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MAT2110 TUTORIAL SHEET 5

Handing-in Date: 9 March 2015

1. Find the rational fraction in lowest terms that is equal to the given infinite repeating decimal fraction.
(i) $0.83\overline{3}$ (ii) $0.77\overline{272}$ (iii) $0.074\overline{074}$

2. Find the sum of the geometric series if the series converges

$$\text{(i)} \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k \quad \text{(ii)} \sum_{k=1}^{\infty}\frac{1}{\left(5-\sqrt{13}\right)^k} \quad \text{(iii)} \sum_{k=1}^{\infty}\frac{\pi^k}{e^{2k}}$$

3. Expand the given polynomial about the point $x = a$
(i) $P(x) = 10 - 20x + 15x^2 - 4x^3,\ a = 1$
(ii) $P(x) = (x-1)^5 - (x-1)^3,\ a = -1$
(iii) $P(x) = (x-3)^5 + (x+1)^5,\ a = 1$

4. Find the Maclaurin series for the given function
(i) $f(x) = \sqrt{e^x}$ (ii) $f(x) = \frac{1}{(1+2x)^2}$ (iii) $f(x) = \cosh x - x\sinh x$

5. Determine the Taylor series of the given function about the given point.
(i) $f(x) = 1/x,\ a = 2$
(ii) $f(x) = \cos x,\ a = \pi/2$
(iii) $f(x) = 1/(1-x)^2,\ a = 2$

$$\begin{array}{r} 99 \\ 7 \\ \hline 693 \\ 72 \\ \hline 775 \end{array}$$

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5. Taylor series of $f(x) = \cos x$ at $a = \pi/2$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x=a)}{n!}(x-a)^2$$

$f^{(0)}\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$f'(x) = -\sin x;\ f'\left(\frac{\pi}{2}\right) = -1$

$f''(x) = -\cos x,\ f''\left(\frac{\pi}{2}\right) = 0$

$f^{(3)}(x) = \sin x,\ f^{(3)}\left(\frac{\pi}{2}\right) = 1$

$f^{(4)}(x) = \cos x,\ f^{(4)}\left(\frac{\pi}{2}\right) = 0$

$f^{(5)}(x) = -\sin x,\ f^{(5)}\left(\frac{\pi}{2}\right) = -1$

$f^{(6)}(x) = -\cos x,\ f^{(6)}\left(\frac{\pi}{2}\right) = 0$

$$f(x) = -(x-a) + \frac{1}{3!}(x-a)^3 + -\frac{(x-a)^3}{5!} + \frac{(x-a)^5}{7!} + \ldots$$

$a = \frac{\pi}{2}$

This can be written as

$$f(x) = \sum \frac{(x-a)^{n}}{n!}$$

$$f(x) = \sum_{k=1,2} (\ldots)$$

Let us introduce a new index k, which has the initial value zero. Then we have

n	1	3	5	7
k	0	1	2	3
sign	-ve	+ve	-ve	+ve

Clearly, $n = 2k+1$, and the sign is $(-1)^{k+1}$

$$f(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k \left(x - \frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \left(x - \frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}$$

MA2110 Tutorial Sheet 5 Solutions

1: We want to express 8.33333.... as a rational fraction in the lowest terms.

$$8.3333 = 8 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$$

$$= 8 + 3\left(\frac{1}{10} + \frac{1}{1000} + \frac{1}{1000} + \cdots\right)$$

We have the geometric series

$a + ar + ar^2 + \cdots$

in the parentheses. This has sum

$$S = \frac{a}{1-r}$$

Here $a = \frac{1}{10} = 0.1$, $r = 0.1$

$$S = \frac{0.1}{1-0.1} = \frac{0.1}{0.9} = \frac{1}{9}$$

$$\therefore 8.333 = 8 + \frac{3}{9} = \frac{72}{9} + \frac{3}{9}$$

$$= \cancel{\frac{72+8}{72}}$$

$$N = \frac{75}{9} = \frac{75}{9}$$

(ii) We have $0.772727272\ldots$

$\therefore 0.77272727\ldots$

$$= \cancel{0.7 + \frac{72}{100} + \frac{72}{1000} + \frac{72}{10^6} + \cdots}$$

$$= \cancel{\frac{7}{10} + 72\left(\frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \cdots\right)}$$

$$= 0.7 + 0.072 + 0.00072 + 0.0000072 + \cdots$$

$$= \frac{7}{10} + \frac{72}{1000} + \frac{72}{10^5} + \frac{72}{10^7}$$

$$= \frac{7}{10} + 72\left(\frac{1}{10^3} + \frac{1}{10^5} + \frac{1}{10^7} + \cdots\right)$$

The geometric series has $a = 10^{-3}$, $r = 10^{-2}$

$$S = \frac{10^{-3}}{1-10^{-2}} = \frac{10^{-3}}{1-0.01} = \frac{10^{-3}}{0.99} = \frac{0.001}{0.99} = \frac{1}{990}$$

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$$\therefore N = \frac{7}{10} + \frac{72}{990} = \frac{7 \times 99}{990} + \frac{72}{990} = \frac{775}{990}$$

(iii) Let $N = 0.074074074\ldots$

$N = 0.074 + 0.000074 + 0.000000074\ldots$

$= \frac{74}{1000} + \frac{74}{10^6} + \frac{74}{10^9}$

$= 74\left(\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9}\right)$

The geometric series has

$a = \frac{1}{10^3}$, $r = \frac{1}{10^3}$.

Its sum is

$$S = \frac{10^{-3}}{1-10^{-3}}$$

$$= \frac{1}{1000}\left(\frac{1}{1-\frac{1}{1000}}\right)$$

$$= \frac{1}{\cancel{1000}}\left(\frac{\cancel{1000}}{1000-1}\right)$$

$$= \frac{1}{999}$$

Hence $N = \frac{74}{999}$ //

(i) $S = \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3$

This GS has $a = \frac{2}{3}$, $r = \frac{2}{3}$

$$S = \frac{a}{1-r} = \frac{2/3}{1-2/3}$$

$$= \frac{2/3}{1/3} = 2 //$$

(ii) $S = \sum_{k=1}^{\infty} \frac{1}{(5-\sqrt{13})^k}$ 2

$$= \frac{1}{5-\sqrt{13}} + \frac{1}{(5-\sqrt{13})^2} + \frac{1}{(5-\sqrt{13})^3} + \ldots$$

This GS has $a = \frac{1}{5-\sqrt{13}}$

and $r = \frac{1}{5-\sqrt{13}}$

$$S = \frac{1/(5-\sqrt{13})}{1-\frac{1}{5-\sqrt{13}}}$$

$$= \frac{1/\cancel{(5-\sqrt{13})}}{(5-\sqrt{13}-1)/\cancel{(5-\sqrt{13})}}$$

$$= \frac{1}{4-\sqrt{13}} = \frac{4+\sqrt{13}}{(4-\sqrt{13})(4+\sqrt{13})}$$

$$= \frac{4+\sqrt{13}}{16-13} = \frac{4+\sqrt{13}}{3} //$$

(iii) $S = \sum_{k=1}^{\infty} \frac{\pi^k}{e^{2k}}$

$= \frac{\pi}{e^2} + \frac{\pi^2}{e^4} + \frac{\pi^3}{e^6} + \dots$

This as has $a = \pi/e^2$ and

$r = \frac{\pi}{e^2}$

$S = \frac{\pi/e^2}{1 - \pi/e^2}$

$= \frac{\pi}{e^2} \cdot \frac{1}{(e^2-\pi)/e^2}$

$= \frac{\pi}{e^2 - \pi}$ //

(3) $P(x) = 10 - 20x + 15x^2 - 4x^3$

$P(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k P}{dx^k}\Big|_{x=a} (x-a)^k$

For $k=0$, $P(x=1)$ gives

$P(1) = 10 - 20 + 15 - 4$

$= 35 - 24 = 11$

$\frac{dP}{dx} = -20 + 30x - 12x^2$

$\frac{dP}{dx}\Big|_{x=1} = -20 + 30 - 12$

$= -42 + 30 = -12$

$\frac{d^2P}{dx^2} = 30 - 24x$ 3

$\frac{d^2P}{dx^2}\Big|_{x=1} = 30 - 24 = 6$

$\frac{d^3P}{dx^3} = -24$; $\frac{d^3P}{dx^3}\Big|_{x=1} = -24$

Hence

$P(x) = 11 - 12(x-1) + \frac{6(x-1)^2}{2} - \frac{24x^3}{6}$

$= 11 - 12(x-1) + 3(x-1)^2 - 4x^3$

(ii) $P(x) = (x-1)^5 - (x-1)^3$ $a = -1$

$P(-1) = (-1-1)^5 - (-1-1)^3$

$= (-2)^5 - (-2)^3$

$= -32 + 8 = -24$

$\frac{dP}{dx} = 5(x-1)^4 - 3(x-1)^2$

$\frac{dP}{dx}\Big|_{x=-1} = 5(-2)^4 - 3(-2)^2$

$= 5\times16 - 3\times4$

$= 80-12 = 68$

$\frac{d^2P}{dx^2} = 20(x-1)^3 - 6(x-1)$

$\left.\frac{d^2P}{dx^2}\right|_{x=-1} = 20(-2)^3 - 6(-2)$

$= -20\times8+12$

$= -160+12 = -148$

$\frac{d^3P}{dx^3} = \cancel{120}\ 60(x-1)^2 - 6$

$\left.\frac{d^3P}{dx^3}\right|_{x=1} = 60(-2)^2 - 6$

$= 240-6 = 234$

$\frac{d^4P}{dx^4} = 120(x-1)$

$\left.\frac{d^4P}{dx^4}\right|_{x=-1} = 120(-2)$

$= -240$

$\frac{d^5P}{dx^5} = 120$

Hence

$P(x) = -24 + 68(x+1) - \frac{148}{2!}(x+1)^2 + \frac{234}{3!}(x+1)^3 - \frac{240}{4!}(x+1)^4 + \frac{120}{5!}(x+1)^5$

$= -24+68(x+1) - 74(x+1)^2 + 39(x+1)^3 + (x+1)^5$ //

(iii) $P(x) = (x-3)^5 + (x+1)^5$

$P(1) = (-2)^5 + (2)^5$

$= -32+32 = 0$

$\frac{dP}{dx} = 5(x-3)^4 + 5(x+1)^4$

$= 5(-2)^4 + 5(2)^4$

$= 10\times16 = 160$

$\frac{d^2p}{dx^2} = 20(x-3)^3 + 20(x+1)^3$

$\left.\frac{d^2P}{dx^2}\right|_{x=1} = 0$

$\frac{d^3P}{dx^3} = \cancel{60}\ 60(x-3)^2 + 60(x+1)^2$

$\left.\frac{d^3P}{dx^3}\right|_{x=1} = 2\times60(2)^2 = 480$

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$$\frac{d^4P}{dx^4} = 120(x-3)+120(x+1)$$

$$\left.\frac{d^4P}{dx^4}\right|_{x=1} = 0$$

$$\frac{d^5P}{dx^5} = 120+120$$

$$\frac{d^5P}{dx^5} = 240$$

$$P(x) = 160(x-1) + \frac{480}{3!}(x-1)^3 + \frac{240}{5!}(x-1)^5$$

$$= 160(x-1) + 80(x-1)^3 + 2(x-1)^5 \ //$$

$\underline{4}$: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f(0) = 1$

$f^{0}(x) = e^{x/2}$

$f'(x) = \frac{1}{2} e^{x/2}$

$f''(x) = \frac{1}{4} e^{x/2}$

$f^{3}(x) = \frac{1}{8} e^{x/2}$

$f^{(k)}(k) = \frac{1}{2^k} e^{x/2}$

Hence $f'(0) = \frac{1}{2}$

$f''(0) = \frac{1}{4}$

$f'''(0) = \frac{1}{8}$

$\vdots$

$f^{(k)}(0) = \frac{1}{2^k}$

$$\therefore f(x) = \sum_{k=0}^{\infty} \frac{x^k}{2^k k!}$$

(ii) $f(x) = \frac{1}{(1+2x)^2}$

$$K = \frac{f''(x)}{\left[1+[f'(x)]^2\right]^{3/2}}$$

$$x^2+y^2=r^2, \quad y^2=r^2-x^2 \quad y=(r^2-x^2)^{1/2}$$

$$f'=y'=\frac{1}{2}(r^2-x^2)^{-1/2}(-2x)=\frac{-x}{(r^2-x^2)^{1/2}}$$

$$=-x(r^2-x^2)^{-1/2}$$

$$f''=-(r^2-x^2)^{-1/2}+\frac{1}{2}x(r^2-x^2)^{-3/2}(-2x)$$

$$=-(r^2-x^2)^{-1/2}-x^2(r^2-x^2)^{-3/2}$$

$$=-\frac{1}{(r^2-x^2)^{1/2}}-\frac{x^2}{(r^2-x^2)^{3/2}}$$

$$=-\frac{1}{(r^2-x^2)^{1/2}}\left[1+\frac{x^2}{r^2-x^2}\right]$$

$$=-\frac{1}{(r^2-x^2)^{1/2}}\left[\frac{r^2-x^2+x^2}{r^2-x^2}\right]$$

$$=-\frac{r^2}{(r^2-x^2)^{3/2}}$$

$$1+f'^2=1+\frac{x^2}{(r^2-x^2)}=\frac{r^2+\cancel{x^2}-\cancel{x^2}}{r^2-x^2}=\frac{r^2}{r^2-x^2}$$

$$K=\frac{-r^2/(r^2-x)^{3/2}}{\left[r2/(r^2-x^2)\right]^{3/2}}=-\frac{r}{r^3}=\frac{1}{r}$$

(iii) $f(x) = \cosh x - x\sinh x$

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$f(x) = \frac{e^{x}+e^{-x}}{2} + x\frac{(e^{x}-e^{-x})}{2}$$

$$= \frac{e^{x}}{2}(x+1) + \frac{e^{-x}}{2}(1-x)$$

Now $e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$f(x) = \frac{1}{2}\sum_{k=0}^{\infty}(1+x)\frac{x^k}{k!}$$

$$+\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k(1-x)\frac{x^k}{k!}$$

$$= \frac{1}{2}\sum\left[1+(-1)^k\right]\frac{x^k}{k!}$$

$$+\frac{1}{2}\sum\left[(-1)^{k+1}+1\right]\frac{x^{k+1}}{k!}$$

$$= f_1(x) + f_2(x)$$

where $f_1(x) = \frac{1}{2}\sum(1+(-1)^k)\frac{x^k}{k!}$

When k is odd, coeff is zero

$$\therefore f_1(x) = \frac{1}{2}\sum_{k=0,2}^{\infty} 2\frac{x^k}{k!}$$

$$= \sum_{k=0,2}^{\infty}\frac{x^k}{k!}$$

Let $j = \frac{k}{2} \Rightarrow k = 2j$

~~Then $f_1(x) =$~~

Then $k = 0, 2, 4, 6, \ldots$ gives $j = 0, 1, 2, 3, \ldots$

$$f_1(x) = \sum_{j=0}^{\infty}\frac{x^{2j}}{(2j)!}$$

Consider

$$f_2(x) = \frac{1}{2}\sum\left[(-1)^{k+1}+1\right]\frac{x^{k+1}}{k!}$$

When k is even, coeff is zero.

$$f_2(x) = \frac{1}{2}\sum_{k=1,2} 2\cdot\frac{x^{k+1}}{k!}$$

$$= \sum_{k1,2}^{\infty}\frac{x^{k+1}}{k!}$$

Hence

$$f_2(x) = \frac{x^2}{1!} + \frac{x^4}{3!} + \frac{x^6}{5!} + \ldots$$

$$f_1(x) = \frac{1}{1!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

$$f(x) = f_1(x) + f_2(x)$$

$$= 1 + \left(\frac{1}{1!} + \frac{1}{2!}\right)x^2 + \left(\frac{1}{3!} + \frac{1}{4!}\right)x^4 + \left(\frac{1}{5!} + \frac{1}{6!}\right)x^6 + \ldots$$

$$= 1 + \sum_{k=2}^{\infty}\left(\frac{1}{(2k-3)!} + \frac{1}{(2k-2)!}\right)x^{2k-2}$$

$$= 1 + \sum_{k=2}^{\infty}\left(\frac{1}{(2k-3)!} + \frac{1}{(2k-2)!}\right)x^{2k-2} \quad //$$

$$f(2) = \frac{1}{2}$$

$$\frac{df}{dx} = -x^{-2};\ \left.\frac{df}{dx}\right|_2 = -\frac{1}{4}$$

$$\frac{d^2f}{dx^2} = 2x^{-3};\ \left.\frac{d^2f}{dx^2}\right|_2 = \frac{2}{8}$$

$$\frac{d^3f}{dx^3} = -6x^{-4};\ \left.\frac{d^3f}{dx^3}\right|_2 = \frac{6}{16}$$

Clearly,

$$\frac{d^{(n)}f}{dx^{(n)}} = (-1)^n n! \frac{1}{x^{n+1}}$$ and $$\left.\frac{d^{(n)}f}{dx^{(n)}}\right|_2 = (-1)^n \frac{n!}{2^{n+1}}$$

$$\therefore \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{2^{n+1}} (x-2)^n \quad //$$

5 (i) $f(x) = \frac{1}{x}$, $a = 2$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^2$$

(ii) $f(x) = \cos x,\ a = \pi/2$

$f\left(\frac{\pi}{2}\right) = 0$

$f'(x) = -\sin x,\ f'\left(\frac{\pi}{2}\right) = -1$

$f''(x) = -\cos x,\ f''\left(\frac{\pi}{2}\right) = 0$

$f^{(3)}(x) = \sin x,\ f^{3}\left(\frac{\pi}{2}\right) = 1$

$f^{4}(x) = \cos x,\ f^{4}\left(\frac{\pi}{2}\right) = 0$

$\vdots$

Hence

$$f(x) = -1\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5 + \frac{1}{7!}\left(x - \frac{\pi}{2}\right)^7 + \cdots$$

$$= \sum_{k=1,2}^{\infty} \text{sign} \cdot \frac{\left(x - \frac{\pi}{2}\right)^k}{k!}$$

Let $k = 2j+1$. We have

j	0	1	2	3
k	1	3	5	7

~~Hence~~ The sign is clearly $(-1)^{j+1}$

$$f(x) = \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\left(x - \frac{\pi}{2}\right)^{2j+1}}{(2j+1)!}$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\left(x - \frac{\pi}{2}\right)^{2k+1}}{(2k+1)!} \quad //$$

(iii) $f(x) = \dfrac{1}{(1-x)^2},\ a = 2$

$f(2) = \dfrac{1}{(1-2)^2} = 1$

$f(x) = (1-x)^{-2}$

$f'(x) = 2(1-x)^{-3}$

$f''(x) = 6(1-x)^{-4}$

$f^{(3)}(x) = 24(1-x)^{-5}$

Also, $f'(2) = -2$

$f''(2) = 6$

$f^{(3)}(2) = -24$

(ii) $f(x) = \frac{1}{(1+2x)^2}$

Consider $G(x) = \frac{1}{1+2x}$

Since

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$= \sum_{k=0}^{\infty} x^k$$

we have

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)}$$

$$= 1 + (-2x) + (-2x)^2 + \cdots$$

$$= \sum_{k=0}^{\infty} (-2x)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k (2x)^k$$

Now,

$$\therefore G(x) = \sum_{k=0}^{\infty} (-1)^k (2x)^k$$

Now $\frac{dG}{dx} = \frac{d}{dx}(1+2x)^{-1}$

$$= -(1+2x)^{-2}(2)$$

$$= -\frac{2}{(1+2x)^2}$$

$$\therefore \frac{1}{(1+2x)^2} = -\frac{1}{2}\frac{dG}{dx}$$

$$\therefore f(x) = -\frac{1}{2}\frac{d}{dx}\sum_{k=0}^{\infty} (-1)^k 2^k x^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{2}(-1)^{k+1} 2^k k x^{k-1}$$

Since the 1st term vanishes we write

$$f(x) = \frac{1}{2}\sum_{k=1}^{\infty} (-1)^{k+1} k 2^k x^{k-1}$$

~~Let $j = k+1$, When $k=1$~~, $j =$

Let $j = k-1$; $k = j+1$

When $k=1$, $j=0$

$$f(x) = \frac{1}{2}\sum_{j=0}^{\infty} (-1)^{j+2} (j+1) 2^{j+1} x^j$$

$$= \frac{2}{2}\sum_{j=0}^{\infty} (-1)^j (j+1) 2^j x^j$$

$$= \sum_{k=0}^{\infty} (-1)^k (k+1)(2x)^k \;//$$

Clearly,

$$f(x) = 1 - 2(x-2) + \frac{6}{2!}(x-2)^2 - \frac{24}{3!}(x-2)^3 + \frac{120}{4!}(x-2)^4 + \cdots$$

$$= 1 - 2(x-2) + 3(x-2)^2 - 4(x-2)^3 + 5(x-2)^4 + \cdots$$

$$= \cancel{\sum_{k=0}^{\infty} (-1)^k k (x-2)^k}$$

$$= \sum_{k=0}^{\infty} (-1)^k (k+1)(x-2)^k$$

//