MAT 2110: ENGINEERING MATHEMATICS I

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Contents

Chapter 1

Analytic geometry

In this chapter, we give some geometric definitions of parabolas, ellipses and hyperbolas and derive their standard equations. They are called conic sections or conics because they are formed from the intersection of the cone with a plane.

1.1 The circle

Let $C = (h, k)$ be a fixed point. Then the locus of points equidistant from C is called the circle. We call $C = (h, k)$ the center of the circle, and the distance from C to any point on the circumference is called the radius of the circle. This means that if $P = (x, y)$ is any point on the circumference of the circle, with center $C = (h, k)$, then $r = \sqrt{(x-h)^2 + (y-k)^2}$, so that $r^2 = (x-h)^2 + (y-k)^2$, is the equation of the circle.

Example 1.1.1. 1. Find the equation of a circle with center at $(2, -3)$, and passes through $(5, -1)$.

2. Find the center of the circle whose equation is $3x^2 + 3y^2 + 6x - 4y - 5 = 0$.

Definition 1.2.1. A parabola is the set of all points in a plane equidistant from a fixed line (D) called the directrix, and a fixed point (F) called the focus in the plane.

Let's derive the algebraic equation for a parabola. Without loss of generality, we can assume the focus is $F(0, p)$ on the positive y-axis and the directrix is the line $y = -p$ (without loss of generality just means that any other situation could be transformed into this case). From the definition of parabola, we must have for an arbitrary point $P(x, y)$ on the parabola:

$$\begin{aligned} \| PF \| &= \| PD \| \\ \text{distance to focus} &= \text{distance to directrix} \\ \sqrt{x^2 + (y-p)^2} &= \sqrt{(y+p)^2} \\ x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 &= 4yp. \end{aligned}$$

This reveals the parabola's symmetry about the y-axis. The point where the parabola crosses its axis is the **vertex**. For $x^2 = 4yp$, the vertex lies at the origin. The positive number p is the parabola's **focal length**.

The standard form for the equation of a parabola is $x^2 = 4yp$. If $p > 0$, the parabola opens upwards, and downward if $p < 0$. By interchanging x and y in $x^2 = 4yp$, we obtain $y^2 = 4xp$, with the parabola opening to the right if $p > 0$ or opening to the left if $p < 0$.

Example 1.2.2. 1. Find the focus and directrix for the parabola

$$x^2 = -12y.$$

2. Find the focus and directrix for the parabola

$$y^2 = 10x.$$

3. Write an equation for a parabola that opens to the left, with vertex $(0, 2)$ and passes through $(-6, -4)$. Hence sketch the graph.

1.3 Ellipses

Definition 1.3.1. An ellipse is the set of all points is the plane the sum of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are called **foci** (plural of focus).

The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and the ellipse cross are the ellipse's **vertices**.

Suppose the foci are on the $x-$axis, at $(c, 0)$ and $(-c, 0)$, so that the origin is halfway between the foci as in the figure below. Let the sum of the distances from the point $P = (x, y)$ on the ellipse to the foci be $2a > 0$. It follows that

$$||PF_1|| + ||PF_2|| = 2a$$
$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining $(a^2 - c^2)x^2 + y^2a^2 = a^4 - a^2c^2$, so that

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Since $||PF_1|| + ||PF_2||$ is greater than $||F_1F_2||$ by the triangle inequality, we have that $2a$ is greater than $2c$. Consequently, $a > c$ and so $a^2 - c^2 > 0$. Thus every point P whose coordinates satisfy an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \text{ with } 0 < c < a$$

also satisfies the equation

$$||PF_1|| + ||PF_2|| = 2a.$$

A point therefore lies on the ellipse if and only if its coordinates satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If $b^2 = a^2 - c^2$, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the equation of an ellipse symmetric with respect to the origin and both coordinate axes. It lies in the rectangle bounded by

$x = \pm a$ and $y = \pm b$.

1.4 Major and Minor axes of an ellipse

The major axis of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $c = \sqrt{a^2 - b^2}$, is the line segment of length $2a$ joining the points $(\pm a, 0)$. The minor axis of the ellipse is the line segment of length $2b$ joining the points $(0, \pm b)$. The number a itself is the semimajor axis, the number b is the semiminor axis. The number c is the **center-to-focus** distance of the ellipse.

In conclusion, the standard-form equation for ellipses centered at the origin is

1. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ with } a > b.$$
   (a) Foci on the x-axis
   (b) Center-to-focus distance: $c = \sqrt{a^2 - b^2}$
   (c) Foci: $(\pm c, 0)$
   (d) Vertices: $(\pm a, 0)$ for the major axis.
   (e) Vertices: $(0, \pm b)$ for the minor axis.

2. $$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \text{ with } a > b.$$
   (a) Foci on the y-axis
   (b) Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

(c) Foci: $(0, \pm c)$

(d) Vertices: $(0, \pm a)$ for the major axis.

(e) Vertices: $(\pm b, 0)$ for the minor axis.

Example 1.4.1. 1. Find the foci, vertices and sketch each of the following

(a) $9x^2 + 16y^2 = 144$.

(b) $\frac{x^2}{9} + \frac{y^2}{16} = 1$

2. Find the equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

1.5 Hyperbolas

Definition 1.5.1. A hyperbola is the set of all points in the plane the difference of whose distances from two fixed points (foci) F_1 and F_2 is a constant.

If we place the foci on the x-axis at $(c, 0)$ and $(-c, 0)$, as in the figure below, so that the origin is halfway between the foci and the constant difference $2a$, then from $|PF_1| - |PF_2| = \pm 2a$, we obtain that a point $P(x, y)$ lies on the hyperbola if and only if

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Since $a < c$ by the triangle inequality, it follows that $a^2 - c^2$ is negative. If we let $b = \sqrt{c^2 - a^2}$, then $a^2 - c^2 = -b^2$ so that $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$ becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

If the foci of the hyperbola are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging the x and y in the above equation. Thus, we have

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1,$$

with vertices at $(0, \pm a)$ and asymptotes $y = \pm \frac{a}{b} x$.

Like the ellipse, the hyperbola is symmetric with respect to the origin and the coordinate axes. The **focal axis**, the **foci**, the **center** and the **vertices** are defined as in an ellipse.

1.6 Asymptotes of hyperbolas-Graphing

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has two asymptotes, the lines $y = \pm \frac{b}{a} x$. They give us the guidance we need to graph hyperbolas quickly.

The standard-form equations for hyperbolas centered at the origin are:

1.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(a) Foci on the x-axis

(b) Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

(c) Foci: $(\pm c, 0)$

(d) Vertices: $(\pm a, 0)$.

(e) Asymptotes: $y = \pm \frac{b}{a} x$.

2.
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

(a) Foci on the y-axis

(b) Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

(c) Foci: $(0, \pm c)$

(d) Vertices: $(0, \pm a)$

(e) Asymptotes: $y = \pm \frac{a}{b} x$.

Example 1.6.1. 1. Find the foci, vertices and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch the graph.

2. Find the foci, vertices and asymptotes for the hyperbola $\frac{y^2}{4} - \frac{x^2}{5} = 1$.

3. Find the foci and the equations of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

1.7 Translation of axes

1.7.1 Introduction

In the last sections, we found equations for parabolas, ellipses, and hyperbolas located with their axes on the coordinate axes and centered at the origin. What happens if we move conics away from the origin while keeping their axes parallel to the coordinate axes?

1.7.2 Translation of axes

A translation of coordinate axes occurs when the new coordinate axes have the same direction as and are parallel to the original coordinate axes. A point P in the plane has two sets of coordinates: (x, y) in the original system and (x', y') in the translated system. If the coordinates of the origin of the translated system are (h, k) relative to the original system, then the old and new coordinates are related as follows

1.
$$\begin{aligned} x &= x' + h \\ y &= y' + k \end{aligned}$$

2.
$$\begin{aligned} x' &= x - h \\ y' &= y - k \end{aligned}$$

It can be shown that these formula hold for (h, k) located anywhere in the original coordinate system.

1.7.3 Standard Equations of Translated Conics

We now proceed to find standard equations of conics translated away from the origin. We do this by first writing the standard equations found earlier sections in the $x'y'$ coordinate system with $0'$ at (h, k). We then use translation equations to find the standard forms relative to the original xy coordinate system. The equations of translation in all cases are

$$x' = x - h$$
$$y' = y - k$$

For parabolas, we have

$$x'^2 = 4py' \qquad (x-h)^2 = 4p(y-k)$$
$$y'^2 = 4px' \qquad (y-k)^2 = 4p(x-h)$$

For ellipses, we have for $a > b > 0$

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \qquad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
$$\frac{x'^2}{b^2} + \frac{y'^2}{a^2} = 1 \qquad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

For hyperbolas, we have

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1 \qquad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
$$\frac{y'^2}{a^2} + \frac{x'^2}{b^2} = 1 \qquad \frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

1.8 Sketching Equations of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$

It can be shown that the graph of

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where A and C are not both zero, is a conic or a degenerate conic or that there is no graph. If we can transform the equation above into one of the standard equation listed above (previous section), then we will be able to identify its graph and sketch it as before. Here the knowledge of completing the square will be vital. The following examples will make the process clear.

Example 1.8.1. Transform each of the following equations into one of the standard forms in the previous section. Identify the conic and graph it.

1. $y^2 - 6y - 4x + 1 = 0.$
2. $9x^2 + 16y^2 + 36x - 32y - 92 = 0$
3. $9x^2 - 4y^2 - 36x - 24y - 35 = 0$

1.9 Rotation of axes

Here we show that the general second-equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be analyzed by rotating the axes so as to eliminate the term Bxy. A point P has coordinates (x, y) in the first system and (X, Y) in the new and rotated coordinate system. We will use the following equations to change from coordinate system to the other.

$$x = X\cos\theta - Y\sin\theta \qquad y = X\sin\theta + Y\cos\theta \tag{1.1}$$

and

$$X = x\cos\theta + y\sin\theta \qquad Y = -x\sin\theta + y\cos\theta$$

Example 1.9.1. 1. If the axes are rotated through $60°$, find the XY-coordinates of the point whose xy-coordinates are $(2, 6)$.

2. If the axes are rotated through $45°$, find the xy-coordinates of the point whose xy-coordinates are $(\sqrt{2}, \sqrt{2})$.

We now determine the angle θ such that the term Bxy in equation 1 disappears when the axes are rotated through the angle θ. If we substitute (1.1) in

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

expanding and collecting like terms we get

$$A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F = 0,$$

where

$$B' = 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) = (C - A)\sin 2\theta + B\cos 2\theta.$$

To eliminate the XY term we choose θ so that $B' = 0$, that is,

$$\cot 2\theta = \frac{A - C}{B}.$$

Example 1.9.2. 1. Show that the graph of $xy = 1$ is a hyperbola.

2. Identify and sketch the curve

$$73x^2 + 72xy + 52y^2 + 30x - 40y - 75 = 0.$$

1.9.3 The Discriminant Test

We can identify the conic section represented by the quadratic curve

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1.2}$$

by using the discriminant $B^2 - 4AC$. Equation (1.2) is

1. a parabola if $B^2 - 4AC = 0$,
2. an ellipse if $B^2 - 4AC < 0$,
3. a hyperbola if $B^2 - 4AC > 0$.

Example 1.9.4. Use the Discriminant to identify each of the following conic sections.

1. $3x^2 - 6xy + 3y^2 + 2x - 7 = 0$
2. $x^2 + xy + y^2 - 1 = 0$
3. $xy - y^2 - 5y + 1 = 0$.

1.10 Classifying Conic Sections by Eccentricity

The **eccentricity** is a number associated with each conic section and reveals the conic section's type.

Definition 1.10.1. The **eccentricity** of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for $a > b$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

We observe that if $a = b$, then $e = 0$, and so the circle has eccentricity 0. Hence $e \in [0, 1)$ for the ellipse. The smaller the eccentricity of an ellipse, the rounder it is and the larger the eccentricity of an ellipse, the more squashed it is.

Definition 1.10.2. The **eccentricity** of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

The larger the eccentricity of the hyperbola, the more squashed it is as well.

For the ellipse and the hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices. For the parabola, the eccentricity is given by $e = \frac{PD}{PF}$, so that $e = 1$.

Example 1.10.3.

1. Locate the vertices of an ellipse of eccentricity 0.8 whose foci lie at the points $(0, \pm 7)$.
2. Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.
3. Find a Cartesian equation for the hyperbola centered at the origin that has focus at $(3, 0)$ and the line $x = 1$ as the corresponding directrix.

1.11 Polar Coordinates

In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates.

Definition 1.11.1. To define polar coordinates, we first fix an origin 0 (called the pole) and an initial ray from 0, see the figure below. Then each point P can be located by assigning to it a polar coordinate pair (r, θ) in which r gives the directed distance from 0 to P and θ gives the directed angle from the initial ray to ray $0P$.

As in trigonometry, θ is positive when measured counter clockwise and negative when measured clockwise. The angle associated with a given point is not unique.

For example, the point two units from the origin along the ray $\theta = \frac{\pi}{6}$ has polar coordinate $r = 2$, $\theta = \frac{\pi}{6}$. It also has coordinates $r = 2$, $\theta = -\frac{11\pi}{6}$.

1.11.2 Negative values of r

There are occasions when we wish to allow r to be negative. That is why we use directed distance in $P = (r, \theta)$. The point $P = (2, \frac{7\pi}{6})$ can be reached by turning $\frac{7\pi}{6}$ counter clockwise from the initial ray and going forward 2 units. It can also be reached by turning $\frac{\pi}{6}$ counter clockwise from the initial ray and going backward 2 units. So the point also has polar coordinates $r = -2$, $\theta = \frac{\pi}{6}$.

Example 1.11.3. 1. Plot the points whose polar coordinates are given.

(a) $(1, \frac{5\pi}{4})$

(b) $(2, 3\pi)$

(c) $(2, \frac{-2\pi}{3})$

(d) $(-3, \frac{3\pi}{4})$.

2. Find all the polar coordinates of the polar $P = (2, \frac{\pi}{6})$.

1.11.4 Cartesian versus Polar coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The two coordinate systems are then related by the following equations.

$$x = r\cos\theta, \;\; y = r\sin\theta, \;\; x^2 + y^2 = r^2, \;\; \frac{y}{x} = \tan\theta.$$

We use these equations to rewrite polar equations in Cartesian form and vise versa.

Example 1.11.5. 1. Find polar coordinates (r, θ) of the point, where $r < \theta$ and $0 \leq \theta \leq 2\pi$.

(a) $(1, -1)$

(b) $(-1, \sqrt{3})$

(c) $(3\sqrt{3}, 3)$

2. Find a polar equation for the circle $x^2 + (y-3)^2 = 9$.

3. (a) Sketch the curve with polar equation

$$r = 2\cos\theta$$

(b) Find the Cartesian equation of this curve. Hence sketch.

4. Sketch the curve

$$r = 1 + \sin\theta.$$

1.11.6 Polar equations for Conic sections

1. Circles

   To find a polar equation for the circle of radius a centered at $P_0 = (r_0, \theta_0)$, we let $P = (r, \theta)$ be a point on the circle and apply the law of cosines to triangle $0P_0P$. This gives

   $$a^2 = r_0^2 + r^2 - 2r_0 r\cos(\theta_0 - \theta).$$

   If the circle passes through the origin, then $r_0 = a$, and $a^2 = r_0^2 + r^2 - 2r_0 r\cos(\theta_0 - \theta)$ simplifies to

   $$r = 2a\cos(\theta_0 - \theta).$$

   If the circle's center lies on the positive x-axis, $\theta_0 = 0$ and $r = 2a\cos(\theta_0 - \theta)$ becomes

   $$r = 2a\cos\theta.$$

   If the center lies on the positive y-axis, $\theta = \frac{\pi}{2}$, $\cos(\theta - \frac{\pi}{2}) = \sin\theta$, and $r = 2a\cos(\theta_0 - \theta)$ becomes

   $$r = 2a\sin\theta.$$

   Equations for circle through the origin centered on the negative x- and y- axes can be obtained from $r = 2a\cos\theta$ and $r = 2a\sin\theta$ by replacing r by $-r$.

2. Ellipses, Parabolas and Hyperbolas

   To find polar equations for ellipses, Parabolas and Hyperbolas, we place one focus at the origin and the corresponding directrix to the right of the origin along the vertices line $x = k$ as shown below. This makes $PF = r$ and $PD = k - FB = k - r\cos\theta$. The conic's focus-directrix equation $PF = e \cdot PD$ becomes

$$r = e(k - r\cos\theta),$$

so that

$$r = \frac{ek}{1 + e\cos\theta}.$$

The equation

$$r = \frac{ek}{1 + e\cos\theta}$$

represents an ellipse if $0 < e < 1$, a Parabola if $e = 1$ and a Hyperbola if $e > 1$. If the directrix is the line $x = -k$ to the left of the origin (the origin still the focus), we get

$$r = \frac{ek}{1 - e\cos\theta}$$

Example 1.11.7. (a) Find the equation for the Hyperbola with eccentricity $\frac{3}{2}$ and directrix $x = 2$.

(b) Find the directrix of the Parabola,

$$r = \frac{25}{10 + 10\cos\theta}.$$

In a similar way, it can easily be shown that if the directrix is $y = k$, then we get

$$r = \frac{ek}{1 + e\sin\theta},$$

and

$$r = \frac{ek}{1 - e\sin\theta}$$

if the directrix is $y = -k$.

Example 1.11.8. 1. Find the polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

2. Find the eccentricity, identify the conic, give an equation of the directrix and hence sketch the conic.

   (a) $r = \frac{12}{2+4\sin\theta}$.

   (b) $r = \frac{10}{3-2\sin\theta}$.

Chapter 2

Differential Calculus

2.1 Introduction

When a function $y = f(x)$ is differentiated with respect to x, the differential coefficient is written as $\frac{dy}{dx}$ or $f'(x)$. From differentiation by first principle of a number of examples we considered in first year, a general rule for differentiating $y = ax^n$ emerged, where a and n are constants. The rule is: if $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$. By using this rule, it is easy to check that the derivative of any constant is zero. In order to differentiate polynomials we recall that the derivative of a sum (difference) of functions is the sum (difference) of their derivatives. Thus, if $f(x) = p(x) + q(x) - r(x)$, (where p, q and r) are functions, then $f'(x) = p'(x) + q'(x) - r'(x)$.

Example 2.1.1. Find $f'(x)$ in each of the following:

1. $f(x) = 3x^2 + 1$
2. $f(x) = 2 - 4x^2 + \frac{2}{x} + x^{\frac{3}{2}}$

If the expression $\frac{dy}{dx}$ or $f'(x)$ is differentiated again, the second differential coefficient is obtained and is written as $\frac{d^2y}{dx^2}$ (pronounced dee two y by dee x squared) or $f''(x)$ (pronounced f double prime x).

2.2 Methods of differentiation

In order to differentiate more complicated functions than the ones we have considered so far, such as products, quotients and composite functions, we will require the following results.

1. When $y = uv$, and $u = u(x)$ and $v = v(x)$ are both functions of x, then
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$
This is known as the **product rule**.

   Find y'' and y''' in each of the following:

   (a) $y = 3x^2(x^{-2} + \sqrt{x})$

   (b) $y = x^2(\frac{1}{x} + x)$

2. When $y = \frac{u}{v}$, and $u = u(x)$ and $v = v(x)$ are both functions of x, then
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$
This is known as the **quotient rule**. Find y'' and y''' in each of the following:

   (a) $y = \frac{3x^2}{x+1}$

   (b) $y = \frac{x-1}{x^2-1}$

3. It is often easier to make a substitution before differentiating. If y is a function of x, we may write $y = u$, where u is the

function of x, so that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This is known as the ‘function of a function’ rule (or sometimes the **chain rule**). Differentiate the following:

(a) $y = (x^2 - 4x + 5)^8$

(b) $y = (x^2 + 1)^2$

Example 2.2.1. Find the second derivative in each of the following:

1. $y = (x^2 + 1)^2$
2. $y = (x^2 - 4x + 5)^8$

Example 2.2.2.
1. Find all derivatives $y^{(n)}$ for the function $y = \pi x^3 - 7x$.
2. Find all derivatives $y^{(n)}$ for the function $y = \sqrt{x + 5}$.
3. Find all derivatives $y^{(n)}$ for the function $y = \frac{1}{3+x}$.

2.3 Trigonometric functions

By using the first principle, it can be shown that if $y = \sin x$, then $\frac{dy}{dx} = \cos x$, and if $y = \cos x$, then $\frac{dy}{dx} = -\sin x$. We use these two differential coefficients to differentiate other trigonometric functions such as $\sec x$ and $\tan x$ among others. In general, if $y = \sin(f(x))$, then $\frac{dy}{dx} = f'(x)\cos(f(x))$, and if $y = \cos(f(x))$, then $\frac{dy}{dx} = -f'(x)\sin(f(x))$.

Example 2.3.1. Find $f''(x)$ in each of the following:

1. $f(x) = \cos(x)$

2. $f(x) = \sin(4x)$.

2.3.2 Exponential Functions and Logarithmic Functions

It can be shown that if $y = e^{f(x)}$ and $y = \ln(f(x))$, then $y' = f'(x)e^{f(x)}$ and $y' = \frac{f'(x)}{f(x)}$, respectively. For a different base other than e, it was shown in MAT 1100 that if $y = a^{f(x)}$ for some constant $a > 0$, then $y' = f'(x)a^{f(x)}\ln a$, and if $y = \log_b(f(x))$, then $y' = \frac{f'(x)}{(f(x))\ln b}$.

Example 2.3.3. Find $f''(x)$ in each of the following:

1. $y = e^x$

2. $y = 6^{x^2}$

3. $y = \ln(x^2 + 2x - 9)$

4. $y = \log_6(x^2 + 2x - 9)$

5. $y = \ln(e^x)$

2.4 Implicit Functions

When an equation can written in the form $y = f(x)$, it is said to be an explicit function of x. For example

1. $y = 2x^3 - 3x + 4$

2. $y = 2x \ln x$ and

3. $y = \frac{3e^x}{\cos x}$ are explicit functions.

In these examples y may be differentiated with respect to x by using standard derivatives, the product rule and quotient rule of differentiation.

Sometimes, it is impossible to make y the subject of the formula. Such equations are called implicit functions and examples include:

1. $y^3 + 2x^2 = y^2 - x$ and
2. $y = x^2 + 2xy$

2.4.1 Differentiation of implicit functions

It is possible to differentiate an implicit function by using the function of a function rule, which we looked at earlier.

Thus, to differentiate y^3 wrt x, the substitution $u = y^3$ is made, so that $\frac{du}{dy} = 3y^2$. Hence

$$\frac{d}{dx}(y^3) = 3y^2 \times \frac{dy}{dx}.$$

In summary,

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}.$$

When differentiating implicit functions containing products and quotients, the product and quotient rules of differentiation must be applied.

Example 2.4.2. 1. Find $\frac{dy}{dx}$ if $3x^2 + y^2 - 5x + y = 2$.

2. Given that $2y^2 - 5x^4 - 2 - 7y^3 = 0$, determine $\frac{dy}{dx}$.

3. If $4x^2 + 2xy^3 - 5y^2 = 0$, find $\frac{dy}{dx}$ at $(1, 2)$.

2.5 Differentiation of Hyperbolic functions

2.5.1 Introduction

Functions which are associated with the geometry of the conic section called a hyperbola are called hyperbolic functions. By definition:

1. Hyperbolic sine of x, $\sinh x = \frac{e^x - e^{-x}}{2}$, pronounced as "shine x" .

2. Hyperbolic cosine of x, $\cosh x = \frac{e^x + e^{-x}}{2}$, pronounced as "kosh x" .

3. Hyperbolic tangent of x, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, pronounced as "than x" .

4. Hyperbolic cotangent of x, $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, pronounced as "koth x " .

In order to differentiate functions involving hyperbolic functions, we recall that if $y = \sinh x = \frac{e^x - e^{-x}}{2}$, then $\frac{dy}{dx} = \cosh x$, and if $y = \cosh x = \frac{e^x + e^{-x}}{2}$, then $\frac{dy}{dx} = \sinh x$.

Thus, using the rules of differentiation, we can easily find the derivatives of other functions such as $\tanh x$.

Example 2.5.2. Find $\frac{dy}{dx}$ if

1. $y = 4\sinh 2x - \frac{3}{7}\cosh 3x$

2. $y = 4\sinh 3x \cosh 4x$.

3. $y = \frac{3}{4}\ln(\tanh(\frac{x}{2}))$.

If $y = 3x-2$, then by transposition, $x = \frac{y+2}{3}$. The function $x = \frac{y+2}{3}$ is called the inverse function of $y = 3x - 2$.

Inverse trigonometric functions are denoted by prefixing the function with 'arc' or, more commonly, by using the $^{-1}$ notation. For the purpose of this course, we will use $^{-1}$. Thus if $y = \sin x$, then $x = \sin^{-1} y$ and if $y = \tan x$, then $x = \tan^{-1} y$.

Similarly, inverse hyperbolic functions are denoted by prefixing the function with 'ar' or, more commonly, by using the $^{-1}$ notation. Again for the purpose of this course, we will use $^{-1}$. Thus if $y = \sinh x$, then $x = \sinh^{-1} y$ and if $y = \tanh x$, then $x = \tanh^{-1} y$.

2.6.1 Inverse trigonometric functions

If $y = \sin^{-1} x$, then $x = \sin y$. Differentiating both sides with respect to y gives $\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y}$. Consequently $\frac{dx}{dy} = \sqrt{1 - x^2}$, so that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

Similarly, if $y = \sin^{-1}(f(x))$, the function of a function rule may be used to show that

$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1 - (f(x))^2}}.$$

Example 2.6.2. 1. Show that if $y = \cos^{-1}(f(x))$, then

$$\frac{dy}{dx} = \frac{-f'(x)}{\sqrt{1 - (f(x))^2}}.$$

2. Given that $y = \sin^{-1} 5x^2$, find $\frac{dy}{dx}$.

3. Given that $y = \cos^{-1} 2x$, find $\frac{dy}{dx}$.

Using the derivatives for $\cos^{-1}$ and $\sin^{-1}$, we can obtain derivatives involving other inverse trigonometric functions such as $\tan^{-1}$, $\sec^{-1}$ and $\csc^{-1}$.

2.6.3 Hyperbolic functions

Using the same arguments as for inverse trigonometric functions, it can be shown easily that

1. If $y = \sinh^{-1}(f(x))$, then
$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{(f(x))^2 + 1}}.$$

2. If $y = \cosh^{-1}(f(x))$, then
$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{(f(x))^2 - 1}}.$$

3. If $y = \tanh^{-1}(f(x))$, then
$$\frac{dy}{dx} = \frac{f'(x)}{1 - (f(x))^2}.$$

Using the derivatives for $\cosh^{-1}$ and $\sinh^{-1}$, we can obtain derivatives involving other inverse hyperbolic functions such as $\coth^{-1}$.

Example 2.6.4. 1. Given that $y = \cosh^{-1}(\sqrt{x^2 + 1})$, find $\frac{dy}{dx}$.

2. Find $\frac{dy}{dx}$ if $y = \sinh^{-1}(2x - 1)$.

In this section, we present some applications of differentiation in our day to day life.

2.7.1 Rates of change

If a quantity y depends on and varies with a quantity x, then the rate of change of y with respect to x is $\frac{dy}{dx}$. Thus, for example, the rate of change of pressure P with height h is $\frac{dP}{dh}$. If the rate of change is with respect to time, we usually call it 'the rate of change', the 'with respect to time', being assumed. Therefore, a rate of change of current, i, is $\frac{di}{dt}$ and the rate of change of temperature, θ, is $\frac{d\theta}{dt}$, and so on.

Example 2.7.2. The luminous intensity I candelas of a lamp at varying voltage V is given by $I = 4 \times 10^{-4}V^2$. Determine the voltage at which the light is increasing at a rate of 0.6 candelas per volt.

Example 2.7.3. Newtons law of cooling is given by $T = \theta_\circ e^{-kt}$, where the excess of temperature at zero is $\theta_{\circ,}$ and at time t seconds is $\theta°C$. Determine the rate of change of temperature after 40s, given that $\theta_\circ = 16°C$, and $k = -0.03$

2.7.4 Turning points

The gradient (or rate of change) of the curve may be positive or negative at a given interval or seen to be zero at a point, say P. Thus, if at P the gradient is zero, and, as x increases, the gradient of the curve changes from positive just before P

to negative just after, then P is called the **maximum point**, and appears as the 'crest of a wave'. On the other hand, if at Q the gradient is zero, and, as x increases, the gradient of the curve changes from negative just before Q to positive just after, then Q is called the **minimum point**, and appears as the 'bottom of a valley'. Points such as P and Q are given a general name of **turning points.**

It is possible to have a turning point, the gradient on either side of which is the same. Such a turning point is given the special name **inflexion point**. Maximum and minimum points and points of inflexion are given the general term of **stationary points**.

Procedure for finding and distinguishing between stationary points.

1. Given a function $y = f(x)$, find $\frac{dy}{dx}$.
2. Let $\frac{dy}{dx} = 0$ and solve for x.
3. Substitute the value(s) of x found in (2), in the original equation $y = f(x)$, to get the corresponding y value(s).
4. To determine the nature of the stationary points. Either
   (a) **Find** $\frac{d^2y}{dx^2}$ and substitute into it the value(s) of x found in (2). If the result is:
      i. positive, the point is a minimum on,
      ii. negative, then the point is a maximum one,
      iii. zero, then the point is a point of inflexion.

   (b) **Use the sign of the gradient** of the curve just before and just after the stationary point.

Example 2.7.5. Locate the turning point on the curve $y = 3x^2 - 6x$ and determine its nature by examining the sign on either side.

Example 2.7.6. Locate the turning point of the following curve and determine whether it is maximum or minimum point, $y = 4\theta + e^{-\theta}$.

Example 2.7.7. Find the maximum and minimum values of the curve $y = x^3 - 3x + 5$.

Example 2.7.8. Determine the coordinates of the maximum and minimum points of the graph $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$ and distinguish them.

Example 2.7.9.

1. If triangle ABC is isosceles with $AB = BC = 20cm$ and $BC = 24cm$. A rectangle $PQRS$ is drawn inside the triangle with PQ on BC, and S and R on AB and AC respectively.

   (a) If $PQ = 2x\,cm$, show that the area $A\,cm^2$ of the rectangle is given by

   $$A = \frac{8x(12 - x)}{3}.$$

   (b) Hence, find the value of x for which A is a maximum.

2. The length of a closed rectangular box is 3 times its width. If the its volume is $972\,cm^3$, find the dimensions of the box if the surface area is to be a minimum.

2.7.10 Tangents and Normals

2.7.11 Tangents

The equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) is given by:

$$(y - y_1) = m(x - x_1),$$

where $m = \frac{dy}{dx}$ =gradient of the curve at (x_1, y_1).

Example 2.7.12. 1. Find the equation of the tangent to the curve $y = x^2 - x - 2$ at the point $(1, -2)$.

2. Find the equations of the tangent lines to the ellipse $9x^2 + 16y^2 = 52$ that are parallel to the line $9x - 8y = 1$.

2.7.13 Normals

The normal at any point on a curve is the line which passes through the point and is at right angles to the tangent. It may be shown that if two lines are at right angles, then the product of their gradients is -1. Thus if m is the gradient of the tangent, then the gradient of the normal is $\frac{-1}{m}$. Hence the equation of the normal at the point (x_1, y_1) is given by:

$$(y - y_1) = \frac{-1}{m}(x - x_1).$$

Example 2.7.14. 1. Find the equation of the normal to the curve $y = x^2 - x - 2$ at the point $(1, -2)$.

2. Find the equation of the normal to the curve $x^2+2xy-3y^2 = 9$ at the point.

2.7.15 Rolle's Theorem

Theorem 2.7.16. *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there exists a number c in (a, b) such that $f'(c) = 0$.*

Example 2.7.17. Find the two x-intercepts of $f(x) = x^2 - 3x + 2$ and show that $f'(c) = 0$ at some point between the two x-intercepts.

2.7.18 The Mean Value Theorem

Theorem 2.7.19. *If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof. The equation for the secant line that passes through the points $(a, f(a)$ and $(b, f(b)$ is

$$y = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a).$$

Let $g(x)$ be the difference between $f(x)$ and y. Then

$$\begin{aligned} g(x) &= f(x) - y \\ &= f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a). \end{aligned}$$

It follows that $g(a) = 0 = g(b)$. Because f is continuous on $[a, b]$, g is also continuous on $[a, b]$. Furthermore, because f is differentiable on (a, b), g is also differentiable on (a, b). By Rolle's Theorem, there exists a number c in (a, b) such that $g'(c) = 0$.

This implies that

$$\begin{aligned} 0 &= g'(c) \\ &= f'(c) - \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

Consequently,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Example 2.7.20. Given that $f(x) = 5 - \frac{4}{x}$, find all the values of c in the open interval $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

2.8 limits

The limit of a function $f(x)$ as x tends to a number say a is defined as the value L of $f(x)$ as x approaches closer and closer to a without actually reaching it, and is denoted by

$$\lim_{x \to a} f(x) = L.$$

Example 2.8.1. Evaluate the limits

1. $\lim_{x \to 2} 3x^2 + x - 5 = 9$
2. $\lim_{x \to 1} \frac{x^2 - 2x + 7}{x + 2} = 2$

Some points to note:

1. We do not evaluate the limit by actually substituting $x = a$ in $f(x)$ in general, although in some simple cases it is possible.

2. The value of the limit can depend on which side it is approached, from above or below, i.e through the values of x less than a or through the values of x greater than a respectively. The two possible values may not be the same in which case the limit does not exist.

3. The limit may not exist at all and even if it does it may not be equal to $f(a)$.

The kind of limits that cause most difficult and which are probably the most important are those arising from so called indeterminate forms. Any expression that yields results of the form $a/0\,0/0$, ∞/∞ or $0 \times \infty$, 0^0, 0^∞, ∞^0 is called an indeterminate form. These include $\lim_{x\to 1}\frac{x^2-1}{x-1} = \frac{0}{0}$ and $\lim_{x\to 0}\frac{\sin x}{x} = \frac{0}{0}$.

Even though the function does not exist at such points, its limit at the point may exist. For example, $\lim_{x\to 1}\frac{x^2-1}{x-1}$ and $\lim_{x\to 0}\frac{\sin x}{x}$.

Theorem 2.8.2. *L'hopital's Rule*

Let $f(x)$ *and* $g(x)$ *be functions so that* $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$. *Then,*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

This will help us compute many limits of fractions. Sometimes we might have to take several derivatives to get to a limit we can evaluate.

Example 2.8.3. Evaluate each of the following:

1. $\lim_{x\to\infty}\frac{\sin^{-1}(x)}{e^x-1}$

2. $\lim_{x\to 0}\frac{\sqrt{x+1}-1-\frac{x}{2}}{x^2}$

3. $\lim_{x\to\infty} \frac{x}{2^x}$

Chapter 3

Integration

3.1 Introduction

We recall that the process of integration reverses the process of differentiation. In differentiation, if $y = f(x) = 2x^2$, then $y' = 4x$. Thus the integral of $4x$ is $2x^2$, i.e. integration is the process of moving from $f'(x)$ to $f(x)$.

3.2 Integrals of the form ax^n

The general solution of integrals of the form $\int ax^n$, where a and n are constants is given by

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c, \; n \neq -1.$$

We recall that the integral of the sum/difference of functions is the sum/difference of their integrals.

Example 3.2.1. Integrate each of the following:

1. $\int 3x^4 \, dx$
2. $\int \frac{2}{x^2} \, dx$
3. $\int \frac{2x^3 - 3x}{4x} \, dx$

3.3 Standard integrals

Since integration is the reverse process of differentiation, the standard integrals listed below may be deduced and readily checked by differentiation.

1. $\int ax^n\, dx = \frac{ax^{n+1}}{n+1} + c$, $n \neq -1$
2. $\int \frac{1}{x}\, dx = \ln x + c$
3. $\int \cos ax\, dx = \frac{1}{a} \sin ax + c$
4. $\int \sin ax\, dx = -\frac{1}{a} \cos ax + c$
5. $\int \sec^2 ax\, dx = \frac{1}{a} \tan ax + c$
6. $\int \csc^2 ax\, dx = -\frac{1}{a} \cot ax + c$
7. $\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax + c$
8. $\int \csc ax \cot ax\, dx = -\frac{1}{a} \csc ax + c$
9. $\int b^{ax}\, dx = \frac{1}{a \ln b} b^{ax} + c$
10. $\int e^{ax}\, dx = \frac{1}{a} e^{ax} + c$

Example 3.3.1. Determine

1. $\int 4 \cos 3x\, dx$
2. $\int 5 \sin 2\theta\, d\theta$
3. $\int 7 \sec^2 4t\, dt$
4. $\int 5^{3x}\, dx$

3.4 Definite integrals

Integrals containing an arbitrary constant c in their result are called *indefinite integrals* since the precise value cannot be determined without further information. *Definite integrals* are those in which limits are applied. If the expression is written in the form $x|_a^b$, then the operation of applying the limits is

$$x|_a^b = (b) - (a).$$

In general,

$$f(x)|_a^b = f(b) - f(a).$$

Example 3.4.1. Evaluate each of the following:

1. $\int_1^4 \frac{\theta+2}{\sqrt{\theta}}\, d\theta$
2. $\int_0^{\frac{\pi}{2}} 3\sin 2\theta\, d\theta$
3. $\int_0^{\frac{\pi}{6}} 3\sec^2 2x\, dx$
4. $\int_1^2 4e^{2x}\, dx$

3.5 Methods of integration

Functions which require integration are not always in the standard form listed in the previous section. However, it is often possible to change a function into a form which can be integrated by using either:

1. An algebraic substitution.
2. A trigonometric substitution or hyperbolic substitution.

3. Partial fractions.

4. The $t = \tan\frac{\theta}{2}$ substitution.

5. Integration by parts.

6. Reduction formula.

3.5.1 Algebraic substitution

With algebraic substitution, the substitution usually made is to let u be equal to $f(x)$ such that $f(u)\,du$ is a standard integral. It is known that integrals of the forms,

$$k\int [f(x)]^n f'(x)\,dx$$

and

$$k\int \frac{f'(x)}{[f(x)]^n}\,dx$$

where k and n are constants, can both be integrated by using substituting u for $f(x)$.

Example 3.5.2. Determine:

1. $\int \cos(3x+7)\,dx$

2. $\int (2x-5)^7\,dx$

3. $\int_0^{\frac{\pi}{6}} 24\sin^5\theta\cos\theta\,d\theta$.

4. $\int_0^1 2e^{6x-1}\,dx$.

3.5.3 Integration using trigonometric and hyperbolic substitutions

The following table gives a summary of the integrals that require the use of trigonometric and hyperbolic substitutions and their application is demonstrated in the following example. We will begin by looking at $\sin^2$, $\cos^2$, $\tan^2$ and $\cot^2$.

Example 3.5.4. Determine:

1. $\int \sin^2 3\theta \, d\theta$.
2. $\int 3\tan^2 2x \, dx$.
3. $\int_0^{\frac{\pi}{4}} 2\cos^2 4t \, dt$.
4. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta \, d\theta$.

In the next set of examples, we look at integrals involving powers of sines and cosines.

Example 3.5.5. Determine

1. $\int \sin^5 \theta \, d\theta$.
2. $\int \sin^2 t \cos^4 t \, dt$.
3. $\int_0^{\frac{\pi}{4}} 4\cos^4 \theta \, d\theta$.

In the next example, we look at some problems involving products of sines and cosines with different angles.

Example 3.5.6. Determine

1. $\int \sin 3t \cos 2t \, dt$.
2. $\int \frac{1}{3}\cos 5x \sin 2x \, dx$.
3. $\int 2\cos 6\theta \cos \theta \, d\theta$.
4. $\int 3\sin 5x \sin 3x \, dx$.

We now look at a set of examples which require a $\sin\theta$ of $\cos\theta$ substitution.

Example 3.5.7. Determine

1. $\int \frac{1}{a^2-x^2}\,dx$.
2. $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$.
3. $\int \sqrt{a^2-x^2}\,dx$.
4. $\int_0^4 \sqrt{16-x^2}\,dx$.

Next, we will look at some examples which require using $\tan\theta$ substitution.

Example 3.5.8. Determine

1. $\int \frac{1}{a^2+x^2}\,dx$.
2. $\int_0^2 \frac{1}{4+x^2}\,dx$.

Certain problems will require the use of $\cosh\theta$ substitution as is the case of the next example.

Example 3.5.9. Find

1. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$.
2. $\int \frac{2x-3}{\sqrt{x^2-9}}\,dx$.
3. $\int_2^3 \frac{1}{\sqrt{x^2-4}}\,dx$.

3.5.10 Integration using partial fractions

The process of expressing a fraction in terms of simpler fractions called *partial fractions* was introduced in MAT 1100. The following list summarizes the forms of partial fractions used

Denominator containing	Expression	Form of partial fraction
Linear factors	$\dfrac{f(x)}{(x+a)(x-b)(x+c)}$	$\dfrac{A}{x+a}+\dfrac{B}{x-b}+\dfrac{C}{x+c}$
Repeated linear factors	$\dfrac{f(x)}{(x+a)^3}$	$\dfrac{A}{x+a}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+a)^3}$
Quadratic factors	$\dfrac{f(x)}{(ax^2+bx+c)(x+d)}$	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{C}{x+d}$

Example 3.5.11. Find

1. $\int \frac{11-3x}{x^2+2x-3}\,dx$.
2. $\int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}\,dx$.
3. $\int \frac{x^3-2x^2-4x-4}{x^2+x-2}\,dx$.
4. $\int \frac{2x+3}{(x-2)^3}\,dx$.
5. $\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}\,dx$.

3.5.12 The $t=\tan\frac{\theta}{2}$ substitution

Integrals of the form

$$\int \frac{1}{a\cos\theta+b\sin\theta+c}\,d\theta$$

where a, b and c are constants, may be determined by using the substitution $t=\tan\frac{\theta}{2}$. By Pythagora's theorem, we get the triangle ABC so that

$$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$$

and

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}.$$

Consequently,

1. $$\sin\theta = \frac{2t}{1+t^2}$$

   and

2. $$\cos\theta = \frac{1-t^2}{1+t^2}.$$

3. $$d\theta = \frac{2\,dt}{1+t^2}$$

Let us now look at some examples of the form

$$\int \frac{1}{a\cos\theta + b\sin\theta + c}\,d\theta,$$

where a, b and c are constants.

Example 3.5.13. Find

1. $\int \frac{1}{\sin\theta}\,d\theta$.
2. $\int \frac{1}{\cos x}\,dx$.
3. $\int \frac{1}{1+\cos x}\,dx$.
4. $\int \frac{1}{5+4\cos\theta}\,d\theta$.

3.5.14 Integration by parts

From the product rule of differentiation

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx},$$

where both u and v are functions of x, we get

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}.$$

Integrating both sides with respect to x gives:

$$\int u\,dv = uv - \int v\,du.$$

This is called *integration by parts* formula and provides a method of integrating such products of simple functions as $\int xe^x\,dx$, $\int t\sin t\,dt$ $\int e^x \cos x\,dx$ and $\int x\ln x\,dx$.

Given a product of two terms to integrate, the initial choice is: which part to make equal to u and which part to make equal to dv. The choice must be such that the 'u part' becomes a constant after successive differentiation, and the 'dv part ' can be integrated from standard integrals.

The following rule holds: If a product to be integrated contains an algebraic term (such as x, t^2 or 3θ), then this term is chosen as the 'u part', except when $\ln x$ term is involved; in this case $\ln x$ is chosen as the 'u part'.

Example 3.5.15. Determine

1. $\int x\cos x\,dx$.
2. $\int 3te^{2t}\,dt$.

3. $\int x^2 \sin x \, dx$.

4. $\int x \ln x \, dx$.

5. $\int e^{ax} \cos bx \, dx$.

3.5.16 Reduction formulae

When integration by parts in the previous section, we noticed that the integral such as $\int x^2 e^x \, dx$ requires integration by parts twice. Thus, integrals such as $\int x^5 e^x \, dx$, $\int x^6 e^x \, dx$ and $\int x^8 e^x \, dx$ for instance would take a long time to determine using integration by parts. Reduction formulae provide a quicker method for determining such integrals. Let us look at the following example.

Example 3.5.17. Determine the reduction formula for

1. $\int x^n e^{ax} \, dx$.

2. $\int x^n \cos bx \, dx$.

3. Use the reduction formula to find $\int x^3 e^{2x} \, dx$ and $\int x^2 \cos 3x \, dx$.

In the next example, we obtain reduction formulae for integral of the form $\int sin^n x \, dx$ and $\int cos^n x \, dx$.

Example 3.5.18. Determine the reduction formula for

1. $\int sin^n x \, dx$ and $\int cos^n x \, dx$

2. Use the reduction formula to find $\int sin^4 x \, dx$ and $\int cos^5 x \, dx$.

We now look at how to use reduction formula for integrals of the form $\int tan^n x \, dx$ and $\int (\ln x)^n \, dx$.

Example 3.5.19. Determine the reduction formula for

1. $\int tan^n x\, dx$ and $\int (\ln x)^n\, dx$.

2. Hence, use the reduction formula to find $\int tan^7 x\, dx$ and $\int (\ln x)^3\, dx$.

3.6 Application of integration

Integration has several applications and this section looks at some of them.

3.6.1 Area of the region between two curves

Consider two functions f and g that are continuous on the interval $[a, b]$. If, as in the figure below, the graphs of both f and g lie above the x-axis, and that the graph of g lies below the graph of f, you can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f.

Thus, if f and g are continuous functions of $[a, b]$ and $g(x) \leq f(x)$ for all $x \in [a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Similarly, if f and g are continuous functions of $[c, d]$ and $g(y) \leq f(y)$ for all $y \in [c, d]$, then the area of the region bounded by the graphs of f and g and the horizontal lines $y = c$ and $y = d$ is

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

Example 3.6.2. 1. Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$ and $x = 1$.

2. Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$, and $g(x) = x$.

3. Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

3.6.3 Volume of solids of revolution

Another important application of integration is its use in finding the volume of a 3 dimensional solid.

1. The disk method

   If a region in the plane is revolved about a line, the resulting solid is a *solid of revolution*. The simplest such solid is a right circular cylinder or disk, which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle. The volume of such a disk is given by

   $$V = (\text{ area of disk})(\text{width of disk}) = \pi R^2 \omega,$$

   where R is the radius of the disk and ω is its width.

   To find the volume of a solid of revolution with the disk method, use the following formula

   $$V = \pi \int_a^b [R(x)]^2 \, dx.$$

   Example 3.6.4. (a) Find the volume of the solid formed by revolving the region bounded by the graph of $y = \sqrt{\sin x}$

and the x-axis ($0 \leq x \leq \pi$).

(b) Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = 2 - x^2$ and $g(x) = 1$, and the line $y = 1$.

2. The washer method

The disk method can be extended to cover solids or revolution with holes by representative disk with a representative *washer*. The washer is formed by revolving the rectangle about an axis. If r and R are the inner and outer radii of the washer and ω is the width of the washer, the volume is given by

$$V = \pi(R^2 - r^2)\omega.$$

To see how this concept can be used to find the volume of a solid of revolution, consider a region bounded by an outer radius $R(x)$ and and inner radius $r(x)$. If the region is revolved about its axis of revolution, the volume of the resulting solid is given by

$$v = \pi \int_a^b [(R(x))^2 - (r(x))^2]\, dx.$$

Note:

The integral involving the inner radius represents the volume of the hole and is subtracted from the integral involving the outer radius.

Example 3.6.5. (a) Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$

and $y = x^2$ about the x-axis.

(b) Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$ and $y = 1$ about the y-axis.

3. The Shell method

To find the volume of a solid of revolution with the shell method, we use one of the following:

(a)

$$V = 2\pi \int_c^d p(y)h(y)\,dy, \qquad \text{horizontal axis of revolution.}$$

(b)

$$V = 2\pi \int_a^b p(x)h(x)\,dx, \qquad \text{vertical axis of revolution.}$$

Example 3.6.6. (a) Find the volume of the solid formed by revolving the region bounded by $y = x - x^3$ and the x-axis ($0 \le x \le 1$) about the y-axis.

(b) Find the volume of the solid formed by revolving the region bounded by $x = e^{-y^2}$ and the y-axis ($0 \le y \le 1$) about the x-axis.

3.6.7 Arc length

In this section, definite integrals are used to find the lengths of curves. In either case, an arc (a segment of a curve) is approximated by straight line segments whose lengths are given by the usual distance formula. However, if f is a continuous function on a closed interval, we use the following definition.

Definition 3.6.8. Let the function given by $y = f(x)$ represent a smooth curve (continuously differentiable) on the interval $[a, b]$. The arc length of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Similarly, for the smooth curve given by $x = g(y)$ on the interval $[c, d]$. The arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example 3.6.9. 1. Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $[\frac{1}{2}, 2]$.

2. Find the arc length of the graph of $(y-1)^3 = x^2$ on the interval $[0, 8]$.

3.7 Area of a surface of revolution

Definition 3.7.1. If the graph of a continuous function is revolved about a line, the resulting surface is a *surface of revolution*.

Let $y = f(x)$ have a continuous derivative on an interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is:

1. $$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx$$

y is a function of x and $r(x)$ is the distance between the graph of f and the axis of revolution.

2.
$$S = 2\pi \int_a^b r(y)\sqrt{1 + [g'(y)]^2}\, dy,$$
when $x = g(y)$ is a smooth function defined $[c, d]$ and $r(y)$ is the distance between the graph of g and the axis of revolution.

Example 3.7.2. 1. Find the area of the surface formed by revolving the graph of $y = x^3$ on the interval $[0, 1]$ about the x-axis.

2. Find the area of the surface formed by revolving the graph of $y = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis.

3.8 Moments and center of mass: one dimensional system

In this section, we will look at some important applications of integration that are related to mass. We recall that mass is a body's resistance to changes in motion, and is independent of the particular gravitational system in which the body is located.

We will consider two types of moments of a mass:

1. The moments about a point and
2. The moment about a line.

To define these two moments, consider an idealized situation in which a mass m is concentrated at a point. If x is the distance between this point mass and another point p, the moment of m

about the point p is

$$\text{Moment } = mx,$$

and x is the length of the moment arm.

The measure of the tendency of the system to rotate about the origin is the moment about the origin, and it is defined as the sum of the n products $m_i x_i$

$$M_0 = m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n.$$

If $m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n = 0$, the system is said to be in equilibrium.

For a system that is not equilibrium, the center of mass is defined as the point $\bar{x}$ at which the fulcrum could be relocated to attain equilibrium. We define $\bar{x}$ as follows

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i},$$

moment of system about origin divided by total mass of a system.

Example 3.8.1. 1. Find the center of mass of the linear system shown below.

3.9 Moments and center of mass: two dimensional system

Let the point masses $m_1, m_2, \ldots, m_n$ be located at

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

1. The moment about the y-axis is $M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.

2. The moment about the x-axis is $M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$.

3. The center mass $(\bar{x}, \bar{y})$ (or center of gravity) is

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m},$$

where m is the total mass of the system.

Example 3.9.1. 1. Find the center of mass of the system of point masses $m_1 = 6$, $m_2 = 3$, $m_3 = 2$ and $m_4 = 9$ located at $(3, -2), (0, 0), \ldots, (-5, 3)$ and $(4, 2)$ respectively.

3.9.2 Moments and Center of mass of a planar lamina

Let f and g be continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, and consider the planar lamina of uniform density ρ bounded by the graphs of $y = f(x)$, and $y = g(x)$, and $a \leq x \leq b$.

1. The moments about the x- and y- axes are

$$M_x = \rho \int_b^a \frac{f(x) + g(x)}{2}[f(x) - g(x)]\ dx.$$

$$M_y = \rho \int_b^a x[f(x) - g(x)]\ dx.$$

2. The center of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m},$$

where $m = \rho \int_a^b [f(x) - g(x)]\ dx$ is the mass of the lamina.

Example 3.9.3. 1. Find the center of mass of the lamina of uniform density ρ bounded by the graph of $f(x) = 4 - x^2$ and x-axis.

2. Find the centroid of the region bounded by the graphs of $f(x) = 4 - x^2$ and $g(x) = x + 2$.

Chapter 4

Infinite series

4.1 Introduction

In this chapter, we will discuss infinite sequences and infinite series. We will learn how to determine whether a sequence/series converges or diverges. We will also learn how to find the Taylor and Maclaurin series and find the radius and interval of convergence of a power series. Mathematically a sequence is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequences by subscript notation rather than the standard function notation. For example, in the sequence

$1, 2, 3, \ldots, n, \ldots, \quad a_1, a_2, a_3, \ldots, a_n, \ldots,$ a_1 represent the first term, a_2 represents the second term and so on. The number a_n, is the n^{th} term of the sequence and the entire sequence is denoted by $\{a_n\}$.

Example 4.1.1. List the terms of each of the following sequences

1.

$$\{a_n\} = \{3 + (-1)^n\}.$$

2.

$$\{b_n\} = \left\{ \frac{n}{1-2n} \right\}.$$

4.2 Sequences

A sequence is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequences by subscript notation rather than by t

4.2.1 Limit of a sequence

Definition 4.2.2. Let L be a real number. The limit of a sequence $\{a_n\}$ is L, written as $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \epsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence converges to L, otherwise the sequence diverges.

Theorem 4.2.3. *Properties of limits and sequences*

Let $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = K$. *We get*

1. $\lim_{n\to\infty}(a_n \pm b_n) = L \pm K$.

2. $\lim_{n\to\infty}(ca_n) = cL,\ c \in \mathbb{R}$.

3. $\lim_{n\to\infty}(a_n b_n) = LK$.

4. $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{L}{K},\ b_n \neq 0$ *and* $K \neq 0$.

Example 4.2.4. Find the limit in each of the following sequences

1.

$$\{a_n\} = \{3 + (-1)^n\}.$$

2.

$$\{b_n\} = \left\{ \frac{n}{1-2n} \right\}.$$

3.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots$$

Example 4.2.5. Show that the sequence whose n^{th} term $a_n = \frac{n^2}{2^n-1}$ converges.

Theorem 4.2.6. *Absolute value theorem*

For the sequence $\{a_n\}$, *if* $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

4.3 Pattern recognition for sequences

Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the term of the sequence. In such cases, you may be required to discover a pattern in the sequence and to describe the n^{th} term. Once the n^{th} term has been specified, you can investigate the convergence or divergence of the sequence.

Example 4.3.1. Find a sequence $\{a_n\}$ whose first five terms are

1.

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \dots.$$

2.

$$-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \dots.$$

The process of determining an n^{th} term from the pattern observed in the first several terms of a sequence is an example of induction reasoning.

One important application of infinite sequences is in representing "infinite summation." If $\{a_n\}$ is an infinite sequence, them

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is an *infinite series.* The numbers $a_1, a_2, a_3, \ldots$ are the terms of the series.

Definition 4.4.1. For the infinite series $\sum_{n=1}^{\infty} a_n$, the n^{th} partial sum is given by $s_n = a_1 + a_2 + a_3 + \cdots + a_n$. If the sequence of partial sums $\{s_n\}$ converges to s, then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit s is called the sum of the series. $s = a_1 + a_2 + a_3 + \cdots + a_n$, $s = \sum_{n=1}^{\infty} a_n$. If $\{s_n\}$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 4.4.2. Determine the convergence or divergence of each of the following

1. $$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

2. $$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

Theorem 4.4.3. *Properties of infinite series*

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be convergent series, and let* A, B *and* c *be real numbers. If* $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$, *then the following series converges to the indicated sums.*

1. $\sum_{n=1}^{\infty} c a_n = cA$

2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

In this section, we will study several convergence tests that apply to series with positive terms.

4.5.1 The integral test

Theorem 4.5.2. *If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\, dx$ either both converge or both diverge.*

Example 4.5.3. Apply the integral test to the following cases

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
2. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

4.5.4 p-series and Harmonic series

We now investigate a second type of series that has a simple arithmetic test for convergence or divergence. A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is a p-series, where p is a positive constant. If $p = 1$, then the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

is the harmonic series.

Theorem 4.5.5. *Convergence of p-series*

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

1. *converges if $p > 1$, and*

2. *diverges if $0 < p \leq 1$.*

Example 4.5.6. Discuss the convergence or divergence of

1. the harmonic series and

2. the p-series with $p = 2$.

4.5.7 Limit Comparison test

Theorem 4.5.8. *Suppose that $a_n > 0$, $b_n > 0$, and*

$$\lim_{n \longrightarrow \infty} \left(\frac{a_n}{b_n} \right) = L,$$

where L is finite and positive.

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge

Example 4.5.9. Determine the convergence or divergence of

1. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

2. $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$

4.5.10 Ratio and Root tests

1. Ratio test

   Theorem 4.5.11. *Let $\sum a_n$ be a series with nonzero terms.*

(a) $\sum a_n$ converges absolutely if

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1.$$

(b) $\sum a_n$ diverges if

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1 \textit{ or } \quad \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty.$$

(c) The ratio test is inconclusive if

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1.$$

Example 4.5.12. Determine the convergence or divergence of

(a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(b) $\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$.

2. Root test

Theorem 4.5.13. *Let $\sum a_n$ be a series.*

(a) $\sum a_n$ converges absolutely if

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1.$$

(b) $\sum a_n$ diverges if

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} > 1 \textit{ or } \quad \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty.$$

(c) The Root test is inconclusive if

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 1.$$

Example 4.5.14. Determine the convergence or divergence of

(a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

(b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$.

4.6 Taylor and Maclaurin polynomial

If f has derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the n^{th} Taylor polynomial for f at c.

If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}(x)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x)^n$$

is called the n^{th} Maclaurin polynomial for f.

Example 4.6.1. Find the n^{th} Taylor polynomial for

1. $f(x) = e^x$ at $c = 0$.
2. $f(x) = \sin x$ at $c = 0$.
3. $f(x) = \ln x$ at $c = 1$.

Example 4.6.2. Find the fourth Maclaurin polynomial to approximate the value of $\ln(1.1)$.

4.6.3 Remainder of a Taylor polynomial

An approximation technique is of little value without some idea of its accuracy. To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial $P_n(x)$, you can use the concept of a remainder $R_n(x)$, defined as follows

$$f(x) = P_n(x) + R_n(x),$$

so that $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the **ERROR** associated with the approximation. That is

$$\text{Error } = |R_n(x)| = |f(x) - P_n(x)|.$$

Theorem 4.6.4. *Taylor's Theorem*

If the function f is differentiable through order $n+1$ in an interval I containing $c, 4then, foreach4x$ in I, there exists z between x and c such that

$$f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+R_n(x),$$

where

$$R_n(x) = \frac{f^{(n)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Example 4.6.5. 1. Find the third Maclaurin polynomial for $\sin x$.

2. Use Taylor's Theorem to approximate $\sin(0.1)$ by $P_3(0.1)$ and determine the accuracy of the approximation.

4.7 Power series

Definition 4.7.1. If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a power series.

More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

is called a power series centered at c, where c is a constant.

4.7.2 Radius and interval of convergence

A power series in x can be viewed as a function of x

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n,$$

where the domain of f is the set of all x for which the power series converges. Determining the domain of a power series is the primary concern in this section. Of course, every power series converges at its center c because

$$f(c) = \sum_{n=0}^{\infty} a_n(c-c)^n = a_0.$$

So c always lies in the domain of f.

Theorem 4.7.3. *For a power series centered at c, precisely one of the following is true.*

1. *The series only converges at c.*
2. *There exists a real number $R > 0$ such that the series converges absolutely for $|x-c| < R$, and diverges for $|x-c| > R$.*
3. *The series converges absolutely for x.*

The number R is the radius of convergence of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

Example 4.7.4. Find the radius of convergence of:

1. $\sum_{n=0}^{\infty} n!x^n$
2. $\sum_{n=0}^{\infty} 3(x-2)^n$

Example 4.7.5. Find the interval of convergence of:

1. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$
2. $\sum_{n=0}^{\infty} \frac{(-1)^n(x+1)^n}{2^n}$.
3. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

4.7.6 Operations with power series

Let $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$.

1. $f(kx) = \sum a_n k^n x^n$
2. $f(x^N) = \sum a_n x^{nN}$
3. $f(x) \pm g(x) = \sum (a_n \pm b_n) x^n$.

4.8 Taylor and Maclaurin series

Theorem 4.8.1. *If f is represented by a power series $f(x) = \sum a_n (x-c)^n$ for all x in an open interval I containing c, then*

$a_n = \frac{f^n(c)}{n!}$ *and*

$$f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots .$$

Definition 4.8.2. If a function f has derivatives of all orders at $x = c$, then the series

$$f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

is called the Taylor series for $f(x)$ at c. If $c = 0$, the series is called the Maclaurin series of f.

Theorem 4.8.3. *If* $\lim_{n\to\infty} R_n = 0$ *for all* x *in the interval* I, *then the Taylor series for* f *converges and equals* $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n.$$

Example 4.8.4. 1. Find the Maclaurin series for $f(x) = \sin x^2$.

2. Find the Taylor series for $f(x) = \frac{1}{x}$, at $c = 1$.

NOTE: THESE NOTES ARE MEANT FOR USE ONLY IN MAT2110 2016/2017 ACADEMIC YEAR.