0.1 Parametric equations for conics and other curves

In this lesson, we examine the position of a particle along a conic section and other curves. Parametric equations for the position of a particle moving in the plane are sometimes called parametric equations for the path traced by the particle.

We will discuss parametric equations through the following examples.

Example 0.1.1

Describe the motion of the particle whose position $P(x, y)$ at time t is given by

$$x = a\cos t,\ \ y = a\sin t,\ \ 0 \leq t \leq 2\pi$$

Solution

Since $x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$, the particle moves on a circle of radius a centered at the origin. The particle begins at the point $(a, 0)$ when $t = 0$ and moves once counter-clockwise around the circle as t increases to 2π. See Figure 1 below.

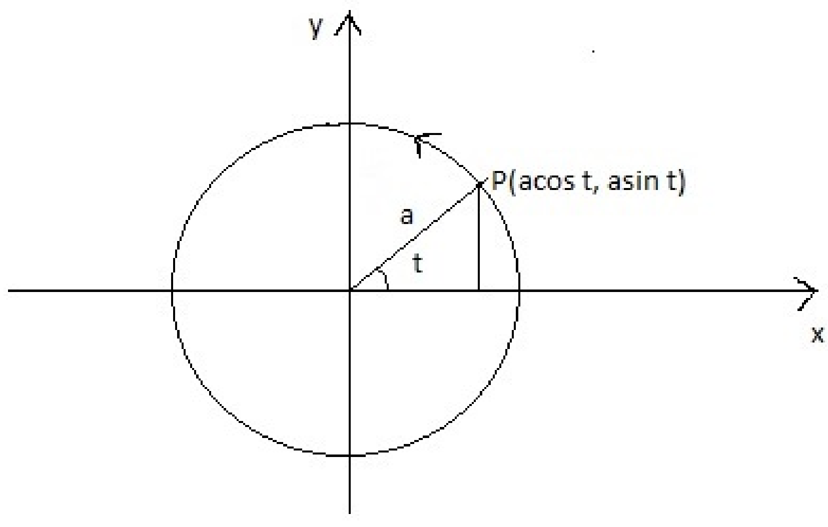

Figure 1: The circle defined by $x = a\cos t, y = a\sin t, 0 \leq t \leq 2\pi$.

Example 0.1.2 *Describe the motion of the particle whose position $P(x, y)$ at time t is given by*

$$x = t^2, \ y = t, \ -\infty < t < \infty$$

Solution

We will find the Cartesian equation by eliminating t . Since

$$y^2 = t^2 = x,$$

we see that the motion takes place on the parabola

$$y^2 = x.$$

As t increases between $-\infty$ and ∞, the particle comes in on the lower half of the parabola, reaches the origin when $t = 0$, and moves out into the first quadrant as t continues to increase. See Figure 2 below.

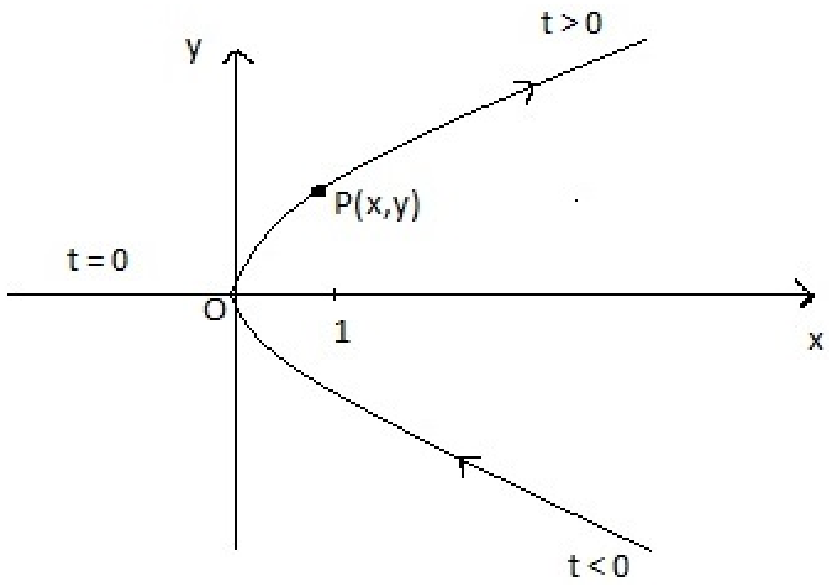

Figure 2: The parabola defined by $x = t^2, y = t, -\infty < t < \infty$

Note that there are many different parametrisations of the parabola. Some are

$$y = (\tan^{-1} t)^2, \ x = \tan^{-1} t, \ -\frac{\pi}{2} < t < \frac{\pi}{2} \text{ (same direction as above)}$$

and

$$x = t^2, \; y = -t, \; -\infty < t < \infty \text{ (direction reversed).}$$

Example 0.1.3 *Describe the motion of the particle whose position $P(x, y)$ at time t is given by*

$$x = a\cos t, \; y = b\sin t, \; 0 \leq t \leq 2\pi$$

Solution

We find the Cartesian equation by eliminating t . Since

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1,$$

the motion takes place on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The particle begins at $(a, 0)$ when $t = 0$, and moves counter-clockwise around the ellipse transversing it exactly once as t moves from 0 to π. See Figure 3 below.

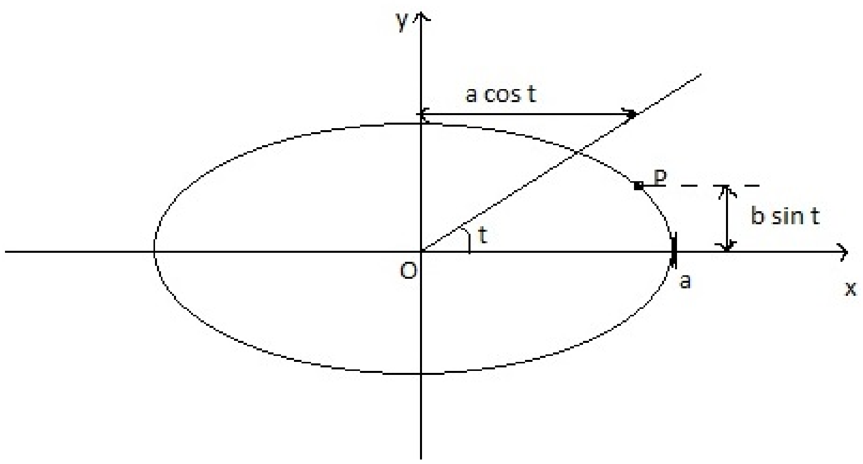

Figure 3: The coordinates at P are $x = a\cos t, y = b\sin t$

Example 0.1.4 *Describe the motion of the particle whose position $P(x, y)$ at time t is given by the equations*

$$y = \tan t, \; x = \sec t, \; -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Solution

Eliminating t we have

$$x^2 - y^2 = \sec^2 t - \tan^2 t = 1$$

from which we see that the motion takes place somewhere on the hyperbola $x^2 - y^2 = 1$. Since $x = \sec t$ is always positive for the parameter values $-\frac{\pi}{2} < t < \frac{\pi}{2}$, the motion takes place on the hyperbola's right-hand branch. As t moves from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, the particle comes in along the lower half of the right-hand branch, reaching the origin at $t = 0$. It then moves into the first quadrant to complete the coverage of the right-hand branch as t approaches $\frac{\pi}{2}$. See Figure 4 below.

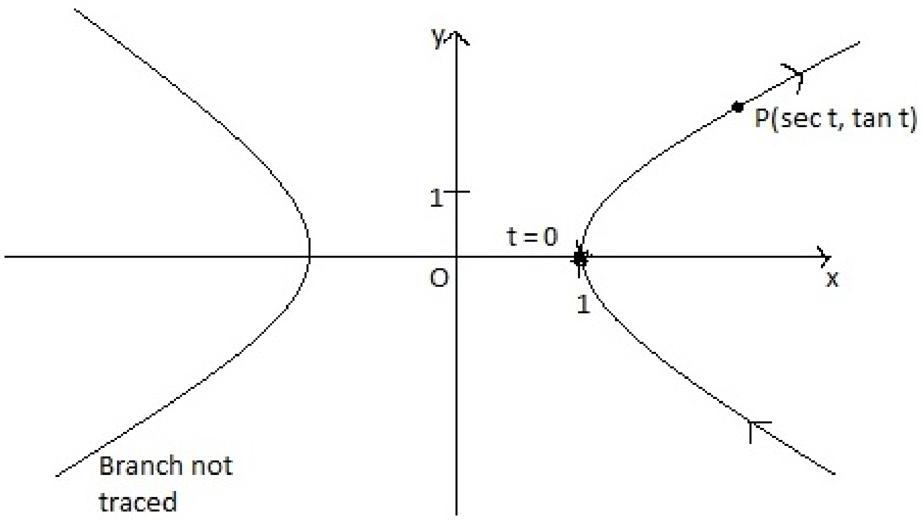

Figure 4: The right-hand branch of the hyperbola defined by $x = \sec t, y = \tan t, \frac{-\pi}{2} < t < \frac{\pi}{2}$.

Example 0.1.5 *Sketch the curve traced by the point $P(x, y)$ whose coordinates satisfy the equations*

$$y = 1 - \cos t,\ x = \cos 2t,\ -\infty < t < \infty.$$

Solution

We first fine the Cartesian equation by eliminating t. We have

$$y = 1 - \cos 2t = 1 - (2\cos^2 t - 1) = 2 - 2\cos^2 t = 2 - 2x^2$$

so the particle traces some portion of the parabola $y = 2 - 2x^2$. Since $|\cos\theta| \leq 1$ for any angle θ, the parametric equations describe only the portion for which

$$-1 \leq x = \cos 2t \leq 1 \text{ and } 0 \leq y = 1 - \cos t \leq 2.$$

Thus we see that $P(x, y)$ starts at $A(1, 0)$ when $t = 0$ and moves up and the the left as t increases, arriving at $B(0, 2)$ when $t = \frac{\pi}{2}$. It continues to $C(-1, 0)$ as t increases to π. See Figure 5 below.

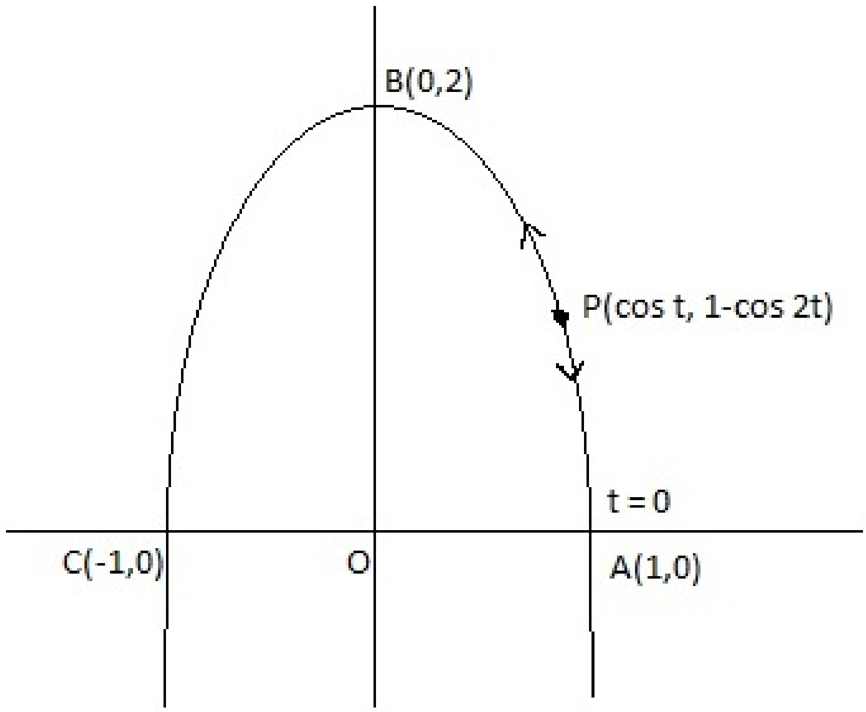

Figure 5: As t varies from $-\infty$ to ∞, the point P traces and retraces the parabolic arch.

As t varies from π to 2π, the point traces the arch CBA back to A. Since x and y are periodic, (x with period 2π and y with period π), any further variation of t results in tracing a portion of the arch.

Example 0.1.6 *A wheel of radius r rolls along a horizontal straight line without slipping. Find the curve traced by the point $P(x, y)$ on a spoke of the wheel b units from its centre. Such a curve is called a trochoid. If $b = a$, P is* *on the circumference and the curve is called a cycloid. This is like the path travelled by a pebble in the thread of the rolling tire.*

Solution

Consider the figure below. We take the x-axis to be the line the wheel rolls

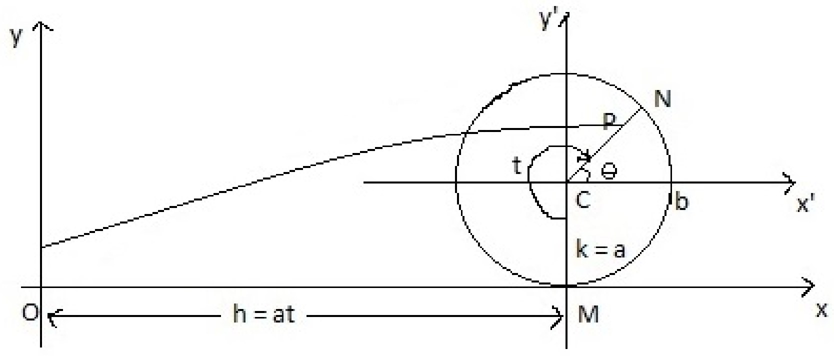

Figure 6: The trochoid $x = at - b\sin t, y = a - b\cos t$, shown for $t \leq 0$.

along, with the y-axis through a low point of the trochoid. Since the circle rolls without slipping, the distance OM is equal to the circular arc $MN = at$. The xy-coordinates of C are therefore

$$h = at, \;\; k = a \tag{1}$$

Now we introduce $x'y'$-axes parallel to the xy-axes and having their origin at C. The xy- and $x'y'$-coordinates at P are related by the equations

$$x = h + x', \;\; y = k + y'. \tag{2}$$

From the figure, we see that

$$x' = b\cos\theta, \;\; y' = b\sin\theta$$

or since $\theta = \frac{3\pi}{2} - t$,

$$x' = -b\sin t, y' = -b\cos t.$$

Substituting these into (1) and (2) we obtain

$$x = at - b\sin t, \;\; y = a - b\cos t \tag{3}$$

as parametric equations of a trochoid.

Taking $b = a$ in (3) we obtain

$$x = a(t - \sin t), \;\; y = a(1 - \cos t)$$

as the parametric equation of the cycloid.