

# Chapter 1

## Analytic geometry (Lecture 1)

In this Lecture, we will review some of the work covered in MAT1100 about the circle. We will define a parabola, with vertex at the origin, derive its equation and look at some examples.

### 1.1 The circle

Let  $C = (h, k)$  be a fixed point. We recall that the locus of points equidistant from  $C$  is called the circle. We call  $C = (h, k)$  the center of the circle, and the distance from  $C$  to any point on the circumference is called the radius of the circle. This means that if  $P = (x, y)$  is any point on the circumference of the circle, with center  $C = (h, k)$ , then  $r = \sqrt{(x - h)^2 + (y - k)^2}$ , so that  $r^2 = (x - h)^2 + (y - k)^2$  or  $x^2 - 2hx + y^2 - 2ky + h^2 + k^2 - r^2 = 0$  is the equation of the circle. The equation  $x^2 - 2hx + y^2 - 2ky + h^2 + k^2 - r^2 = 0$  is generally written as  $x^2 + y^2 + Cx + Dy + E = 0$ , and is called the *GENERAL EQUATION* of the circle, while  $r^2 = (x - h)^2 + (y - k)^2$  is known as the *STANDARD EQUATION* of the circle.

**Example 1.1.1.** 1. Find the equation of a circle with center at  $(2, -3)$ , and passes through  $(5, -1)$ .

2. Find the center of the circle whose equation is  $3x^2 + 3y^2 + 6x - 4y - 5 = 0$ .

## 1.2 Parabolas

**Definition 1.2.1.** A parabola is the set of all points in a plane equidistant from a fixed line ( $D$ ) called the directrix, and a fixed point ( $F$ ) called the focus in the plane.

Let's derive the algebraic equation for a parabola. Without loss of generality, we can assume the focus is  $F(0, p)$  on the positive  $y$ -axis and the directrix is the line  $y = -p$  (without loss of generality just means that any other situation could be transformed into this case). From the definition of parabola, we must have for an arbitrary point  $P(x, y)$  on the parabola:

$$\| PF \| = \| PD \|$$

distance to focus = distance to directrix

$$\sqrt{x^2 + (y - p)^2} = \sqrt{(y + p)^2}$$

$$x^2 + (y - p)^2 = (y + p)^2$$

$$x^2 = 4yp.$$

This reveals the parabola's symmetry about the  $y$ -axis. The point where the parabola crosses its axis is the **vertex**. For  $x^2 = 4yp$ , the vertex lies at the origin. The positive number  $p$  is the parabola's **focal length**.

The standard form for the equation of a parabola is  $x^2 = 4yp$ . If  $p > 0$ , the parabola opens upwards, and downward if  $p < 0$ . By interchanging  $x$  and  $y$  in  $x^2 = 4yp$ , we obtain  $y^2 = 4xp$ , with the parabola opening to the right if  $p > 0$  or opening to the left if  $p < 0$ .

**Example 1.2.2.** 1. Find the focus and directrix for the parabola

$$x^2 = -12y.$$

2. Find the focus and directrix for the parabola

$$y^2 = 10x.$$

3. Write an equation for a parabola that opens to the left, with vertex  $(0, 2)$  and passes through  $(-6, -4)$ . Hence sketch the graph.

**Note:** Check the course outline for references.

# Chapter 1

## Analytic geometry (Lecture 2)

### 1.1 Ellipses

**Definition 1.1.1.** An ellipse is the set of all points in the plane the sum of whose distances from two fixed points  $F_1$  and  $F_2$  is a constant. These two fixed points are called **foci** (plural of focus).

The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and the ellipse cross are the ellipse's **vertices**.

Suppose the foci are on the  $x$ -axis, at  $(c, 0)$  and  $(-c, 0)$ , so that the origin is halfway between the foci as in the figure below. Let the sum of the distances from the point  $P = (x, y)$  on the ellipse to the foci be  $2a > 0$ . It follows that

$$\begin{aligned} \|PF_1\| + \|PF_2\| &= 2a \\ \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} &= 2a \end{aligned}$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again,

obtaining  $(a^2 - c^2)x^2 + y^2a^2 = a^4 - a^2c^2$ , so that

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Since  $\|PF_1\| + \|PF_2\|$  is greater than  $\|F_1F_2\|$  by the triangle inequality, we have that  $2a$  is greater than  $2c$ . Consequently,  $a > c$  and so  $a^2 - c^2 > 0$ . Thus every point  $P$  whose coordinates satisfy an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \text{ with } 0 < c < a$$

also satisfies the equation

$$\|PF_1\| + \|PF_2\| = 2a.$$

A point therefore lies on the ellipse if and only if its coordinates satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If  $b^2 = a^2 - c^2$ , then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the equation of an ellipse symmetric with respect to the origin and both coordinate axes. It lies in the rectangle bounded by  $x = \pm a$  and  $y = \pm b$ .

### 1.1.2 Major and Minor axes of an ellipse

The major axis of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where  $c = \sqrt{a^2 - b^2}$ , is the line segment of length  $2a$  joining the points  $(\pm a, 0)$ . The minor axis of the ellipse is the line segment of length  $2b$

joining the points  $(0, \pm b)$ . The number  $a$  itself is the semimajor axis, the number  $b$  is the semiminor axis. The number  $c$  is the **center-to-focus** distance of the ellipse.

In conclusion, the standard-form equation for ellipses centered at the origin is

1.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ with } a > b.$$

(a) Foci on the  $x$ -axis

(b) Center-to-focus distance:  $c = \sqrt{a^2 - b^2}$

(c) Foci:  $(\pm c, 0)$

(d) Vertices:  $(\pm a, 0)$  for the major axis.

(e) Vertices:  $(0, \pm b)$  for the minor axis.

2.

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \text{ with } a > b.$$

(a) Foci on the  $y$ -axis

(b) Center-to-focus distance:  $c = \sqrt{a^2 - b^2}$

(c) Foci:  $(0, \pm c)$

(d) Vertices:  $(0, \pm a)$  for the major axis.

(e) Vertices:  $(\pm b, 0)$  for the minor axis.

**Example 1.1.3.** 1. Find the foci, vertices and sketch each of the following

(a)  $\frac{x^2}{9} + \frac{y^2}{16} = 1$

(b)  $9x^2 + 16y^2 = 144$ .

2. Find the equation of the ellipse with foci  $(0, \pm 2)$  and vertices  $(0, \pm 3)$ .

**Note: Check the course outline for references.**

# Chapter 1

## Analytic geometry (Lecture 3)

### 1.1 Hyperbolas

**Definition 1.1.1.** A hyperbola is the set of all points in the plane the difference of whose distances from two fixed points (foci)  $F_1$  and  $F_2$  is a constant.

If we place the foci on the  $x$ -axis at  $(c, 0)$  and  $(-c, 0)$ , as in the figure below, so that the origin is halfway between the foci and the constant difference is  $2a$ , then from  $|PF_1| - |PF_2| = \pm 2a$ , we obtain that a point  $P(x, y)$  lies on the hyperbola if and only if

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Since  $a < c$ , it follows that  $a^2 - c^2$  is negative. If we let  $b = \sqrt{c^2 - a^2}$ , then  $a^2 - c^2 = -b^2$  so that  $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$  becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

If the foci of the hyperbola are located on the  $y$ -axis at  $(0, \pm c)$ , then we can find its equation by interchanging the  $x$  and  $y$  in the above equation. Thus, we have

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$$

with vertices at  $(0, \pm a)$  and asymptotes  $y = \pm \frac{a}{b}x$ .

Like the ellipse, the hyperbola is symmetric with respect to the origin and the coordinate axes. The **focal axis**, the **foci**, the **center** and the **vertices** are defined as in an ellipse.

### 1.1.2 Asymptotes of hyperbolas-Graphing

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has two asymptotes, the lines  $y = \pm \frac{b}{a}x$ . They give us the guidance we need to graph hyperbolas quickly.

The standard-form equations for hyperbolas centered at the origin are:

1.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(a) Foci on the  $x$ -axis

(b) Center-to-focus distance:  $c = \sqrt{a^2 + b^2}$

(c) Foci:  $(\pm c, 0)$

(d) Vertices:  $(\pm a, 0)$ .

(e) Asymptotes:  $y = \pm \frac{b}{a}x$ .

2.

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

(a) Foci on the  $y$ -axis

(b) Center-to-focus distance:  $c = \sqrt{a^2 + b^2}$

(c) Foci:  $(0, \pm c)$

(d) Vertices:  $(0, \pm a)$

(e) Asymptotes:  $y = \pm \frac{a}{b}x$ .

**Example 1.1.3.** 1. Find the foci, vertices and asymptotes of the hyperbola  $9x^2 - 16y^2 = 144$  and sketch the graph.

2. Find the foci, vertices and asymptotes for the hyperbola  $\frac{y^2}{4} - \frac{x^2}{5} = 1$ .

3. Find the foci and the equations of the hyperbola with vertices  $(0, \pm 1)$  and asymptote  $y = 2x$ .

## 1.2 Translation of axes

### Introduction

In the previous sections, we found equations for parabolas, ellipses, and hyperbolas located with their axes on the coordinate axes and centered at the origin. What happens if we move conics away from the origin while keeping their axes parallel to the coordinate axes?

#### 1.2.1 Translation of axes

A translation of coordinate axes occurs when the new coordinate axes have the same direction as and are parallel to the original coordinate axes. A point  $P$  in the plane has two sets of coordinates:  $(x, y)$  in the original system and  $(x', y')$  in the translated system. If the coordinates of the origin of the translated system are  $(h, k)$  relative to the original system, then the old and new coordinates are related as follows

1.

$$x = x' + h$$

$$y = y' + k$$

2.

$$x' = x - h$$

$$y' = y - k$$

It can be shown that these formula hold for  $(h, k)$  located anywhere in the original coordinate system.

### 1.2.2 Standard Equations of Translated Conics

We now proceed to find standard equations of conics translated away from the origin. We do this by first writing the standard equations found in earlier sections in the  $x'y'$  coordinate system with  $O'$  at  $(h, k)$ . We then use translation equations to find the standard forms relative to the original  $xy$  coordinate system. The equations of translation in all cases are

$$x' = x - h$$

$$y' = y - k$$

For parabolas, we have

$$x'^2 = 4py' \qquad (x - h)^2 = 4p(y - k)$$

$$y'^2 = 4px' \qquad (y - k)^2 = 4p(x - h)$$

For ellipses, we have for  $a > b > 0$

$$\begin{array}{l} \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \\ \frac{x'^2}{b^2} + \frac{y'^2}{a^2} = 1 \end{array} \qquad \begin{array}{l} \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \\ \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \end{array}$$

For hyperbolas, we have

$$\begin{array}{l} \frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1 \\ \frac{y'^2}{a^2} - \frac{x'^2}{b^2} = 1 \end{array} \qquad \begin{array}{l} \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \\ \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \end{array}$$

**Note: Check the course outline for references.**

# Chapter 1

## Analytic geometry (Lecture 4)

### 1.1 Sketching Equations of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$

It can be shown that the graph of

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where  $A$  and  $C$  are not both zero, is a conic or a degenerate conic or that there is no graph. If we can transform the equation above into one of the standard equations listed above (previous section), then we will be able to identify its graph and sketch it as before. Here, the knowledge of completing the square will be useful tool. The following examples will make the process clear.

**Example 1.1.1.** Transform each of the following equations into one of the standard forms in the previous section. Identify the conic and graph it.

1.  $y^2 - 6y - 4x + 1 = 0.$

2.  $9x^2 + 16y^2 + 36x - 32y - 92 = 0$

3.  $9x^2 - 4y^2 - 36x - 24y - 35 = 0$

## 1.2 Rotation of axes

Here we show that the general second-equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be analyzed by rotating the axes so as to eliminate the term  $Bxy$ . A point  $P$  has coordinates  $(x, y)$  in the first system and  $(X, Y)$  in the new and rotated coordinate system. We will use the following equations to change from coordinate system to the other.

$$x = X \cos \theta - Y \sin \theta \qquad y = X \sin \theta + Y \cos \theta \qquad (1.1)$$

and

$$X = x \cos \theta + y \sin \theta \qquad Y = -x \sin \theta + y \cos \theta$$

**Example 1.2.1.** 1. If the axes are rotated through  $60^\circ$ , find the  $XY$ -coordinates of the point whose  $xy$ -coordinates are  $(2, 6)$ .

2. If the axes are rotated through  $45^\circ$ , find the  $xy$ -coordinates of the point whose  $xy$ -coordinates are  $(\sqrt{2}, \sqrt{2})$ .

We now determine the angle  $\theta$  such that the term  $Bxy$  in equation 1 disappears when the axes are rotated through the angle  $\theta$ . If we substitute (1.1) in

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

expanding and collecting like terms we get

$$A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F = 0,$$

where

$$B' = 2(C - A) \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) = (C - A) \sin 2\theta + B \cos 2\theta.$$

To eliminate the  $XY$  term we choose  $\theta$  so that  $B' = 0$ , that is,

$$\cot 2\theta = \frac{A - C}{B}.$$

**Example 1.2.2.** 1. Show that the graph of  $xy = 1$  is a hyperbola.

2. Identify and sketch the curve

$$73x^2 + 72xy + 52y^2 + 30x - 40y - 75 = 0.$$

### 1.2.3 The Discriminant Test

We can identify the conic section represented by the quadratic curve

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad (1.2)$$

by using the discriminant  $B^2 - 4AC$ . Equation (1.2) is

1. a parabola if  $B^2 - 4AC = 0$ ,
2. an ellipse if  $B^2 - 4AC < 0$ ,
3. a hyperbola if  $B^2 - 4AC > 0$ .

**Example 1.2.4.** Use the Discriminant to identify each of the following conic sections.

1.  $3x^2 - 6xy + 3y^2 + 2x - 7 = 0$
2.  $x^2 + xy + y^2 - 1 = 0$
3.  $xy - y^2 - 5y + 1 = 0$ .

## 0.1 Parametric equations for conics and other curves

In this lesson, we examine the position of a particle along a conic section and other curves. Parametric equations for the position of a particle moving in the plane are sometimes called parametric equations for the path traced by the particle.

We will discuss parametric equations through the following examples.

### Example 0.1.1

Describe the motion of the particle whose position  $P(x, y)$  at time  $t$  is given by

$$x = a \cos t, \quad y = a \sin t, \quad 0 \leq t \leq 2\pi$$

### Solution

Since  $x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2$ , the particle moves on a circle of radius  $a$  centered at the origin. The particle begins at the point  $(a, 0)$  when  $t = 0$  and moves once counter-clockwise around the circle as  $t$  increases to  $2\pi$ . See Figure 1 below.

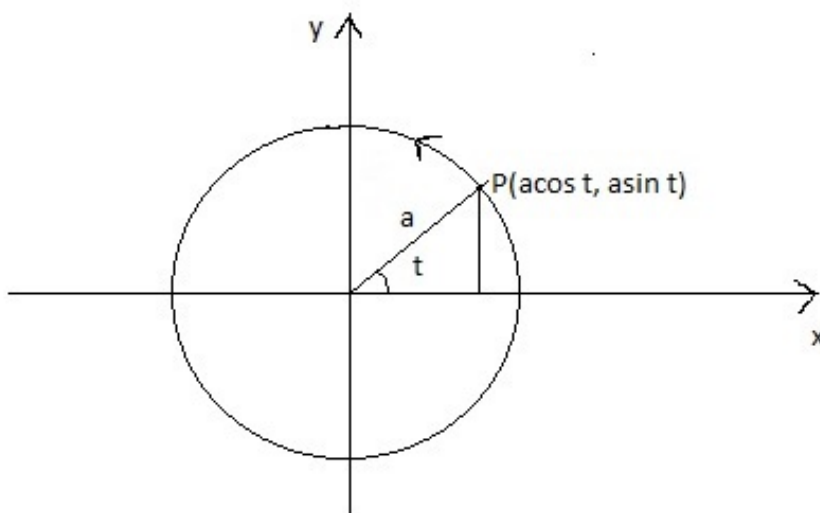


Figure 1: The circle defined by  $x = a \cos t, y = a \sin t, 0 \leq t \leq 2\pi$ .

**Example 0.1.2** Describe the motion of the particle whose position  $P(x, y)$  at time  $t$  is given by

$$x = t^2, \quad y = t, \quad -\infty < t < \infty$$

**Solution**

We will find the Cartesian equation by eliminating  $t$ . Since

$$y^2 = t^2 = x,$$

we see that the motion takes place on the parabola

$$y^2 = x.$$

As  $t$  increases between  $-\infty$  and  $\infty$ , the particle comes in on the lower half of the parabola, reaches the origin when  $t = 0$ , and moves out into the first quadrant as  $t$  continues to increase. See Figure 2 below.

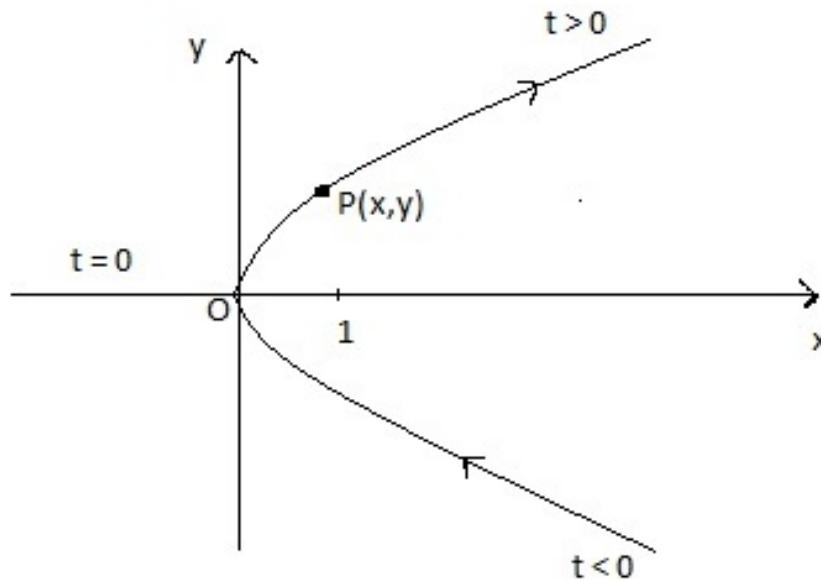


Figure 2: The parabola defined by  $x = t^2, y = t, -\infty < t < \infty$

Note that there are many different parametrisations of the parabola. Some are

$$y = (\tan^{-1} t)^2, \quad x = \tan^{-1} t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \quad (\text{same direction as above})$$

and

$$x = t^2, y = -t, -\infty < t < \infty \text{ (direction reversed).}$$

**Example 0.1.3** Describe the motion of the particle whose position  $P(x, y)$  at time  $t$  is given by

$$x = a \cos t, y = b \sin t, 0 \leq t \leq 2\pi$$

**Solution**

We find the Cartesian equation by eliminating  $t$ . Since

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2 \cos^2 t}{a^2} + \frac{b^2 \sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1,$$

the motion takes place on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The particle begins at  $(a, 0)$  when  $t = 0$ , and moves counter-clockwise around the ellipse transversing it exactly once as  $t$  moves from 0 to  $\pi$ . See Figure 3 below.

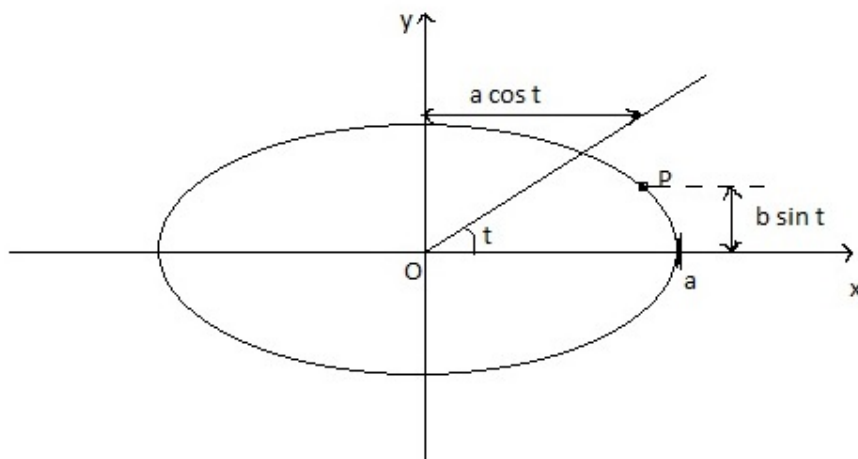


Figure 3: The coordinates at  $P$  are  $x = a \cos t, y = b \sin t$

**Example 0.1.4** Describe the motion of the particle whose position  $P(x, y)$  at time  $t$  is given by the equations

$$y = \tan t, x = \sec t, -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

### Solution

Eliminating  $t$  we have

$$x^2 - y^2 = \sec^2 t - \tan^2 t = 1$$

from which we see that the motion takes place somewhere on the hyperbola  $x^2 - y^2 = 1$ . Since  $x = \sec t$  is always positive for the parameter values  $-\frac{\pi}{2} < t < \frac{\pi}{2}$ , the motion takes place on the hyperbola's right-hand branch. As  $t$  moves from  $-\frac{\pi}{2}$  to  $\frac{\pi}{2}$ , the particle comes in along the lower half of the right-hand branch, reaching the origin at  $t = 0$ . It then moves into the first quadrant to complete the coverage of the right-hand branch as  $t$  approaches  $\frac{\pi}{2}$ . See Figure 4 below.

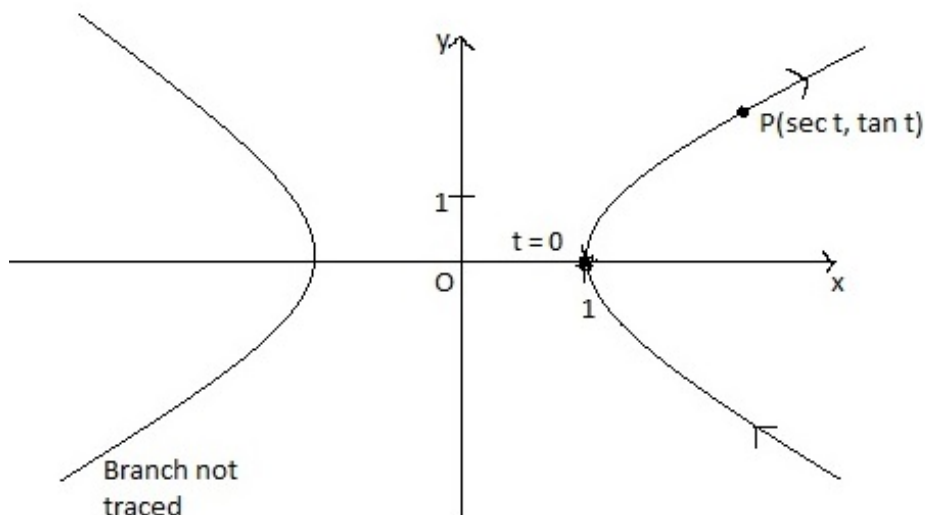


Figure 4: The right-hand branch of the hyperbola defined by  $x = \sec t$ ,  $y = \tan t$ ,  $-\frac{\pi}{2} < t < \frac{\pi}{2}$ .

**Example 0.1.5** Sketch the curve traced by the point  $P(x, y)$  whose coordinates satisfy the equations

$$y = 1 - \cos t, \quad x = \cos 2t, \quad -\infty < t < \infty.$$

### Solution

We first find the Cartesian equation by eliminating  $t$ . We have

$$y = 1 - \cos 2t = 1 - (2 \cos^2 t - 1) = 2 - 2 \cos^2 t = 2 - 2x^2$$

so the particle traces some portion of the parabola  $y = 2 - 2x^2$ . Since  $|\cos \theta| \leq 1$  for any angle  $\theta$ , the parametric equations describe only the portion for which

$$-1 \leq x = \cos 2t \leq 1 \text{ and } 0 \leq y = 1 - \cos t \leq 2.$$

Thus we see that  $P(x, y)$  starts at  $A(1, 0)$  when  $t = 0$  and moves up and the the left as  $t$  increases, arriving at  $B(0, 2)$  when  $t = \frac{\pi}{2}$ . It continues to  $C(-1, 0)$  as  $t$  increases to  $\pi$ . See Figure 5 below.

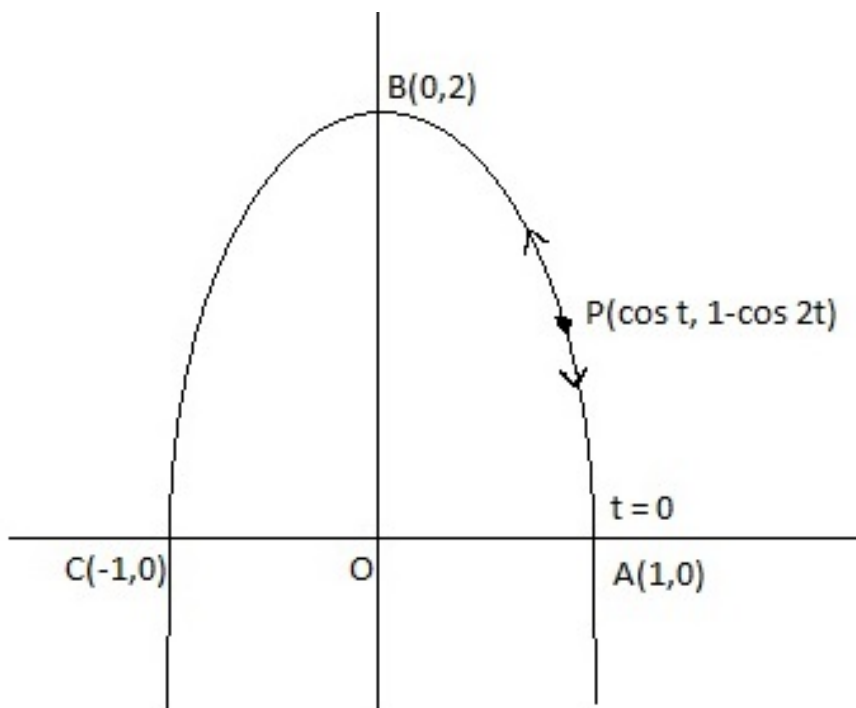


Figure 5: As  $t$  varies from  $-\infty$  to  $\infty$ , the point  $P$  traces and retraces the parabolic arch.

As  $t$  varies from  $\pi$  to  $2\pi$ , the point traces the arch  $CBA$  back to  $A$ . Since  $x$  and  $y$  are periodic, ( $x$  with period  $2\pi$  and  $y$  with period  $\pi$ ), any further variation of  $t$  results in tracing a portion of the arch.

**Example 0.1.6** *A wheel of radius  $r$  rolls along a horizontal straight line without slipping. Find the curve traced by the point  $P(x, y)$  on a spoke of the wheel  $b$  units from its centre. Such a curve is called a trochoid. If  $b = a$ ,  $P$  is*

on the circumference and the curve is called a cycloid. This is like the path travelled by a pebble in the thread of the rolling tire.

### Solution

Consider the figure below. We take the  $x$ -axis to be the line the wheel rolls

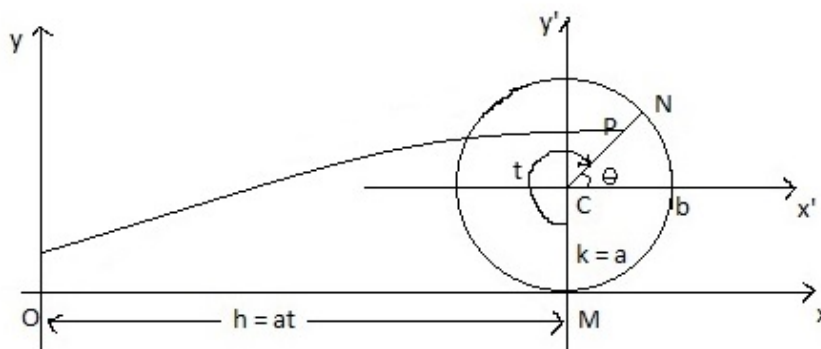


Figure 6: The trochoid  $x = at - b \sin t, y = a - b \cos t$ , shown for  $t \leq 0$ .

along, with the  $y$ -axis through a low point of the trochoid. Since the circle rolls without slipping, the distance  $OM$  is equal to the circular arc  $MN = at$ . The  $xy$ -coordinates of  $C$  are therefore

$$h = at, \quad k = a \quad (1)$$

Now we introduce  $x'y'$ -axes parallel to the  $xy$ -axes and having their origin at  $C$ . The  $xy$ - and  $x'y'$ -coordinates at  $P$  are related by the equations

$$x = h + x', \quad y = k + y'. \quad (2)$$

From the figure, we see that

$$x' = b \cos \theta, \quad y' = b \sin \theta$$

or since  $\theta = \frac{3\pi}{2} - t$ ,

$$x' = -b \sin t, \quad y' = -b \cos t.$$

Substituting these into (1) and (2) we obtain

$$x = at - b \sin t, \quad y = a - b \cos t \quad (3)$$

as parametric equations of a trochoid.

Taking  $b = a$  in (3) we obtain

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

as the parametric equation of the cycloid.

In this section we discuss tangents and normals to parametric curves. We will also look at polar coordinates for conic sections.

## 0.1 Tangents and Normals to parametric curves

**Theorem 0.1.1** *Let  $C$  be the parametric curve  $x = f(t), y = g(t)$  where  $f'(t)$  and  $g'(t)$  are continuous on an interval  $I$ . If  $f'(t) \neq 0$  on  $I$ , then  $C$  is smooth and has at  $t$  a tangent with slope*

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$

*If  $g'(t) \neq 0$  on  $I$ , then  $C$  is smooth and has at each  $t$  a normal line with slope*

$$-\frac{dx}{dy} = -\frac{f'(t)}{g'(t)}.$$

*Thus  $C$  is smooth except possibly at points where  $f'(t)$  and  $g'(t)$  are both 0.*

**Proof.** If  $f'(t) \neq 0$  on  $I$ , the  $f$  is either increasing or decreasing on  $I$  and so is one-to-one and invertible. The part of  $C$  corresponding to values of  $t$  in  $I$  has ordinary equation  $y = g(f^{-1}(x))$  and hence slope

$$\frac{dy}{dx} = g'(f^{-1}(x)) \frac{d}{dx} f^{-1}(x) = \frac{g'(f^{-1}(x))}{f'(f^{-1}(x))} = \frac{g'(t)}{f'(t)}.$$

We have used here the formula

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

for the derivative of the inverse function. The slope is a continuous function of  $t$ , so the tangent to  $C$  turns continuously for  $t$  in  $I$ .

The proof for  $g'(t) \neq 0$  is similar. In this case, the slope of the normal is a continuous function of  $t$ , so the normal turns continuously. Therefore, so does the tangent. ■

**Note 0.1.2** *If  $f$  and  $g$  are continuous and both vanish at  $t = t_0$ , then the curve  $x = f(t), y = g(t)$  may or may not be smooth at  $t_0$ .*

**Definition 0.1.3** *If  $f'$  and  $g'$  are continuous and not both 0 at  $t_0$ , then the parametric equations*

$$\begin{cases} x = f(t_0) + f'(t_0)(t - t_0) & (-\infty < t < \infty) \\ y = g(t_0) + g'(t_0)(t - t_0) \end{cases}$$

represents the tangent line to the parametric curve  $x = f(t), y = g(t)$  at the point  $(f(t_0), g(t_0))$ . The normal line there has parametric equations

$$\begin{cases} x = f(t_0) + g'(t_0)(t - t_0) & (-\infty < t < \infty) \\ y = g(t_0) - f'(t_0)(t - t_0) \end{cases}$$

Both lines pass through  $(f(t_0), g(t_0))$  when  $t = t_0$ .

**Example 0.1.4** Find equations of the tangent and normal lines to the parametric curve  $x = t^2 - t, y = t^2 + t$  at the point when  $t = 2$ .

**Solution**

At  $t = 2$ , we have  $x = 2, y = 6$  and  $\frac{dx}{dt} = 2t - 1 = 3, \frac{dy}{dt} = 2t + 1 = 5$ . Hence,

$$\text{Tangent: } \begin{cases} x = 2 + 3(t - 2) = 3t - 4 \\ y = 6 + 5(t - 2) = 5t - 4 \end{cases}$$

and

$$\text{Normal: } \begin{cases} x = 2 + 5(t - 2) = 5t - 8 \\ y = 6 - 3(t - 2) = -3t + 12 \end{cases}$$

## Concavity of a parametric equation

The concavity is determined by calculating  $\frac{d^2y}{dx^2}$  using the chain rule:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \frac{dy}{dx} = \frac{d}{dx} \frac{g'(t)}{f'(t)} = \frac{d}{dt} \frac{g'(t)}{f'(t)} \frac{dt}{dx} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^2} \cdot \frac{1}{f'(t)}$$

Thus, on an interval where  $f'(t) \neq 0$ , the parametric curve  $x = f(t), y = g(t)$  has concavity given by

$$\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}$$

Derivatives provide useful information about the shape of the parametric curve. At points where  $dy/dt = 0$  but  $dx/dt \neq 0$ , the tangent is horizontal; and at points where  $dx/dt = 0$  but  $dy/dt \neq 0$ , the tangent is vertical. For points where  $dy/dt = dx/dt = 0$ , anything can happen, it is wise to calculate the left- and the right-hand limits of the slope  $dy/dx$  as the parameter  $t$  approaches one of these points.

## 0.2 Polar coordinates for conics

To define polar coordinates, we first fix an origin  $O$  called the pole and an initial ray from  $O$  as shown in the figure below. Then each point  $P$  can be

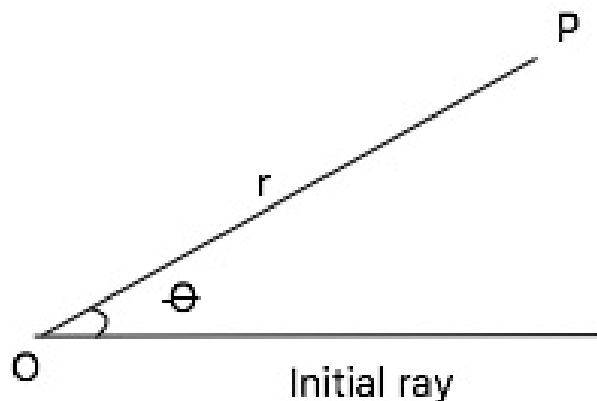


Figure 1: Initial Ray

located by assigning to it a polar coordinate pair  $(r, \theta)$ , where  $r$  gives the directed distance from  $O$  to  $P$  and  $\theta$  gives the directed angle from the initial ray to the segment  $OP$ .

When we want to consider both rectangular and polar coordinate systems in the same plane, we choose the positive  $x$ -axis as the initial ray and the origin of the Cartesian system as the pole. See the diagram below.

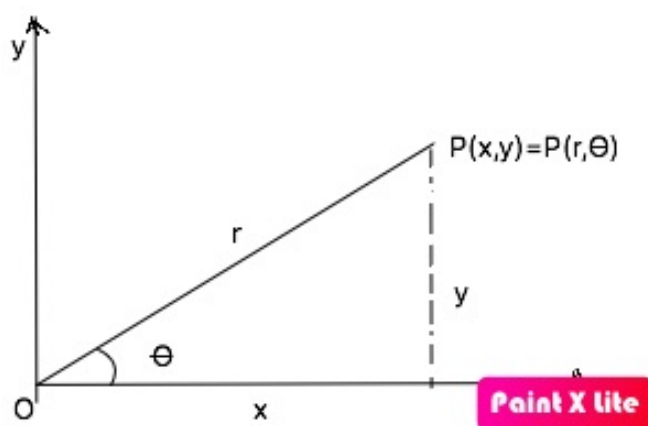


Figure 2: Cartesian-Polar

The two sets of coordinates are related by the equations

$$x = r \cos \theta, y = r \sin \theta \text{ or } x^2 + y^2 = r^2, \tan \theta = y/x.$$

**Example 0.2.1** Find a Cartesian equation for the curve  $r \cos(\theta - \frac{\pi}{3}) = 3$ .

**Solution**

We use the identity  $\cos(A - B) = \cos A \cos B + \sin A \sin B$  with  $A = \theta$  and  $B = \frac{\pi}{3}$ . Thus we have

$$\begin{aligned} r \cos\left(\theta - \frac{\pi}{3}\right) &= 3 \\ \Leftrightarrow r\left(\cos \theta \cos \frac{\pi}{3} + \sin \theta \sin \frac{\pi}{3}\right) &= 3 \\ \Leftrightarrow r \cos \theta \cdot \frac{1}{2} + r \sin \theta \cdot \frac{\sqrt{3}}{2} &= 3 \\ \Leftrightarrow \frac{1}{2}x + \frac{\sqrt{3}}{2}y &= 3 \\ \Leftrightarrow x + \sqrt{3}y &= 6. \end{aligned}$$

**Example 0.2.2** Find a Cartesian equation for the curve  $r^2 = 2ar \cos \theta$  and identify it.

**Solution**

Using the Cartesian-Polar relationship we have

$$r^2 = 2ar \cos \theta \Leftrightarrow x^2 + y^2 = 2ax \Leftrightarrow (x - a)^2 + y^2 = a^2$$

where the last equivalence is found by completing the square. Thus the curve is a circle of radius  $a$  with center  $(a, 0)$ .

**Example 0.2.3** Find a Cartesian equation for the curve

$$r = \frac{4}{2 \cos \theta - \sin \theta}$$

and hence, identify it.

### Solution

We have

$$\begin{aligned}r &= \frac{4}{2 \cos \theta - \sin \theta} \\ \Leftrightarrow r(2 \cos \theta - \sin \theta) &= 4 \\ \Leftrightarrow 2r \cos \theta - r \sin \theta &= 4 \\ \Leftrightarrow 2x - y &= 4 \\ \Leftrightarrow y &= 2x - 4.\end{aligned}$$

Thus the curve is a line with slope 2 and  $y$ -intercept  $-4$ .

## Polar equations for conic sections and other curves

**Example 0.2.4** suppose that the perpendicular from the origin to the line  $L$  at the point  $N(p, \beta)$ . find the polar equation of  $L$ .

### Solution

Let  $P(r, \theta)$  be a typical point on  $L$ .

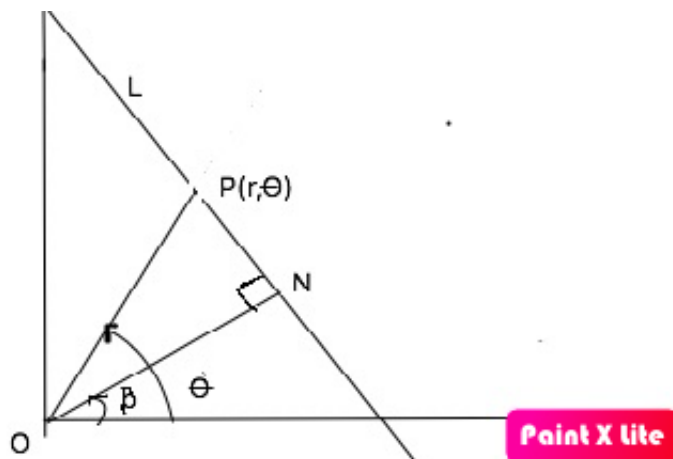


Figure 3: Polar equation of a line

We have from  $\triangle ONP$  that

$$\cos(\theta - \beta) = \frac{p}{r} \text{ or } r \cos(\theta - \beta) = p$$

as the required polar equation. If  $L$  was perpendicular to the  $x$ -axis,  $\beta$  would be 0 and the polar equation would reduce to

$$r \cos \theta = p \text{ or } x = p$$

**Example 0.2.5** Find a polar equation for the circle of radius  $a$  with centre at  $(b, \beta)$ .

**Solution**

Let  $P(r, \theta)$  be a representative point on the circle and apply the law of cosines to the triangle  $OCP$  to obtain

$$a^2 = b^2 + r^2 - 2br \cos(\theta - \beta)$$

If the circle passes through the origin, then  $b = a$  and the equation takes

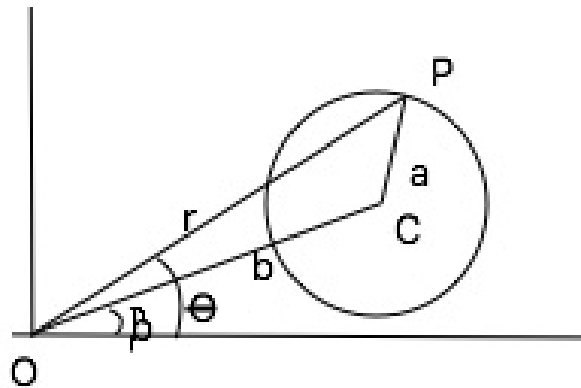


Figure 4: Polar equation of a circle

the simpler form

$$r(r - 2a \cos(\theta - \beta)) = 0 \text{ or } r = 2a \cos(\theta - \beta)$$

If  $\beta = 0$ , then the equation becomes

$$r = 2a \cos \theta.$$

If  $\beta = 90^\circ$ , so the centre of the circle lies on the  $y$ -axis, the equation reduces to

$$r = 2a \sin \theta.$$

**Example 0.2.6** Find a polar equation for the conic section of eccentricity  $e$  if the focus lies at the origin and the associated directrix is the line  $x = -k$ .

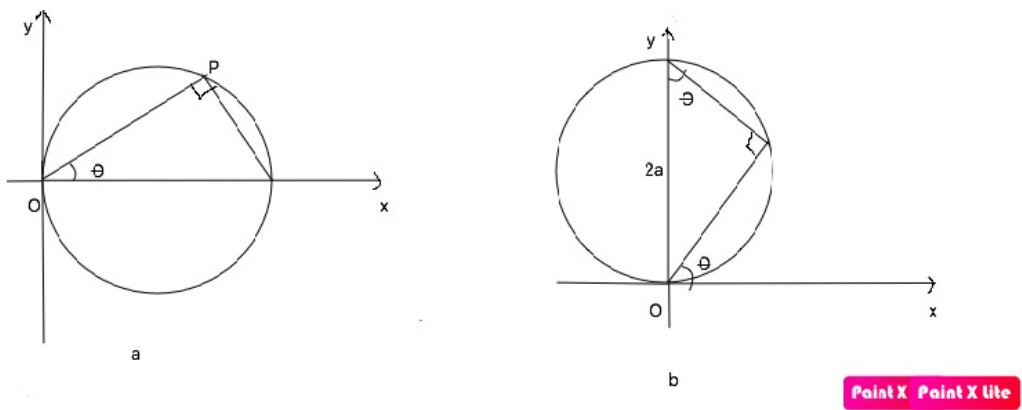


Figure 5: Polar equation of a circle through the origin

**Solution**

Consider the diagram

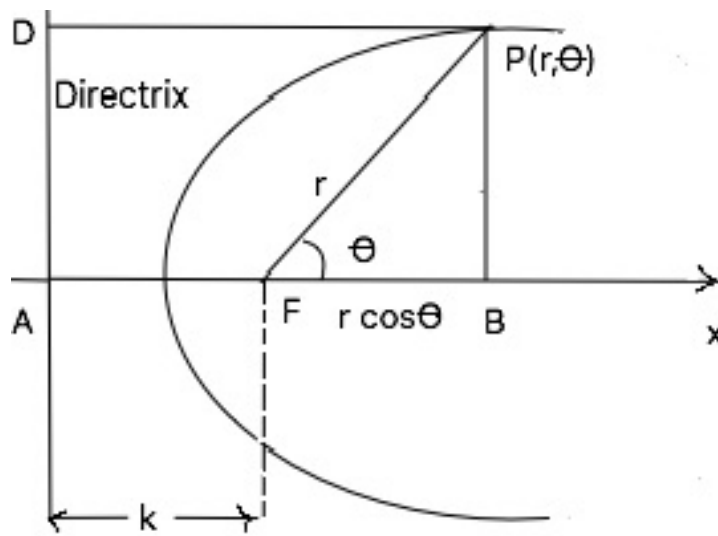


Figure 6: Polar equation of a conic

We use the focus-directrix property

$$PF = e \cdot PD \tag{1}$$

By taking the origin at the focus  $F$ , we have

$$PF + r$$

while

$$PD = AB = AF + FB = k + r \cos \theta$$

Then (1) is same as

$$r = e(k + r \cos \theta) \tag{2}$$

Solving (2) for  $r$  we get the polar equation

$$r = \frac{ke}{1 - e \cos \theta} \tag{3}$$

Here , one focus is at the origin, directrix vertical to the left of the origin,  $k$  = distance from origin to the directrix and  $e$  = eccentricity.

From (3) we have the following:

$$e = \frac{1}{2} : \text{ Ellipse, } r = \frac{k}{2 - \cos \theta} \tag{4}$$

$$e = 1 : \text{ Parabola, } r = \frac{k}{1 - \cos \theta} \tag{5}$$

$$e = 2 : \text{ Hypabola, } r = \frac{2k}{1 - 2 \cos \theta} \tag{6}$$

The denominator in (4) is never less than 1 , so  $r$  is never greater than  $k$ . But  $r \rightarrow \infty$  as  $\theta \rightarrow 0$  in (5) and as  $\theta \rightarrow \frac{\pi}{3}$  in (6).

Now consider the diagram

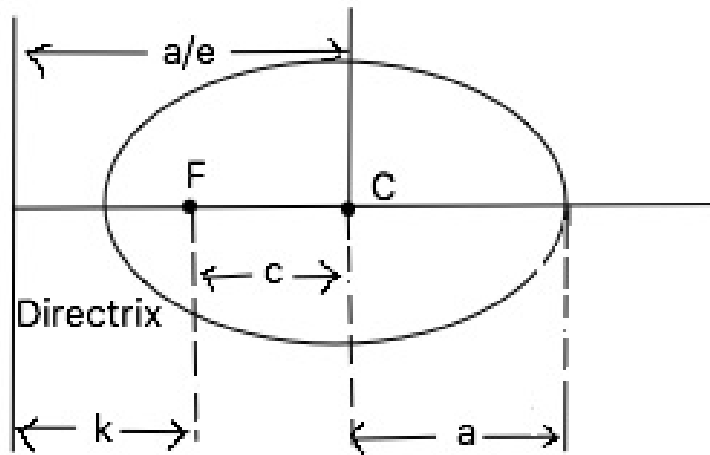


Figure 7: polar equation for an ellipse

We see that  $k$  is related to  $e$  and  $a$  by the equation

$$k = \frac{a}{e} - ea$$

From this we get

$$ke = a(1 - e^2),$$

and therefore the equation for the ellipse with semimajor axis  $a$  and eccentricity  $e$  is

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta} \quad (7)$$

If the directrix of a conic is  $x = k$ , the equation (1) takes the form  $r = e(k - r \cos \theta)$  and solving for  $r$  we get

$$r = \frac{ke}{1 + e \cos \theta}. \quad (8)$$

Notice that this is equation (3) except for the change of sign in the denominator.

**Example 0.2.7** Find a polar equation for the ellipse with semimajor axis 39.44 astronomical units (AU) and eccentricity 0.25. This is approximately the size and shape of the orbit of Pluto around the sun. One astronomical unit is the length of the semimajor axis of the Earth's orbit.

**solution**

Using equation (7) we have

$$r = \frac{39.44(1 - (0.25)^2)}{1 - 0.25 \cos \theta} = \frac{147.9}{4 - \cos \theta}$$

At the point of closest approach, Pluto is

$$r = \frac{147.9}{4 + 1} = 29.58AU$$

from the sun. At its most distance point, Pluto is

$$r = \frac{147.9}{4 - 1} = 49.3AU$$

from the sun.

**Example 0.2.8** Find the distance from one focus of the ellipse in Example 0.2.7 to the associated directrix.

**solution**

We have  $k = a(\frac{1}{e} - e) = 39.44(\frac{1}{0.25} - 0.25) = 147.9AU$ .

**Example 0.2.9** Find an equation for the hyperbola with eccentricity  $\frac{3}{2}$  and directrix  $x = 2$ .

**solution**

Since  $k = 2$  is positive, we use (8) to get

$$r = \frac{2(\frac{3}{2})}{1 + \frac{3}{2} \cos \theta} \text{ or } r = \frac{6}{2 + 3 \cos \theta}.$$

**Example 0.2.10** *Find the directrix of the parabola*

$$r = \frac{25}{10 + 10 \cos \theta}$$

**solution**

We first put the equation in standard form by dividing the equation by 10; that is

$$r = \frac{\frac{5}{2}}{1 + \cos \theta}$$

which implies that

$$r = \frac{ke}{1 + e \cos \theta}$$

with  $k = \frac{5}{2}$ ,  $e = 1$  and directrix  $x = \frac{5}{2}$ .

# Chapter 2

## Differential Calculus

### 2.1 Higher order derivatives

If the derivative  $y' = f'(x)$  of a function  $y = f(x)$  is itself differentiable at  $x$ , we can calculate its derivative, which we call the **second derivative** of  $f$  and denoted by  $y'' = f''(x)$ . Other notations used include

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx} \frac{d}{dx} f(x) = \frac{d^2}{dx^2} f(x) = D_x^2 y = D_x^2 f(x).$$

In general, the  $n^{\text{th}}$  derivative of  $y = f(x)$  is

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} f(x) = D_x^n y = D_x^n f(x)$$

and is defined to be the derivative of the  $(n-1)^{\text{st}}$  derivative. For  $n = 1, 2, 3$ , primes are still normally used:  $f^{(2)}(x) = f''(x)$ ,  $f^{(3)}(x) = f'''(x)$ . Sometimes we denote  $f^{(0)}(x) = f(x)$ .

**Example 2.1.1** *The velocity of a moving object is the rate of change of the position of the object with respect to time. If the object moves along the  $x$ -axis and is at position  $x = f(t)$  at time  $t$ , then its velocity at that time is given by*

$$v = \frac{dx}{dt} = f'(t).$$

*Similarly, the acceleration of the object is the rate of change of velocity. Thus, the acceleration is the second derivative of the position:*

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = f''(t).$$

**Example 2.1.2** If  $y = x^3$ , then  $y' = 3x^2$ ,  $y'' = 6x$ ,  $y''' = 6$ ,  $y^{(4)} = 0$  and all higher derivatives are zero. In general, if  $y = x^n$ , then

$$\begin{aligned} y^{(k)}(x) &= n(n-1)(n-2)\dots(n-(k-1))x^{n-k} \\ &= \begin{cases} \frac{n!}{(n-k)!}x^{n-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases} \end{aligned}$$

where  $n!$  (called  $n$  **factorial**) is defined by

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 1! \times 2 = 1 \times 2 = 2 \\ 3! &= 2! \times 3 = 2 \times 3 = 6 \\ 4! &= 3! \times 4 = 6 \times 4 = 24 \\ &\vdots \\ n! &= (n-1)!n = 1 \times 2 \times 3 \times \dots \times (n-1) \times n. \end{aligned}$$

It follows that if  $P$  is a polynomial of degree  $n$ ,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where  $a_n, a_{n-1}, \dots, a_1, a_0$  are constants, then  $P^{(k)}(x) = 0$  for  $k > n$ . For  $n \leq k$ ,  $P^{(k)}$  is a polynomial of degree  $n - k$ ; in particular,  $P^{(n)}(x) = n!a_n$ , a constant function.

**Example 2.1.3** Show that, if  $A, B$  and  $\kappa$  are constants, then the function  $y = A \cos(\kappa t) + B \sin(\kappa t)$  is a solution of a second-order differential equation of simple harmonic motion

$$\frac{d^2 y}{dt^2} + \kappa^2 y = 0.$$

### Solution

To be a solution, the function  $y(t)$  must satisfy the equation identically, i.e.,

$$\frac{d^2 y(t)}{dt^2} + \kappa^2 y(t) = 0. \quad (2.1)$$

must hold for every real number  $t$ . We calculate the first and second derivatives, i.e.,

$$\frac{dy}{dt} = -A\kappa \sin(\kappa t) + B\kappa \cos(\kappa t)$$

and

$$\frac{d^2y}{dt^2} = -A\kappa^2 \cos(\kappa t) - B\kappa^2 \sin(\kappa t) = -\kappa^2 y(t)$$

Substituting this in equation (2.1), we get

$$-A\kappa^2 \cos(\kappa t) - B\kappa^2 \sin(\kappa t) + \kappa^2(A \cos(\kappa t) + B \sin(\kappa t)) = 0.$$

Thus,  $y(t)$  is a solution to the given equation.

**Example 2.1.4** Find the  $n^{\text{th}}$  derivative of  $y = \frac{1}{1+x}$ .

### Solution

First note that

$$\frac{1}{1+x} = (1+x)^{-1}.$$

We begin by calculation a few derivatives:

$$\begin{aligned}y' &= (-1)(1+x)^{-2} \\y'' &= (-1)(-2)(1+x)^{-3} = 2(1+x)^{-3} \\y''' &= 2(-3)(1+x)^{-4} = -3!(1+x)^{-4} \\y^{(4)} &= -3!(-4)(1+x)^{-5} = 4!(1+x)^{-5}\end{aligned}$$

We are now able to predict the  $n^{\text{th}}$  derivative from the above pattern, which is

$$y^{(n)} = (-1)^n n!(1+x)^{-n-1}.$$

However, this does not prove that this formula is true for every  $n$ . So to complete the proof, we use mathematical induction on  $n$ . We see from the first derivative above that the formula is true for  $n = 1$ . Suppose that the formula is valid for  $n = k$ , where  $k$  is some positive integer. Consider  $y^{k+1}$ :

$$\begin{aligned}y^{k+1} &= \frac{d}{dx} y^{(k)} = \frac{d}{dx} ((-1)^k k!(1+x)^{-k-1}) \\&= (-1)^k k!(-k-1)(1+x)^{-k-2} \\&= (-1)^{k+1} (k+1)!(1+x)^{-(k+1)-1}.\end{aligned}$$

This is what the formula predicted for the  $(k+1)^{\text{st}}$  derivative. Thus, if the formula is true for  $n = k$ , then it is true for  $n = k+1$ . Therefore, by induction, the formula is true for all  $n \geq 1$ .

**Example 2.1.5** Find a formula for  $f^{(n)}(x)$ , given that  $f(x) = \sin(ax + b)$ .

### Solution

We again begin by calculating a few derivatives and see if we can predict the  $n^{\text{th}}$  derivative.

$$\begin{aligned}f'(x) &= a \cos(ax + b) \\f''(x) &= -a^2 \sin(ax + b) = -a^2 f'(x) \\f'''(x) &= -a^3 \cos(ax + b) = -a^2 f''(x) \\f^{(4)}(x) &= a^4 \sin(ax + b) = a^4 f'(x) \\f^{(5)}(x) &= a^5 \cos(ax + b) = a^4 f''(x)\end{aligned}$$

We can see from the pattern above that each derivative is  $-a^2$  times the second previous one. Therefore, a formula that gives all derivatives is

$$f^{(n)}(x) = \begin{cases} (-1)^k a^n \sin(ax + b) & \text{if } n = 2k \\ (-1)^k a^n \cos(ax + b) & \text{if } n = 2k + 1 \end{cases} \quad (k = 0, 1, 2, \dots)$$

**Example 2.1.6** Calculate  $f'$ ,  $f''$ , and  $f'''$  for  $f(x) = \sqrt{x^2 + 1}$ . Can you predict  $f^{(4)}$  from the pattern?

### Solution

Since  $f(x) = (x^2 + 1)^{\frac{1}{2}}$  we have

$$\begin{aligned}f'(x) &= \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}(2x) = x(x^2 + 1)^{-\frac{1}{2}}, \\f''(x) &= (x^2 + 1)^{-\frac{1}{2}} + x\left(-\frac{1}{2}\right)(x^2 + 1)^{-\frac{3}{2}}(2x) = (x^2 + 1)^{-\frac{3}{2}}, \\f'''(x) &= -\frac{3}{2}(x^2 + 1)^{-\frac{5}{2}}(2x) = -3x(x^2 + 1)^{-\frac{5}{2}}.\end{aligned}$$

The pattern is not obvious to predict  $f^{(4)}$  without calculating it. In fact,

$$f^{(4)} = 3(4x^2 - 1)(x^2 + 1)^{-\frac{7}{2}}.$$

So the pattern doesn't become clearer even at this stage.

## 2.2 Implicit Differentiation

Curves are generally the graphs of equations in two variables. Such equations can be written in the form

$$F(x, y) = 0$$

where  $F(x, y)$  denotes an expression involving the two variables  $x$  and  $y$ . For example, a circle of radius 5 and centre at the origin has equation

$$F(x, y) = x^2 + y^2 - 5.$$

To find the derivative of  $F$ , we use a technique called **implicit differentiation**. In this technique, we differentiate  $F$  with respect to  $x$  while treating  $y$  as a function of  $x$  having derivative  $\frac{dy}{dx}$  or  $y'$ .

**Example 2.2.1** Find  $\frac{dy}{dx}$  if  $y^2 = x$ .

### Solution

We differentiate both sides of the equation with respect to  $x$  using the chain rule:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x) \Rightarrow 2y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y} (= \frac{1}{2\sqrt{x}} \text{ or } -\frac{1}{2\sqrt{x}})$$

Note that  $y^2 = x$  implies  $y_1 = \sqrt{x}$  and  $y_2 = -\sqrt{x}$ . So

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \text{ and } \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}$$

which agrees with the above derivative.

**Example 2.2.2** Find the slope of the circle  $x^2 + y^2 = 25$  at  $(3, -4)$ .

### Solution

This equation of the circle combines two graphs namely  $y_1 = \sqrt{25 - x^2}$  and  $y_2 = -\sqrt{25 - x^2}$ . Observe that the point  $(3, -4)$  lies on the graph of  $y_2$ . So we can find the slope by calculating explicitly

$$\frac{dy_2}{dx} \Big|_{x=3} = -\frac{-2x}{2\sqrt{25 - x^2}} \Big|_{x=3} = \frac{6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

But we can also differentiate implicitly with respect to  $x$ :

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25) \Rightarrow 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}.$$

Therefore, the slope at  $(3, -4)$  is

$$\frac{-x}{y} \Big|_{(3, -4)} = -\frac{-3}{-4} = \frac{3}{4}.$$

So why implicit differentiation if we can explicitly differentiate with respect to  $x$ ? Well, it is not all functions that we can easily express  $y$  as a function of  $x$  as the following example shows.

**Example 2.2.3** Find  $\frac{dy}{dx}$  if  $y \sin x = x^3 + \cos y$ .

### Solution

Here, we can only use implicit differentiation. We have

$$\begin{aligned}\frac{d}{dx}(y \sin x) &= \frac{d}{dx}(x^3) + \frac{d}{dx}(\cos y) \\ \Rightarrow \sin x \frac{dy}{dx} + y \cos x &= 3x^2 - \sin y \frac{dy}{dx} \\ \Rightarrow (\sin x + \sin y) \frac{dy}{dx} &= 3x^2 - y \cos x \\ \Rightarrow \frac{dy}{dx} &= \frac{3x^2 - y \cos x}{\sin x + \sin y}.\end{aligned}$$

**Example 2.2.4** Find an equation of the tangent to  $x^2 + xy + 2y^3 = 4$  at  $(-2, 1)$ .

### Solution

Note that  $(-2, 1)$  lies on the curve. To find the slope of the tangent, we differentiate implicitly with respect to  $x$  to get:

$$2x + y + xy' + 6y^2y' = 0.$$

Substituting  $x = -2$ ,  $y = 1$  and solving for  $y'$  we get

$$-4 + 1 - 2y' + 6y' = 0 \Rightarrow y' = \frac{3}{4}.$$

Thus, the slope of the tangent at  $(-2, 1)$  is  $\frac{3}{4}$ , and its equation is

$$y = \frac{3}{4}(x + 2) + 1 \text{ or } 3x - 4y = -10$$

**Example 2.2.5** Find  $\frac{d^2y}{dx^2}$  if  $xy + y^2 = 2x$ .

### Solution

We implicitly differentiate twice both sides of the equation. Thus

$$y + xy' + 2yy' = 2 \tag{2.2}$$

and solving (2.2) for  $y'$  we get

$$y' = \frac{2 - y}{x + 2y}. \tag{2.3}$$

Implicitly differentiating (2.2) again with respect to  $x$  we get

$$y' + y' + xy'' + 2(y')^2 + 2yy'' = 0 \Rightarrow (x + 2y)y'' = -2y' - 2(y')^2$$

so that

$$y'' = \frac{-2y' - 2(y')^2}{x + 2y}. \quad (2.4)$$

Substituting (2.3) into (2.4) we get

$$\begin{aligned} y'' &= \frac{-2\left(\frac{2-y}{x+2y}\right) - 2\left(\frac{2-y}{x+2y}\right)^2}{x + 2y} \\ &= -2 \left( \frac{2-y}{(x+2y)^2} \right) - 2 \left( \frac{(2-y)^2}{(x+2y)^3} \right) \\ &= \frac{-2(2-y)(x+2y+2-y)}{(x+2y)^3} \\ &= \frac{-2(2-y)(x+y+2)}{(x+2y)^3} \\ &= \frac{-2(2x-xy+2y-y^2+4-2y)}{(x+2y)^3} \\ &= \frac{-2(2x-xy-y^2+4)}{(x+2y)^3} \\ &= \frac{-8}{(x+2y)^3} \quad (\text{since } 2x-xy-y^2=0) \end{aligned}$$

(We have used the given equation (in the question) to simplify the numerator.)

## 2.3 Application of derivatives

### Linear approximations

**Definition 2.3.1** The *linearization* of the function  $f$  about  $a$  is the function  $L$  defined by

$$L(x) = f(a) + f'(a)(x - a).$$

We say that  $f(x) \approx L(x)$  provides **linear approximation** for values of  $f$  near  $a$ .

**Example 2.3.2** Find the linearization of

a)  $f(x) = \sqrt{1+x}$  about  $x = 0$

b)  $g(t) = \frac{1}{t}$  about  $t = \frac{1}{2}$

### Solution

a) We have  $f(0) = \sqrt{1+0} = 1$  and since  $f'(x) = \frac{1}{2\sqrt{1+x}}$ ,

$$f'(0) = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}.$$

Therefore, the linearization of  $f$  about  $x = 0$  is

$$L(x) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

b) We have  $g(1/2) = 2$  and since  $g'(t) = \frac{-1}{t^2}$ ,  $g'(1/2) = -4$ . Therefore, then linearization of  $g$  about  $t = 1/2$  is

$$L(t) = 2 - 4(t - 1/2) = 4 - 4t.$$

**Example 2.3.3** Use the linearization for  $\sqrt{x}$  about  $x = 25$  to find an approximate value of  $\sqrt{26}$ .

## Solution

If  $f(x) = \sqrt{x}$ , then  $f'(x) = \frac{1}{2\sqrt{x}}$ . Since  $f(25) = 5$  and  $f'(25) = \frac{1}{10}$ , the linearization of  $\sqrt{x}$  about  $x = 25$  is

$$L(x) = 5 + \frac{1}{10}(x - 25).$$

Putting  $x = 26$ , we get

$$\sqrt{26} = f(26) \approx L(26) = 5 + \frac{1}{10}(26 - 25) = 5.1$$

That is  $\sqrt{26} \approx 5.1$ .

## Error Analysis

In any approximation, the error is defined by

$$\text{error} = \text{true value} - \text{approximate value}$$

If  $f(x) \approx L(x) = f(a) + f'(a)(x - a)$ , then the error  $E(x)$  in this approximation is

$$E(x) = f(x) - L(x) = f(x) - f(a) - f'(a)(x - a).$$

The following theorem and its corollaries give a way to estimate this error if we know bounds for the second derivative of  $f$ .

**Theorem 2.3.4** *If  $f''(t)$  exists for all  $t$  in an interval containing  $a$  and  $x$ , then there exists some point  $s$ ,  $a \leq s \leq x$  such that the error*

$$E(x) = f(x) - L(x)$$

*in the linear approximation  $f(x) \approx L(x)$  satisfies*

$$E(x) = \frac{f''(s)}{2}(x - a)^2.$$

**Corollary 2.3.5** *If  $f''(t)$  has constant sign (positive or negative) between  $a$  and  $x$ , then  $E(x)$  in the approximation  $f(x) \approx L(x)$  in the theorem above has the same sign. If  $f''(t) > 0$  between  $a$  and  $x$ , then  $f(x) > L(x)$ ; if  $f''(t) < 0$  between  $a$  and  $x$ , then  $f(x) < L(x)$ .*

**Corollary 2.3.6** *If  $|f''(t)| < K$  for all  $t$  between  $a$  and  $x$ , then*

$$|E(x)| < \frac{K}{2}(x - a)^2.$$

**Corollary 2.3.7** If  $f''(t)$  satisfies  $M < f''(t) < N$  for all  $t$  between  $a$  and  $x$ , then

$$L(x) + \frac{M}{2}(x - a)^2 < f(x) < L(x) + \frac{N}{2}(x - a)^2.$$

If  $M$  and  $N$  have the same sign, a better approximation of  $f(x)$  is given by the midpoint of this interval containing  $f(x)$ :

$$f(x) \approx L(x) + \frac{M + N}{4}(x - a)^2.$$

For this approximation, then error is less than half the length of the interval:

$$|E(x)| < \frac{N - M}{4}(x - a)^2.$$

**Example 2.3.8** Determine the sign and estimate the size of the error in the approximation  $\sqrt{26} \approx 5.1$  in Example 2.3.3 above. Use these to find an interval that you can be sure contains  $\sqrt{26}$ .

**Solution**

For  $f(t) = t^{\frac{1}{2}}$ , we have  $f'(t) = \frac{1}{2}t^{-\frac{1}{2}}$  and  $f''(t) = -\frac{1}{4}t^{-\frac{3}{2}}$ . For  $25 < t < 26$ , we have  $f''(t) < 0$  and so by Corollary 2.3.5 we have  $\sqrt{26} = f(26) < L(26) = 5.1$ . Also,  $t^{\frac{3}{2}} > 25^{\frac{3}{2}} = 125$ , so

$$|f''(t)| = \left| \frac{1}{4}t^{-3/2} \right| < \left( \frac{1}{4} \right) \left( \frac{1}{125} \right) = \frac{1}{500} = K \text{ and}$$

$$|E(26)| < \left( \frac{1}{2} \right) \left( \frac{1}{500} \right) (26 - 25)^2 = \frac{1}{1000} = 0.001$$

Therefore,  $f(26) > L(26) - 0.001 = 5.1 - 0.001 = 5.099$ , and  $\sqrt{26}$  is in the interval  $(5.099, 5.1)$ .

We can also use Corollary 2.3.7 and the fact that  $\sqrt{26} < 5.1$  to find a better (smaller) interval containing  $\sqrt{26}$  as follows: If  $25 < t < 26$ , then

$$125 = 25^{3/2} < t^{3/2} < 26^{3/2} < 5.1^3$$

Thus

$$M = -\frac{1}{4 \times 125} < f''(t) < -\frac{1}{4 \times 5.1^3} = N,$$

$$\sqrt{26} \approx L(26) + \frac{M + N}{4} = 5.1 - \frac{1}{4} \left( \frac{1}{4 \times 125} + \frac{1}{4 \times 5.1^3} \right) \approx 5.0990288,$$

and

$$|E(26)| < \frac{N - M}{4}(x - a)^2 = \frac{1}{16} \left( -\frac{1}{5.1^3} + \frac{1}{125} \right) \approx 0.0000288.$$

Thus  $\sqrt{26}$  lie in the interval (5.09900, 5.09906)

**Example 2.3.9** Use a suitable linearization to find an approximate value for  $\cos(36^\circ) = \cos(\pi/5)$ . Is the true value greater than or less than your approximation? Estimate the size of the error and give an interval that you can be sure contains  $\cos(36^\circ)$ .

### Solution

Let  $f(t) = \cos t$ , then  $f'(t) = -\sin t$  and  $f''(t) = -\cos t$ . Now, we know the value of  $\cos 30^\circ = \cos(\pi/6)$  and  $30^\circ$  is the nearest angle to  $36^\circ$ . So we use the linearization about the point  $t = \pi/6$ . Thus

$$L(t) = \cos(\pi/6) - \sin(\pi/6)(t - \pi/6) = \frac{\sqrt{3}}{2} - \frac{1}{2} \left( t - \frac{\pi}{6} \right).$$

Since  $\pi/5 - \pi/6 = \pi/30$ , our approximation is

$$\cos(36^\circ) = \cos(\pi/5) \approx L(\pi/5) = \frac{\sqrt{3}}{2} - \frac{1}{2} \left( \frac{\pi}{30} \right) \approx 0.81367.$$

Now, if  $\pi/6 < t < \pi/5$ , then  $f''(t) < 0$  and  $|f''(t)| < \cos(\pi/6) = \frac{\sqrt{3}}{2}$ . Therefore,  $\cos(36^\circ) < 0.81367$ .

For the error, we have

$$|E(36^\circ)| < \frac{\sqrt{3}}{2} \left( \frac{\pi}{30} \right)^2 < 0.00475.$$

Thus

$$0.81367 - 0.00475 < \cos(36^\circ) < 0.81367$$

and so  $\cos(36^\circ)$  lies in the interval (0.80892, 0.81367).

## Taylor Polynomials

If  $f^{(n)}(x)$  exists in an open interval containing  $x = a$ , then the  $n^{\text{th}}$  **Taylor polynomial** for  $f$  about  $a$  is given by

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The Taylor polynomials about  $x = 0$  are called **Maclaurin polynomials**.

**Example 2.3.10** Find the Taylor polynomial  $P_3(x)$  for  $f(x) = \ln x$  about  $x = e$ . (Here,  $e \approx 2.72$ )

### Solution

We calculate the first three derivatives of  $f$ . We have  $f'(x) = \frac{1}{x}$ ,  $f''(x) = -\frac{1}{x^2}$  and  $f'''(x) = \frac{2}{x^3}$ . Thus the Taylor polynomial is

$$\begin{aligned} P_3(x) &= f(e) + f'(e)(x - e) + \frac{f''(e)}{2!}(x - e)^2 + \frac{f'''(e)}{3!}(x - e)^3 \\ &= 1 + \frac{1}{e}(x - e) - \frac{2}{2e^2}(x - e)^2 + \frac{1}{3e^3}(x - e)^3 \end{aligned}$$

**Example 2.3.11** Find the  $n^{\text{th}}$ -order Maclaurin polynomial  $P_n(x)$  for  $e^x$ .

### Solution

Since every derivative of  $e^x$  is  $e^x$  and so is equal to 1 at  $x = 0$ , then  $n^{\text{th}}$ -order Maclaurin polynomial for  $e^x$  is

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

## The Mean Value Theorem

The Mean Value Theorem and its generalizations is more influential in calculus. It provides the mathematics we need to estimate the amount of error involved in linear approximations.

The key to the Mean Value Theorem is an early version of it, called Rolle's Theorem, which we now state.

**Theorem 2.3.12 (Rolle's Theorem)** Suppose that  $y = f(x)$  is continuous at every point of the closed interval  $[a, b]$  and differentiable at every point of its interior  $(a, b)$ . If  $f(a) = f(b) = 0$ , then there is at least one number  $c$  between  $a$  and  $b$  at which  $f'(c) = 0$ .

**Example 2.3.13** The polynomial  $y = x^3 - 4x = f(x)$  is continuous and differentiable for all  $x$ ,  $-\infty < x < \infty$ . If we take  $a = -2$  and  $b = 2$ , the hypotheses of Rolle's Theorem are satisfied since  $f(-2) = 0 = f(2)$ . Thus  $f'(x) = 3x^2 - 4$  must be zero at least once between  $-2$  and  $2$ . In fact, we find  $3x^2 - 4 = 0$  at  $x = c_1 = -\frac{2\sqrt{3}}{3}$  and  $x = c_2 = \frac{2\sqrt{3}}{3}$ .

**Example 2.3.14** Show that the equation  $x^3 + 3x + 1$  has exactly one real solution.

### Solution

We observe that the function  $f(x) = x^3 + 3x + 1$  is differentiable at every value of  $x$  and the derivative  $f'(x) = 3x^2 + 3$  is never zero. If  $f$  has as many as two zeros, by Rolle's Theorem  $f'$  would have a zero between them. Hence,  $f$  has at most one zero. On the other hand,  $f$  has at least one zero because  $f(-1) = -3$  is negative,  $f(1) = 5$  is positive and  $f$  is continuous. That is, the curve of  $f$  crosses the  $x$ -axis. Therefore,  $f$  has exactly one zero.

**Theorem 2.3.15 (Mean Value Theorem)** *If  $y = f(x)$  is continuous at every point of the closed interval  $[a, b]$  and differentiable at every point of its interior  $(a, b)$ , then there is at least one number  $c$  between  $a$  and  $b$  at which*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Example 2.3.16** *Let  $f(x) = x^3$ ,  $a = -2$  and  $b = 2$ . Show that  $f(x)$  satisfies the hypotheses of the Mean Value Theorem. Hence, find  $c$ .*

### Solution

Since  $f$  is continuous on  $[-2, 2]$  and is differentiable on  $(-2, 2)$ , it satisfies the Mean Value Theorem hypotheses.

Since  $f'(x) = 3x^2$ ,  $f'(c) = 3c^2$ ,  $f(b) = 2^3 = 8$  and  $f(a) = (-2)^3 = -8$  we have by the Mean Value Theorem

$$3c^2 = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{8 - (-8)}{2 - (-2)} = \frac{16}{4} = 4.$$

Solving for  $c$  gives  $c = \pm \frac{2}{3}\sqrt{3}$ . Therefore, there are two values of  $c$  between  $a = -2$  and  $b = 2$  where the tangent to the curve  $y = f(x) = x^3$  is parallel to the chord through  $A(-2, -8)$  and  $B(2, 8)$ .

**Example 2.3.17** *Show that the function  $y = \sqrt{1 - x^2}$  satisfies the hypotheses of the Mean Value Theorem on the interval  $[-1, 1]$ .*

### Solution

The function  $y = \sqrt{1 - x^2}$  is continuous at each point of the closed interval, and its derivative

$$y' = \frac{-x}{\sqrt{1 - x^2}}$$

is defined at each point of the interior  $(-1, 1)$ . Hence, the hypotheses of the Mean Value Theorem are satisfied. The graph has a horizontal tangent at  $x = 0$ . Notice that the function is not differentiable at  $x = -1$  and  $x = 1$ . It does not need to be for the theorem to apply.

**Example 2.3.18** Show that the Mean Value Theorem does not apply to the function  $f(x) = x^{\frac{2}{3}}$  on the interval  $[-8, 8]$ .

**Solution**

The derivative of  $f(x) = x^{\frac{2}{3}}$  is

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

Suppose that  $f(x)$  satisfies the hypotheses of the Mean Value Theorem, then there is a  $c$  in  $(-8, 8)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{8^{\frac{2}{3}} - (-8)^{\frac{2}{3}}}{8 - (-8)} = \frac{4 - 4}{16} = 0$$

But  $f'(x) = \frac{2}{3\sqrt[3]{x}}$  is never zero for any value  $c$  in the interval  $(-8, 8)$ . Observe also that  $f'$  does not exist at  $x = 0$ . Therefore, the Mean Value Theorem does not apply to the given function on a given closed interval.

## Indeterminate forms and L'Hopital's Rule

If  $f$  and  $g$  are continuous at  $x = a$  but  $f(a) = g(a) = 0$ , then the limit

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

cannot be evaluated by substituting  $x = a$  since this produces  $\frac{0}{0}$ , a meaningless expression known as an indeterminate form. L'Hopital's rule gives an explicit connection between derivatives and limits that leads to indeterminate forms.

**Theorem 2.3.19 (First form of L'Hopital's Rule)** *Suppose that  $f(a) = g(a) = 0$ ,  $f'(a)$  and  $g'(a)$  exist, and that  $g'(a) \neq 0$ . Then*

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \Big|_{x=a}$$

**Theorem 2.3.20 (Stronger form of L'Hopital's Rule)** *Suppose that  $f(x_0) = g(x_0) = 0$  and that the functions  $f$  and  $g$  are both differentiable on an open interval  $(a, b)$  that contains the point  $x_0$ , suppose also that  $g' \neq 0$  at every point in  $(a, b)$  except possibly at  $x_0$ . Then*

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

*provided the limit on the right exists.*

**Example 2.3.21** *Using L'Hopital's Rule we have*

a)  $\lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1} \Big|_{x=0} = 2$

b)  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \frac{\frac{1}{2\sqrt{x+1}}}{1} \Big|_{x=0} = \frac{1}{2}$

c)  $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \frac{1 - \cos x}{3x^2}$ , but this is still  $\frac{0}{0}$  form. So we apply the stronger form of L'Hopital's Rule:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \quad (\text{still } \frac{0}{0}, \text{ apply the rule again}) \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{6x} \quad (\text{still } \frac{0}{0}, \text{ apply the rule again}) \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{6} \\ &= \frac{1}{6} \end{aligned}$$

Another indeterminate form is  $\frac{\infty}{\infty}$ .

**Example 2.3.22** Using the stronger form of L'Hopital's Rule, we evaluate the following limits that lead to  $\frac{\infty}{\infty}$  indeterminate forms:

$$a) \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan x}{1 + \tan x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x} = 1$$

$$b) \lim_{x \rightarrow \infty} \frac{x - 2x^2}{3x^2 + 5x} = \lim_{x \rightarrow \infty} \frac{1 - 4x}{6x + 8} \text{ (still } \frac{\infty}{\infty}, \text{ apply the rule again)} = \lim_{x \rightarrow \infty} \frac{-4}{6} = \frac{-2}{3}$$

If you get indeterminate forms  $0 \cdot \infty$  or  $\infty \cdot 0$  or  $\infty \pm \infty$ , try to change them into the form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ .

**Example 2.3.23** a)  $\lim_{x \rightarrow \infty} x \cdot \sin\left(\frac{1}{x}\right)$  leads to the form  $\infty \cdot 0$ . We can change to the form  $\frac{0}{0}$  by writing  $x = \frac{1}{t}$  and letting  $t \rightarrow 0^+$ :

$$\begin{aligned} \lim_{x \rightarrow \infty} x \cdot \sin\left(\frac{1}{x}\right) &= \lim_{x \rightarrow 0^+} \frac{1}{t} \cdot \sin(t) \\ &= \lim_{x \rightarrow 0^+} \frac{\sin t}{t} \quad \left(\frac{0}{0} \text{ form}\right) \\ &= \frac{\cos 0}{1} \\ &= 1. \end{aligned}$$

b)  $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$  leads the  $\infty - \infty$  form if  $x \rightarrow 0^+$ , and leads to  $-\infty + \infty$  if  $x \rightarrow 0^-$  which are both indeterminate. But we may also write

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x}$$

which now leads to a  $\frac{0}{0}$  form and we can apply L'Hopital's Rule. Thus,

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} \quad \left(\frac{0}{0} \text{ form}\right) \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} \quad \left(\frac{0}{0} \text{ form}\right) \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x - x \sin x} \\ &= \frac{0}{2} \\ &= 0. \end{aligned}$$

## Rates of Change

When two or more quantities that change with time are linked by an equation, that equation can be differentiated with respect to time to produce an equation linking the rates of change of the quantities. Any one of these rates may then be determined when the others, and the values of the quantities themselves, are known

**Example 2.3.24** *An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and the radio beacon increasing 1 min after the aircraft passes 5 km directly above the beacon?*

### Solution

Let  $C$  be the point on the aircraft's path directly above the beacon  $B$ . Let  $A$  be the position of the aircraft  $t$  minutes after it is at  $C$ , and let  $x$  and  $s$  be the distances  $CA$  and  $BA$  respectively.

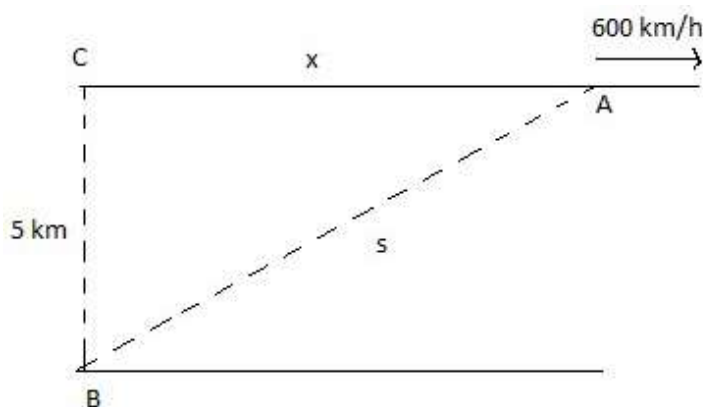


Figure 2.1:

From the right triangle  $BCA$  we have

$$s^2 = x^2 + 5^2.$$

Differentiating this equation implicitly with respect to time we obtain

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt}.$$

We are given that

$$\frac{dx}{dt} = 600 \text{ km/h} = 10 \text{ km/min};$$

hence,  $x = 10\text{km}$  at time  $t = 1$ . At that time,  $s = \sqrt{10^2 + 5^2} = 5\sqrt{5}$  and is increasing at the rate

$$\frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt} = \frac{10}{5\sqrt{5}}(600) = \frac{1200}{\sqrt{5}} \approx 536.7\text{km/h}.$$

**Example 2.3.25** *How fast is the area of a rectangle changing if one side is 10cm long and is increasing at a rate of 2cm/s and the other side is 8 cm long and is decreasing at a rate of 3cm/s.*

**Solution**

Let  $x$  cm and  $y$  cm be the lengths of the sides of the rectangle at time  $t$ , respectively. Then the area at time  $t$  is  $xy\text{cm}^2$ . We want to know the value of  $\frac{dA}{dt}$  when  $x = 10$  and  $y = 8$  given that  $\frac{dx}{dt} = 2$  and  $\frac{dy}{dt} = -3$  (the negative sign indicates that  $y$  is decreasing).

Differentiating  $A = xy$  implicitly, we get

$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt} = 2(8) + 10(-3) = -14.$$

So at the time in question, the area of the rectangle is decreasing at a rate of  $14\text{cm}^2/\text{s}$ .

**Example 2.3.26** *A lighthouse L is located on a small island 2 km from the nearest point A on a long, straight shoreline. If the lighthouse lamp rotates at 3 revolutions per minute, how fast is the illuminated spot P on the shoreline moving along the shoreline when it is 4 km from A?*

**Solution**

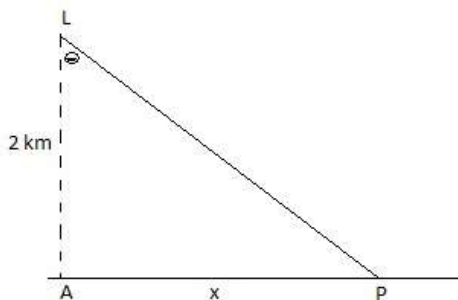


Figure 2.2:

Let  $x$  be the distance  $AP$  and let  $\theta$  be the angle  $PLA$ . Then  $x = 2 \tan \theta$  and

$$\frac{dx}{dt} = 2 \sec^2 \theta \frac{d\theta}{dt}.$$

Now

$$\frac{d\theta}{dt} = 3 \text{ rev/min} \times 2\pi \text{ rad/rev} = 6\pi \text{ rad/min}.$$

Note that it is necessary to change revolutions per minute to radians per minute. When  $x = 4$ , we have  $\tan \theta = 2$  and  $\sec^2 \theta = 1 + \tan^2 \theta = 5$ . Thus

$$\frac{dx}{dt} = 2 \times 5 \times 6\pi = 60\pi \approx 188.5.$$

The spot of light is moving along the shoreline at a rate of about 189 km/min when it is 4 km from  $A$ .

**Example 2.3.27** *A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of  $\frac{1}{12}$  m<sup>3</sup>/min. How fast is the water level in the tank dropping at that time?*

**Solution**

Let  $r$  and  $h$  denote the surface radius and depth of water in the tank at time  $t$ , respectively.

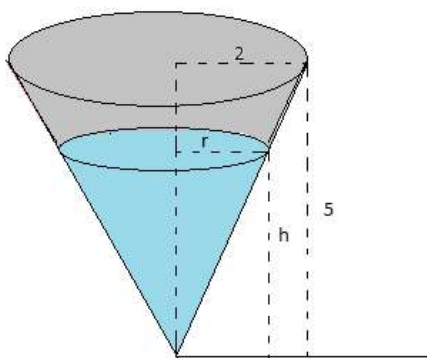


Figure 2.3:

Thus the volume  $V$  of water in the tank at time  $t$  is

$$V = \frac{1}{3}\pi r^2 h.$$

We find a relationship between  $r$  and  $h$  using similar triangles:  $\frac{r}{h} = \frac{2}{5}$ , so that  $r = \frac{2h}{5}$  and

$$V = \frac{1}{3}\pi \left(\frac{2h}{5}\right)^2 h = \frac{4\pi}{75}h^3.$$

Implicitly differentiating this equation with respect to  $t$  we get

$$\frac{dV}{dt} = \frac{4\pi}{25}h^2 \frac{dh}{dt}.$$

Since  $dV/dt = -1/12$  when  $h = 4$ , we have

$$-\frac{1}{12} = \frac{4\pi}{25}(4)^2 \frac{dh}{dt}, \quad \text{so that } \frac{dh}{dt} = \frac{-25}{768\pi}.$$

When the water in the tank is 4 m deep, its level is dropping at a rate of  $\frac{25}{768\pi}$  m/min or about 1.036 cm/min.

**Example 2.3.28** *At a certain instant an aircraft flying due east at 400 km/h passes directly over a car travelling due Southeast at 100 km/h on straight level road. If the aircraft is flying at an altitude of 1 km, how fast is the distance between the aircraft and the car increasing 36 seconds after the aircraft passes directly over the car?*

### Solution

Let  $t$  be measured in hours from the time the aircraft was at position  $A$  directly above the car at position  $C$ . Let  $X$  and  $Y$  be the positions of the aircraft and car respectively at time  $t$ . Let  $x$  be the distance  $AX$ ,  $y$  the distance  $CY$ , and  $s$  the distance  $XY$ , all measured in kilometres. Let  $Z$  be the point 1 km above  $Y$ . (See Figure 2.4)

Since  $XAZ = 45^\circ$ , the Pythagoras theorem and the cosine law yield

$$s^2 = 1 + (ZX)^2 = 1 + x^2 + y^2 - 2xy \cos 45^\circ = 1 + x^2 + y^2 - \sqrt{2}xy.$$

Thus

$$\begin{aligned} 2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} - \sqrt{2} \frac{dx}{dt} y - \sqrt{2} x \frac{dy}{dt} \\ &= 400(2x - \sqrt{2}y) + 100(2y - \sqrt{2}x) \end{aligned}$$

When  $t = \frac{1}{100}$  (i.e, 36 s after  $t = 0$ ), we have  $x = 4$  and  $y = 1$ . Hence,

$$s^2 = 1 + 16 + 1 - 4\sqrt{2} = 18 - 4\sqrt{2}$$



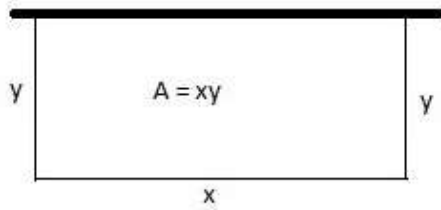


Figure 2.5:

So the area  $A$  is given by

$$A = xy = (100 - 2y)y = 100y - 2y^2.$$

Since  $x \geq 0$ , we have that  $y \leq 50$ , otherwise,  $x$  would be negative which wouldn't make sense. Thus we must maximize the function  $A(y) = 100y - 2y^2$  on the interval  $[0, 50]$ . Observe that  $A(0) = A(50) = 0$  and  $A(y) > 0$  for  $0 < y < 50$ . Hence,  $A(y)$  has maximum. This maximum must occur at a critical point. To find critical points, we solve

$$0 = A'(y) = 100 - 4y \Rightarrow y = 25.$$

Thus the maximum value occurs at  $y = 25$ . Thus the greatest possible area for the enclosure is therefore

$$A(25) = 100(25) - 2(25)^2 = 1250 \text{ m}^2.$$

**Example 2.3.30** *A lighthouse  $L$  is located on a small island 5 km north of a point  $A$  on a straight east-west shoreline. A cable is to be laid from  $L$  to point  $B$  on the shoreline 10 km east of  $A$ . The cable will be laid through the water in a straight line from  $L$  to a point  $C$  on the shoreline between  $A$  and  $B$ , and from there to  $B$  along the shoreline.*

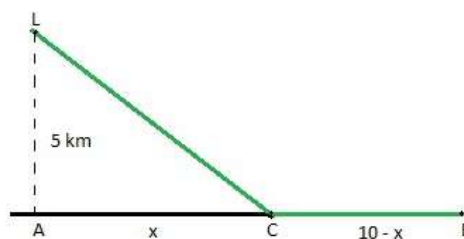


Figure 2.6:

*The part of the cable lying in the water costs K50,000/km, and the part along the shoreline costs K30,000/km.*

a) Where should  $C$  be chosen to minimize the total cost of the cable.

b) Where should  $C$  be chosen if  $B$  is only 3 km from  $A$ .

### Solution

a) Let  $C$  be  $x$  km from  $A$  toward  $B$ . Thus  $0 \leq x \leq 10$ . The length of  $LC$  is  $\sqrt{25 + x^2}$  km and the length of  $CB$  is  $(10 - x)$  km (see Figure 2.6). Hence, the total cost of the cable in kwacha is  $T$ , where

$$T = T(x) = 50,000\sqrt{25 + x^2} + 30,000(10 - x).$$

Note that  $T$  is continuous on  $[0, 10]$ , so it has a minimum value that can occur at one of the endpoints or a critical point in the interval  $(0, 10)$ . For critical points we have

$$\begin{aligned} 0 &= \frac{dT}{dx} = \frac{50,000x}{\sqrt{25 + x^2}} - 30,000 \\ &\Rightarrow 50,000x = 30,000\sqrt{25 + x^2} \\ &\Rightarrow 5x = 3\sqrt{25 + x^2} \quad (\text{squaring both sides}) \\ &\Rightarrow 25x^2 = 9(25 + x^2) \\ &\Rightarrow 25x^2 = 225 + 9x^2 \\ &\Rightarrow 16x^2 = 225 \\ &\Rightarrow x^2 = \frac{225}{16} = \frac{15^2}{4^2}. \end{aligned}$$

So the critical points are  $x = \pm\frac{15}{4}$ . Observe that only  $x = \frac{15}{4}$  lies in the interval  $(0, 10)$ . Since

$$T(0) = 50,000(5) - 30,000(10) = 550,000,$$

$$T\left(\frac{15}{4}\right) = 50,000\left(\sqrt{25 + \frac{225}{16}}\right) - 30,000\left(100 - \frac{15}{4}\right) = 500,000$$

and

$$T(10) = 50,000(\sqrt{25 + 100}) + 0 = 559,016$$

the critical point determines the minimum value of  $T(x)$ . So for minimum cost,  $C$  should be  $\frac{15}{4} = 3.75$  km from  $A$ .

b) If  $B$  is 3 km from  $A$ , we have

$$T(x) = 50,000\sqrt{25 + x^2} + 30,000(3 - x) \quad (0 \leq x \leq 3)$$

which differs from  $T(x)$  in a) by an added constant. Hence, it has the same critical points  $x = \pm \frac{15}{4}$  (verify this) neither of which lies in the interval  $(0, 3)$ . Now,  $T(0) = 340,000$  and  $T(3) = 291,548$ . So to minimize the cost, we chose  $x = 3$ ; that is, the cable should go straight from  $L$  to  $B$ .

**Example 2.3.31** Find the length of the shortest ladder that can extend from a vertical wall, over a fence 2 m high located 1 m away from the wall, to a point on the ground outside the fence.

**Solution**

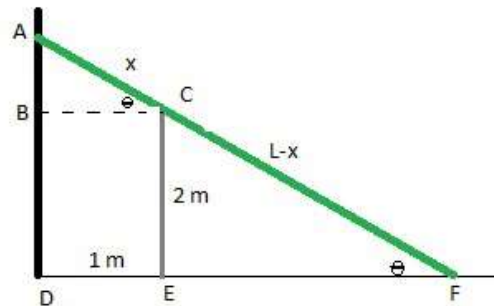


Figure 2.7:

Let  $\theta$  be the angle of inclination of the ladder  $x$  be the length of  $AC$  as shown in Figure 2.7 and  $L$  be the length of the ladder  $AF$ . Using the two right-angled triangles  $ABC$  and  $CEF$ , we obtain the length  $L$  of the ladder as a function of  $\theta$ :

$$x = \frac{1}{\sin \theta}, \quad L - x = \frac{2}{\sin \theta}$$

and solving for  $L$  we get

$$L = L(\theta) = \frac{1}{\cos \theta} + \frac{2}{\sin \theta}, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Since  $\lim_{\theta \rightarrow (\pi/2)^-} L(\theta) = \infty$  and  $\lim_{\theta \rightarrow (0)^+} L(\theta) = \infty$ ,  $L(\theta)$  must have a minimum value on the interval  $(0, \frac{\pi}{2})$  occurring at a critical point. To find critical points, we solve

$$0 = L'(\theta) = \frac{\sin \theta}{\cos^2 \theta} - \frac{2 \cos \theta}{\sin^2 \theta} = \frac{\sin^3 \theta - 2 \cos^3 \theta}{\cos^2 \theta \sin^2 \theta}.$$

So any critical point satisfies  $\sin^3 \theta = 2 \cos^3 \theta$  or equivalently,  $\tan^3 \theta = 2$ . We don't need to solve this equation for  $\theta$  since it is really the corresponding value of  $L(\theta)$  that we want. Observe that

$$\frac{1}{\cos^2 \theta} = \sec^2 \theta = 1 + \tan^2 \theta = 1 + 2^{\frac{2}{3}}.$$

It follows that  $\cos \theta = \frac{1}{(1+2^{\frac{2}{3}})^{\frac{1}{2}}}$  and  $\sin \theta = \tan \theta \cos \theta = \frac{2^{\frac{1}{3}}}{(1+2^{\frac{2}{3}})^{\frac{1}{2}}}$ . Therefore, the minimal value for  $L(\theta)$  is

$$\frac{1}{\cos \theta} + \frac{2}{\sin \theta} = (1 + 2^{\frac{2}{3}})^{\frac{1}{2}} + 2 \frac{(1 + 2^{\frac{2}{3}})^{\frac{1}{2}}}{2^{\frac{1}{3}}} = (1 + 2^{\frac{2}{3}})^{\frac{3}{2}} \approx 4.16$$

Therefore, the length of the shortest ladder is 4.16 m long.

# Chapter 3

## Integration

In first year, you were introduced to some integration techniques. Here we review some and introduce new ones. We first give the table of some elementary integrals.

---

1. $\int 1 dx = x + C$	2. $\int x dx = \frac{x^2}{2} + C$
3. $\int x^2 dx = \frac{x^3}{3} + C$	4. $\int \frac{1}{x^2} = \frac{-1}{x} + C$
5. $\int \sqrt{x} dx = \frac{2}{3}x^{\frac{3}{2}} + C$	6. $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$
7. $\int x^r dx = \frac{1}{r+1}x^{r+1} + C$ ( $r \neq -1$ )	8. $\int \frac{1}{x} dx = \ln x  + C$
9. $\int \cos ax dx = \frac{1}{a} \sin ax + C$	10. $\int \sin ax dx = -\frac{1}{a} \cos ax + C$
11. $\int \sec^2 ax dx = \frac{1}{a} \tan ax + C$	12. $\int \csc ax dx = -\frac{1}{a} \cot ax + C$
13. $\int \sec ax \tan ax dx = \frac{1}{a} \sec ax + C$	14. $\int \csc ax \cot ax dx = -\frac{1}{a} \cot ax + C$
15. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + C$ ( $a > 0$ )	16. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$ ( $a > 0$ )
17. $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$	18. $\int b^{ax} dx = \frac{1}{a \ln b} b^{ax} + C$
19. $\int \cosh ax dx = \frac{1}{a} \sinh ax + C$	20. $\int \sinh ax dx = \frac{1}{a} \cosh ax + C$

---

The linearity formula

$$\int (Af(x) + Bg(x))dx = A \int f(x)dx + B \int g(x)dx$$

makes it possible to integrate sums and constant multiples of functions.

### 3.1 Substitution

The method of substitution is the most important technique of integration. It is the integral version of the chain rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

and in integral form we get

$$\int f'(g(x))g'(x)dx = f(g(x)) + C.$$

To obtain this, let  $u = g(x)$ , then  $\frac{du}{dx} = g'(x)$ , or in differential form,  $du = g'(x)dx$ . Thus

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(u) + C = f(g(x)) + C.$$

**Example 3.1.1** Evaluate the indefinite integrals a)  $\int \frac{x}{x^2+1}dx$  b)  $\int \frac{\sin(3 \ln x)}{x}dx$  c)  $\int e^x \sqrt{1+e^x}dx$ .

**Solution**

- a) Let  $u = x^2 + 1$ , then  $du = 2xdx$  so that  $xdx = \frac{1}{2}du$ . Substituting this in the given integral we have

$$\int \frac{x}{x^2+1}dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 1| + C.$$

- b) Let  $u = 3 \ln x$ , then  $du = \frac{3}{x}dx$  so that  $\frac{dx}{x} = \frac{1}{3}du$ . Substituting this in the given integral we have

$$\int \frac{\sin(3 \ln x)}{x}dx = \frac{1}{3} \int \sin u du = -\frac{1}{3} \cos u + C = -\frac{1}{3} \cos(3 \ln x) + C.$$

c) Let  $v = 1 + e^x$ , then  $dv = e^x dx$ . Substituting this in the given integral we have

$$\int e^x \sqrt{1 + e^x} dx = \int \sqrt{v} dv = \int v^{\frac{1}{2}} dv = \frac{2}{3} v^{\frac{3}{2}} + C = \frac{2}{3} (1 + e^x)^{\frac{3}{2}} + C.$$

Sometimes appropriate substitutions are not obvious. So there is need to manipulate the integrand algebraically to put it into a better form for substitution as we do in the next example.

**Example 3.1.2** Evaluate  $\int \frac{1}{x^2+4x+5} dx$ .

**Solution**

We first complete the square of the denominator in the integrand

$$\begin{aligned} x^2 + 4x + 5 &= (x^2 + 4x + 2^2) - 2^2 + 5 \\ &= (x^2 + 4x + 4) - 4 + 5 \\ &= (x + 2)^2 + 1 \end{aligned}$$

Now we can do the substitution; let  $t = x + 2$ , then  $dt = dx$  and so

$$\begin{aligned} \int \frac{1}{x^2 + 4x + 5} dx &= \int \frac{dx}{(x + 2)^2 + 1} \\ &= \int \frac{dt}{t^2 + 1} \\ &= \tan^{-1} t + C \\ &= \tan^{-1}(x + 2) + C. \end{aligned}$$

We now consider integrals of the form

$$\int \sin^m x \cos^n x dx.$$

If either  $m$  or  $n$  is odd, positive integer, the integrand can be done easily by substitution. If say  $n = 2k + 1$  where  $k$  is an integer, then we can use the identity  $\sin^2 x + \cos^2 x = 1$  to rewrite the integrand as follows

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x \cos^{2k} x \cos x dx \\ &= \int \sin^m x (\cos^2 x)^k \cos x dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x dx \end{aligned}$$

which can be integrated using the substitution  $u = \sin x$ . Similarly,  $u = \cos x$  can be used if  $m$  is odd.

**Example 3.1.3** Evaluate a)  $\int \sin^3 x \cos^8 x dx$ , b)  $\int \cos^5(ax) dx$ .

### Solution

a) Setting  $I = \int \sin^3 x \cos^8 x dx$ , we have

$$I = \int \sin^2 x \cos^8 x \sin x dx = \int (1 - \cos^2 x) \cos^8 x \sin x dx.$$

Now, let  $u = \cos x$ , then  $du = -\sin x dx$ . Thus

$$\begin{aligned} I &= - \int (1 - u^2) u^8 du \\ &= \int (u^{10} - u^8) du \\ &= \frac{u^{11}}{11} - \frac{u^9}{9} + C \\ &= \frac{1}{11} \cos^{11} x - \frac{1}{9} \cos^9 x + C \end{aligned}$$

b) Setting  $J = \int \cos^5(ax) dx$ , we have

$$J = \int \cos^4(ax) \cos(ax) dx = \int (1 - \sin^2(ax))^2 \cos(ax) dx.$$

Let  $u = \sin(ax)$ , then  $du = a \cos(ax) dx$ . Thus

$$\begin{aligned} J &= \frac{1}{a} \int (1 - u^2)^2 du \\ &= \frac{1}{a} \int (1 - 2u^2 + u^4) du \\ &= \frac{1}{a} \left( u - \frac{2}{3} u^3 + \frac{1}{5} u^5 \right) + C \\ &= \frac{1}{a} \left( \sin(ax) - \frac{2}{3} \sin^3(ax) + \frac{1}{5} \sin^5(ax) \right) + C. \end{aligned}$$

If both  $m$  and  $n$  are even, we can use the double angle formulas:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

**Example 3.1.4** Evaluate  $\int \sin^4 x dx$ .

### Solution

We apply the double angle formula twice.

$$\begin{aligned}\int \sin^4 x dx &= \int \sin^2 x)^2 dx \\ &= \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 dx \\ &= \frac{1}{4} \int (1 - \cos 2x)^2 dx \\ &= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4} \int 1 dx - \frac{2}{4} \int \cos 2x dx + \frac{1}{4} \int \cos^2 2x dx \text{ (apply the formula again)} \\ &= \frac{x}{4} - \frac{1}{4} \sin 2x + \frac{1}{4} \int \left(\frac{1}{2}(1 + \cos 4x)\right) dx \\ &= \frac{x}{4} - \frac{1}{4} \sin 2x + \frac{1}{8} \int (1 + \cos 4x) dx \\ &= \frac{x}{4} - \frac{1}{4} \sin 2x + \frac{x}{8} + \frac{1}{32} \sin 4x + C \\ &= \frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C.\end{aligned}$$

Similarly, integrals of the form

$$\int \sec^m x \tan^n x dx \text{ and } \int \csc^m x \cot^n x dx$$

for both  $m$  and  $n$  even positive numbers can be respectively evaluated by using the identities

$$\sec^2 x = 1 + \tan^2 x \text{ and } \csc^2 x = 1 + \cot^2 x.$$

If either  $m$  or  $n$  is odd, the integrals cannot be handled by substitution.

## 3.2 Integration by parts

Suppose that  $U(x)$  and  $V(x)$  are two differentiable functions. According to the product rule,

$$\frac{d}{dx}(U(x)V(x)) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}.$$

Integrating both sides, we get

$$\int \frac{d}{dx}(U(x)V(x))dx = \int U(x)\frac{dV}{dx}dx + \int V(x)\frac{dU}{dx}dx$$

which implies that

$$\int U(x)\frac{dV}{dx}dx = U(x)V(x) - \int V(x)\frac{dU}{dx}dx$$

or simply

$$\int UdV = UV - \int VdU.$$

This is the formula for carrying out integration by parts. We now use the formula to solve the following examples:

**Example 3.2.1** Evaluate a)  $\int xe^x dx$ , b)  $\int \ln x$ , c)  $\int x^2 \sin x dx$  and d)  $\int x \tan^{-1} x dx$

**Solution**

a) Let  $U = x$  and  $dV = e^x dx$ , then  $dU = dx$  and

$$V = \int dV = \int e^x dx = e^x + K.$$

Thus

$$\begin{aligned}\int UdV &= UV - \int VdU \\ &= x(e^x + K) - \int (e^x + K)dx \\ &= xe^x + Kx - e^x - Kx + C \\ &= xe^x - e^x + C.\end{aligned}$$

Notice that the constant of integration  $K$  cancels out at the end. So we usually leave it out as we shall do from now on.

b) Let  $U = \ln x$  and  $dV = dx$ , then  $dU = \frac{1}{x}dx$  and  $V = x$ . By the integration by parts formula we have

$$\int \ln x = UV - \int VdU = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C.$$

- c) We do integration by parts twice: Let  $U = x^2$  and  $dV = \sin x dx$ , then  $dU = 2x dx$  and  $V = -\cos x$ . Thus

$$\int x^2 \sin x dx = -x^2 \cos x + 2 \int x \cos x dx.$$

We again apply the formula on the second term on the right hand side. Let  $U = x$  and  $dV = \cos x$ , then  $dU = dx$  and  $V = \sin x$ . So

$$\begin{aligned} \int x^2 \sin x dx &= -x^2 \cos x + 2(x \sin x - \int \sin x dx) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C \\ &= (2 - x^2) \cos x + 2x \sin x + C. \end{aligned}$$

- d) Let  $U = \tan^{-1} x$  and  $dV = x dx$ , then  $dU = \frac{dx}{1+x^2}$  and  $V = \frac{1}{2}x^2$ . So

$$\begin{aligned} \int x \tan^{-1} x dx &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2} \tan^{-1} x + C \\ &= \frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{1}{2}x + C. \end{aligned}$$

The following are useful rules for choosing  $U$  and  $dV$ :

- i) If the integrand involves a polynomial multiplied by an exponential, a sine or a cosine, or some other readily integrable function, try  $U$  equals the polynomial and  $dV$  equals the rest.
- ii) If the integrand involves a logarithm, an inverse trigonometric function or some other function which is not readily integrable but whose derivative is readily calculated, try that function for  $U$  and let  $dV$  equals the rest.

### 3.3 Reduction Formula

Consider the problem of finding  $\int x^n e^{-x} dx$ . We can of course use integration by parts, but it is repetitive and tedious. We can use the following approach: For  $n \geq 0$ , let

$$I_n = \int x^n e^{-x} dx.$$

We want to find  $I_4$ . First we integrate  $I_n$  by parts to get the general formula: Let  $U = x^n$  and  $dV = e^{-x}$ , then  $dU = nx^{n-1}dx$  and  $V = -e^{-x}$ . Thus

$$I_n = -xe^{-x} + n \int x^{n-1}e^{-x}dx = -x^n e^{-1} + nI_{n-1}.$$

The formula

$$I_n = -x^n e^{-1} + nI_{n-1}$$

is called a reduction formula because it gives the value of the integral  $I_n$  in terms of  $I_{n-1}$ , an integral corresponding to a reduced value of the exponent  $n$ .

We now evaluate  $\int x^4 e^{-x} dx$  by finding  $I_4$  using the reduction formula. Starting with

$$I_0 = \int x^0 e^{-x} dx = \int e^{-x} = -e^{-x} + C$$

we apply the reduction formula four times to get

$$I_1 = -xe^{-x} + I_0 = -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C_1$$

$$I_2 = -x^2 e^{-x} + 2I_1 = -x^2 e^{-x} + 2(e^{-x}(x+1) + C) = -e^{-x}(x^2 + 2x + 2) + C_2$$

$$I_3 = -x^3 e^{-x} + 3I_2 = -e^{-x}(x^3 + 3x^2 + 6x + 6) + C_3$$

$$I_4 = -x^4 e^{-x} + 4I_3 = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + C_4$$

**Example 3.3.1** Obtain and use a reduction formula to evaluate

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx \quad (n = 0, 1, 2, 3, \dots)$$

**Solution**

Observe first that  $I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$  and  $I_1 = \int_0^{\frac{\pi}{2}} \cos x dx = \sin x \Big|_0^{\frac{\pi}{2}} = 1$ . Now, let  $n \geq 2$ :

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x dx$$

We apply integration by parts: Let  $U = \cos^{n-1} x$  and  $dV = \cos x dx$ , then  $dU = -(n-1) \cos^{n-2} x \sin x dx$  and  $V = \sin x$ . Thus

$$\begin{aligned} I_n &= \cos^{n-1} x \sin x \Big|_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx \\ &= 0 - 0 + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x (1 - \cos^2 x) dx \\ &= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x dx - (n-1) \int_0^{\frac{\pi}{2}} \cos^n x dx \\ &= (n-1)I_{n-2} - (n-1)I_n. \end{aligned}$$

Solving for  $I_n$ , we obtain  $nI_n = (n-1)I_{n-2}$  or

$$I_n = \frac{n-1}{n}I_{n-2}$$

which is the required reduction formula. This formula is valid for  $n \geq 2$ , which was used to ensure that  $\cos^{n-1}(\frac{\pi}{2}) = 0$ .

If  $n \geq 2$  is an even integer, we have

$$\begin{aligned} I_n &= \frac{n-1}{n}I_{n-2} = \frac{n-1}{n} \cdot \frac{n-3}{n-2}I_{n-4} \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \end{aligned}$$

For instance,

$$I_4 = \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}.$$

If  $n \geq 3$  is an odd integer, we have

$$\begin{aligned} I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \end{aligned}$$

For example

$$I_7 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{48}{105}.$$

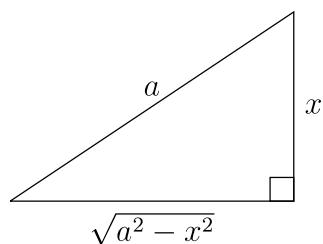
### 3.4 Inverse trigonometric substitution

The substitution we have seen so far is where we replace an expression in the integrand with a single variable. In this section, we consider the reverse approach. We replace a variable of integration with a function of a new variable. Such substitutions, called inverse substitutions may appear on the surface to make the integral more complicated. However, as we will see, such substitutions can actually simplify and transform the integral into one that can be evaluated by inspection or to which other techniques can readily be applied.

We now outline some very useful inverse substitutions:

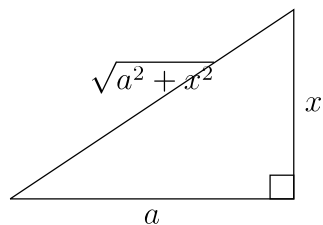
1. Integrals involving  $\sqrt{a^2 - x^2}$  (where  $a > 0$ ) can frequently be reduced to a simpler form by means of the substitution

$$x = a \sin \theta \text{ or equivalently } \theta = \sin^{-1} \frac{x}{a}$$



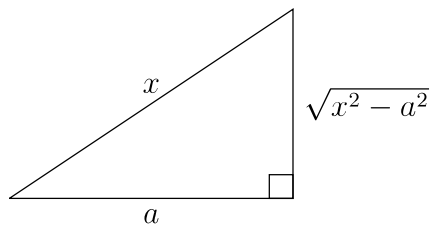
2. For integrals involving  $\sqrt{a^2 + x^2}$  or  $\frac{1}{x^2+a^2}$  (where  $a > 0$ ) substitute

$$x = a \tan \theta \text{ or equivalently } \theta = \tan^{-1} \frac{x}{a}$$



3. For integrals involving  $\sqrt{x^2 - a^2}$ ,  $a > 0$ , use the substitution

$$x = a \sec \theta \text{ or equivalently } \theta = \sec^{-1} \frac{x}{a}.$$



These substitutions are valid for  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .

4. For integrals involving  $\sqrt{ax + b}$  use the substitution  $ax + b = u^2$  and integrals involving  $\sqrt[n]{ax + b}$  use the substitution  $ax + b = u^n$ .

**Example 3.4.1** Evaluate  $\int \frac{1}{(5-x^2)^{\frac{3}{2}}} dx$ .

### Solution

Observe that the integral is in the form described in 1. with  $a = \sqrt{5}$ . Let  $x = \sqrt{5} \sin \theta$ . Then  $dx = \sqrt{5} \cos \theta d\theta$  and

$$\begin{aligned}(5 - x^2)^{\frac{3}{2}} &= (5 - (\sqrt{5} \sin \theta)^2)^{\frac{3}{2}} \\ &= (5 - 5 \sin^2 \theta)^{\frac{3}{2}} \\ &= (5(1 - \sin^2 \theta))^{\frac{3}{2}} \\ &= (5 \cos^2 \theta)^{\frac{3}{2}} \\ &= 5^{\frac{3}{2}} \cos^3 \theta.\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{1}{(5 - x^2)^{\frac{3}{2}}} dx &= \int \frac{\sqrt{5} \cos \theta d\theta}{5^{\frac{3}{2}} \cos^3 \theta} \\ &= \frac{1}{5} \int \sec^2 \theta d\theta \\ &= \frac{1}{5} \tan \theta + C \\ &= \frac{1}{5} \frac{x}{\sqrt{5 - x^2}} + C. \text{ (Refer to the triangle in 1.)}\end{aligned}$$

**Example 3.4.2** Evaluate  $\int \frac{1}{\sqrt{4+x^2}} dx$ .

### Solution

The integral is in the form described in 2. with  $a = 2$ . Let  $x = 2 \tan \theta$ , then  $dx = 2 \sec^2 \theta d\theta$  and

$$\sqrt{4 + x^2} = \sqrt{4 + 4 \tan^2 \theta} = \sqrt{4(1 + \tan^2 \theta)} = \sqrt{4 \sec^2 \theta} = 2 \sec \theta.$$

Thus

$$\begin{aligned}\int \frac{1}{\sqrt{4 + x^2}} dx &= \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta} \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C \text{ (Refer to the triangle in 2.)} \\ &= \ln |\sqrt{4 + x^2} + x| - \ln 2 + C \\ &= \ln |\sqrt{4 + x^2} + x| + C_1\end{aligned}$$

where  $C_1 = C - \ln 2$ .

**Example 3.4.3** Evaluate  $I = \int \frac{dx}{\sqrt{x^2 - a^2}}$ , where  $a > 0$ .

**Solution**

For the moment, assume that  $x \geq a$ . If  $x = a \sec \theta$ , then  $dx = a \sec \theta \tan \theta d\theta$  and  $\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2(\sec^2 \theta - 1)} = a \tan \theta$ . Thus

$$\begin{aligned} I &= \int \frac{a \sec \theta \tan \theta d\theta}{a \tan \theta} \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C \quad (\text{Refer to the triangle in 3.}) \\ &= \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + C \\ &= \ln |x + \sqrt{x^2 - a^2}| - \ln a + C \\ &= \ln |x + \sqrt{x^2 - a^2}| + C_1 \end{aligned}$$

where  $C_1 = C - \ln a$ .

If  $x \leq a$ , let  $u = -x$  so that  $u \geq a$  and  $du = -dx$ . Thus we have

$$\begin{aligned} I &= - \int \frac{du}{\sqrt{u^2 - a^2}} \\ &= - \ln |u + \sqrt{u^2 - a^2}| + C_1. \quad (\text{From the } x \geq a \text{ case above}) \\ &= - \ln | -x + \sqrt{(-x)^2 - a^2} | + C_1 \\ &= \ln \left| \frac{1}{-x + \sqrt{x^2 - a^2}} \right| + C_1 \\ &= \ln \left| \frac{1}{-x + \sqrt{x^2 - a^2}} \cdot \frac{x + \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}} \right| + C_1 \\ &= \ln \left| \frac{x + \sqrt{x^2 - a^2}}{-a^2} \right| + C_1 \\ &= \ln |x + \sqrt{x^2 - a^2}| - \ln | -a^2 | + C_1 \\ &= \ln |x + \sqrt{x^2 - a^2}| + C_2 \end{aligned}$$

where  $C_2 = C_1 - 2 \ln a$ .

Thus in either case, we have seen that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| + C.$$

**Example 3.4.4** Evaluate  $\int \frac{1}{1+\sqrt{2x}} dx$ .

**Solution**

We use the substitution in 4. Let  $2x = u^2$ , then  $2dx = 2udu$  or  $dx = udu$ . Thus

$$\begin{aligned} \int \frac{1}{1+\sqrt{2x}} dx &= \int \frac{u}{1+u} du \\ &= \int \frac{1+u-1}{1+u} \\ &= \int \left(1 - \frac{1}{1+u}\right) du \quad (\text{Let } v = 1+u, dv = du) \\ &= u - \int \frac{dv}{v} \\ &= u - \ln |v| + C \\ &= u - \ln |1+u| + C \\ &= \sqrt{2x} - \ln(1 + \sqrt{2x}) + C. \end{aligned}$$

where  $n$  is even and  $\Delta x = \frac{b-a}{n}$ .

**Example 3.1.13**

Use Simpson's rule with  $n=10$  to approximate  $\int_1^2 \frac{1}{x} dx$ .

**Solution**

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = 0.1$$

Thus, intervals are

$$[1,1.1], [1.1,1.2], [1.2,1.3], [1.3,1.4], [1.4,1.5], [1.5,1.6], [1.6,1.7], [1.7,1.8], [1.8,2], [1.9,2]$$

and so

$$x_0 = 1.0, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4, x_5 = 1.5, x_6 = 1.6, x_7 = 1.7, x_8 = 1.8, x_9 = 1.9, x_{10} = 2.0$$

$$\begin{aligned} \therefore \int_1^2 \frac{1}{x} dx &\approx \frac{\Delta x}{3} [f(1.0) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + \dots + 2f(1.8) + 4f(1.9) + f(2.0)] \\ &= \frac{0.1}{3} \left[ \frac{1}{1.0} + 4 \cdot \frac{1}{1.1} + 2 \cdot \frac{1}{1.2} + 4 \cdot \frac{1}{1.3} + 2 \cdot \frac{1}{1.4} + 4 \cdot \frac{1}{1.5} + 2 \cdot \frac{1}{1.6} + 4 \cdot \frac{1}{1.7} + 2 \cdot \frac{1}{1.8} + 4 \cdot \frac{1}{1.9} + \frac{1}{2.0} \right] \\ &\approx 0.693150. \end{aligned}$$



**3.1.8 IMPROPER INTEGRALS**

In the definition of a definite integral,  $f(x)$  was assumed to be bounded in the interval  $a \leq x \leq b$ . If  $f(x)$  is not bounded in  $a \leq x \leq b$ , then the integral

$$\int_a^b f(x) dx$$

is said to be an improper integral. In other words, integrals with infinite limits of integration are improper integrals. In addition, integrals of functions that become infinite at a point within the interval of integration are improper integrals. When the limits involved exist, we evaluate such integrals with the following definitions:

1. If  $f$  is continuous on  $[a, \infty)$ , then

$$\int_a^{\infty} f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx.$$

2. If  $f$  is continuous on  $(-\infty, b]$ , then

$$\int_{-\infty}^b f(x)dx = \lim_{a \rightarrow -\infty} \int_a^b f(x)dx.$$

3. If  $f$  is continuous on  $(a, b]$ , then

$$\int_a^b f(x)dx = \lim_{c \rightarrow a^+} \int_c^b f(x)dx.$$

4. If  $f$  is continuous on  $[a, b)$ , then

$$\int_a^b f(x)dx = \lim_{c \rightarrow b^-} \int_a^c f(x)dx.$$

5. If  $f$  becomes infinite at an interior point  $d$  of  $[a, b]$ , then

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx.$$

The integral from  $a$  to  $b$  converges if the integrals from  $a$  to  $d$  and from  $d$  to  $b$  both converge. Otherwise, the integral from  $a$  to  $b$  diverges.

6. If  $f$  is continuous on  $(-\infty, \infty)$ , and if  $\int_{-\infty}^a f(x)dx$  and  $\int_a^{\infty} f(x)dx$  both converge, we say that  $\int_{-\infty}^{\infty} f(x)dx$  converges and

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^{\infty} f(x)dx.$$

If either or both of the integrals on the right hand side diverges, the integral from  $-\infty$  to  $\infty$  is said to diverge as well.

### **Example 3.1.4**

Evaluate

(a)  $\int_2^{\infty} \frac{1}{x^3} dx$  (b)  $\int_{-\infty}^0 \tanh x dx$  (c)  $\int_0^1 \frac{1}{\sqrt{x}} dx$  (d)  $\int_0^1 \frac{1}{1-x} dx$  (e)  $\int_0^6 \frac{2x}{(x^2-4)^{3/2}} dx$  (f)  $\int_0^6 \frac{2x}{x^2-4} dx$

(g)  $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$

### Solution

$$(a) \int_2^{\infty} \frac{1}{x^3} dx = \lim_{b \rightarrow \infty} \int_2^b x^{-3} dx = \lim_{b \rightarrow \infty} \left( -\frac{1}{2} x^{-2} \Big|_2^b \right) = -\frac{1}{2} \lim_{b \rightarrow \infty} \left( \frac{1}{b^2} - \frac{1}{4} \right) = \frac{1}{8}.$$

$$(b) \int_{-\infty}^0 \tanh x dx = \lim_{a \rightarrow -\infty} \int_a^0 \tanh x dx = \lim_{a \rightarrow -\infty} \left( (\ln |\cosh x|) \Big|_a^0 \right) = \lim_{a \rightarrow -\infty} (\ln 2 - \ln \cosh a) = -\infty.$$

$\therefore$  The integral diverges.

(c) The function  $\frac{1}{\sqrt{x}}$  is discontinuous at  $x=0$ .

$$\therefore \int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{c \rightarrow 0^+} \int_c^1 x^{-\frac{1}{2}} dx = \lim_{c \rightarrow 0^+} \left( 2x^{\frac{1}{2}} \Big|_c^1 \right) = \lim_{c \rightarrow 0^+} \left( 2 - 2c^{\frac{1}{2}} \right) = 2.$$

(d) The function  $\frac{1}{1-x}$  is discontinuous at  $x=1$ .

$$\therefore \int_0^1 \frac{1}{1-x} dx = \lim_{c \rightarrow 1^-} \int_0^c \frac{1}{1-x} dx = \lim_{c \rightarrow 1^-} \left( (-\ln |1-x|) \Big|_0^c \right) = -\lim_{c \rightarrow 1^-} (\ln |1-c|) = \infty$$

$\therefore$  The integral diverges.

(e) In the interval  $[0, 6]$ , the function  $\frac{2x}{(x^2-4)^{\frac{2}{3}}}$  is discontinuous at  $x=2$ .

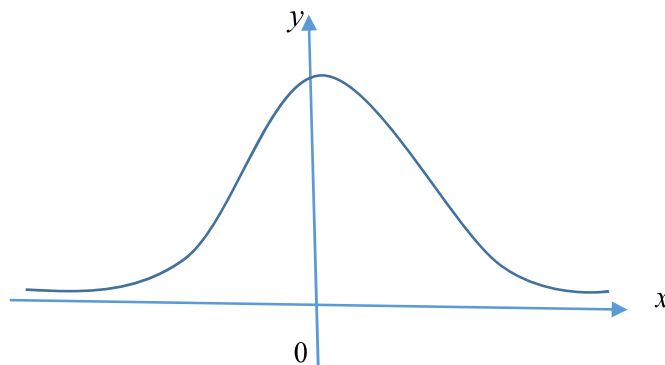
$$\begin{aligned} \therefore \int_0^6 \frac{2x}{(x^2-4)^{\frac{2}{3}}} dx &= \int_0^2 \frac{2x}{(x^2-4)^{\frac{2}{3}}} dx + \int_2^6 \frac{2x}{(x^2-4)^{\frac{2}{3}}} dx = \lim_{c \rightarrow 2^-} \int_0^c \frac{2x}{(x^2-4)^{\frac{2}{3}}} dx + \lim_{c \rightarrow 2^+} \int_c^6 \frac{2x}{(x^2-4)^{\frac{2}{3}}} dx \\ &= \lim_{c \rightarrow 2^-} \left( 3(x^2-4) \Big|_0^c \right) + \lim_{c \rightarrow 2^+} \left( 3(x^2-4)^{\frac{1}{3}} \Big|_c^6 \right) \\ &= \lim_{c \rightarrow 2^-} \left[ 3(c^2-4)^{\frac{1}{3}} - 3(-4)^{\frac{1}{3}} \right] + \lim_{c \rightarrow 2^+} \left[ 3(32)^{\frac{1}{3}} - 3(c^2-4)^{\frac{1}{3}} \right] \\ &= 3 \cdot 4^{\frac{1}{3}} + 3 \cdot (32)^{\frac{1}{3}} \\ &= 9 \cdot 4^{\frac{1}{3}}. \end{aligned}$$

(f) The integrand is discontinuous at  $x=2$ . Thus,

$$\begin{aligned}
\int_0^6 \frac{2x}{x^2-4} dx &= \lim_{c \rightarrow 2^-} \int_0^c \frac{2x}{(x^2-4)^{\frac{3}{2}}} dx + \lim_{c \rightarrow 2^+} \int_c^6 \frac{2x}{(x^2-4)^{\frac{3}{2}}} dx \\
&= \lim_{c \rightarrow 2^-} \left( (\ln |x^2-4|) \Big|_0^c \right) + \lim_{c \rightarrow 2^+} \left( (\ln |x^2-4|) \Big|_c^6 \right) \\
&= \lim_{c \rightarrow 2^-} (\ln |c^2-4| - \ln 4) + \lim_{c \rightarrow 2^+} (\ln 32 - \ln |c^2-4|).
\end{aligned}$$

Since  $\lim_{c \rightarrow 2^-} (\ln |c^2-4| - \ln 4) = -\infty$  and  $\lim_{c \rightarrow 2^+} (\ln 32 - \ln |c^2-4|) = \infty$ , the integral diverges.

(g) We try to sketch the graph of  $\frac{1}{1+x^2}$ .



The graph is symmetric at  $x=0$ . So

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{a \rightarrow -\infty} \int_a^0 \frac{1}{1+x^2} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx \\
&= \lim_{a \rightarrow -\infty} (\arctan x \Big|_a^0) + \lim_{b \rightarrow \infty} (\arctan x \Big|_0^b) = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi.
\end{aligned}$$



### 3.2 APPLICATION OF INTEGRATION

The use of the integral as a limit of a sum enables us to solve many physical and geometrical problems such as determination of areas, volumes, arc length, surface area, centre of mass and moments of inertia.

#### 3.2.1 THE AREA BETWEEN TWO CURVES

Suppose that we have two curves  $y_1 = f_1(x)$  and  $y_2 = f_2(x)$  and we want to find the area  $A$  of the region  $R$  bounded by these two curves and two vertical lines  $x = a$  and  $x = b$ . We assume that the curve  $y_2 = f_2(x)$  lies above the curve  $y_1 = f_1(x)$  such that  $f_2(x) \geq f_1(x)$ , for  $a \leq x \leq b$ . Then

$$A = \int_a^b [f_2(x) - f_1(x)] dx.$$

Similarly, if  $R$  is bounded by two curves  $x_1 = f_1(y)$  and  $x_2 = f_2(y)$  and two horizontal lines  $y = c$  and  $y = d$  in such a way that  $f_2(y) \geq f_1(y)$ , for  $c \leq y \leq d$ , then

$$A = \int_c^d [f_2(y) - f_1(y)] dy.$$

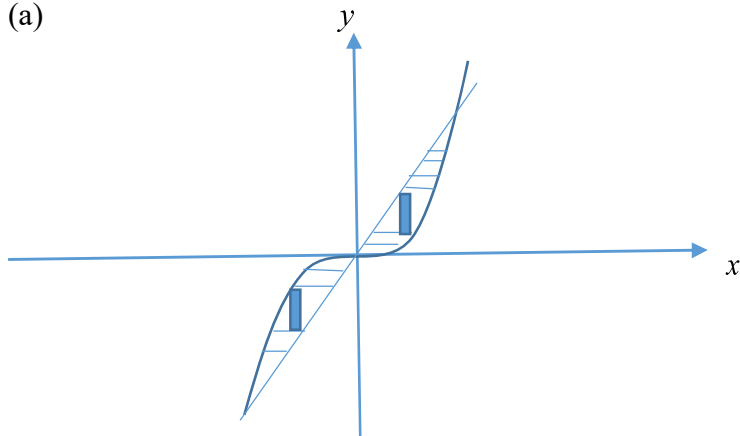
**Example 3.2.1**

Find the area bounded by the curves  $x = y$  and  $y = \frac{x^5}{16}$  using

- (a) vertical strips
- (b) horizontal strips.

**Solution**

(a)

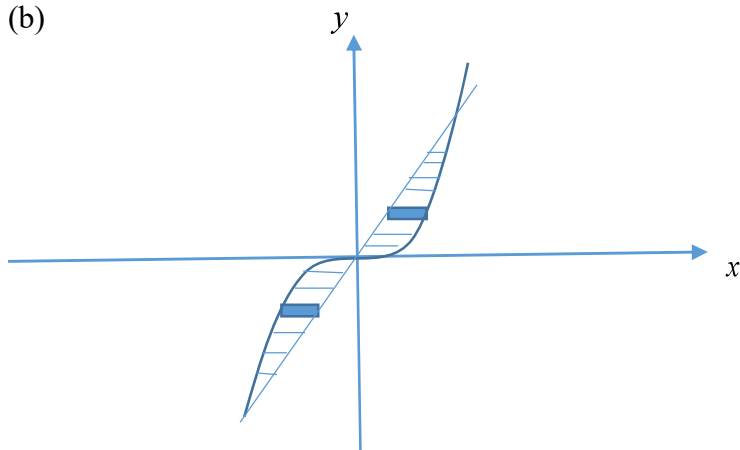


Solving the two equations simultaneously, we find

$$16x = x^5 \text{ or } x(16 - x^4) = 0 \Rightarrow x = 0 \text{ or } x = \pm 2.$$

$$\therefore A = \int_{-2}^0 \left( \frac{x^5}{16} - x \right) dx + \int_0^2 \left( x - \frac{x^5}{16} \right) dx = \left( \frac{x^6}{96} - \frac{x^2}{2} \right) \Big|_{-2}^0 + \left( \frac{x^2}{2} - \frac{x^6}{96} \right) \Big|_0^2 = \frac{8}{3}.$$

(b)



$$x = y, \quad x = (16y)^{\frac{1}{5}}$$

$$\therefore A = \int_{-2}^0 (y - (16y)^{\frac{1}{5}}) dy + \int_0^2 ((16y)^{\frac{1}{5}} - y) dy = \left( \frac{y^2}{2} - \frac{5}{6} \cdot (16)^{\frac{1}{5}} y^{\frac{6}{5}} \right) \Big|_{-2}^0 + \left( \frac{5}{6} \cdot (16)^{\frac{1}{5}} y^{\frac{6}{5}} - \frac{y^2}{2} \right) \Big|_0^2 = \frac{8}{3}.$$



### 3.2.2 VOLUMES OF SOLIDS OF REVOLUTION

If a region in the plane is revolved about a given line, the resulting solid is a solid of revolution and the line is called axis of revolution.

#### 1. DISC METHOD

The simplest solid of revolution is a right circular cylinder which is found by revolving a rectangle about an axis adjacent to one side of the rectangle. In this case, the cross sections of the solid (slice or disc) are perpendicular to the x-axis of revolution. Two cases arise:

(i) The volume of the solid generated by a region under  $y = f(x)$  bounded by the x-axis and vertical lines  $x = a$  and  $x = b$  which is revolved about the x-axis is

$$V = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \pi \int_a^b [f(x)]^2 dx.$$

Here the disc is with respect to  $x$  and  $r = y = f(x)$ .

(ii) The volume of the solid generated by a region under  $x = f(y)$  bounded by the y-axis and horizontal lines  $y = c$  and  $y = d$  which is revolved about the y-axis is

$$V = \pi \int_c^d [f(y)]^2 dy.$$

#### 2. WASHER METHOD

Suppose that we are dealing with a hollowed object and two functions have been given instead of one. Then we extend the disc method and take into account the difference of the two given functions. This is the washer method and we have two cases:

(i) If the washer is with respect to  $x$  then the volume of the solid generated by a region between  $y_1 = f(x)$  and  $y_2 = g(x)$  bounded by the vertical lines  $x = a$  and  $x = b$ , which is revolved about the x-axis is

$$V = \int_a^b \pi (R^2 - r^2) dx = \pi \int_a^b \left[ [f(x)]^2 - [g(x)]^2 \right] dx,$$

where  $R$  is the outer radius and  $r$  is the inner radius.

(ii) If the washer being considered is with respect to  $y$ , then the volume of the solid generated by a region between  $x_1 = f(y)$  and  $x_2 = g(y)$  bounded by the horizontal lines  $y = c$  and  $y = d$ , which is revolved about the  $y$ -axis is

$$V = \pi \int_a^b \left[ [f(y)]^2 - [g(y)]^2 \right] dy.$$

NOTE: The axis of revolution does not necessarily need be the  $x$ - or  $y$ -axis.

**Example 3.2.2**

1. Find the volume of the solid of revolution obtained by rotating the region under the curve  $y = x^{\frac{2}{3}}$  between  $x = 0$  and  $x = 8$

(a) about the  $x$ -axis

(b) about the  $y$ -axis.

2. Determine the volume of the solid generated by rotating the region bounded by  $y = \sqrt[3]{x}$  and  $y = \frac{x}{4}$  that lies in the first quadrant.

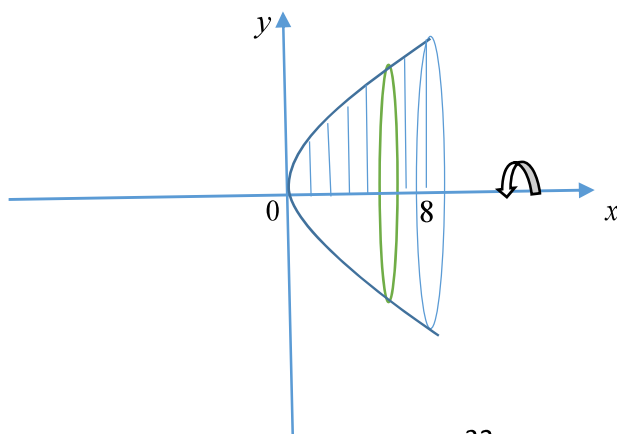
(a) about the  $x$ -axis

(b) about the  $y$ -axis.

3. Determine the volume of the solid obtained by rotating the region bounded by  $y = x^2 - 2x$  and  $y = x$  about the line  $y = 4$ .

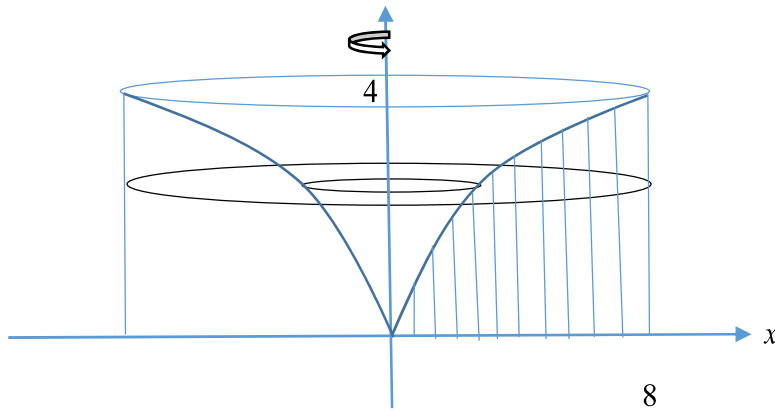
**Solution**

1. (a)



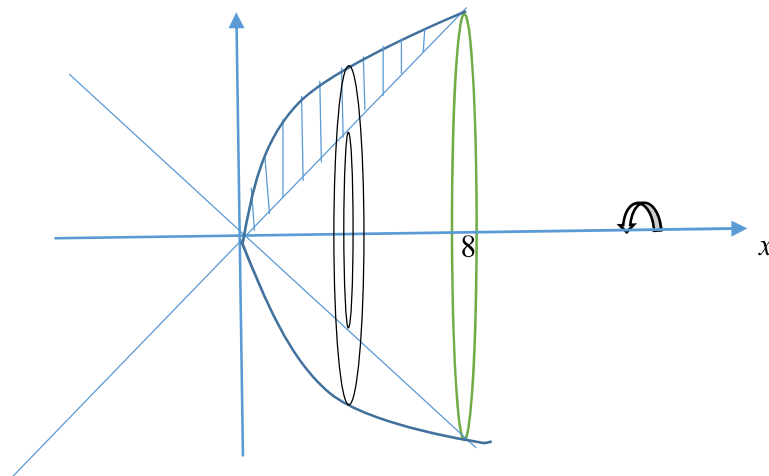
$$V = \pi \int_0^8 \left(x^{\frac{2}{3}}\right)^2 dx = \pi \cdot \frac{3}{7} x^{\frac{7}{3}} \Big|_0^8 = \frac{384\pi}{7}.$$

(b) When  $x = 8$ ,  $y = 4$  and  $x = y^{\frac{3}{2}}$



$$V = \pi \int_0^4 \left(8^2 - \left(y^{\frac{3}{2}}\right)\right) dy = \pi \left(64y - \frac{y^4}{4}\right) \Big|_0^4 = 192\pi.$$

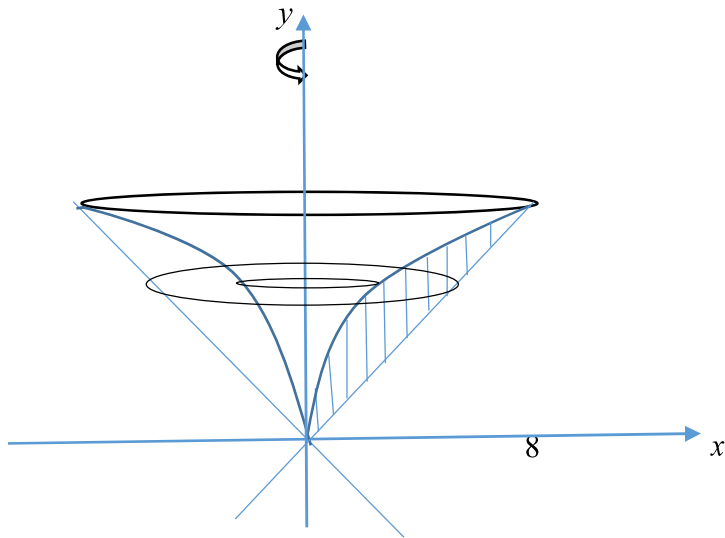
2. (a)



$$y = \frac{x}{4}, \quad y = x^{\frac{1}{3}} \Rightarrow \frac{x}{4} = x^{\frac{1}{3}} \Rightarrow x = 0 \text{ or } x = \pm 8.$$

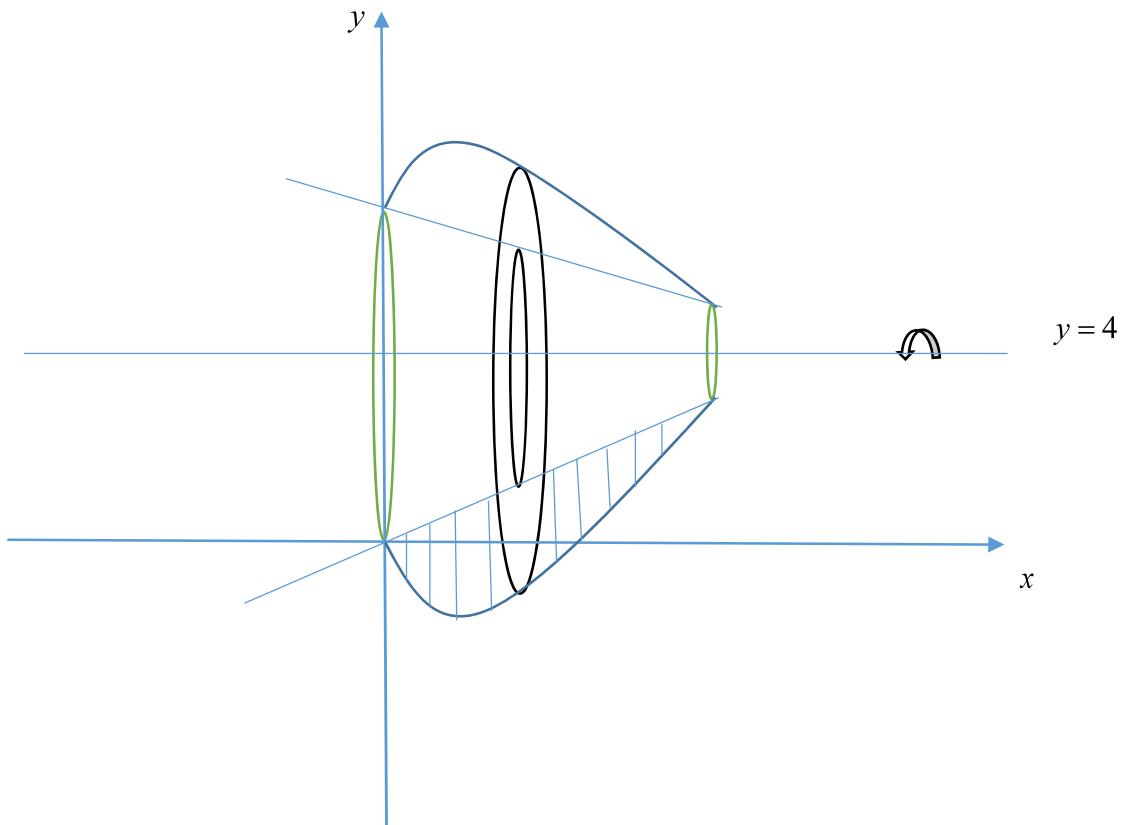
$$V = \pi \int_0^8 (R^2 - r^2) dx = \pi \int_0^8 \left( \left(x^{\frac{1}{3}}\right)^2 - \left(\frac{x}{4}\right)^2 \right) dx = \pi \left( \frac{3}{5} x^{\frac{5}{3}} - \frac{x^3}{48} \right) \Big|_0^8 = \frac{128\pi}{15}.$$

(b)  $y = \frac{x}{4}$ ,  $y = x^{\frac{1}{3}} \Rightarrow x = 4y$  and  $x = y^3$ . When  $x = 8$ ,  $y = 2$ .



$$V = \pi \int_0^2 \left[ (4y)^2 - (y^3)^2 \right] dy = \pi \left( \frac{16y^3}{3} - \frac{y^7}{7} \right) \Big|_0^2 = \frac{512\pi}{21}.$$

3.



$$y = x, y = x^2 - 2x \Rightarrow x = x^2 - 2x = x \Rightarrow x^2 - 3x = 0 \Rightarrow x = 0 \text{ or } x = 3.$$

$$V = \pi \int_0^3 (R^2 - r^2) dx = \pi \int_0^3 \left[ (4 - (x^2 - 2x))^2 - (4 - x)^2 \right] dx = \pi \left( \frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2 \right) \Big|_0^3 = \frac{153\pi}{5}.$$



### 3. SHELL METHOD

The shell method is a method of calculating the volume of a solid of revolution when integrating along an axis parallel to the axis of revolution. The idea is that a “representative rectangle” can be rotated about the axis of revolution, thus generating a hollow cylinder—a shell.

Then

$$\text{Volume}(V) = \int 2\pi \cdot \text{radius} \cdot \text{height} \cdot \text{thickness}$$

(i) The volume of the solid generated by a region bounded by the x-axis and vertical lines  $x = a$  and  $x = b$ , which is revolved about the y-axis is

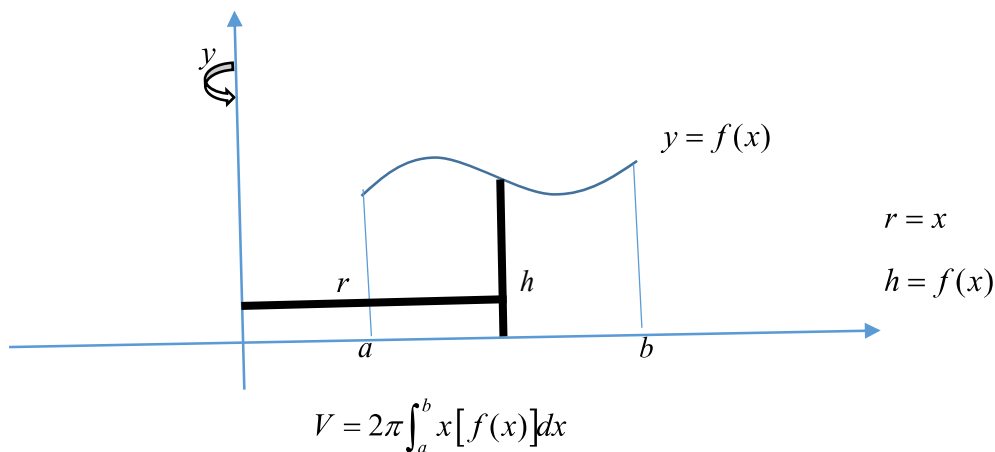
$$V = \int_a^b 2\pi xy dx = 2\pi \int_a^b x[f(x)] dx,$$

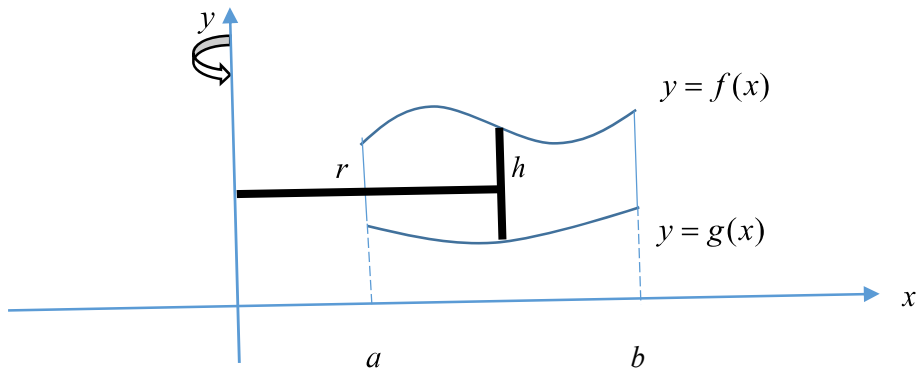
where *radius* ( $r$ ) =  $x$ , *height* ( $h$ ) =  $y = f(x)$  and *thickness* =  $dx$ .

(ii) The volume of the solid generated by a region bounded by the y-axis and horizontal lines  $y = c$  and  $y = d$ , which is revolved about the x-axis is

$$V = 2\pi \int_c^d y[f(y)] dy.$$

**NOTE:** Sometimes two functions are functions are involved and the axis of revolutions is not always the x- or y-axis. These and other cases discussed above are summarised in the following graphs below:

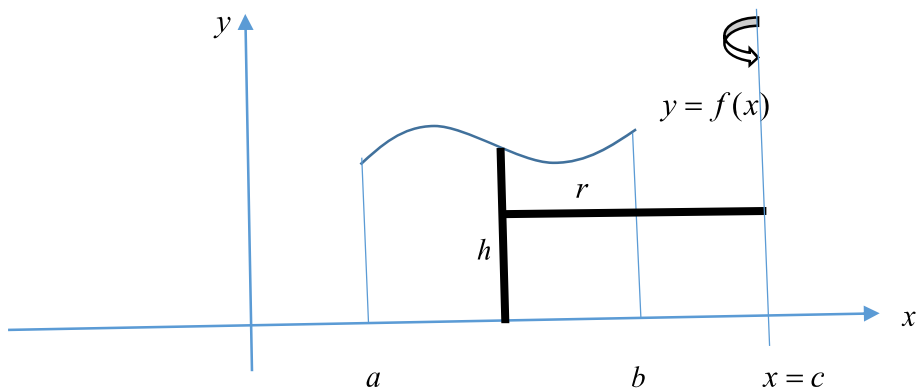




$$r = x$$

$$h = f(x) - g(x)$$

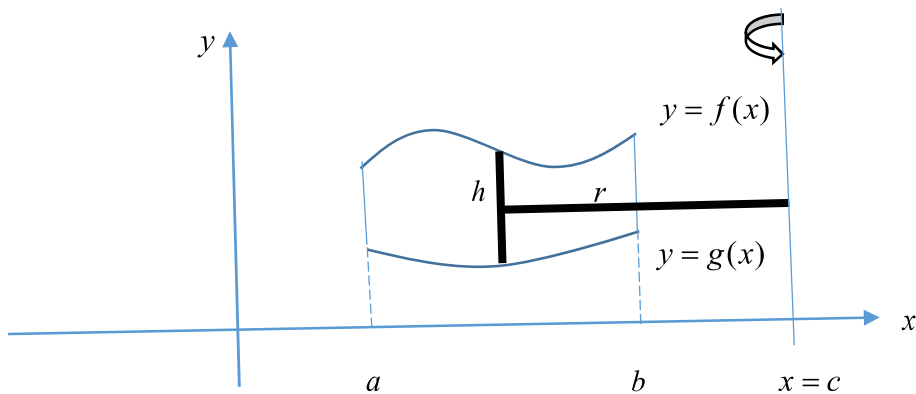
$$V = 2\pi \int_a^b x [f(x) - g(x)] dx$$



$$r = c - x$$

$$h = f(x)$$

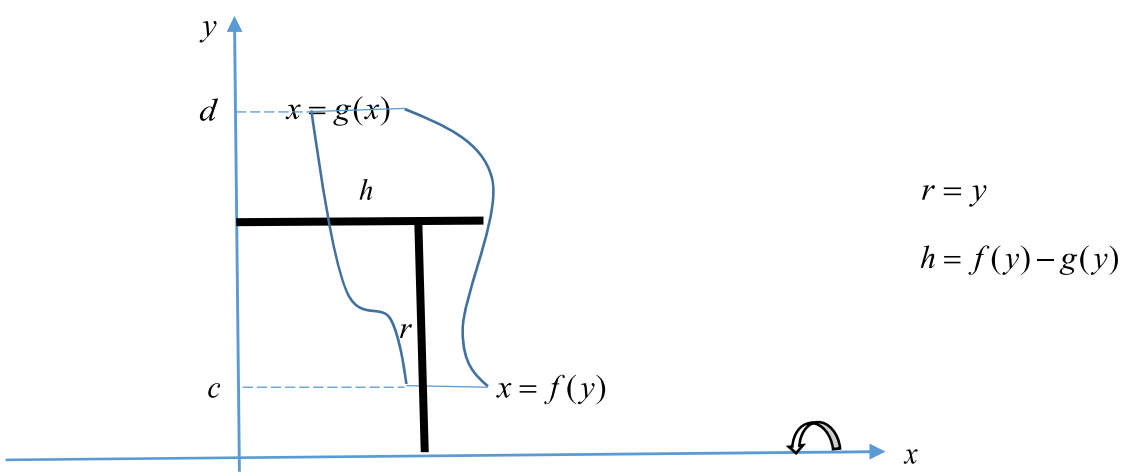
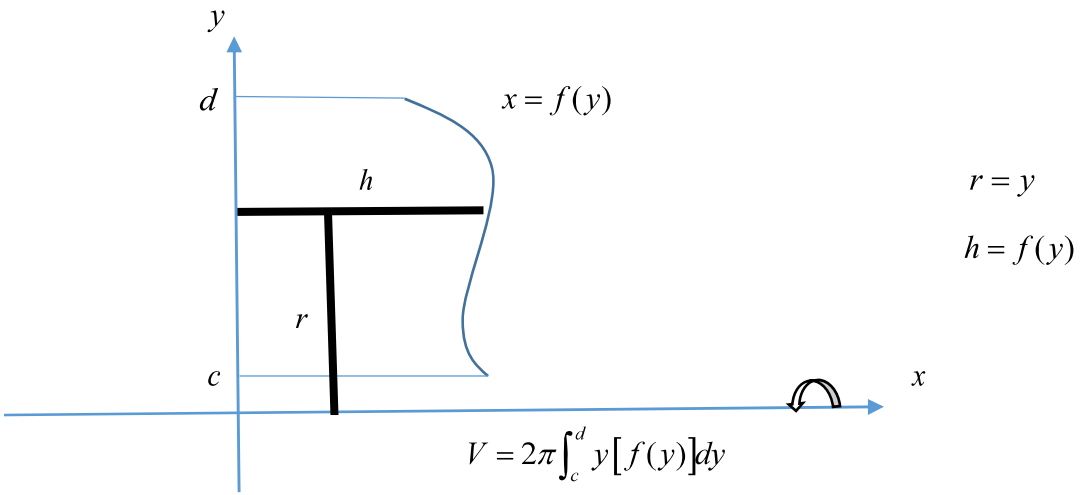
$$V = 2\pi \int_a^b (c - x) [f(x)] dx$$



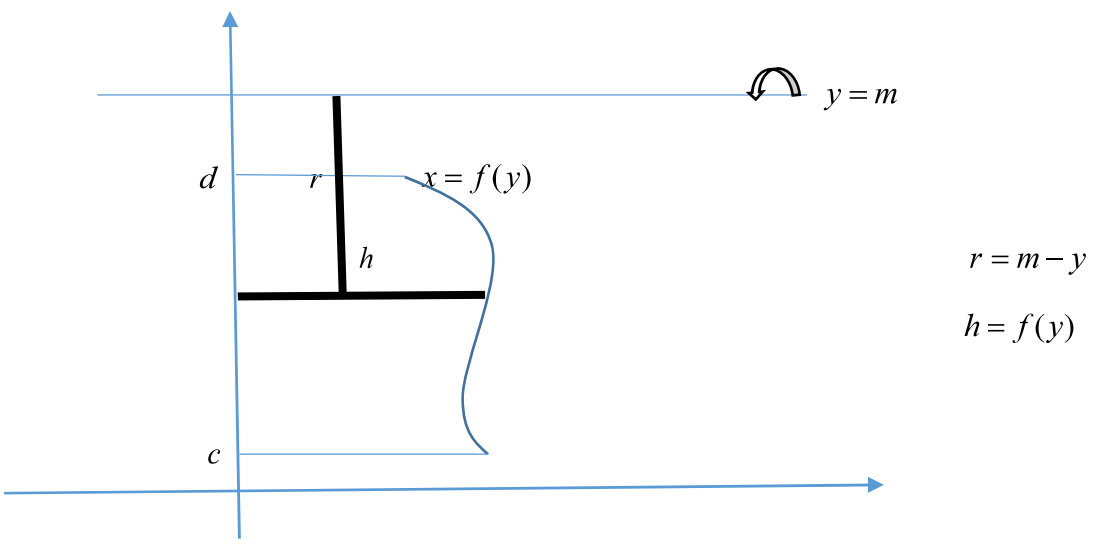
$$r = c - x$$

$$h = f(x) - g(x)$$

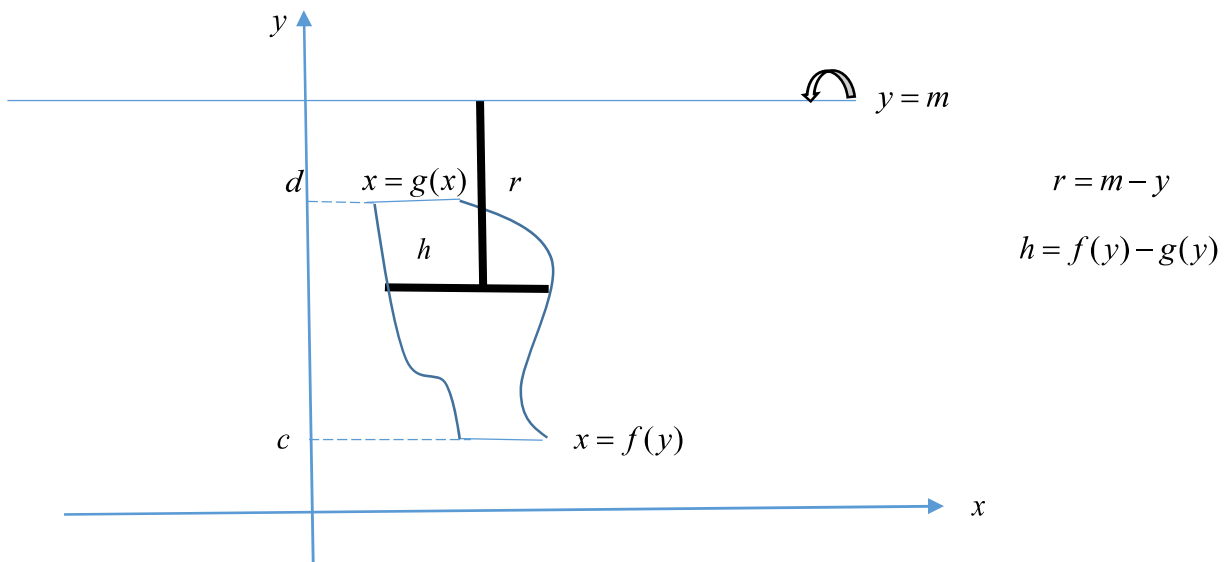
$$V = 2\pi \int_a^b (c - x) [f(x) - g(x)] dx$$



$$V = 2\pi \int_c^d y[f(y) - g(y)]dy$$



$$V = 2\pi \int_c^d (m - y)[f(y)]dy$$



$$V = 2\pi \int_c^d (m - y)[f(y) - g(y)] dy$$

**Example 3.2.3**

1. Find the Volume of the solid of revolution formed by rotating the finite region bounded by the graphs of  $y = \sqrt{x-1}$  and  $y = (x-1)^2$

(a) about the x-axis (b) about the y-axis

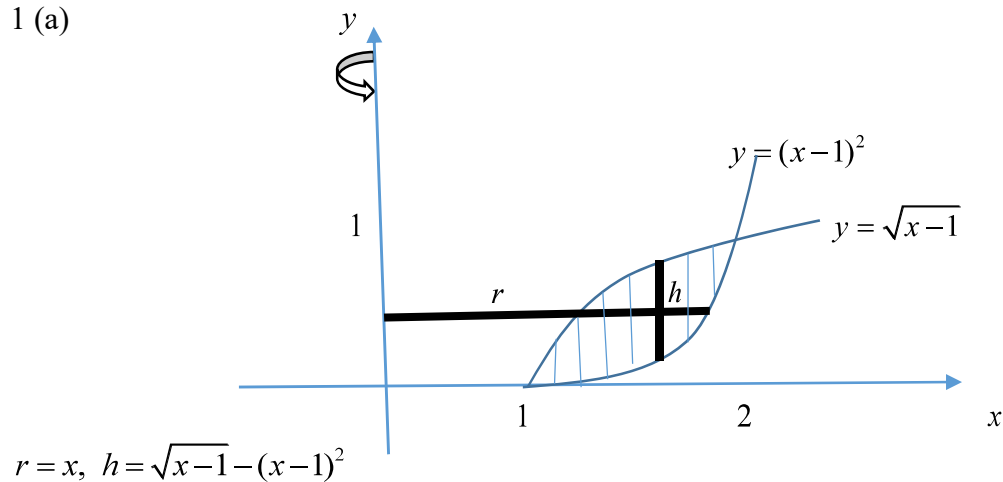
2. A region is bounded by

$$y = \cos x, y = \sin x, x = 0, x = \frac{\pi}{2}.$$

Find the volume when this region is rotated about the line  $x = 2$ .

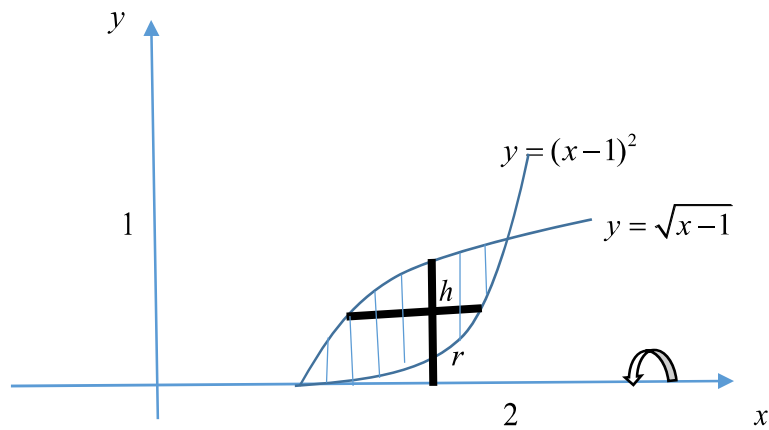
**Solution**

1 (a)



$$\begin{aligned}
\therefore V &= 2\pi \int_1^2 x(\sqrt{x-1} - (x-1)^2) dx = 2\pi \left[ \int_1^2 x\sqrt{x-1} dx - \int_1^2 (x^3 - 2x + x) dx \right] \\
&= 2\pi \left[ \left( \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} \right) \Big|_1^2 - \left( \frac{x^4}{4} - \frac{2}{3}x^3 + \frac{1}{2}x^2 \right) \Big|_1^2 \right] \\
&= 2\pi \left( \frac{29}{60} \right) \\
&= \frac{29\pi}{30}.
\end{aligned}$$

(b)

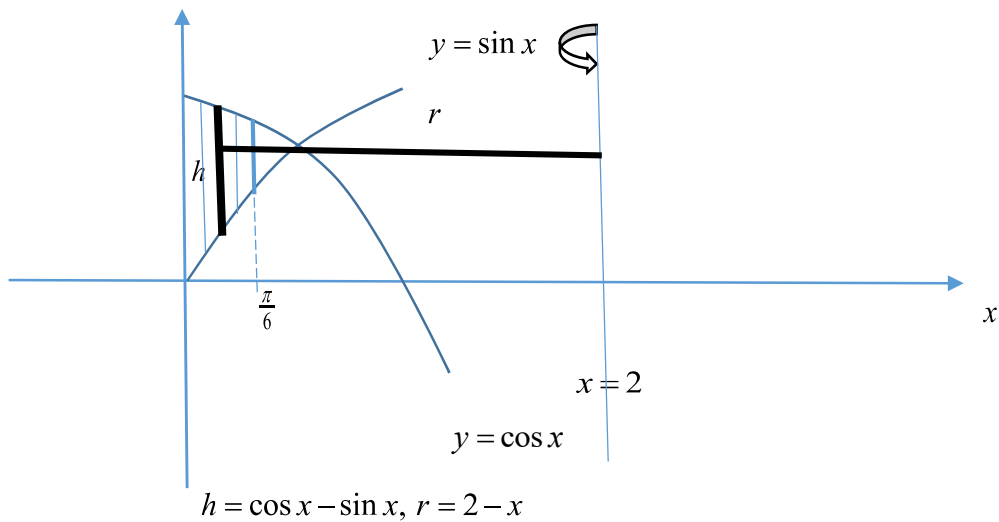


$$r = y, \quad h = y^{\frac{1}{2}} + 1 - (y^2 + 1) = y^{\frac{1}{2}} - y^2$$

$$y = \sqrt{x-1} \Rightarrow x = y^2 + 1, \quad y = (x-1)^2 \Rightarrow x = y^{\frac{1}{2}} + 1.$$

$$\therefore V = 2\pi \int_0^1 y(y^{\frac{1}{2}} - y^2) dy = 2\pi \int_0^1 (y^{\frac{3}{2}} - y^3) dy = \frac{3\pi}{10}.$$

2.



$$h = \cos x - \sin x, \quad r = 2 - x$$

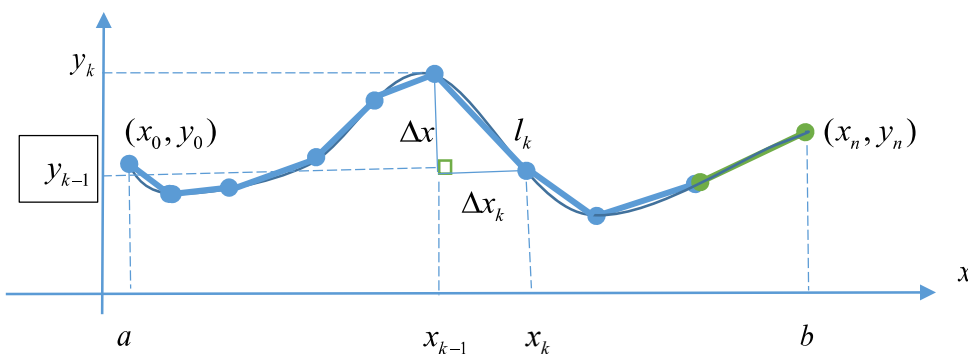
$$\therefore V = 2\pi \int \left( 2 - x \right) (\cos x - \sin x) dx = 2\pi \left[ (2 - x)(\sin x + \cos x) \Big|_0^{\frac{\pi}{6}} + (\sin x - \cos x) \Big|_0^{\frac{\pi}{6}} \right]$$

$$= 2\pi \left[ \left(2 - \frac{\pi}{6}\right) \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) - (2-1) \right] = \frac{\pi [6 + 12\sqrt{3} - \pi - \pi\sqrt{3}]}{6}. \quad \triangle$$

### 3.2.3 LENGTH OF THE ARC

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a differentiable Function such that  $f'$  is continuous. Such a Function is said to be smooth and its graph is said to be a smooth curve. The arc of a curve is said to be simple if it does not intersect itself.

Suppose we wish to find the length of a simple arc of a smooth curve.



If we partition the interval  $[a, b]$  then the length of each line segment  $l_k$  can be found by Pythagorean Theorem

$$l_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

Thus, the length of the arc is obtained by summing up all the  $l_k$ 's and finding the limit as  $n \rightarrow \infty$ . Therefore,

$$L = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \right) \Delta x_k = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

#### Theorem 3.2.1

If  $f'(x)$  is continuous in  $a \leq x \leq b$ , then the length of the arc of the curve  $y = f(x)$  between  $x = a$  and  $x = b$  is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

If a plane curve is given by  $\{(x(t), y(t)) : t \in [a, b]\}$ , where  $x$  and  $y$  are continuous functions, then

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

If a curve is given in polar form  $r = f(\theta)$ ,  $f(\theta) > 0$ ,  $\alpha \leq \theta \leq \beta$ , then  $x(\theta) = r(\theta) \sin \theta$ ,  
 $y(\theta) = r(\theta) \cos \theta$  and

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

□

**Example 3.2.4**

1. Find the length of the arc of the curve  $y = 2\sqrt{x^3}$  between  $x = \frac{1}{3}$  and  $x = \frac{5}{3}$ .
2. Find the arc length of the cardioid  $r = 2 + 2 \cos \theta$ ,  $0 \leq \theta \leq 2\pi$ .

**Solution**

1.  $y = \sqrt{x^3}$ ,  $x \geq 0$  and  $\frac{dy}{dx} = 3\sqrt{x}$

$$\begin{aligned} \therefore L &= \int_{\frac{1}{3}}^{\frac{5}{3}} \sqrt{1 + [y'(x)]^2} dx = \int_{\frac{1}{3}}^{\frac{5}{3}} \sqrt{1 + [3\sqrt{x}]^2} dx \\ &= \int_{\frac{1}{3}}^{\frac{5}{3}} \sqrt{1 + 9x^2} dx \\ &= \frac{1}{9} \cdot \frac{2}{3} (1 + 9x)^{\frac{3}{2}} \Big|_{\frac{1}{3}}^{\frac{5}{3}} \\ &= \frac{112}{27}. \end{aligned}$$

2.  $r = 2 + 2 \cos \theta \Rightarrow \frac{dr}{d\theta} = -2 \sin \theta$

$$\begin{aligned} \therefore L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(2 + 2 \cos \theta)^2 + (-2 \sin \theta)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{8 + 8 \cos \theta} d\theta \\ &= 2 \int_0^{2\pi} \sqrt{2 + 2 \cos \theta} d\theta \\ &= 2 \int_0^{2\pi} \sqrt{4 \cos^2 \frac{\theta}{2}} d\theta \\ &= 4 \int_0^{2\pi} \cos \frac{\theta}{2} d\theta \\ &= 8 \int_0^{\pi} \cos \frac{\theta}{2} d\theta \end{aligned}$$

$$= 16 \sin \frac{\theta}{2} \Big|_0^\pi$$

$$= 16.$$



### 3.2.4 AREA OF A SURFACE OF REVOLUTION

If a curve is rotated about an axis it generates a surface called a surface of revolution. We want to find the area of such a surface.

#### Theorem 3.2.2

Suppose that  $f(x) \geq 0$  and  $f'(x)$  is continuous in the interval  $[a, b]$ . Then, the area of the surface of revolution generated by rotating the curve  $y = f(x)$  between  $x = a$  and  $x = b$  about the x-axis is given by

$$\delta = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

If  $x = g(y)$  is the curve between  $y = c$  and  $y = d$  rotated about the y-axis, then

$$\delta = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

NOTE: The axis of rotation could be any line. If  $y = n$  and  $x = m$  are axes of rotation, then

$$\delta = \int_a^b 2\pi(n - y) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

and

$$\delta = \int_c^d 2\pi(m - x) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

respectively.



#### Example 3.2.5

The arc of the curve  $y = x^3$  lying between  $x = 0$  and  $x = 2$  is rotated about the x-axis. Find the area of the surface generated.

### Solution

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$$

$$\therefore \delta = 2\pi \int_0^2 x^3 \sqrt{1+(3x^2)^2} dx = 2\pi \int_0^2 x^3 \sqrt{1+9x^4} dx = \frac{2\pi}{36} \cdot \frac{2}{3} (1+9x^4)^{\frac{3}{2}} \Big|_0^2 = \frac{\pi}{27} [(145)^{\frac{3}{2}} - 1]. \quad \triangle$$

### 3.2.5 CENTER OF MASS AND MOMENTS OF INERTIA

We consider two important quantities in Physics and Mechanics and how integration can be used to compute them. The term “system of particles” will be used to mean a collection of objects in which the mass of each object is regarded as concentrated at a point.

#### Definition 3.2.1

Let  $\mathcal{P}$  be a plane system of particles of masses  $m_1, m_2, \dots, m_n$  located at the points  $p_1, p_2, \dots, p_n$ , respectively. Let the line  $\mathcal{L}$  in the plane be taken as an axis and let  $l_1, l_2, \dots, l_n$  be the directed distances from the line  $\mathcal{L}$  of the points  $p_1, p_2, \dots, p_n$ , respectively. Then, the moment of this system of particles about the axis  $\mathcal{L}$ , denoted by  $M_{\mathcal{L}}$ , is given by

$$M_{\mathcal{L}} = \sum_{k=1}^n m_k l_k = m_1 l_1 + m_2 l_2 + \dots + m_n l_n.$$

Thus, moments about x- and y-axes are given by

$$M_x = \sum_{k=1}^n m_k y_k$$

and

$$M_y = \sum_{k=1}^n m_k x_k,$$

respectively.

For example, for the points  $p_1(1, -1)$ ,  $p_2(-1, 1)$ ,  $p_3(1, 2)$ , and  $p_4(2, 3)$  with their respective masses  $m_1 = 2$ ,  $m_2 = 1$ ,  $m_3 = 4$  and  $m_4 = 7$ , we have that

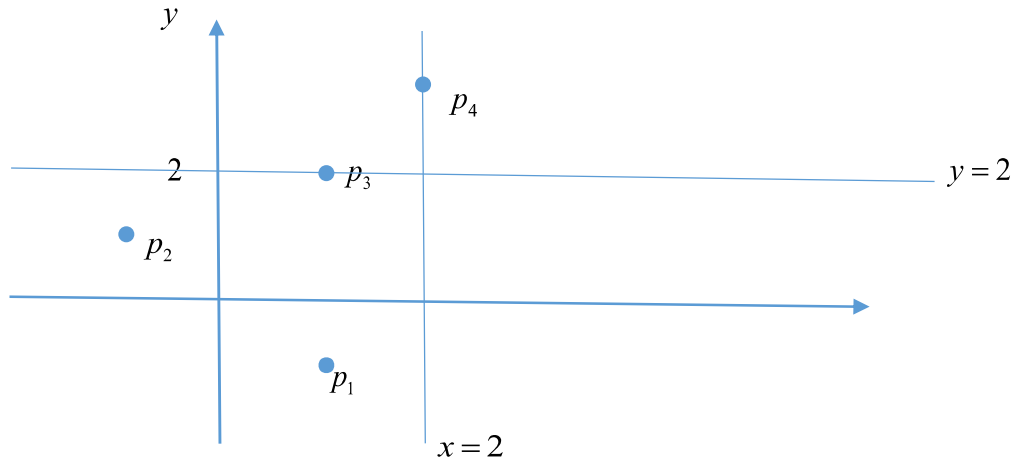
$$M_x = 2(-1) + 1(1) + 4(2) + 7(3) = 28$$

$$M_y = 2(1) + 1(-1) + 4(1) + 7(2) = 19.$$

If  $\mathcal{L}_1$  is the line  $x = 2$  and  $\mathcal{L}_2$  is the line  $y = 2$ , then

$$M_{\mathcal{L}_1} = 2(-1) + 1(-3) + 4(-1) + 7(0) = -1$$

$$M_{\mathcal{L}_2} = 2(-3) + 1(-1) + 4(0) + 7(1) = 0$$



### Definition

Let  $\mathcal{P}$  be a plane system of particles and let  $\bar{\mathcal{P}}$  be the system obtained by concentrating the total mass of the system at a single point  $\bar{\mathcal{P}}(\bar{x}, \bar{y})$ . Let  $M_{\mathcal{L}}$  and  $\bar{M}_{\mathcal{L}}$  denote the moments about  $\mathcal{L}$  of the systems  $\mathcal{P}$  and  $\bar{\mathcal{P}}$  respectively. If  $M_{\mathcal{L}} = \bar{M}_{\mathcal{L}}$  for each axis  $\mathcal{L}$ , then  $\bar{\mathcal{P}}(\bar{x}, \bar{y})$  is called the centre of mass of the system  $\mathcal{P}$ .

If the moment of the system  $\bar{\mathcal{P}}$  is equal to the moment of the original system, then we must have  $m\bar{x} = M_y$  and  $m\bar{y} = M_x$ , where  $m$  is the total mass of the system. Thus,

### Theorem 3.2.3

If  $\bar{\mathcal{P}}(\bar{x}, \bar{y})$  is called the centre of mass of a system of particles of mass at  $p_k(x_k, y_k)$ ,  $k = 1, 2, \dots, n$ , then

$$\bar{x} = \frac{M_y}{m}$$

and

$$\bar{y} = \frac{M_x}{m},$$

where  $m = m_1 + m_2 + \dots + m_n$ .

□

From our previous example,

$$m = 2 + 1 + 4 + 7 = 14.$$

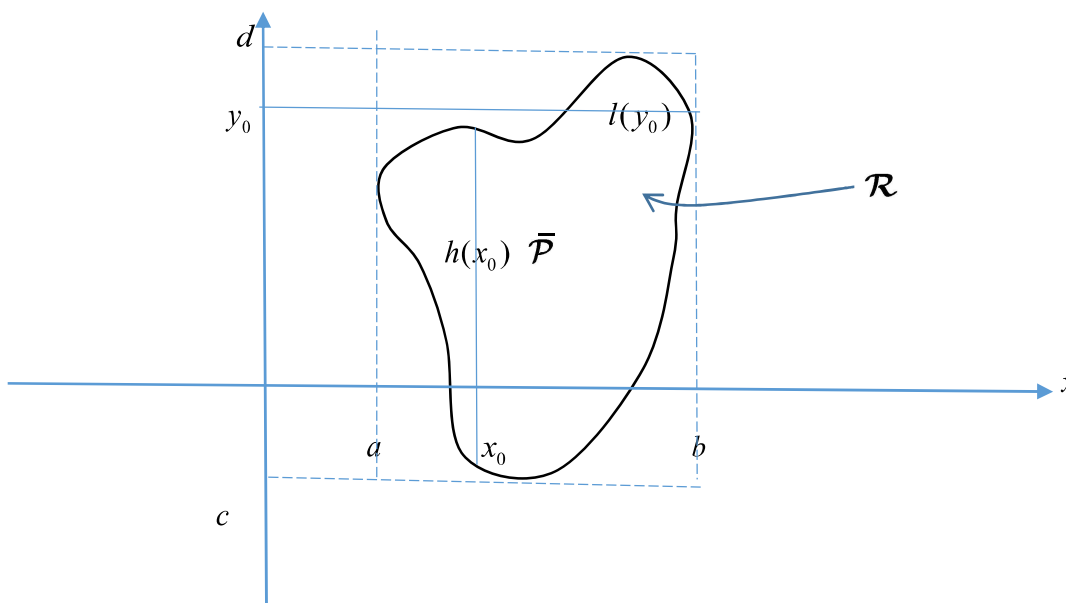
Hence,

$$\bar{x} = \frac{M_y}{m} = \frac{19}{14}$$

$$\bar{y} = \frac{M_x}{m} = \frac{28}{14} = 2.$$

Instead of a plane system of particles  $\mathcal{P}$ , suppose that we have a plane region of particles  $\mathcal{R}$ .

Then, the centre of mass will be called centroid of the region.



### Definition 3.2.3

Let  $\mathcal{R}$  be a plane region lying in the rectangle  $a \leq x \leq b$ ,  $c \leq y \leq d$ . Let the line  $x = x_0$  intersect this region in a line segment of length  $h(x_0)$  for each  $x_0$  in  $a \leq x \leq b$ , and let the line  $y = y_0$  intersect this region in a line segment of length  $l(y_0)$  for each  $y_0$  in  $c \leq y \leq d$ . Then,  $M_x$  and  $M_y$ , the moments of  $\mathcal{R}$  about the x- and the y-axes respectively, are given by

$$M_x = \int_c^d y l(y) dy,$$

$$M_y = \int_a^b xh(x)dx$$

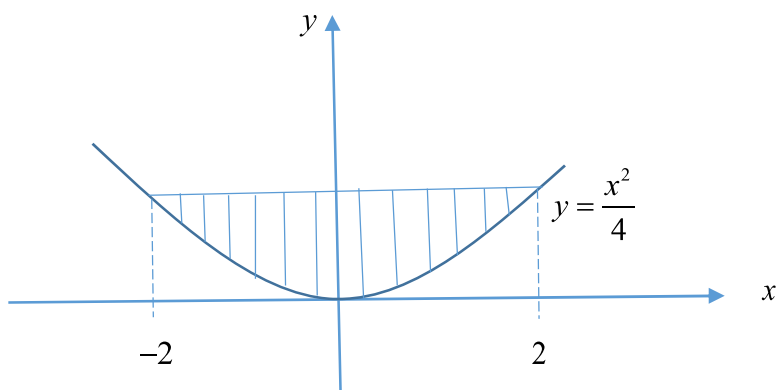
and the centroid  $\bar{P}(\bar{x}, \bar{y})$  of the region is such that

$$\bar{x} = \frac{M_y}{A} \text{ and } \bar{y} = \frac{M_x}{A}, \text{ where } A \text{ is the area of } \mathcal{R}.$$

**Example 3.2.6**

Find the centroid of the region bounded above by  $y = 1$  and below by  $y = \frac{x^2}{4}$ .

Solution



$$A = \int_{-2}^2 \left(1 - \frac{x^2}{4}\right) dx = 2 \int_0^2 \left(1 - \frac{x^2}{4}\right) dx = 2 \left(x - \frac{x^3}{12}\right) \Big|_0^2 = \frac{8}{3}$$

$$\therefore M_x = \int_0^1 y l(y) dy = \int_0^1 y (2\sqrt{y} + 2\sqrt{y}) dy = 4 \int_0^1 y^{\frac{3}{2}} dy = 4 \cdot \frac{2}{5} \cdot y^{\frac{5}{2}} \Big|_0^1 = \frac{8}{5}$$

$$M_y = \int_{-2}^2 x \left(1 - \frac{x^2}{4}\right) dx = \left(\frac{x^2}{2} - \frac{x^4}{16}\right) \Big|_{-2}^2 = 0.$$

$$\therefore \bar{x} = \frac{M_y}{A} = 0.$$

This is clear from the fact that the region is symmetric with respect to the y-axis.

$$\bar{y} = \frac{M_x}{A} = \frac{8}{5} \cdot \frac{3}{8} = \frac{3}{5}.$$

$$\therefore \bar{P}(\bar{x}, \bar{y}) = \bar{P}\left(0, \frac{3}{5}\right).$$



### Definition 3.2.4

For a moment of a particle  $ml$ , the second moment  $ml^2$  is referred to as the moment of inertia.

Thus, the moment of inertia (I) of the region  $\mathcal{R}$  about the x-axis and the y-axis is given by

$$I_x = \int_c^d y^2 l(y) dy$$

and

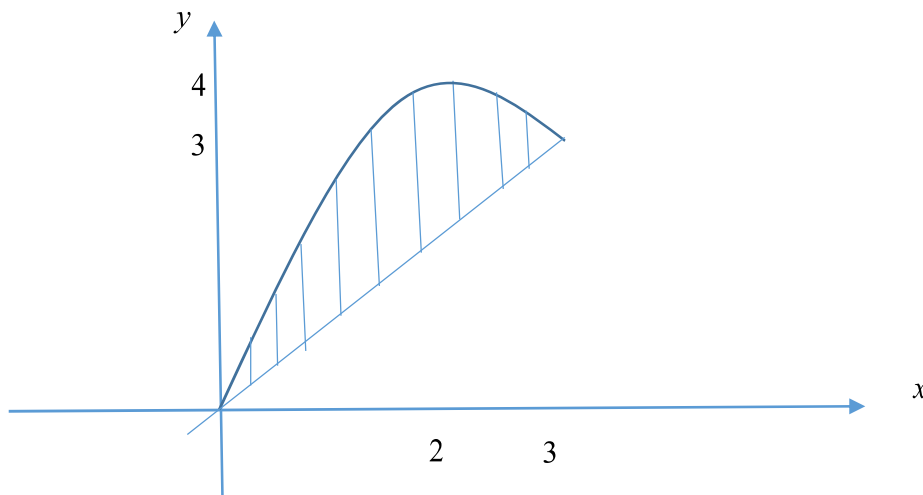
$$I_y = \int_a^b x^2 h(x) dx,$$

respectively.

### Example 3.2.7

Find the moment of inertia for the region bounded by the parabola  $y = 4x - x^2$  and the line  $y = x$ .

### Solution



$$\begin{aligned} I_x &= \int_0^3 y^2 (y - 2 + \sqrt{4 - y}) dy + \int_3^4 y^2 (2 + \sqrt{4 - y} - 2 + \sqrt{4 - y}) dy \\ &= \int_0^3 (y^3 - 2y^2 + y^2 \sqrt{4 - y}) dy + 2 \int_3^4 y^2 \sqrt{4 - y} dy \\ &= \left( \frac{y^4}{4} - \frac{2}{3} y^3 - \frac{32}{3} (4 - y)^{\frac{3}{2}} + \frac{32}{5} (4 - y)^{\frac{5}{2}} - \frac{2}{7} (4 - y)^{\frac{7}{2}} \right) \Big|_0^3 + 2 \left( -\frac{32}{3} (4 - y)^{\frac{3}{2}} + \frac{32}{5} (4 - y)^{\frac{5}{2}} - \frac{2}{7} (4 - y)^{\frac{7}{2}} \right) \Big|_3^4 \\ &= -\frac{30167}{420}. \end{aligned}$$

$$I_y = \int_0^3 x^2 h(x) dx = \int_0^3 x^2 (4x - x^2 - x) dx = \int_0^3 (3x^3 - x^4) dx = \left( \frac{3}{4} x^4 - \frac{1}{5} x^5 \right) \Big|_0^3 = \frac{243}{20}.$$

