)$

This is a GS with $a = \frac{1}{5}$

and $r = \frac{1}{5}$

$S = \frac{a}{1-r}$

$= \frac{1/5}{1 - 1/5}$

$= \frac{1/5}{4/5} = 0.25$ //

(c) ~~sinx = Σ(f)~~

(i) $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x=0)x^n}{n!}$

$f(x) = \sin x, f(0) = 0$

$f'(x) = \cos x, f'(0) = 1$

$f''(x) = -\sin x, f''(0) = 0$

$f^{(3)}(x) = -\cos x, f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x, f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x, f^{(5)}(0) = 1$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

$+ \cdots$

3(a) $f(x) = \frac{x}{1-x^2}$

$f(x) = \frac{x}{(1-x)(1+x)}$

$= \frac{A}{1-x} + \frac{B}{1+x}$

$\therefore x = A(1+x) + B(1-x)$

When $x = -1$,

$-1 = B(1-(-1))$

$2B = -1$, $B = -\frac{1}{2}$

When $x = 1$;

$1 = 2A$; $A = \frac{1}{2}$

$f(x) = \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$\frac{1}{1+x} = 1 + (-x) + (-x)^2 + (-x^3)$

$\frac{1}{2(1-x)} = \frac{1}{2} + \frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{2} + \dots$

$-\frac{1}{2}\frac{1}{1+x} = -\frac{1}{2} + \frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} \dots$

Add

$f(x) = \frac{1}{2} + \frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2} + \dots$

$+ \; -\frac{1}{2} + \frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} - \frac{x^4}{2} + \dots$

$= x + x^3 + x^5 + x^7 \dots$

$f(x) = \sum_{k=0}^{\infty} x^{2k+1}$ //

(b) $3x - 4y + z = 10$ (1)

$2x + y + z = 12$ (2)

Subtract one from the other

$x - 5y = -2$

~~Let x = t~~

$x = 5y - 2$

Let $y = t$

$x = 5t - 2$

$z = 12 - 2x - y$

$z = 12 - 2(5t-2) - t$

$= 12 - 10t + 4 - t$

$= 16 - 11t$

Hence the line is

$x = 5t-2, y = t, z = 16 - 11t$

or $\frac{x+2}{5} = y = \frac{z-16}{-11}$ //

(ii) $\vec{N_1} = (3, -4, 1)$

$\vec{N_2} = (2, 1, 1)$

$\vec{N_1} \cdot \vec{N_2} = 6 - 4 + 1 = 3$

$\vec{N_1} \cdot \vec{N_2} = |\vec{N_1}||\vec{N_2}| \cos\theta$

$\vec{N_1} = \sqrt{9+16+1} = \sqrt{26}$

$\vec{N_2} = \sqrt{4+1+1} = \sqrt{6}$

$\cos\theta = \frac{3}{\sqrt{26 \times 6}}$

$= \frac{3}{\sqrt{156}}$

$= 0.2402$

$\theta = 76.1°$

(c) $\vec{R} = (a+mt)\vec{i} + (b+nt)\vec{j}$

$x = a + mt$

$y = b + nt$

$t = \frac{x-a}{m}$

$\therefore y = b + n\left(\frac{x-a}{m}\right)$

$= b + \frac{n}{m}x - \frac{na}{m}$

$= \frac{n}{m}x + b - \frac{na}{m}$

$\therefore y = Mx + C$

This is a st. line.

(ii) $\frac{ds}{dt} = |\vec{R}'(t)|$

Here $\vec{R} = (a+mt)\vec{i} + (b+nt)\vec{j}$

$\vec{R}' = m\vec{i} + n\vec{j}$

$|\vec{R}'(t)| = \sqrt{m^2+n^2}$

$ds = \sqrt{m^2+n^2}\,dt$

$s = \int_{t_0}^{t} \sqrt{m^2+n^2}\,du$

Now $\vec{R}(t_0) = P_0(a,b)$

$\therefore a + mt_0 = a$

$b + nt_0 = b$

Hence $t_0 = 0$

$s = \sqrt{m^2+n^2}\int_0^t du$

$= \sqrt{m^2+n^2}\,t$

Hence $t = \frac{s}{\sqrt{m^2+n^2}}$

Hence $\vec{R}(s) = \left(a + \frac{ms}{\sqrt{m^2+n^2}}\right)\vec{i} + \left(b + \frac{ns}{\sqrt{m^2+n^2}}\right)\vec{j}$

We expect that

$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$

Here $x = a + \frac{ms}{\sqrt{m^2+n^2}}$

$y = b + \frac{ns}{\sqrt{m^2+n^2}}$

$\frac{dx}{ds} = \frac{m}{\sqrt{m^2+n^2}}$

$\frac{dy}{ds} = \frac{n}{\sqrt{m^2+n^2}}$

$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = \frac{m^2+n^2}{\left(\sqrt{m^2+n^2}\right)^2}$

$= \frac{m^2+n^2}{m^2+n^2} = 1$ //

4 (a).

$$W = \int_{\vec{r_1}}^{\vec{r_2}} \vec{F} \cdot d\vec{r}$$

$$= \int (2x, 3e^{-y}, -2z^2) \cdot (dx, dy, dz)$$

$$= \int_0^1 2x\,dx + \int_0^1 3e^{-y}\,dy + \int_0^1 (-2z^2)\,dz$$

$$= \frac{2x^2}{2}\Big|_0^1 - 3e^{-y}\Big|_0^1 - \frac{2z^3}{3}\Big|_0^1$$

$$= 1 - 3(e^{-1} - 1) - \frac{2}{3} \times 1$$

$$= \frac{1}{3} - 3e^{-1} + 3$$

$$= \frac{4}{3} - 3e^{-1} //$$

$$\vec{F} = m\vec{a}$$

$$\vec{a} = \frac{\vec{F}}{m} = \frac{1}{3}(2x, 3e^{-y}, -2z^2)$$

At $(0.5, 1.5, 1)$

$$\vec{a} = \frac{1}{3}(2 \times 0.5, 3e^{-1.5}, -2)$$

$$= \left(\frac{1}{3}, e^{-1.5}, -\frac{2}{3}\right) m/s^2 //$$

1 (b) We have

$$y = x$$

$$r\sin\theta = r\cos\theta$$

$$\sin\theta = \cos\theta$$

$$\tan\theta = 1$$

Hence $\theta = \pi/4$ is the eqn of the line.

(ii) $r = \dfrac{10}{3 - 4\cos\theta}$

At the intersection, $\theta = \frac{\pi}{4}$

$$r = \frac{10}{3 - 4\cos\frac{\pi}{4}}$$

$$= \frac{10}{3 - 4 \times \frac{1}{\sqrt{2}}}$$

$$= \frac{10}{3 - \frac{2}{\sqrt{2}}}$$

$$= \frac{10}{3 - \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}}}$$

$$= \frac{10}{3 - \sqrt{2}}$$

The intersection in polar coordinates is

$$(r,\theta)=\left(\frac{10}{3-\sqrt{2}}, \frac{\pi}{4}\right)$$

Now $\frac{10}{3-\sqrt{2}} = \frac{10(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}$

$$= \frac{30+10\sqrt{2}}{9-2}$$

$$= \frac{30+10\sqrt{2}}{7}$$

$$x = r\cos\theta$$

$$= \frac{30+10\sqrt{2}}{7} \cdot \cos\frac{\pi}{4}$$

$$= \frac{30+10\sqrt{2}}{7} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{(30+10\sqrt{2})\sqrt{2}}{7\times\sqrt{2}\sqrt{2}}$$

$$= \frac{(30+10\sqrt{2})\sqrt{2}}{7\times 2}$$

$$= \frac{(15+5\sqrt{2})\sqrt{2}}{7}$$

$$= \frac{15\sqrt{2}+10}{7}$$

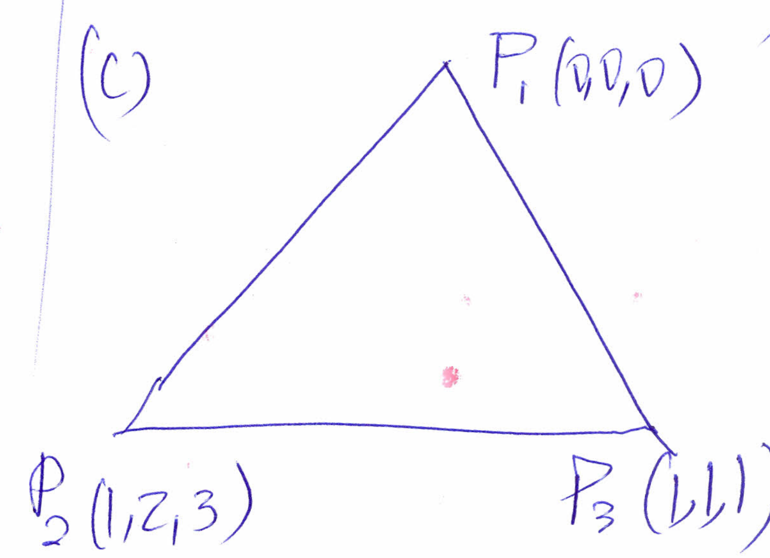
$$y = r\sin\theta = r\cos\theta$$

$$= \frac{15\sqrt{2}+10}{7}$$

Hence the intersection point is

$$\left(\frac{10+15\sqrt{2}}{7}, \frac{10+15\sqrt{2}}{7}\right)$$ //

(c)



The sides are $\overrightarrow{P_1P_2}$

$$\overrightarrow{P_1P_2} = (1,2,3)-(0,0,0)$$
$$= (1,2,3)$$

$$\overrightarrow{P_2P_3} = (1,1,1)-(1,2,3)$$

$= (0,-1,-2)$

$\overrightarrow{P_1P_3} = (1,1,1)-(0,0,0)$

$= (1,1,1)$

The lengths are

$|\overrightarrow{P_1P_2}| = \sqrt{1+4+9} = \sqrt{14}$

$|\overrightarrow{P_2P_3}| = \sqrt{0+1+4} = \sqrt{5}$

$|\overrightarrow{P_1P_3}| = \sqrt{1+1+1} = \sqrt{3}$

$\frac{h}{\sqrt{5}} = \sin\theta$

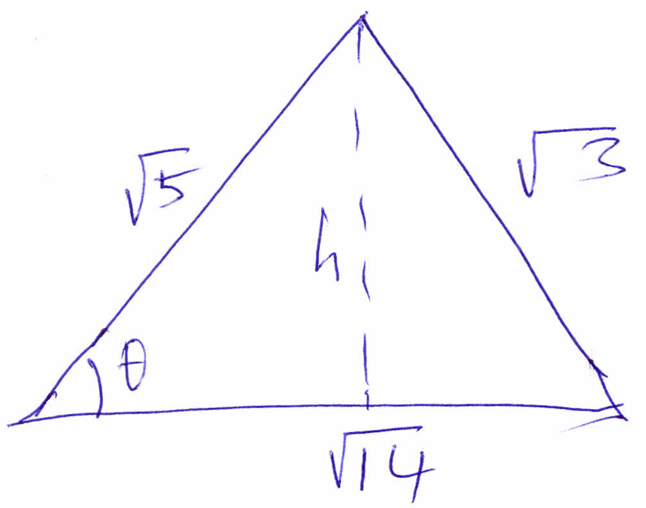
$h = \sqrt{5}\sin\theta$

$= \sqrt{5}\times 0.2928$

$= 0.6547$

$A = \frac{1}{2}\sqrt{14}\times 0.6547$

$= 1.2248$ Units



$(\sqrt{3})^2 = (\sqrt{5})^2 + (\sqrt{14})^2 - 2\sqrt{5}\sqrt{14}\cos\theta$

$3 = 5 + 14 - 2\sqrt{70}\cos\theta$

$\cos\theta = \frac{19-3}{2\sqrt{70}} = \frac{16}{2\sqrt{70}}$

$= 17.02^0$

2.23 1.73

3.74