



# Lecture A4

CEE 2219: STATICS AND INTRODUCTION TO MECHANICS OF MATERIALS

# MOMENTS, COUPLES & RESULTANT FORCES

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# CEE 2219 – STATICS & INTRODUCTION TO MECHANICS OF MATERIALS

## Lecture A4

- ❖ INTRODUCTION TO MOMENTS, COUPLES & RESULTANT FORCES
- ✓ CROSS PRODUCT
- ✓ TRIPLE SCALAR PRODUCT
- ✓ INTERNAL & EXTERNAL FORCES





# LECTURE OBJECTIVES

- ❖ To discuss the concept of the moment of a force and show how to calculate it in 2D & 3D.
- ❖ To provide a method for finding the moment of a force about a specified axis.
- ❖ To define the moment of a couple.
- ❖ To show how to find the resultant effect of a non-concurrent force system.





# CHAPTER INTRODUCTION

- Treatment of a body as a single particle is not always possible. In general, the size of the body and the specific points of application of the forces must be considered
- Most bodies in elementary mechanics are assumed to be rigid, i.e., the actual deformations are small and do not affect the conditions of equilibrium or motion of the body
- In this chapter you will study the effect of forces exerted on a rigid body, and you will learn how to replace a given system of forces by a **simpler equivalent system**.





# CHAPTER INTRODUCTION

- In this lecture you will learn how the forces can be replaced with a simpler equivalent system:
  - moment of a force about a point
  - moment of a force about an axis
  - moment due to a couple
- *Note that the determination of these quantities ( $M$ ) involves the computation of vector products and [scalar products] of two vectors,*





# CHAPTER INTRODUCTION

- Another concept introduced in this lecture is that of a couple, any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple
- This basic system is called a **force-couple system**
- In the case of concurrent, coplanar, or parallel forces, the equivalent force-couple system can be further reduced to a single force, called the **resultant of the system**, or to a **single couple**, called the **resultant couple of the system.**





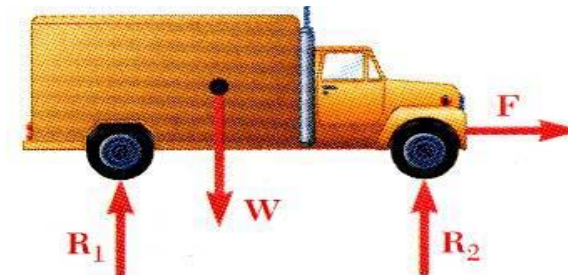
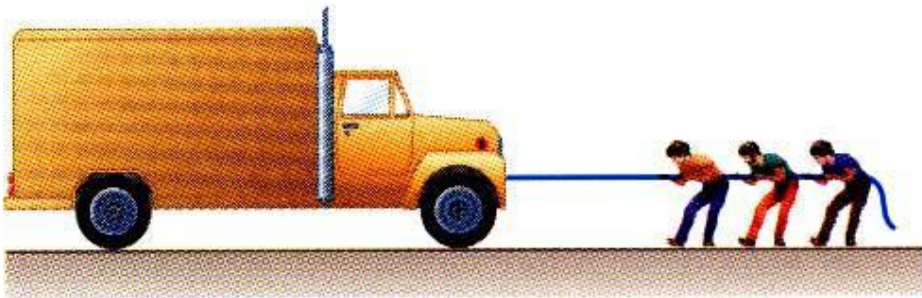
# INTERNAL AND EXTERNAL FORCES

- Forces acting on rigid bodies are divided into two groups: **External & Internal forces**
- The **internal forces** are the forces which hold together the particles forming the rigid body.
- If the rigid body is structurally composed of several parts, the forces holding the component parts together are also defined as internal forces.
- Internal forces will be considered in other upcoming lectures.



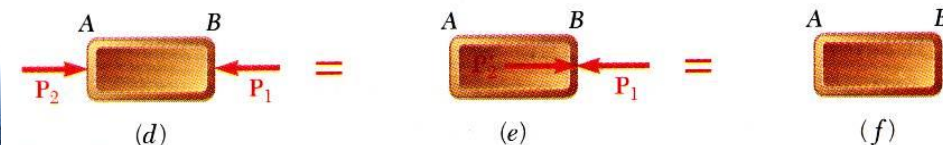
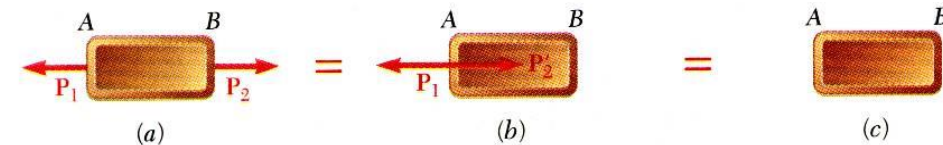
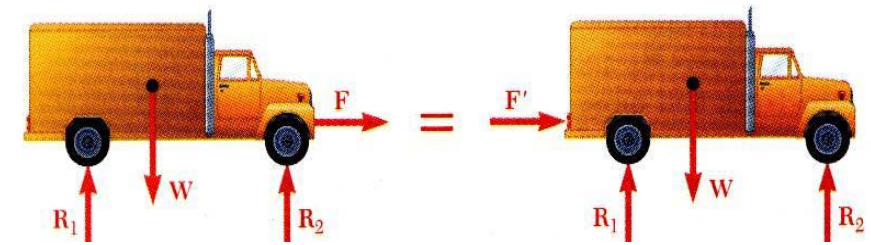
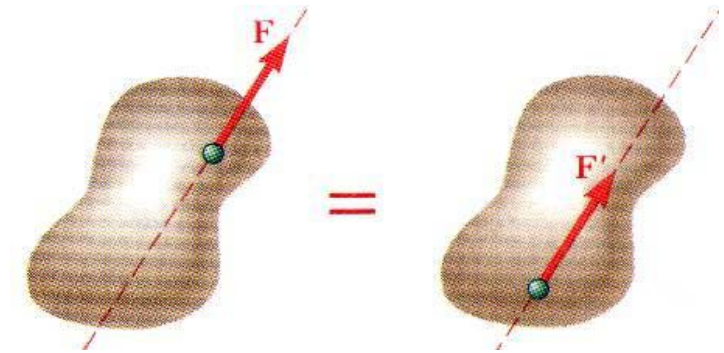
## INTERNAL AND EXTERNAL FORCES

- The **external forces** represent the action of other bodies on the rigid body under consideration as shown using a FBD.
- They are entirely responsible for the external behavior of the rigid body. They will either cause it to move or ensure that it remains at rest.
- If unopposed, each external force can impart a motion of translation or rotation, or both
- **We shall be concerned only with external forces in this lecture**



# PRINCIPLE OF TRANSMISSIBILITY

- The principle states that the conditions of equilibrium or motion of a rigid body will remain unchanged if a force  $F$  acting at a given point of the rigid body is replaced by a force  $F'$  of the same magnitude and same direction, but acting at a different point, provided that the two forces have the same line of action
- Moving the point of application of the force  $F$  to the rear bumper does not affect the motion or the other forces acting on the truck
- Principle of transmissibility may not always apply in determining internal forces and deformations

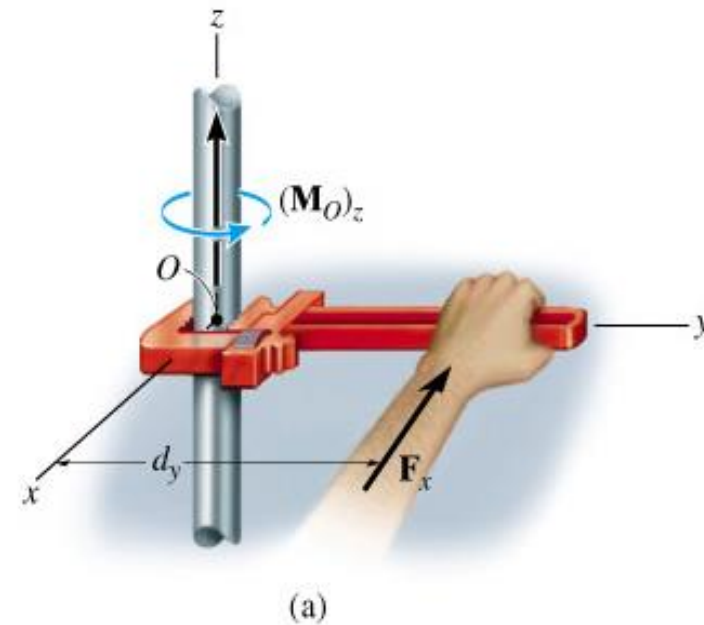


# MOMENT OF A FORCE

## Scalar Formulation

- The tendency of a force to rotate a rigid body about any defined axis is called the **Moment of the force** about the axis
- The moment,  $M$ , of a force about a point provides a measure of the tendency for rotation (sometimes called a torque).
- The Moment of Force ( $\mathbf{F}$ ) about an axis through Point  $O$ , the Moment of  $\mathbf{F}$  about  $O$ , is the product of the magnitude of the force and the perpendicular distance between  $O$  and the line of action of Force ( $\mathbf{F}$ )

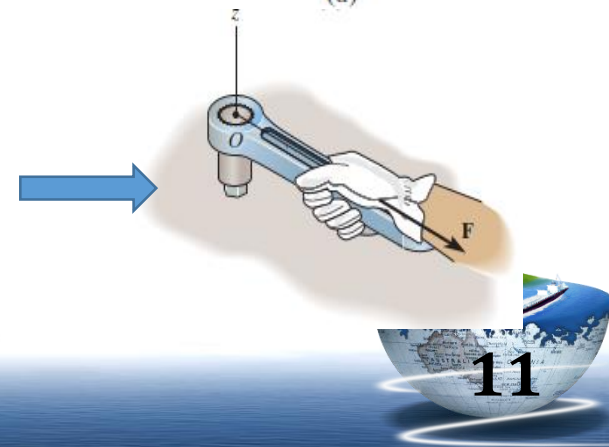
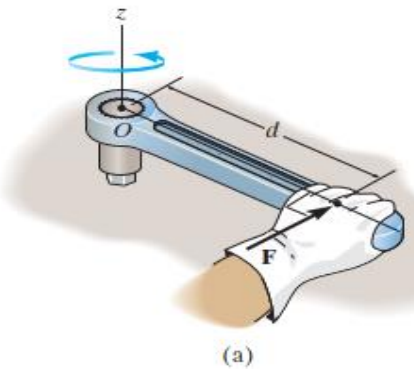
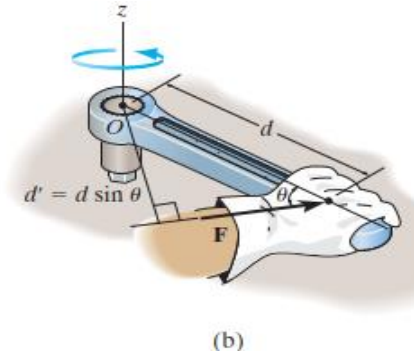
$$M = F_x \times d_y$$



# MOMENT OF A FORCE

## Scalar Formulation

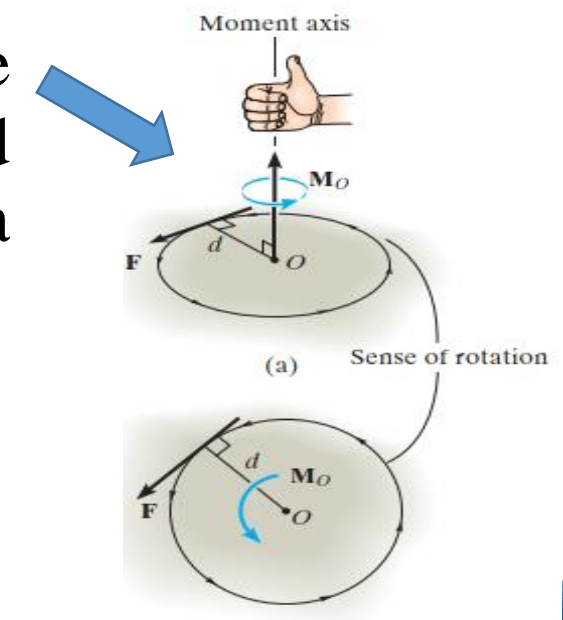
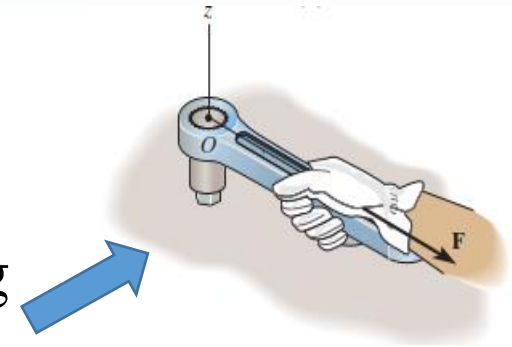
- Consider a wrench used to unscrew the bolt in (a) If a force is applied to the handle of the wrench it will tend to turn the bolt about point O.
- The magnitude of the moment is directly proportional to the magnitude of F and the perpendicular distance or moment arm d.
- Max. moment or turning effect occurs at an angle  $90^\circ$  in (a)
- If F is applied along the wrench, Fig. c, its moment arm will be zero since the line of action of F will intersect point O (the z axis).

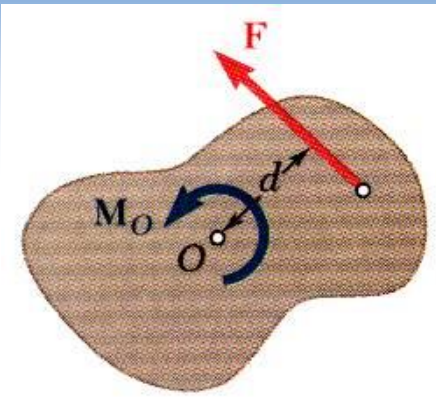


# MOMENT OF A FORCE

## Scalar Formulation

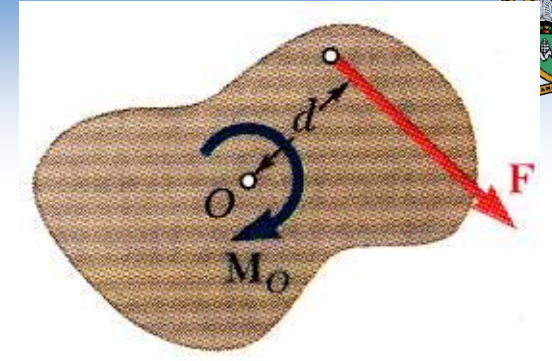
- As a result, the moment of  $F$  about  $O$  is also zero and no turning can occur.
- We can generalize the above discussion and consider the force  $F$  and point  $O$  which lie in the shaded plane as shown in Fig. a. The moment  $M_O$  about point  $O$ , or about an axis passing through  $O$  and perpendicular to the plane, is a **vector quantity** since it has a specified **magnitude and direction**.
- **The magnitude of  $M_O = Fd$**





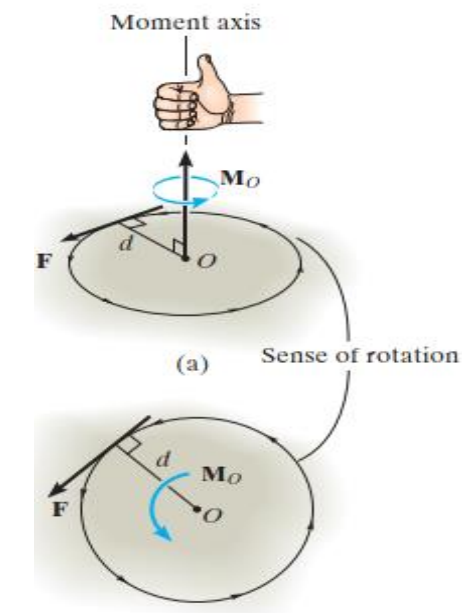
(a)  $M_O = +Fd$

## MOMENT OF A FORCE Scalar Formulation



(b)  $M_O = -Fd$

- **The direction of  $M_O$**  is defined by its moment axis, which is perpendicular to the plane that contains the force  $F$  and its moment arm  $d$ .
- The right-hand rule is used to establish the sense of direction of  $M_O$ .
- Counter-clockwise rotation means the moment vector is positive.



# MOMENT OF A FORCE

## Scalar Formulation

- **Resultant Moment.** For two-dimensional problems, where all the forces lie within the x–y plane, Fig. 4–3, the resultant moment ( $M_R$ ) about point O can be determined by finding the algebraic sum of the moments caused by all the forces in the system.
- As a convention, we will generally consider positive moments as counter-clockwise & clockwise moments will be negative.
- Doing this, the directional sense of each moment can be represented by a plus or minus sign.

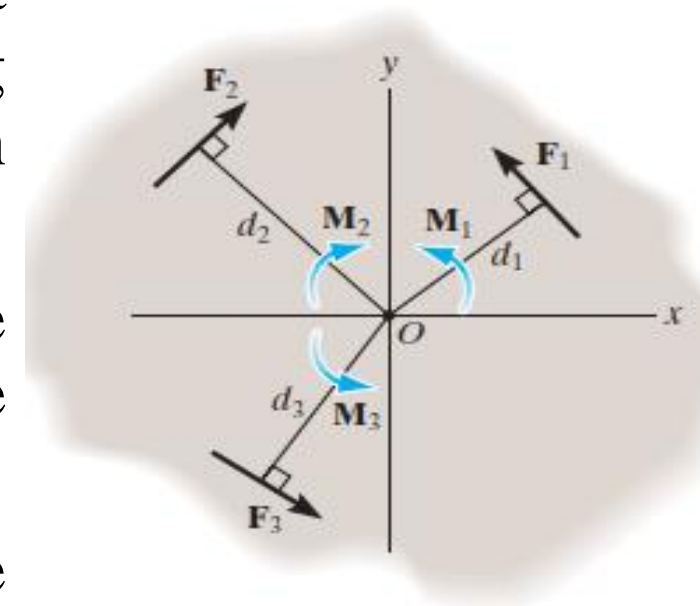


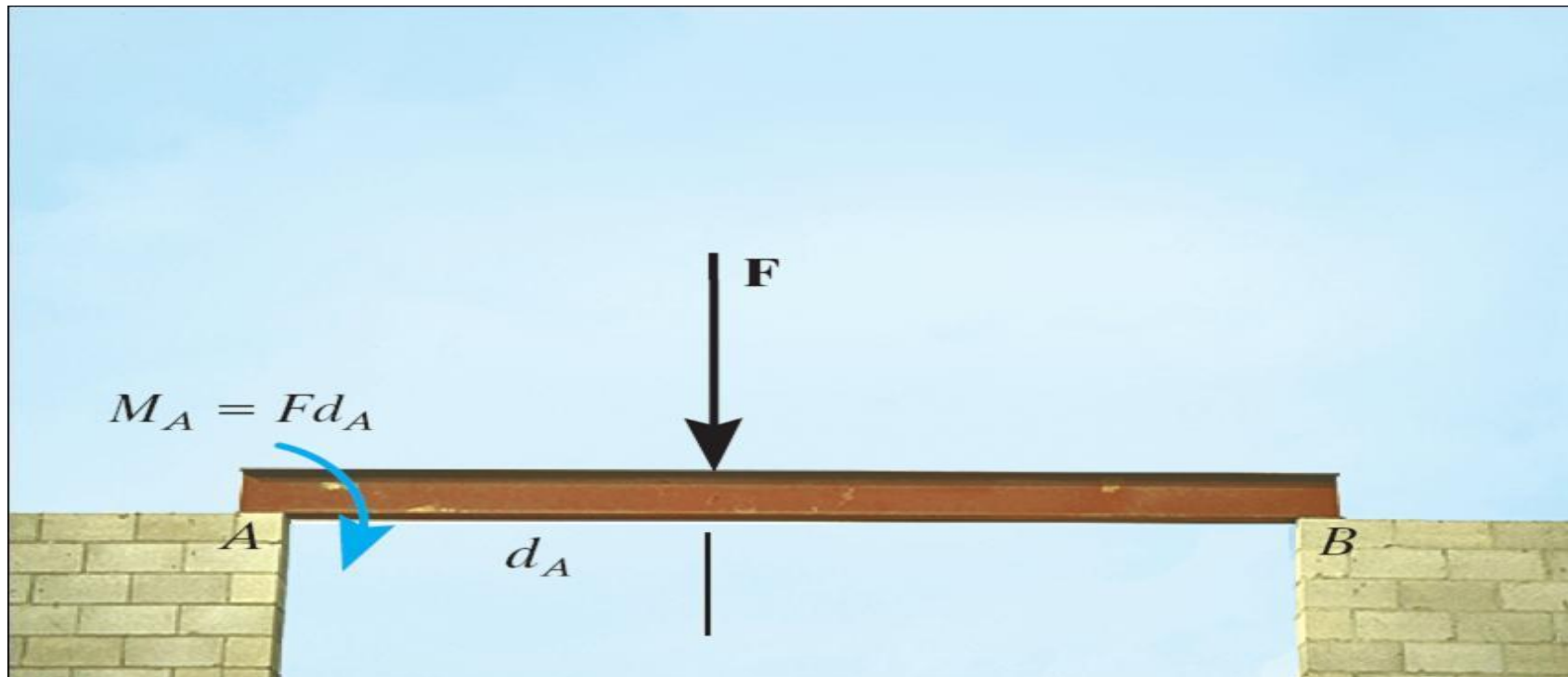
Fig. 4–3

$$\zeta + (M_R)_O = \sum Fd; \quad (M_R)_O = F_1d_1 - F_2d_2 + F_3d_3$$



# MOMENT OF A FORCE

## Application

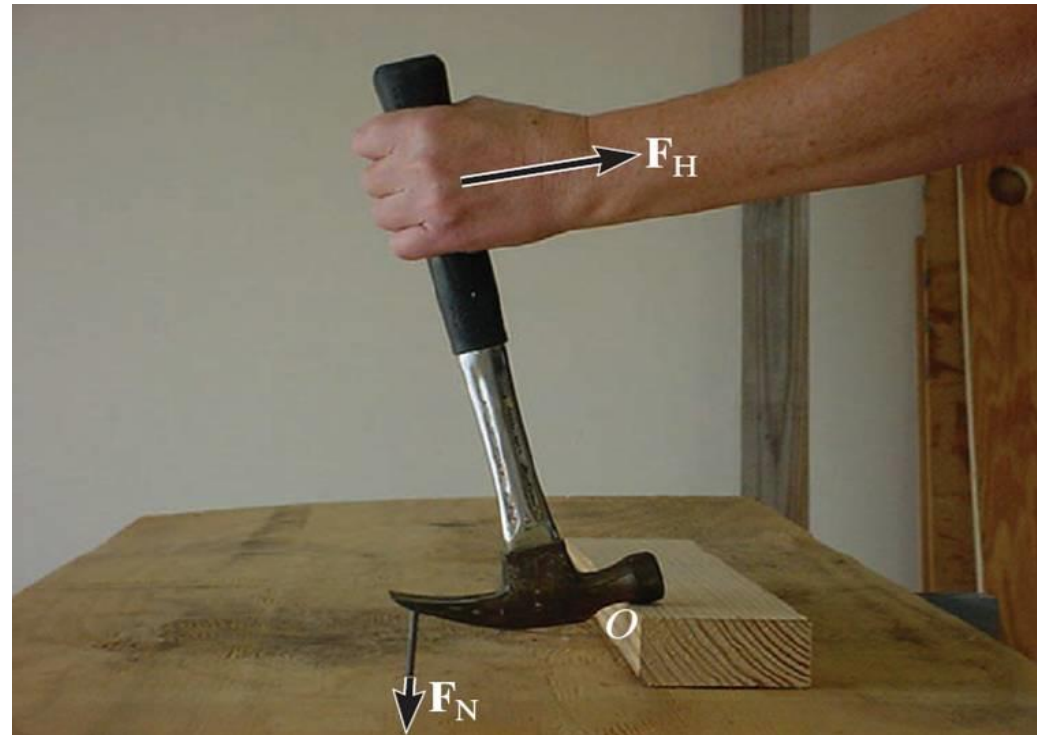


- Beams are often used to bridge gaps in walls. We have to know what the effect of the force on the beam will have on the beam supports.
- What do you think those impacts are at points A and B?



# MOMENT OF A FORCE

## Application



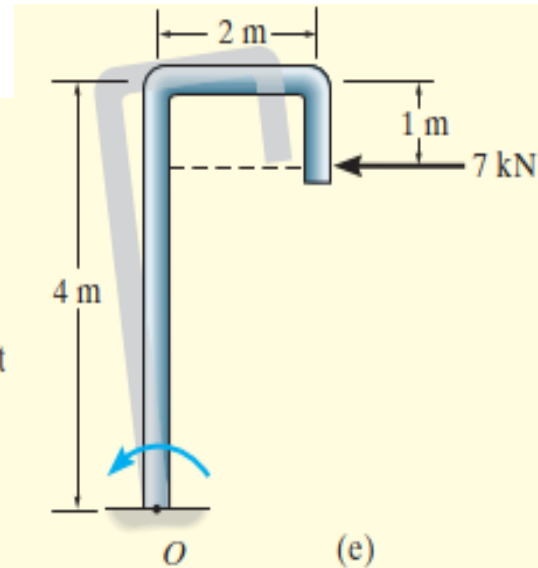
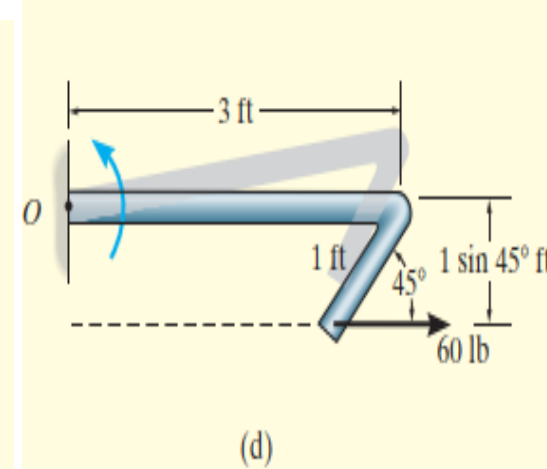
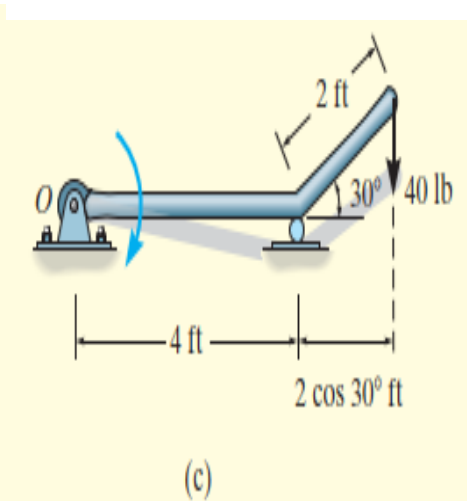
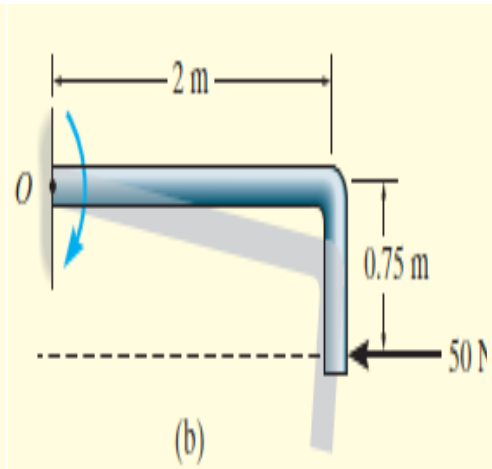
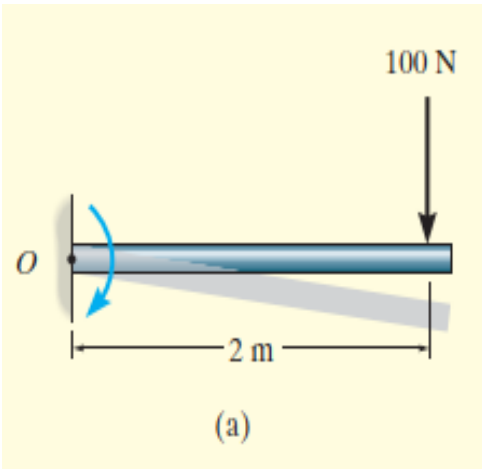
Carpenters often use a hammer in this way to pull a stubborn nail. Through what sort of action does the force  $F_H$  at the handle pull the nail? How can you mathematically model the effect of force  $F_H$  at point O?

# MOMENT OF A FORCE

## Example 4.1

### Question

Determine the moment of the force about point O





## MOMENT OF A FORCE

### Example 4.1

#### Solution

- The line of action of each force is extended as a dashed line in order to establish the moment arm  $d$ . Also illustrated is the tendency of rotation of the member as caused by the force. Furthermore, the orbit of the force about  $O$  is shown as a colored curl.

Thus

$$a \quad M_O = (100 \text{ N})(2 \text{ m}) = 200 \text{ N} \cdot \text{m} \curvearrowright \quad \text{Ans.}$$

$$b \quad M_O = (50 \text{ N})(0.75 \text{ m}) = 37.5 \text{ N} \cdot \text{m} \curvearrowright \quad \text{Ans.}$$

$$c \quad M_O = (40 \text{ lb})(4 \text{ ft} + 2 \cos 30^\circ \text{ ft}) = 229 \text{ lb} \cdot \text{ft} \curvearrowright \quad \text{Ans.}$$

$$d \quad M_O = (60 \text{ lb})(1 \sin 45^\circ \text{ ft}) = 42.4 \text{ lb} \cdot \text{ft} \curvearrowleft \quad \text{Ans.}$$

$$e \quad M_O = (7 \text{ kN})(4 \text{ m} - 1 \text{ m}) = 21.0 \text{ kN} \cdot \text{m} \curvearrowleft \quad \text{Ans.}$$

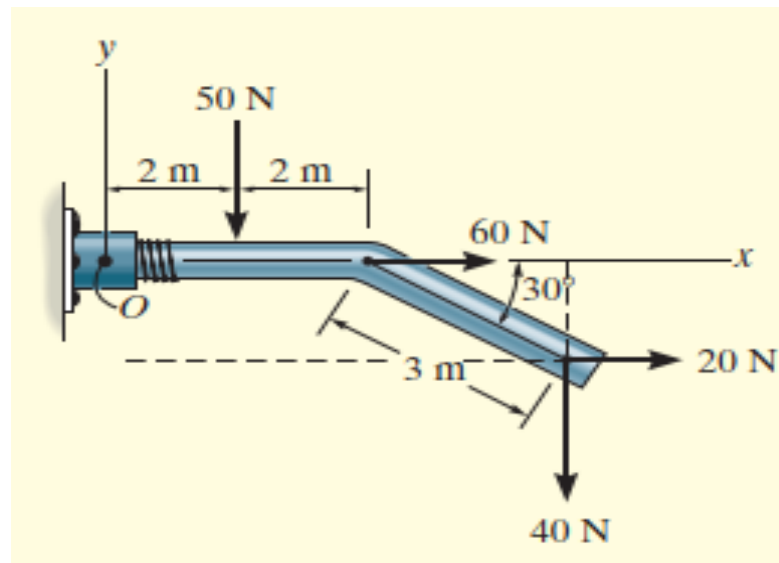


# MOMENT OF A FORCE

## Example 4.2

### Question

Determine the resultant moment of the four forces acting on the rod shown about point O.





## MOMENT OF A FORCE

### Example 4.2

#### Solution

- Assuming that positive moments act in the + k direction, i.e., Counter-clockwise, we have

$$\curvearrowleft + (M_R)_O = \sum Fd;$$

$$(M_R)_O = -50 \text{ N}(2 \text{ m}) + 60 \text{ N}(0) + 20 \text{ N}(3 \sin 30^\circ \text{ m}) \\ -40 \text{ N}(4 \text{ m} + 3 \cos 30^\circ \text{ m})$$

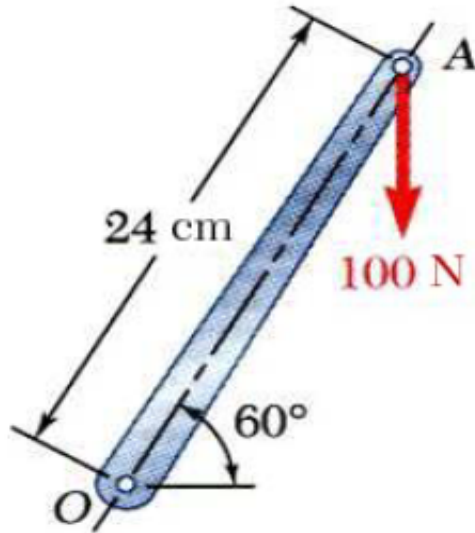
$$(M_R)_O = -334 \text{ N} \cdot \text{m} = 334 \text{ N} \cdot \text{m} \curvearrowright$$

- For this calculation, note how the moment-arm distances for the 20-N and 40-N forces are established from the extended (dashed) lines of action of each of these forces.



# MOMENT OF A FORCE

## Example 4.3



A 100-N vertical force is applied to the end of a lever which is attached to a shaft at  $O$ .

Determine:

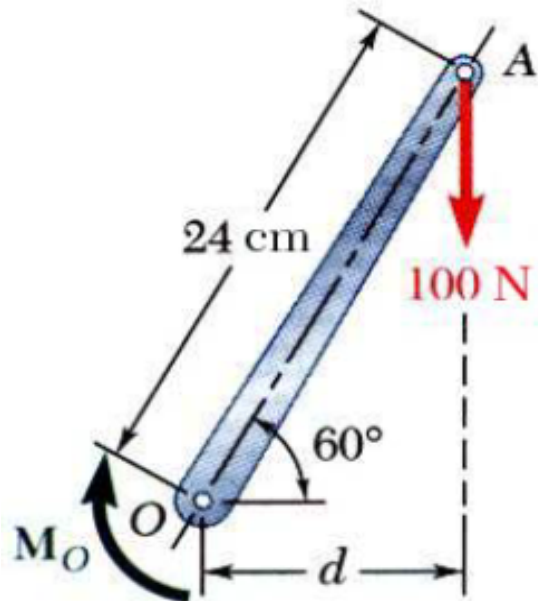
- moment about  $O$ ,
- horizontal force at  $A$  which creates the same moment,
- smallest force at  $A$  which produces the same moment,
- location for a 240-N vertical force to produce the same moment,
- whether any of the forces from b, c, and d is equivalent to the original force.



## MOMENT OF A FORCE

### Example 4.3

Solution



- a) Moment about  $O$  is equal to the product of the force and the perpendicular distance between the line of action of the force and  $O$ . Since the force tends to rotate the lever clockwise, the moment vector is into the plane of the paper.

$$M_O = Fd$$

$$d = (24 \text{ cm}) \cos 60^\circ = 12 \text{ cm}$$

$$M_O = (100 \text{ N})(12 \text{ cm})$$

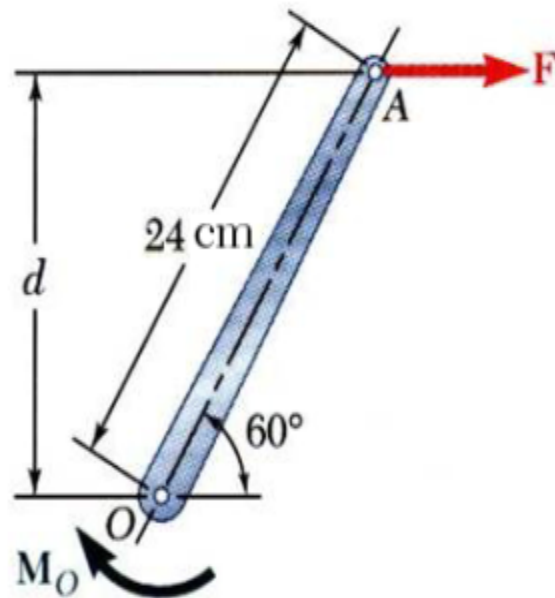
$$M_O = 1200 \text{ N} \cdot \text{cm}$$



# MOMENT OF A FORCE

## Example 4.3

Solution



c) Horizontal force at  $A$  that produces the same moment,

$$d = (24 \text{ cm}) \sin 60^\circ = 20.8 \text{ cm}$$

$$M_O = Fd$$

$$1200 \text{ N} \cdot \text{cm} = F(20.8 \text{ cm})$$

$$F = \frac{1200 \text{ N} \cdot \text{cm}}{20.8 \text{ cm}}$$

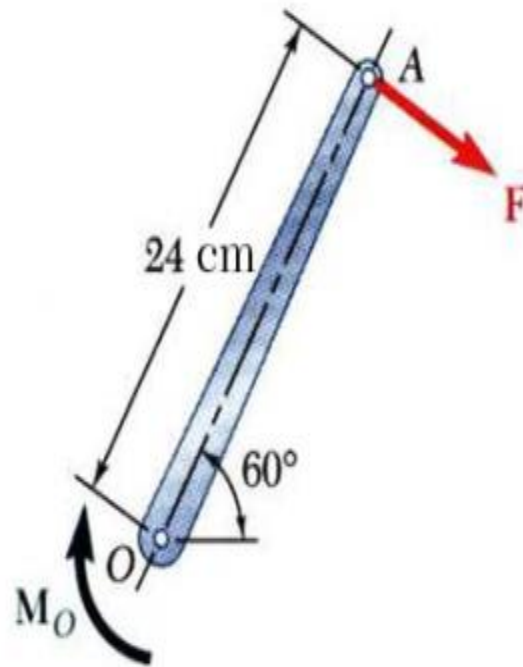
$$F = 57.7 \text{ N}$$



## MOMENT OF A FORCE

### Example 4.3

Solution



- c) The smallest force  $A$  to produce the same moment occurs when the perpendicular distance is a maximum or when  $F$  is perpendicular to  $OA$ .

$$M_O = Fd$$

$$1200 \text{ N} \cdot \text{cm} = F(24 \text{ cm})$$

$$F = \frac{1200 \text{ N} \cdot \text{cm}}{24 \text{ cm}}$$

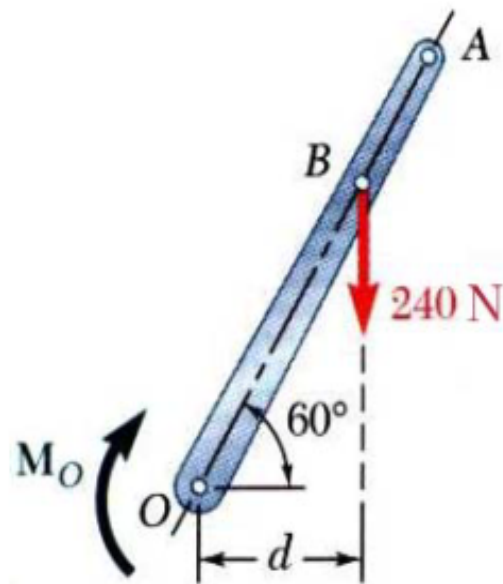
$$F = 50 \text{ N}$$



## MOMENT OF A FORCE

### Example 4.3

Solution



- d) To determine the point of application of a 240 lb force to produce the same moment,

$$M_O = Fd$$

$$1200 \text{ N} \cdot \text{cm} = (240 \text{ N})d$$

$$d = \frac{1200 \text{ N} \cdot \text{cm}}{240 \text{ N}} = 5 \text{ cm}$$

$$OB \cos 60^\circ = 5 \text{ cm}$$

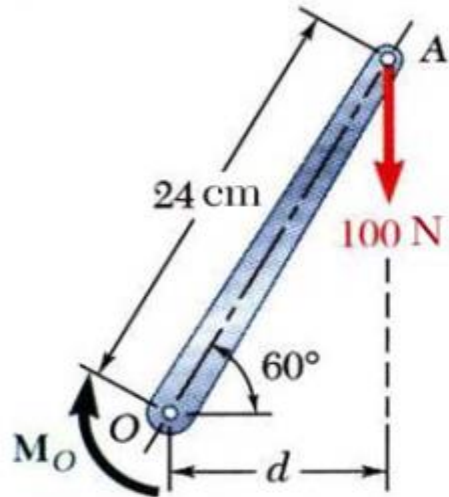
$$OB = 10 \text{ cm}$$



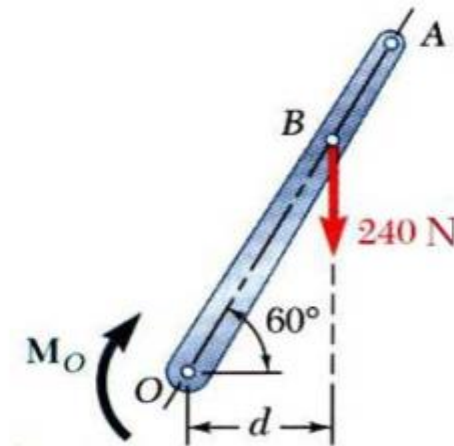
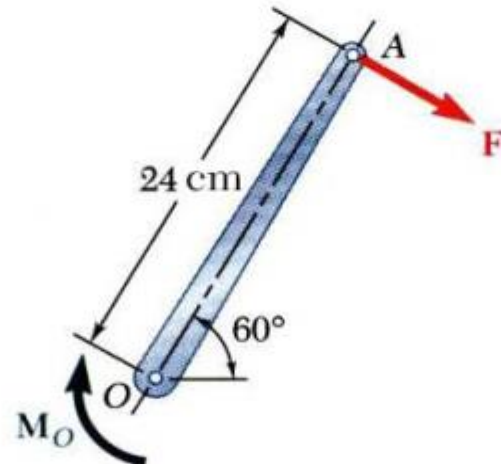
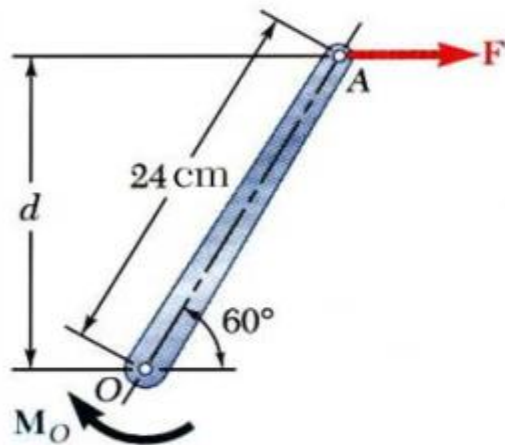
# MOMENT OF A FORCE

## Example 4.3

Solution



e) Although each of the forces in parts b), c), and d) produces the same moment as the 100 N force, none are of the same magnitude and sense, or on the same line of action. None of the forces is equivalent to the 100 N force.





## THE VECTOR PRODUCT OR CROSS PRODUCT

- Before formulating vector moments, it is first necessary to expand our knowledge of vector algebra and introduce the cross-product method of vector multiplication
- Concept of moments of a force about a point is more easily understood through application of cross product
- The cross product of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  yields the vector  $\mathbf{V}$ , which is written

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q}$$

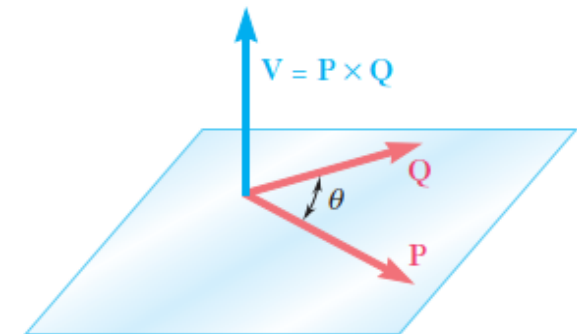
- and is read “ $\mathbf{V}$  equals  $\mathbf{P}$  cross  $\mathbf{Q}$ ”



# THE VECTOR PRODUCT OR CROSS PRODUCT

- Vector product of two vectors is defined as the vector  $V$  which satisfies the following:

1. Line of action of  $V$  is perpendicular to plane
2. Magnitude of  $V = P \times Q = PQ \sin \theta$
3. Direction of  $V$  is obtained from the right-hand rule



(a)

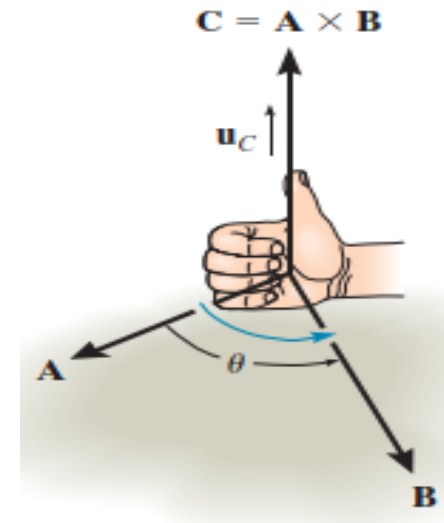


(b)



## THE VECTOR PRODUCT OR CROSS PRODUCT

- The magnitude of Vector product  $\mathbf{C}$  can be defined as the product of the magnitudes of  $\mathbf{A}$  and  $\mathbf{B}$  and the sine of the angle  $\theta$  between their tails ( $180^\circ \geq \theta \geq 0^\circ$ ) Thus,  $\mathbf{C} = AB \sin \theta$
- Vector product has a direction that is perpendicular to the plane containing  $\mathbf{A}$  and  $\mathbf{B}$  such that  $\mathbf{C}$  is specified by the right-hand rule
- Knowing both the magnitude and direction of  $\mathbf{C}$ , we can write  $\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin \theta) \mathbf{u}_C$
- where the scalar  $AB \sin \theta$  defines the magnitude of  $\mathbf{C}$  and the unit vector  $\mathbf{u}_C$  defines the direction of  $\mathbf{C}$ .



# THE VECTOR PRODUCT OR CROSS PRODUCT

- Laws of Operation.

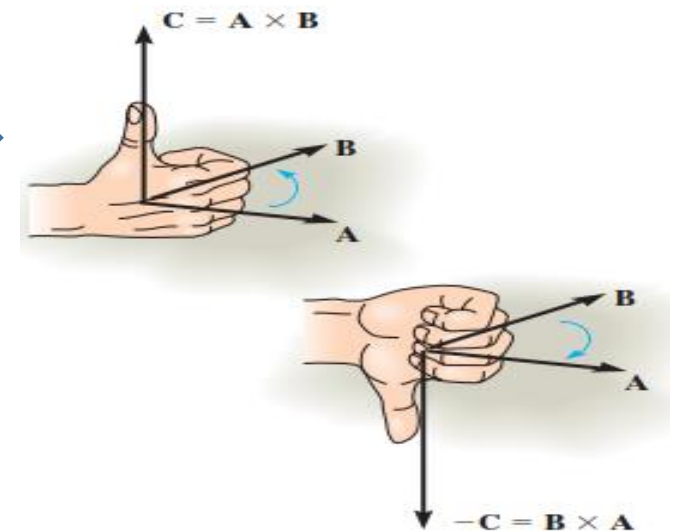
- Are not commutative  $A \times B = -(B \times A)$



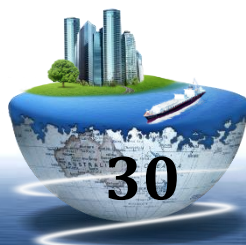
- Are distributive  $A \times (B_1 + B_2) = A \times B_1 + A \times B_2$

- Are not associative (only by a scalar)  $(A \times B) \times D \neq A \times (B \times D)$

- $a(A \times B) = (aA) \times B = A \times (aB) = (A \times B)a$



- *It is important to note that proper order of the cross products must be maintained, since they are not commutative & not associative for cross product.*





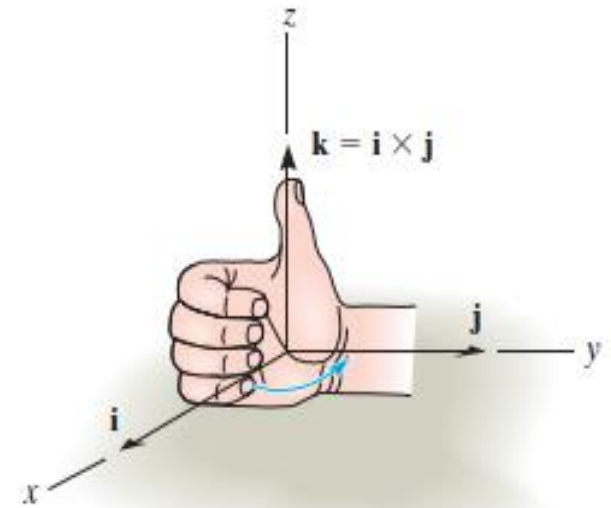
# THE VECTOR PRODUCT OR CROSS PRODUCT

## Cartesian Vector Formulation

- Eqn. shown may be used to find the cross product of any pair of Cartesian unit vectors.

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin \theta)\mathbf{u}_C$$

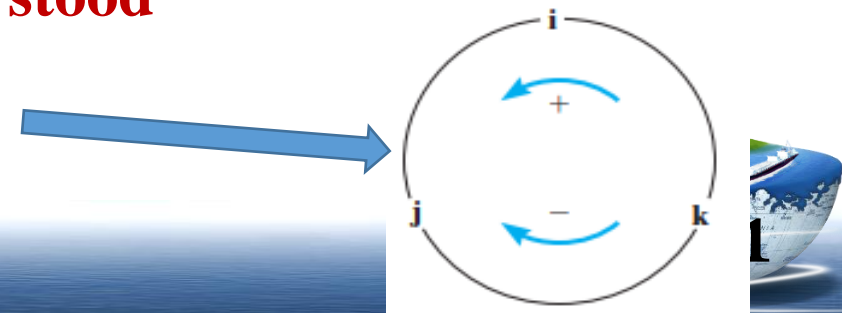
- For example, to find  $\mathbf{i} \times \mathbf{j}$ , the magnitude of the resultant vector is  $(\mathbf{i})(\mathbf{j})(\sin 90^\circ) = (1)(1)(1) = 1$ , and its direction is determined using the right-hand rule.



- Note that the resultant vector points in the +k direction.
- Thus,  $\mathbf{i} \times \mathbf{j} = (1)\mathbf{k}$ . In a similar manner

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} & \mathbf{i} \times \mathbf{i} = 0 \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{j} \times \mathbf{j} = 0 \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{k} \times \mathbf{k} = 0 \end{array}$$

- These results should not be memorized; rather, just understood the right-hand rule and the definition of the cross product**



# THE VECTOR PRODUCT OR CROSS PRODUCT

## Cartesian Vector Formulation

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin \theta)\mathbf{u}_C$$

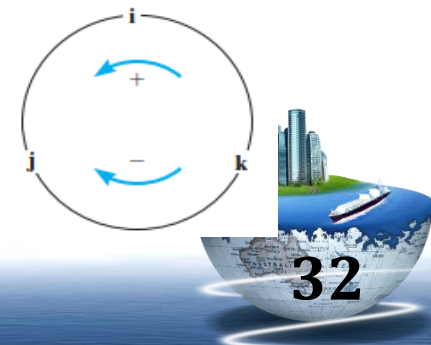
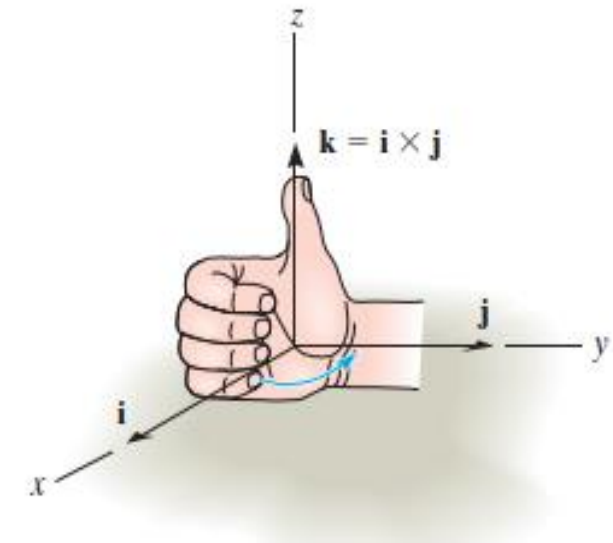
- Let us now consider the cross product of two general vectors  $\mathbf{A}$  and  $\mathbf{B}$  which are expressed in Cartesian vector form.

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x(\mathbf{i} \times \mathbf{i}) + A_xB_y(\mathbf{i} \times \mathbf{j}) + A_xB_z(\mathbf{i} \times \mathbf{k}) \\ &\quad + A_yB_x(\mathbf{j} \times \mathbf{i}) + A_yB_y(\mathbf{j} \times \mathbf{j}) + A_yB_z(\mathbf{j} \times \mathbf{k}) \\ &\quad + A_zB_x(\mathbf{k} \times \mathbf{i}) + A_zB_y(\mathbf{k} \times \mathbf{j}) + A_zB_z(\mathbf{k} \times \mathbf{k}) \end{aligned}$$

- Carrying out the cross-product operations and combining terms yields

- The rectangular components  $C_x = A_yB_z - A_zB_y$ ,  $C_y = A_zB_x - A_xB_z$ , and  $C_z = A_xB_y - A_yB_x$  are thus found to be

- $C_x = A_yB_z - A_zB_y$      $C_y = A_zB_x - A_xB_z$      $C_z = A_xB_y - A_yB_x$





# THE VECTOR PRODUCT OR CROSS PRODUCT

## Cartesian Vector Formulation

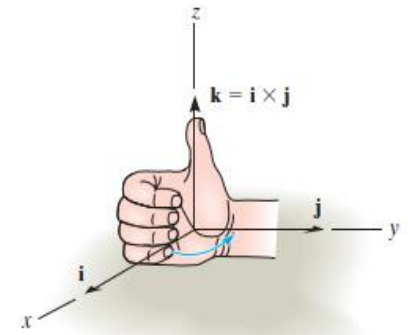
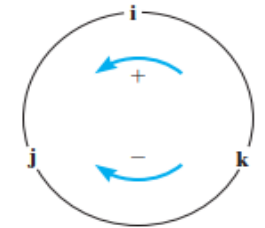
$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin \theta)\mathbf{u}_C$$

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

- This equation may also be written in a more compact determinant form which is more easily memorized

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

- Note that expanded form of  $\mathbf{A} \times \mathbf{B}$  shows the first row of elements consisting of the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  and whose second and third rows represent the  $x$ ,  $y$ ,  $z$  components of the two vectors  $\mathbf{A}$  and  $\mathbf{B}$ , respectively





# THE VECTOR PRODUCT OR CROSS PRODUCT

## Cartesian Vector Formulation

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin \theta)\mathbf{u}_C$$

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

- **For a 3 x 3 determinant**, such as the one shown, the three minors can be generated in accordance with the following scheme:

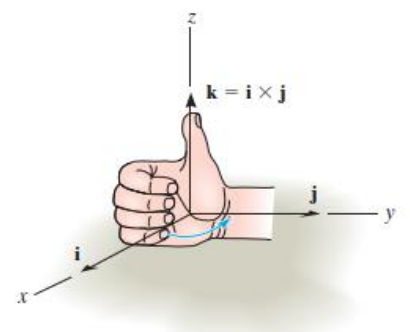
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

For element i:  $\begin{vmatrix} \oplus & & \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(A_y B_z - A_z B_y)$

For element j:  $\begin{vmatrix} & \oplus & \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\mathbf{j}(A_x B_z - A_z B_x)$

Remember the negative sign

For element k:  $\begin{vmatrix} & & \oplus \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{k}(A_x B_y - A_y B_x)$



- $\mathbf{C}_i = \mathbf{i}(A_y B_z - A_z B_y)$
- Or for vector product  $\mathbf{P} \times \mathbf{Q}$

$$\bar{\mathbf{V}} = \bar{\mathbf{P}} * \bar{\mathbf{Q}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} = \mathbf{i} \begin{vmatrix} P_y & P_z \\ Q_y & Q_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} P_x & P_z \\ Q_x & Q_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} P_x & P_y \\ Q_x & Q_y \end{vmatrix}$$

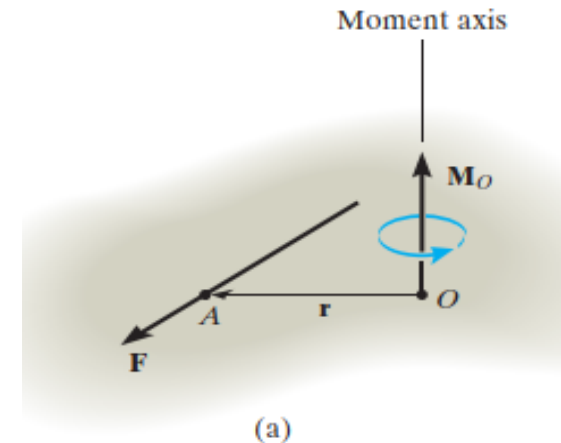


## MOMENT OF A FORCE Vector Formulation

- The moment of a force  $F$  about point  $O$ , or actually about the moment axis passing through  $O$  and perpendicular to the plane containing  $O$  and  $F$ , Fig. (a) can be expressed using the vector cross product, namely,

$$M_O = r \times F$$

- Here  $r$  represents a position vector directed from  $O$  to any point on the line of action of  $F$ .



## MOMENT OF A FORCE

### Vector Formulation

- The magnitude of the cross product is defined as

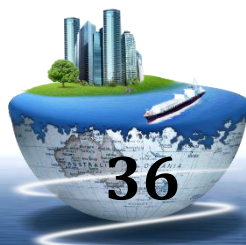
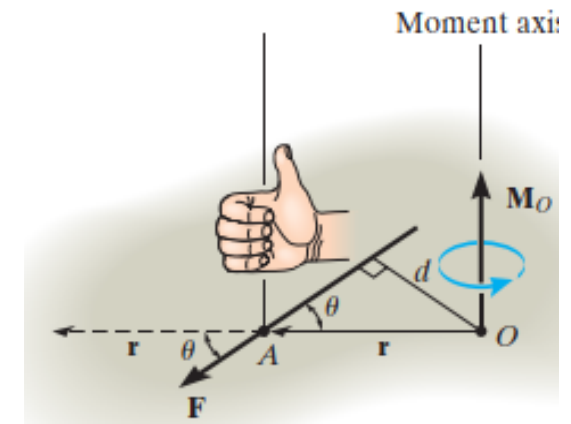
$$r \times F = M_O = rF \sin \theta$$

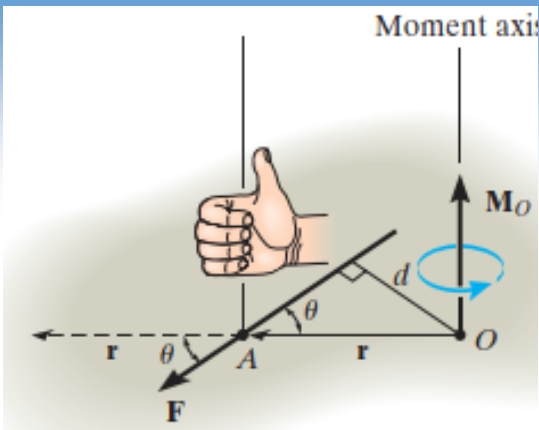
- where the angle  $\theta$  is measured between the tails of  $r$  and  $F$ .

- Since the moment arm  $d = r \sin \theta$ , then

$$M_O = rF \sin \theta = F(r \sin \theta) = Fd$$

- which agrees with  $M_O = Fd$

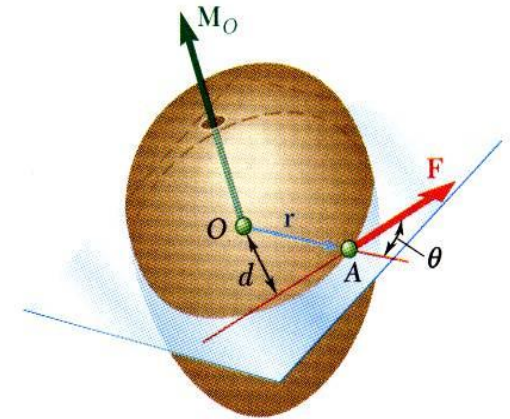




## MOMENT OF A FORCE

### Vector Formulation

- The direction and sense of  $\mathbf{M}_O$  are determined by the right-hand rule.
- Since the cross product does not obey the commutative law, the order of  $\mathbf{r} \times \mathbf{F}$  must be maintained to produce the correct sense of direction for  $\mathbf{M}_O$ .
- The moment vector  $\mathbf{M}_O$  is perpendicular to the plane containing  $O$  and the force  $\mathbf{F}$ .



(a)



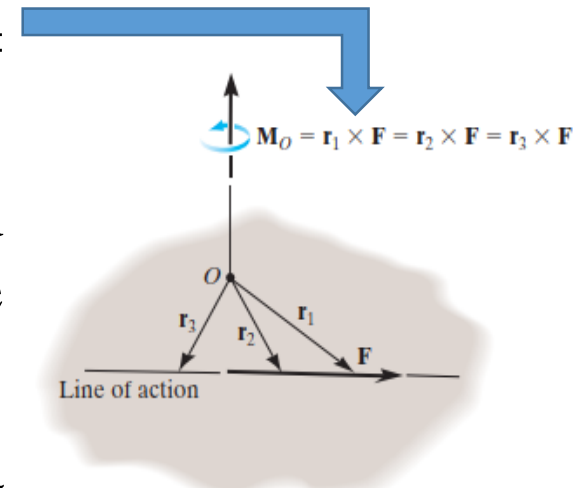
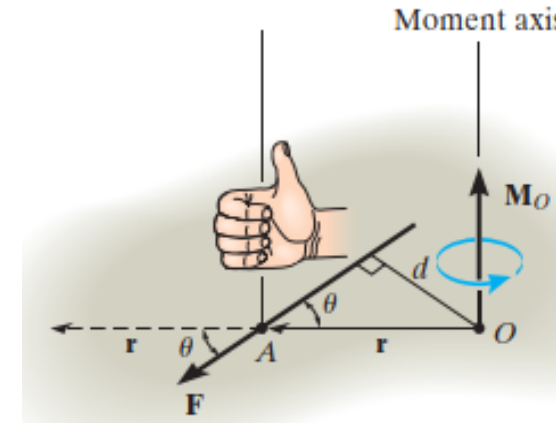
(b)

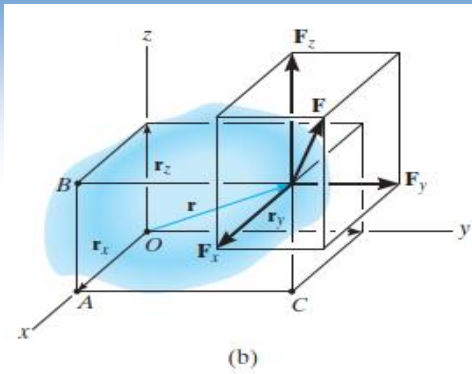


# MOMENT OF A FORCE

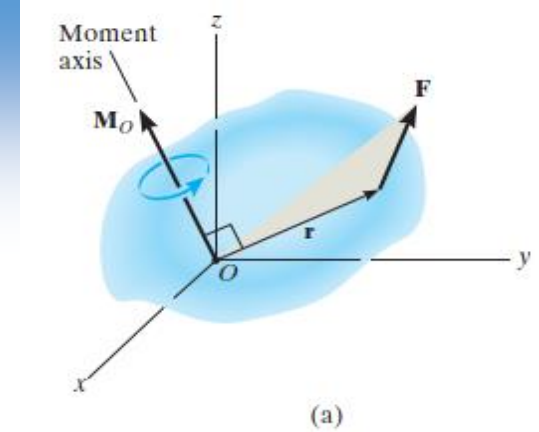
## Vector Formulation

- The cross product operation is often used in three dimensions since the perpendicular distance ( $d$ ) or moment arm from point  $O$  to the line of action of the force is not needed.
- We can use any position vector  $r$  measured from point  $O$  to any point on the line of action of the force  $F$ . Thus,  $M_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} \dots$
- Since  $F$  can be applied at any point along its line of action and still create this same moment about point  $O$ , then  $F$  can be considered a **sliding vector**.
- This property is called the **principle of transmissibility** of a force.





## MOMENT OF A FORCE Vector Formulation

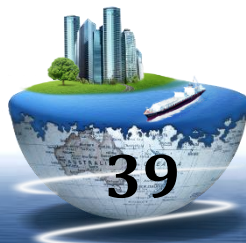


- The position vector  $\mathbf{r}$  and force  $\mathbf{F}$  can be expressed in cartesian vector formulation.

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

- Where  $\mathbf{r}_x$ ,  $\mathbf{r}_y$ ,  $\mathbf{r}_z$  represent the x, y, z components of the position vector drawn from point O to any point on the line of action of the force &  $F_x$ ,  $F_y$ ,  $F_z$  represent the x, y, z components of the force vector
- If the determinant is expanded, then we have

$$\mathbf{M}_O = (r_y F_z - r_z F_y)\mathbf{i} - (r_x F_z - r_z F_x)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k}$$



# MOMENT OF A FORCE

## Vector Formulation

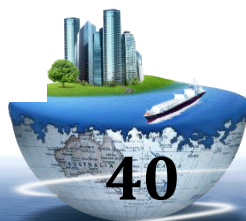
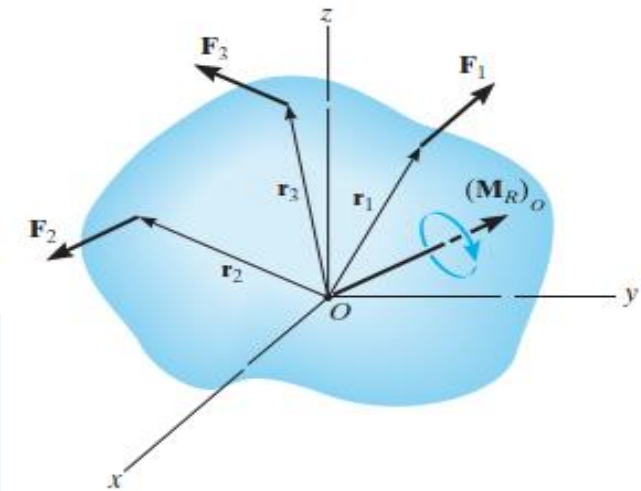
$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

- **Resultant Moment of a System of Forces.**

- If a body is acted upon by a system of forces, the resultant moment of the forces about point O can be determined by vector addition of the moment of each force.
- This resultant can be written as

$$(\mathbf{M}_R)_O = \sum (\mathbf{r} \times \mathbf{F})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}_1 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}_2 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}_3$$



$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

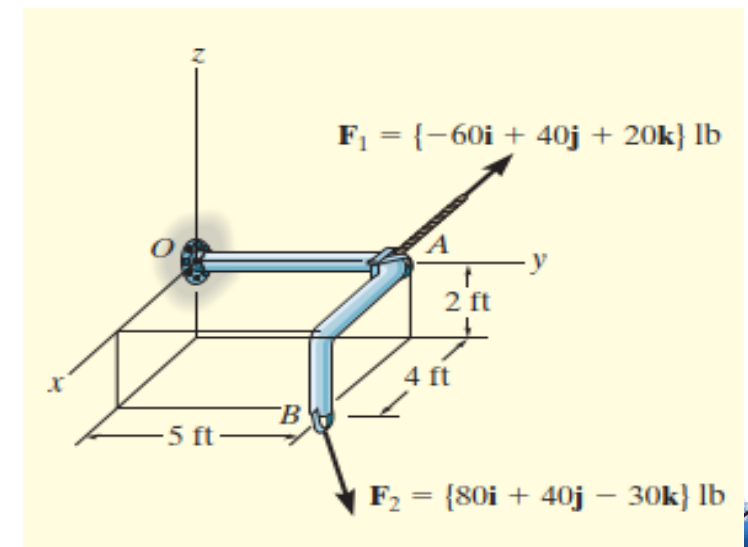
## MOMENT OF A FORCE

### Example 4.4

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

### Question

- Two forces act on the rod shown in Fig.
- Determine the resultant moment they create about the flange at O. Express the result as a Cartesian vector.



$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

# MOMENT OF A FORCE

## Example 4.4

Position vectors are directed from point  $O$  to each force as shown in Fig. 4-15b. These vectors are

$$\mathbf{r}_A = \{5\mathbf{j}\} \text{ ft}$$

$$\mathbf{r}_B = \{4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}\} \text{ ft}$$

The resultant moment about  $O$  is therefore

$$(\mathbf{M}_R)_O = \Sigma(\mathbf{r} \times \mathbf{F})$$

$$= \mathbf{r}_A \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}$$

$$= [5(20) - 0(40)]\mathbf{i} - [0]\mathbf{j} + [0(40) - (5)(-60)]\mathbf{k}$$

$$+ [5(-30) - (-2)(40)]\mathbf{i} - [4(-30) - (-2)(80)]\mathbf{j} + [4(40) - 5(80)]\mathbf{k}$$

$$= \{30\mathbf{i} - 40\mathbf{j} + 60\mathbf{k}\} \text{ lb} \cdot \text{ft}$$

*Ans.*

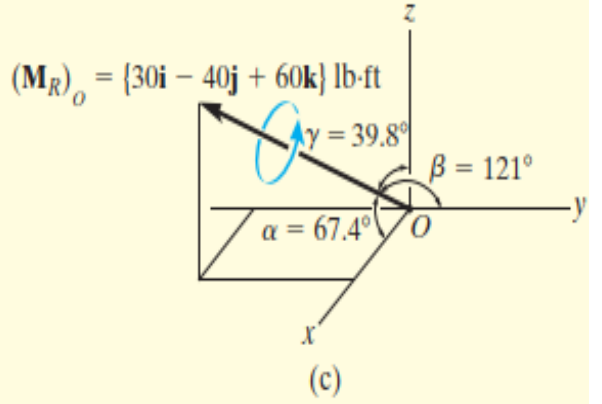
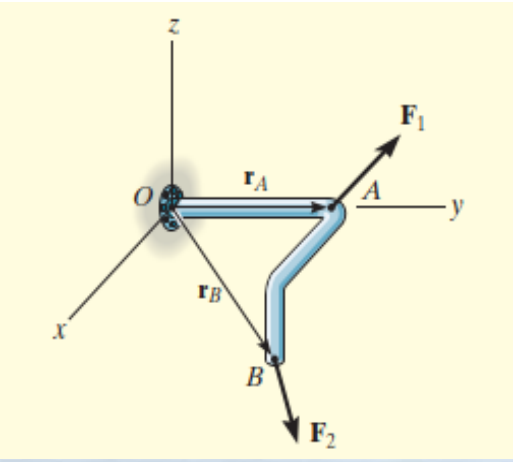
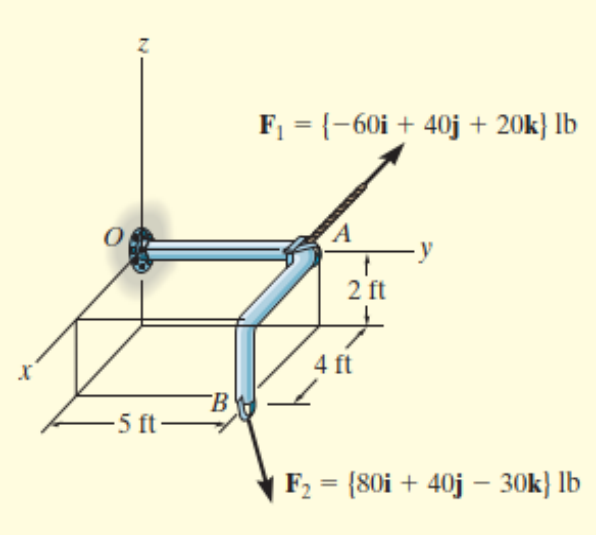


Fig. 4-15





$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

## MOMENT OF A FORCE

### Example 4.5

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

### Question

• Determine the moment of the force  $\mathbf{F}$  about point  $O$  and  $P$ . Express the result as a Cartesian vector.

• Answer  $\mathbf{r}_{OA} = \{1\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}\}$  m &  $\mathbf{r}_{PA} = (1 - 0)\mathbf{i} + (-2 - 4)\mathbf{j} + (6 - 3)\mathbf{k} = \{1\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}\}$  m

$$\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{F}$$

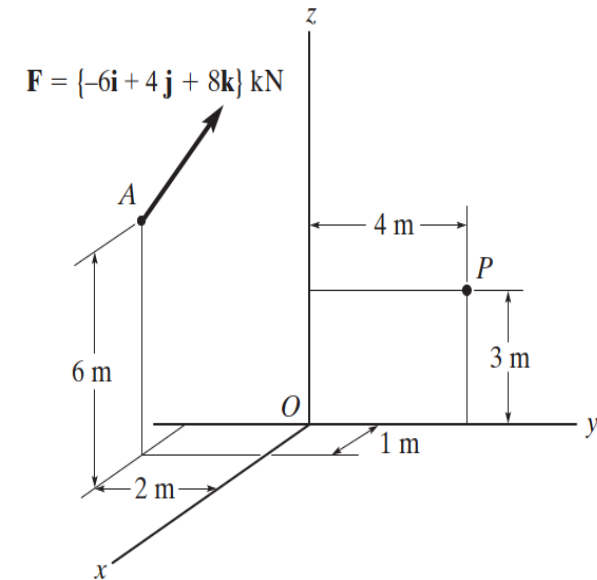
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 6 \\ -6 & 4 & 8 \end{vmatrix}$$

$$\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{F} = \{-40\mathbf{i} - 44\mathbf{j} - 8\mathbf{k}\} \text{ kN.m}$$

$$\mathbf{M}_P = \mathbf{r}_{PA} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -6 & 3 \\ -6 & 4 & 8 \end{vmatrix}$$

$$\mathbf{M}_P = \mathbf{r}_{PA} \times \mathbf{F} = \{-60\mathbf{i} - 26\mathbf{j} - 32\mathbf{k}\} \text{ kN.m}$$



$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

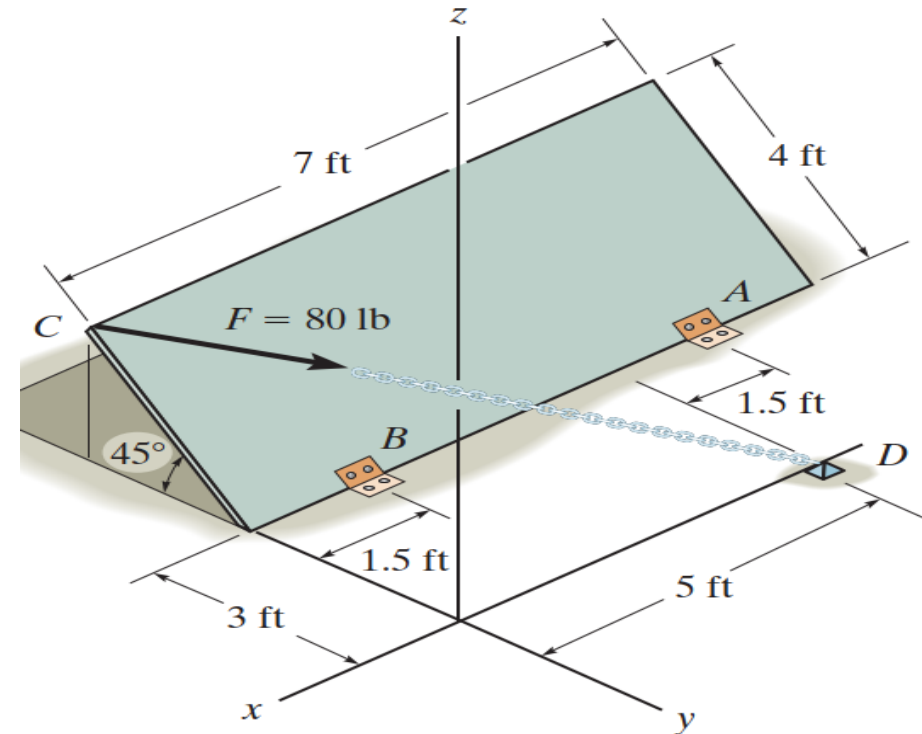
## MOMENT OF A FORCE

### Example 4.6

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

### Question

- Determine the moment of the force  $F$  about the door hinges at  $A$  and  $B$ . Express the result as a Cartesian vector.





$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

## MOMENT OF A FORCE

### Example 4.6

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

## Solution

**Position Vectors And Force Vector.** The coordinates of points  $A$ ,  $C$  and  $D$  are  $A(-6.5, -3, 0)$  ft,  $C[0, -(3 + 4 \cos 45^\circ), 4 \sin 45^\circ]$  ft and  $D(-5, 0, 0)$  ft, respectively. Thus,

$$\begin{aligned} \mathbf{r}_{AC} &= [0 - (-6.5)]\mathbf{i} + [-(3 + 4 \cos 45^\circ) - (-3)]\mathbf{j} + (4 \sin 45^\circ - 0)\mathbf{k} \\ &= \{6.5\mathbf{i} - 2.8284\mathbf{j} + 2.8284\mathbf{k}\} \text{ ft} \end{aligned}$$

$$\mathbf{r}_{AD} = [-5 - (-6.5)]\mathbf{i} + [0 - (-3)]\mathbf{j} + (0 - 0)\mathbf{k} = \{1.5\mathbf{i} + 3\mathbf{j}\} \text{ ft}$$

$$\begin{aligned} \mathbf{r}_{CD} &= (-5 - 0)\mathbf{i} + \{0 - [-(3 + 4 \cos 45^\circ)]\}\mathbf{j} + (0 - 4 \sin 45^\circ)\mathbf{k} \\ &= \{-5\mathbf{i} + 5.8284\mathbf{j} - 2.8284\mathbf{k}\} \text{ ft} \end{aligned}$$

$$\begin{aligned} \mathbf{F} &= F \left( \frac{\mathbf{r}_{CD}}{r_{CD}} \right) = 80 \left( \frac{-5\mathbf{i} + 5.8284\mathbf{j} - 2.8284\mathbf{k}}{\sqrt{(-5)^2 + 5.8284^2 + (-2.8284)^2}} \right) \\ &= \{-48.88\mathbf{i} + 56.98\mathbf{j} - 27.65\mathbf{k}\} \text{ lb} \end{aligned}$$

**Moment of  $F$  About Point  $A$ .**

$$\mathbf{M}_A = \mathbf{r}_{AC} \times \mathbf{F}$$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6.5 & -2.8284 & 2.8284 \\ -48.88 & 56.98 & -27.65 \end{vmatrix} \\ &= \{-82.9496\mathbf{i} + 41.47\mathbf{j} + 232.10\mathbf{k}\} \text{ lb} \cdot \text{ft} \\ &= \{-82.9\mathbf{i} + 41.5\mathbf{j} + 232\mathbf{k}\} \text{ lb} \cdot \text{ft} \end{aligned}$$

OR

$$\mathbf{M}_A = \mathbf{r}_{AD} \times \mathbf{F}$$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.5 & 3 & 0 \\ -48.88 & 56.98 & -27.65 \end{vmatrix} \\ &= \{-82.9496\mathbf{i} + 41.47\mathbf{j} + 232.10\mathbf{k}\} \text{ lb} \cdot \text{ft} \\ &= \{-82.9\mathbf{i} + 41.5\mathbf{j} + 232\mathbf{k}\} \text{ lb} \cdot \text{ft} \end{aligned}$$





$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

## MOMENT OF A FORCE

### Example 4.6

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

## Solution

**Position Vectors And Force Vector.** The coordinates of points  $B$ ,  $C$  and  $D$  are  $B(-1.5, -3, 0)$  ft,  $C[0, -(3 + 4 \cos 45^\circ), 4 \sin 45^\circ]$  ft and  $D(-5, 0, 0)$  ft, respectively. Thus,

$$\begin{aligned} \mathbf{r}_{BC} &= [0 - (-1.5)]\mathbf{i} + [-(3 + 4 \cos 45^\circ) - (-3)]\mathbf{j} + (4 \sin 45^\circ - 0)\mathbf{k} \\ &= \{1.5\mathbf{i} - 2.8284\mathbf{j} + 2.8284\mathbf{k}\} \text{ ft} \end{aligned}$$

$$\mathbf{r}_{BD} = [-5 - (-1.5)]\mathbf{i} + [0 - (-3)]\mathbf{j} + (0 - 0)\mathbf{k} = \{-3.5\mathbf{i} + 3\mathbf{j}\} \text{ ft}$$

$$\begin{aligned} \mathbf{r}_{CD} &= (-5 - 0)\mathbf{i} + \{0 - [-(3 + 4 \cos 45^\circ)]\}\mathbf{j} + (0 - 4 \sin 45^\circ)\mathbf{k} \\ &= \{-5\mathbf{i} + 5.8284\mathbf{j} - 2.8284\mathbf{k}\} \text{ ft} \end{aligned}$$

$$\begin{aligned} \mathbf{F} &= F \left( \frac{\mathbf{r}_{CD}}{r_{CD}} \right) = 80 \left( \frac{-5\mathbf{i} + 5.8284\mathbf{j} - 2.8284\mathbf{k}}{\sqrt{(-5)^2 + 5.8284^2 + (-2.8284)^2}} \right) \\ &= \{-48.88\mathbf{i} + 56.98\mathbf{j} - 27.65\mathbf{k}\} \text{ lb} \end{aligned}$$

**Moment of  $F$  About Point  $B$ .**

$$\mathbf{M}_B = \mathbf{r}_{BC} \times \mathbf{F}$$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.5 & -2.8284 & 2.8284 \\ -48.88 & 56.98 & -27.65 \end{vmatrix} \\ &= \{-82.9496\mathbf{i} - 96.77\mathbf{j} - 52.78\mathbf{k}\} \text{ lb} \cdot \text{ft} \\ &= \{-82.9\mathbf{i} - 96.8\mathbf{j} - 52.8\mathbf{k}\} \text{ lb} \cdot \text{ft} \end{aligned}$$

or

$$\mathbf{M}_B = \mathbf{r}_{BD} \times \mathbf{F}$$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3.5 & 3 & 0 \\ -48.88 & 56.98 & -27.65 \end{vmatrix} \\ &= \{-82.9496\mathbf{i} - 96.77\mathbf{j} - 52.78\mathbf{k}\} \text{ lb} \cdot \text{ft} \\ &= \{-82.9\mathbf{i} - 96.8\mathbf{j} - 52.8\mathbf{k}\} \text{ lb} \cdot \text{ft} \end{aligned}$$

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

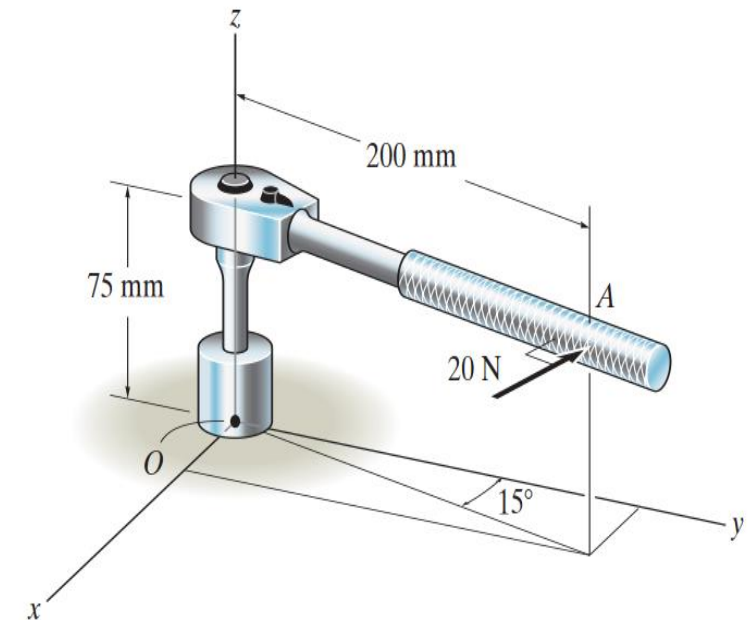
## MOMENT OF A FORCE

### Example 4.7

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

### Question

- A 20-N horizontal force is applied perpendicular to the handle of the socket wrench. Determine the magnitude and the coordinate direction angles of the moment created by this force about point O.



$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

# MOMENT OF A FORCE

## Example 4.7

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

### Solution

$$\begin{aligned} \mathbf{r}_A &= 0.2 \sin 15^\circ \mathbf{i} + 0.2 \cos 15^\circ \mathbf{j} + 0.075 \mathbf{k} \\ &= 0.05176 \mathbf{i} + 0.1932 \mathbf{j} + 0.075 \mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{F} &= -20 \cos 15^\circ \mathbf{i} + 20 \sin 15^\circ \mathbf{j} \\ &= -19.32 \mathbf{i} + 5.176 \mathbf{j} \end{aligned}$$

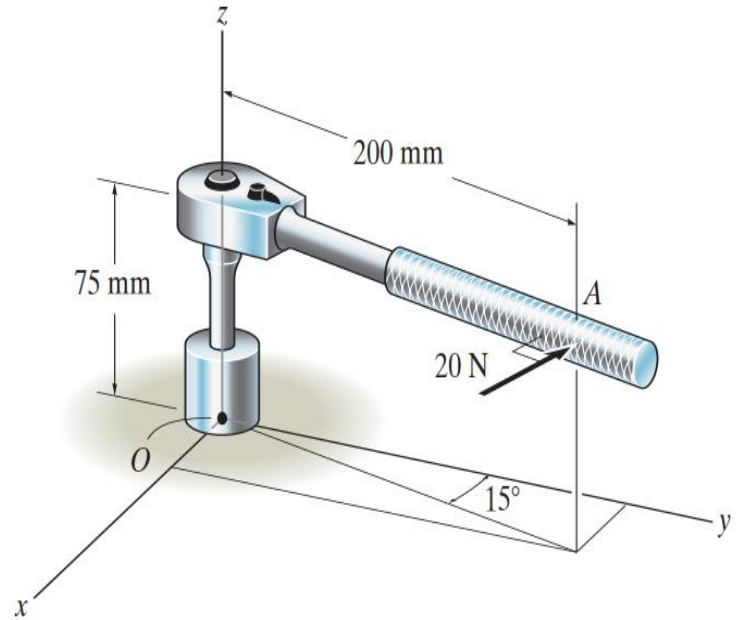
$$\begin{aligned} \mathbf{M}_O &= \mathbf{r}_A \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.05176 & 0.1932 & 0.075 \\ -19.32 & 5.176 & 0 \end{vmatrix} \\ &= \{-0.3882 \mathbf{i} - 1.449 \mathbf{j} + 4.00 \mathbf{k}\} \text{ N} \cdot \text{m} \end{aligned}$$

$$M_O = 4.272 = 4.27 \text{ N} \cdot \text{m}$$

$$\alpha = \cos^{-1} \left( \frac{-0.3882}{4.272} \right) = 95.2^\circ$$

$$\beta = \cos^{-1} \left( \frac{-1.449}{4.272} \right) = 110^\circ$$

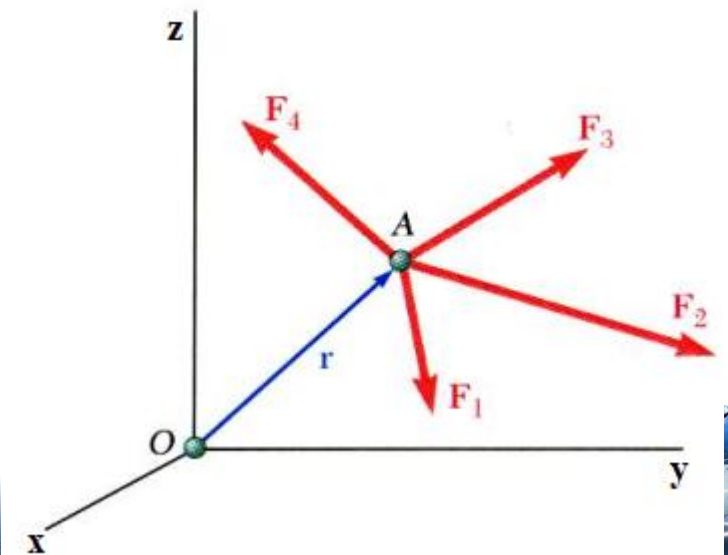
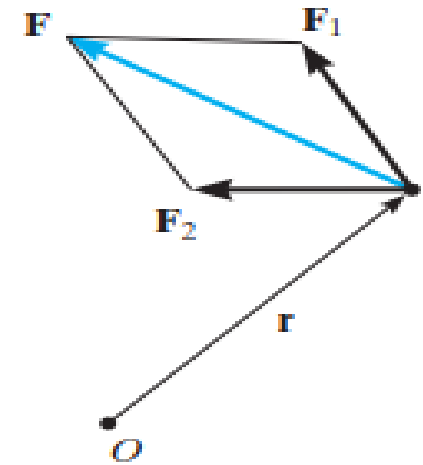
$$\gamma = \cos^{-1} \left( \frac{4}{4.272} \right) = 20.6^\circ$$



# MOMENT OF A FORCE

## Varignon's Theorem (Principles of Moments)

- It states that the moment of a force about a point is equal to the sum of the moments of the components of the force about the point
- Or the moment about a give point O of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point O
- This theorem can be proven easily using the vector cross product since the cross product obeys the distributive law:
- $M_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \dots) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \dots$



$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

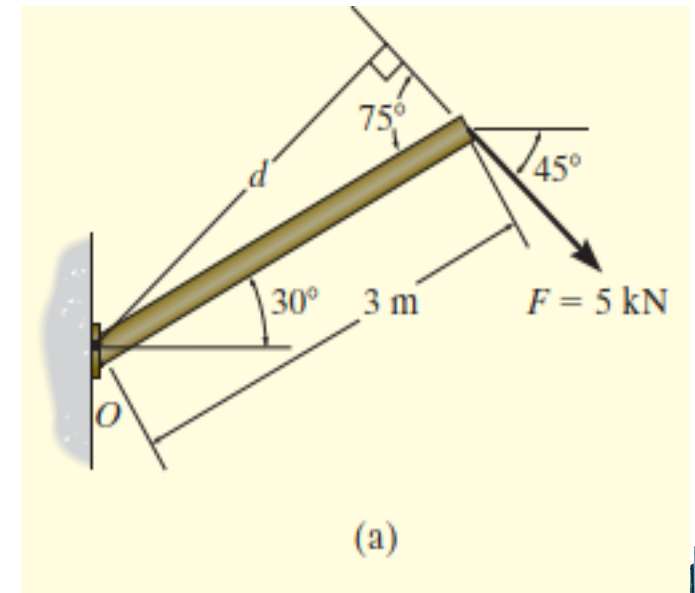
## MOMENT OF A FORCE

### Example 4.8

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

### Question

- Determine the moment of the force in Fig. 4–18a about point O.
- Three different type of solutions are available!



# MOMENT OF A FORCE

## Example 4.8

### Solution

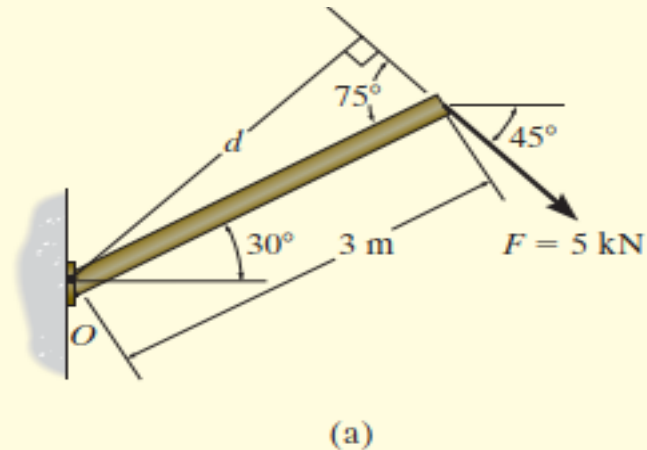
The moment arm  $d$  in Fig. 4-18a can be found from trigonometry.

$$d = (3 \text{ m}) \sin 75^\circ = 2.898 \text{ m}$$

Thus,

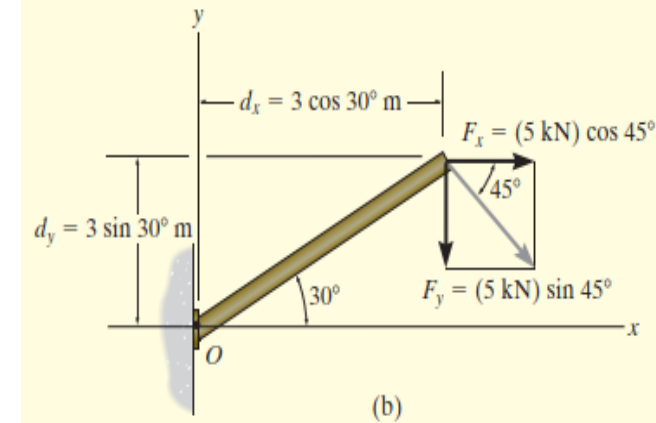
$$M_O = Fd = (5 \text{ kN})(2.898 \text{ m}) = 14.5 \text{ kN} \cdot \text{m} \quad \text{Ans.}$$

Since the force tends to rotate or orbit clockwise about point  $O$ , the moment is directed into the page.



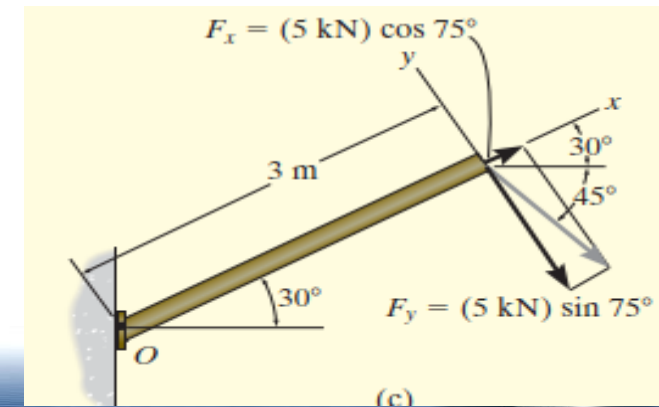
The  $x$  and  $y$  components of the force are indicated in Fig. 4-18b. Considering counterclockwise moments as positive, and applying the principle of moments, we have

$$\begin{aligned} \zeta + M_O &= -F_x d_y - F_y d_x \\ &= -(5 \cos 45^\circ \text{ kN})(3 \sin 30^\circ \text{ m}) - (5 \sin 45^\circ \text{ kN})(3 \cos 30^\circ \text{ m}) \\ &= -14.5 \text{ kN} \cdot \text{m} = 14.5 \text{ kN} \cdot \text{m} \quad \text{Ans.} \end{aligned}$$

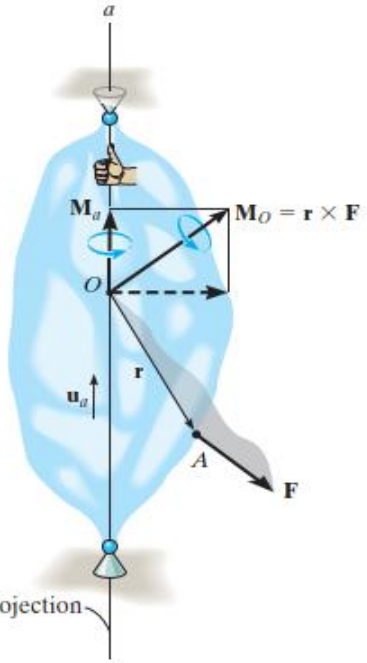


The  $x$  and  $y$  axes can be set parallel and perpendicular to the rod's axis as shown in Fig. 4-18c. Here  $F_x$  produces no moment about point  $O$  since its line of action passes through this point. Therefore,

$$\begin{aligned} \zeta + M_O &= -F_y d_x \\ &= -(5 \sin 75^\circ \text{ kN})(3 \text{ m}) \\ &= -14.5 \text{ kN} \cdot \text{m} = 14.5 \text{ kN} \cdot \text{m} \quad \text{Ans.} \end{aligned}$$



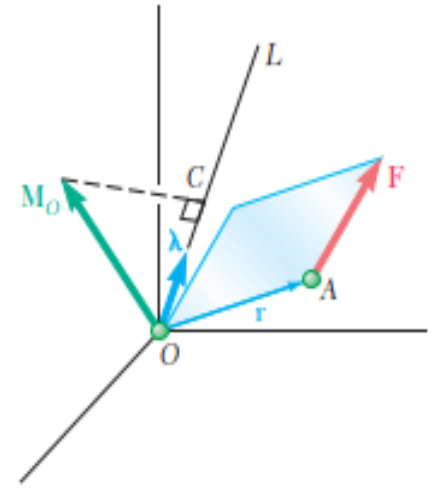
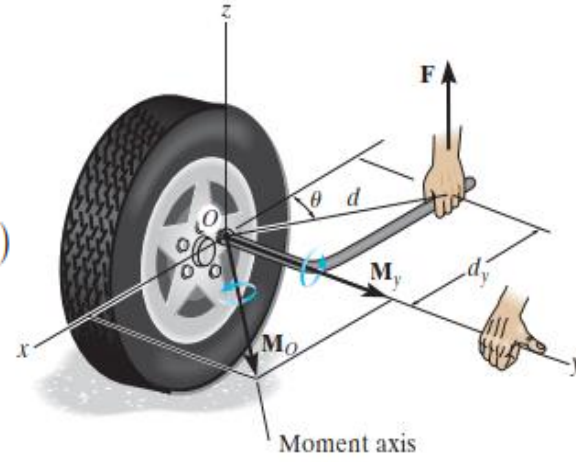
# MOMENT OF A FORCE ABOUT A SPECIFIED AXIS



$$M_a = [u_a \mathbf{i} + u_a \mathbf{j} + u_a \mathbf{k}] \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= u_a(r_y F_z - r_z F_y) - u_a(r_x F_z - r_z F_x) + u_a(r_x F_y - r_y F_x)$$

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{ax} & u_{ay} & u_{az} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

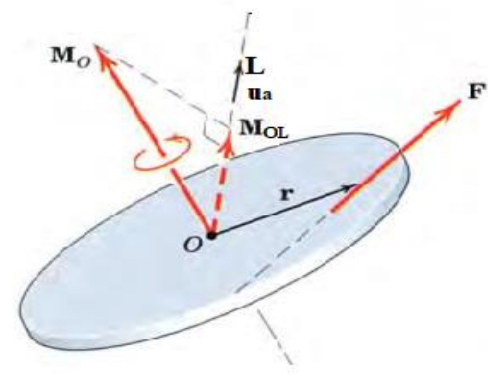


$$\mathbf{M}_a = M_a \mathbf{u}_a$$

← Triple Scalar Product →

$$M_{OL} = \lambda \cdot M_O = \lambda \cdot (\mathbf{r} \times \mathbf{F})$$

- $u_{ax}, u_{ay}, u_{az}$  represent the  $x, y, z$  components of the unit vector defining the direction of the  $a$  axis
- $r_x, r_y, r_z$  represent the  $x, y, z$  components of the position vector extended from *any point*  $O$  on the  $a$  axis to *any point*  $A$  on the line of action of the force
- $F_x, F_y, F_z$  represent the  $x, y, z$  components of the force vector.

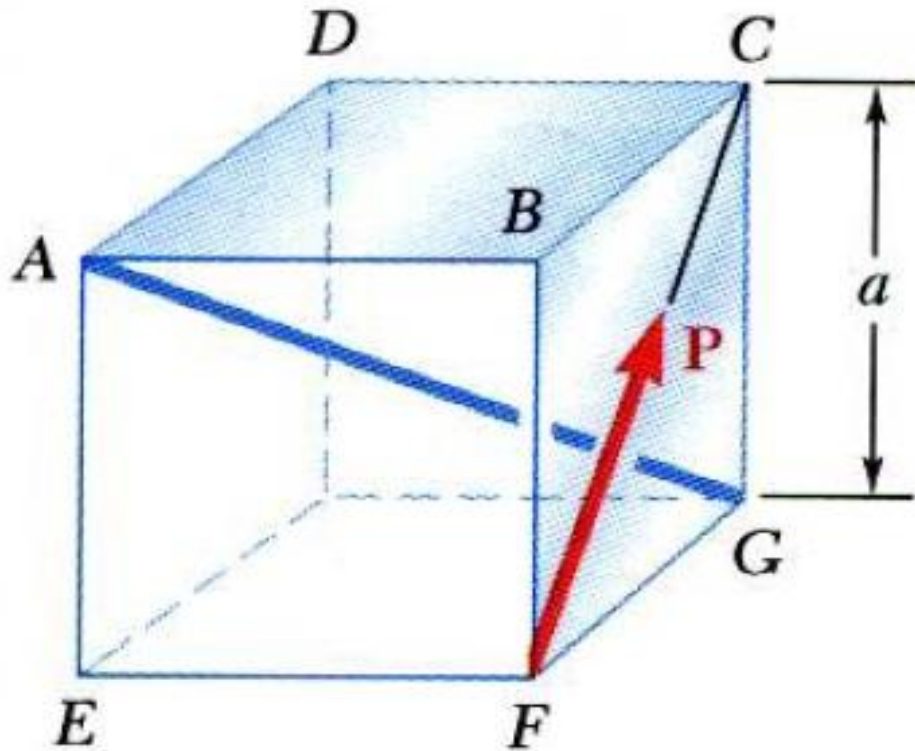


$$M_{OL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

# MOMENT OF A FORCE ABOUT A SPECIFIED AXIS Example 4.9

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

## Question



A cube is acted on by a force  $P$  as shown. Determine the moment of  $P$

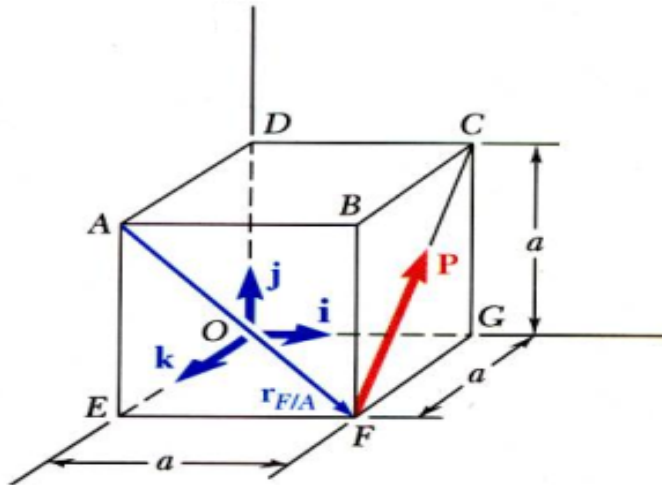
- a) about  $A$
- b) about the edge  $AB$  and
- c) about the diagonal  $AG$  of the cube.
- d) Determine the perpendicular distance between  $AG$  and  $FC$ .

# MOMENT OF A FORCE ABOUT A SPECIFIED AXIS

## Example 4.9

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

Solution



- Moment of  $\mathbf{P}$  about  $A$ ,

$$\vec{M}_A = \vec{r}_{F/A} \times \vec{P}$$

$$\vec{r}_{F/A} = a\vec{i} - a\vec{j} = a(\vec{i} - \vec{j})$$

$$\vec{P} = P(\sqrt{2}\vec{i} + \sqrt{2}\vec{j}) = P\sqrt{2}(\vec{i} + \vec{j})$$

$$\vec{M}_A = a(\vec{i} - \vec{j}) \times P\sqrt{2}(\vec{i} + \vec{j})$$

$$\vec{M}_A = (aP\sqrt{2})(\vec{i} + \vec{j} + \vec{k})$$

- Moment of  $\mathbf{P}$  about  $AB$ ,

$$M_{AB} = \vec{i} \cdot \vec{M}_A$$

$$= \vec{i} \cdot (aP\sqrt{2})(\vec{i} + \vec{j} + \vec{k})$$

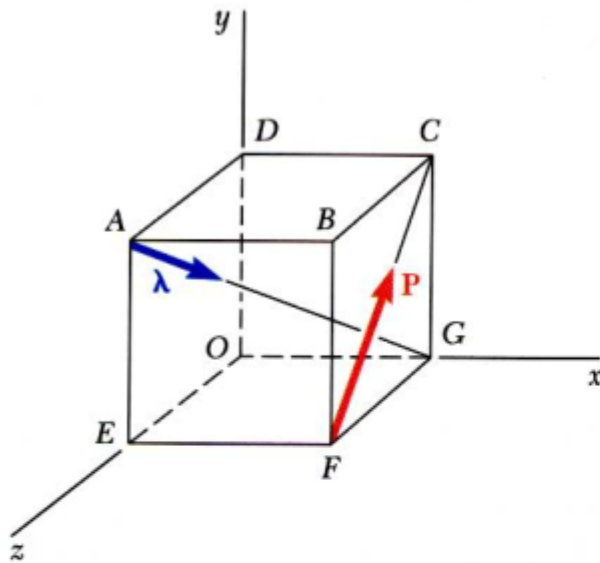
$$M_{AB} = aP\sqrt{2}$$

# MOMENT OF A FORCE ABOUT A SPECIFIED AXIS

## Example 4.9

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

Solution



- Moment of  $\mathbf{P}$  about the diagonal  $AG$ ,

$$M_{AG} = \vec{\lambda} \bullet \vec{M}_A$$

$$\vec{\lambda} = \frac{\vec{r}_{A/G}}{r_{A/G}} = \frac{a\vec{i} - a\vec{j} - a\vec{k}}{a\sqrt{3}} = \frac{1}{\sqrt{3}}(\vec{i} - \vec{j} - \vec{k})$$

$$\vec{M}_A = \frac{aP}{\sqrt{2}}(\vec{i} + \vec{j} + \vec{k})$$

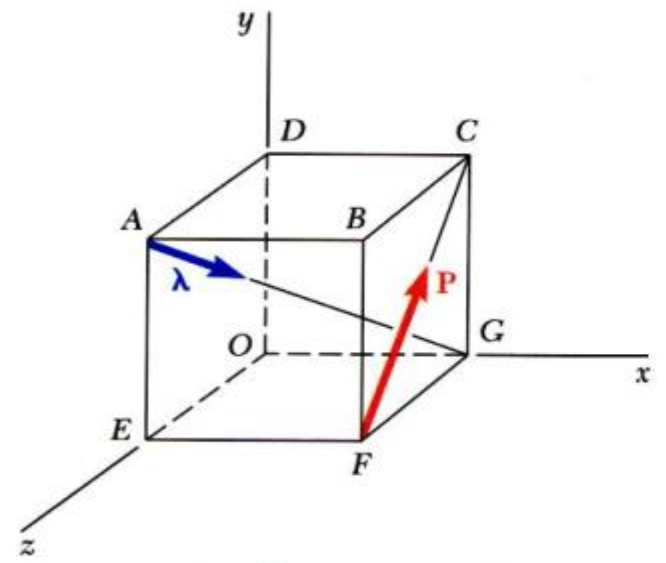
$$\begin{aligned} M_{AG} &= \frac{1}{\sqrt{3}}(\vec{i} - \vec{j} - \vec{k}) \bullet \frac{aP}{\sqrt{2}}(\vec{i} + \vec{j} + \vec{k}) \\ &= \frac{aP}{\sqrt{6}}(1 - 1 - 1) \end{aligned}$$

$$M_{AG} = -\frac{aP}{\sqrt{6}}$$

# MOMENT OF A FORCE ABOUT A SPECIFIED AXIS Example 4.9

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

Solution



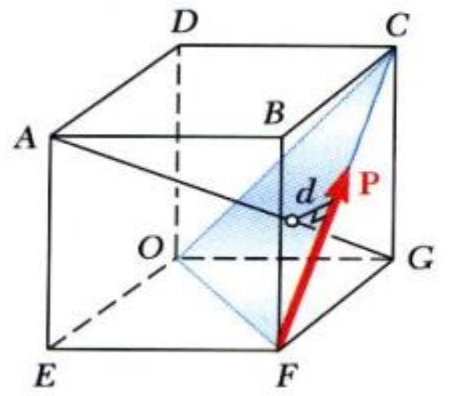
- Perpendicular distance between  $AG$  and  $FC$ ,

$$\begin{aligned} \vec{P} \cdot \vec{\lambda} &= \frac{P}{\sqrt{2}} (\vec{j} - \vec{k}) \cdot \frac{1}{\sqrt{3}} (\vec{i} - \vec{j} - \vec{k}) = \frac{P}{\sqrt{6}} (0 - 1 + 1) \\ &= 0 \end{aligned}$$

Therefore,  $P$  is perpendicular to  $AG$ .

$$|M_{AG}| = \frac{aP}{\sqrt{6}} = Pd$$

$$d = \frac{a}{\sqrt{6}}$$



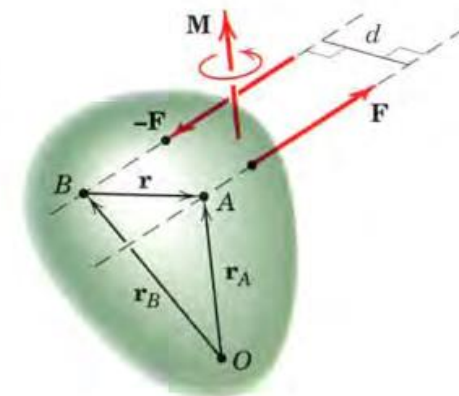
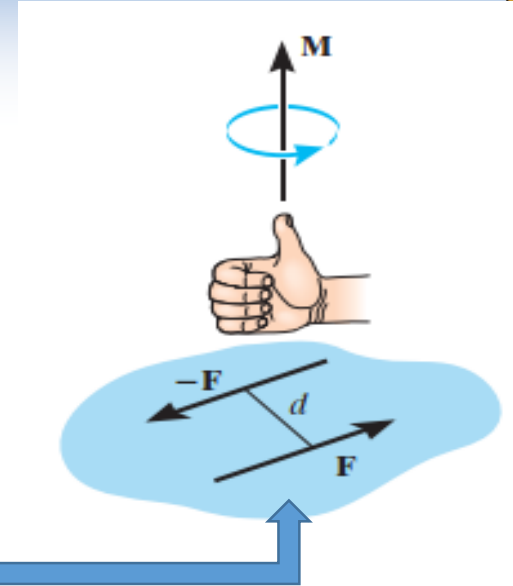


## MOMENT OF A COUPLE

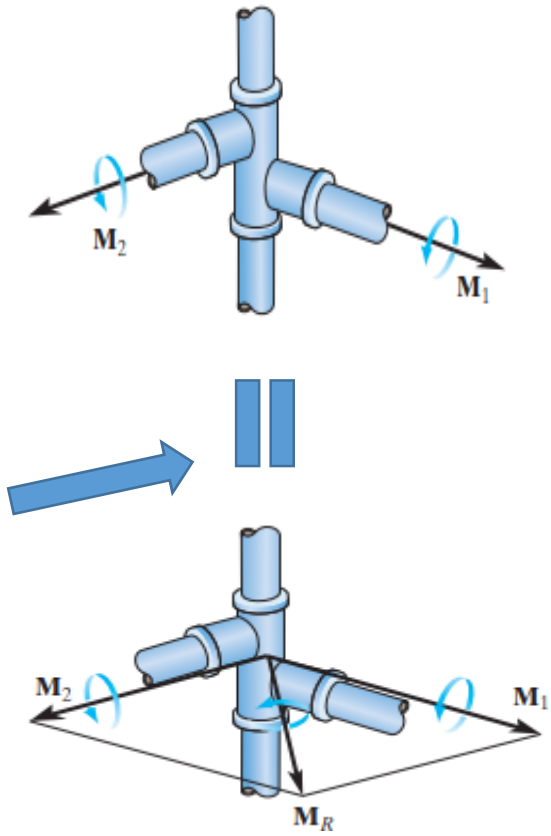
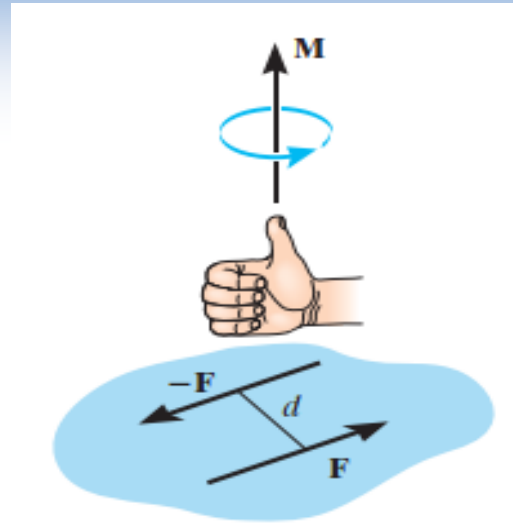
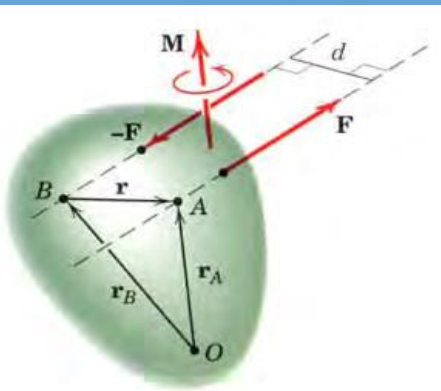
- A couple is defined as two parallel forces that have the same magnitude, but opposite directions, and are separated by a perpendicular distance  $d$ .
- Since the resultant force is zero, the only effect of a couple is to produce an actual rotation, or if no movement is possible, there is a tendency of rotation in a specified direction
- The couple moment determined about  $O$  is therefore:

$$\mathbf{M} = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times -\mathbf{F} = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$

- However  $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$ , so that  $\mathbf{M} = \mathbf{r} \times \mathbf{F}$



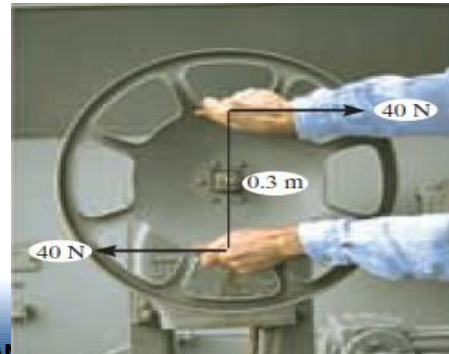
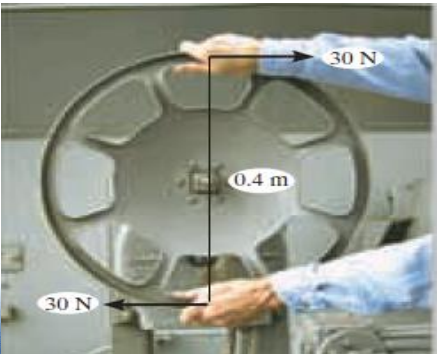
# MOMENT OF A COUPLE



- Scalar formulation  $M = Fd$
- Vector formulation  $M = \mathbf{r} * \mathbf{F}$

• **Resultant Couple Moment.** Since couple moments are vectors, their resultant can be determined by vector addition  $\mathbf{M}_R = \Sigma(\mathbf{r} \times \mathbf{F})$

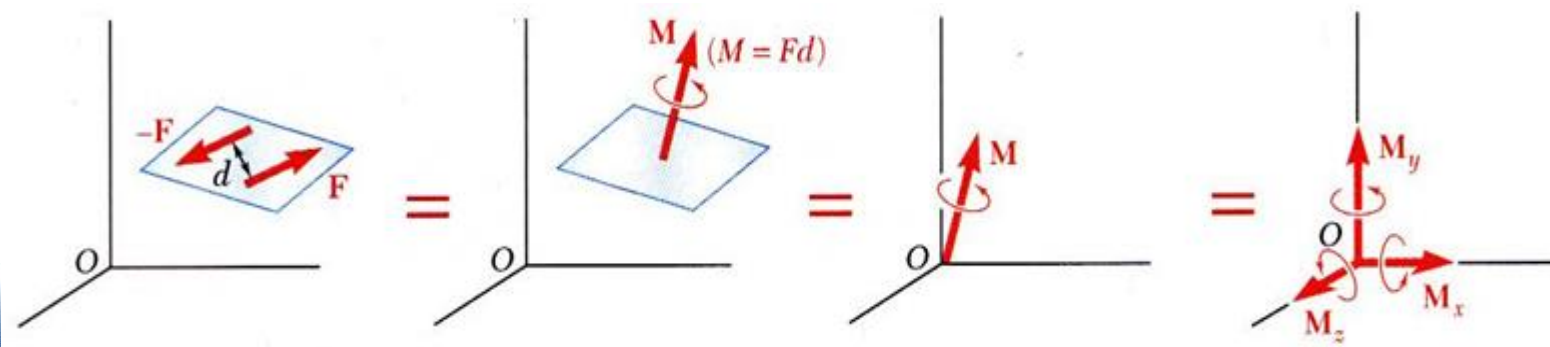
• **Equivalent Couples.** If two couples produce a moment with the same magnitude and direction, then these two couples are equivalent.



# MOMENT OF A COUPLE

## Representing Couple Moment as Vector

- A couple can be represented by a vector with *magnitude* and *direction* equal to the moment of the couple.
- *Couple moment vectors* obey the law of addition of vectors.
- **Couple moment vectors are free vectors, i.e., the point of application is not significant**
- Couple moment vectors may be resolved into component vectors x, y & z

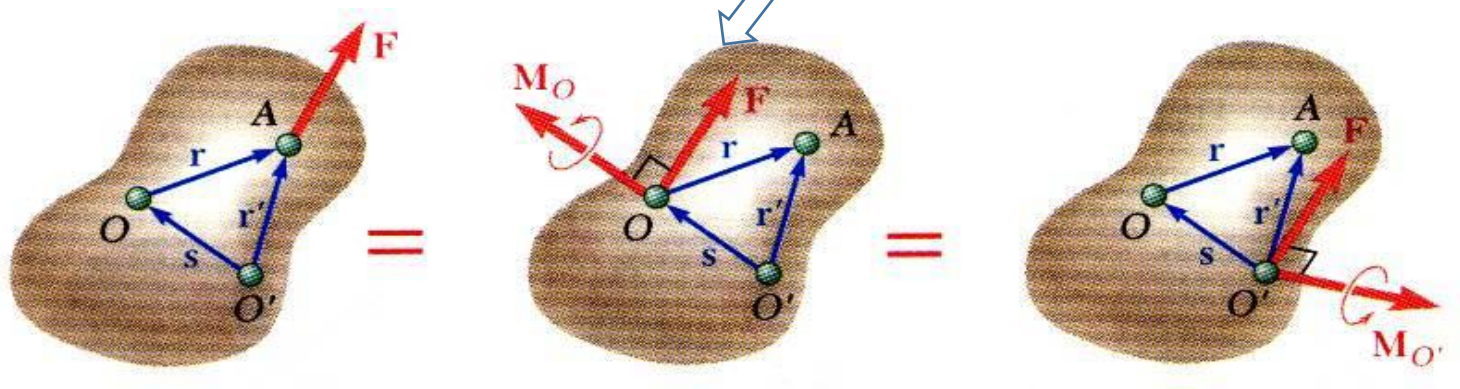
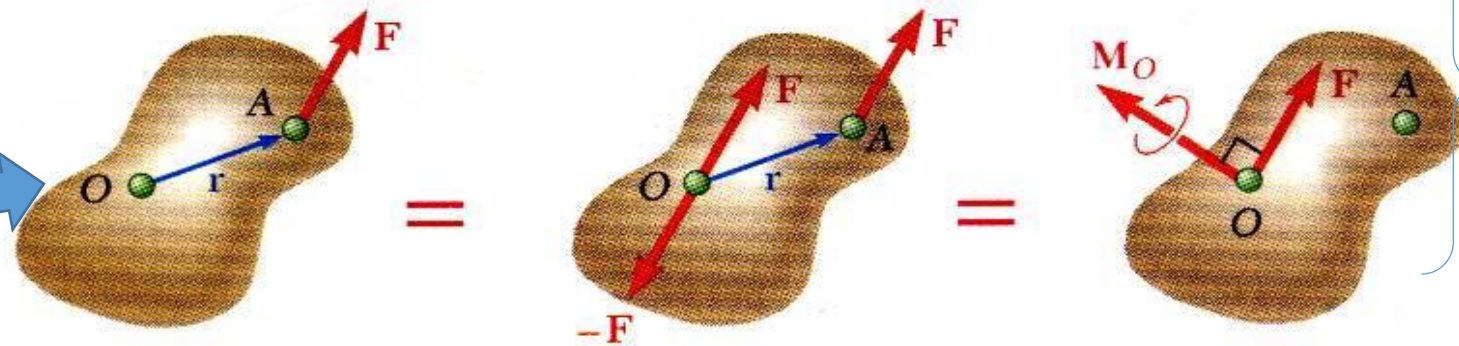


# MOMENT OF A COUPLE

## Resolution of a Force Into a Force at $O$ and a Couple

• Force vector  $F$  can simply be moved to  $O$  by modifying its action on the body if not along the line of action

▶ Attaching equal and opposite force vectors at  $O$  produces no net effect on the body



▶ The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*



## MOMENT OF A COUPLE

# Resolution of a Force and a Couple System into Equivalent Resultant Force and Couple Moments

- A system of several forces and couple moments acting on a body can be reduced to an equivalent single resultant force acting at a point O and a resultant couple moment by applying the following eqns:

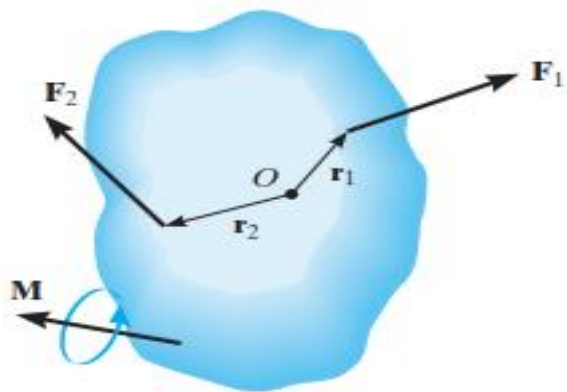
$$\mathbf{F}_R = \Sigma \mathbf{F}$$

$$(\mathbf{M}_R)_O = \Sigma \mathbf{M}_O + \Sigma \mathbf{M}$$

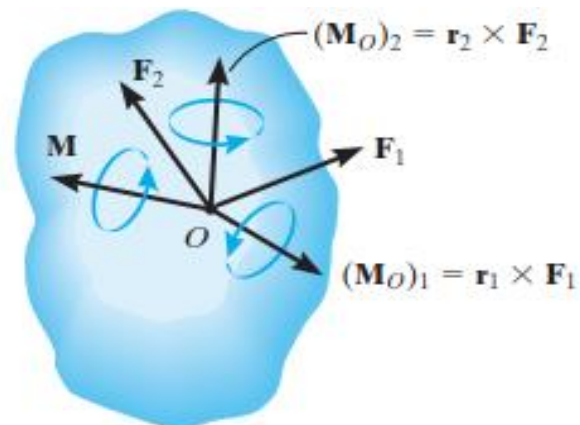
$$(F_R)_x = \Sigma F_x$$

$$(F_R)_y = \Sigma F_y$$

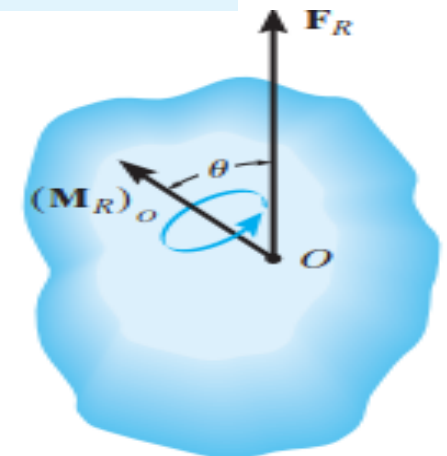
$$(M_R)_O = \Sigma M_O + \Sigma M$$



=



=



# MOMENT OF A COUPLE

## Example 4.10

### EXAMPLE 4.10

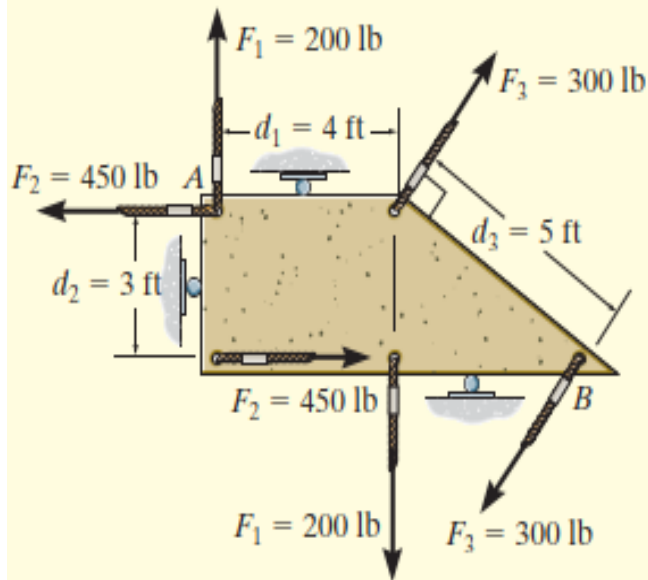


Fig. 4-30

Determine the resultant couple moment of the three couples acting on the plate in Fig. 4-30.

#### SOLUTION

As shown the perpendicular distances between each pair of couple forces are  $d_1 = 4$  ft,  $d_2 = 3$  ft, and  $d_3 = 5$  ft. Considering counterclockwise couple moments as positive, we have

$$\begin{aligned} \zeta + M_R &= \Sigma M; M_R = -F_1d_1 + F_2d_2 - F_3d_3 \\ &= -(200 \text{ lb})(4 \text{ ft}) + (450 \text{ lb})(3 \text{ ft}) - (300 \text{ lb})(5 \text{ ft}) \\ &= -950 \text{ lb} \cdot \text{ft} = 950 \text{ lb} \cdot \text{ft} \quad \text{Ans.} \end{aligned}$$

The negative sign indicates that  $M_R$  has a clockwise rotational sense.



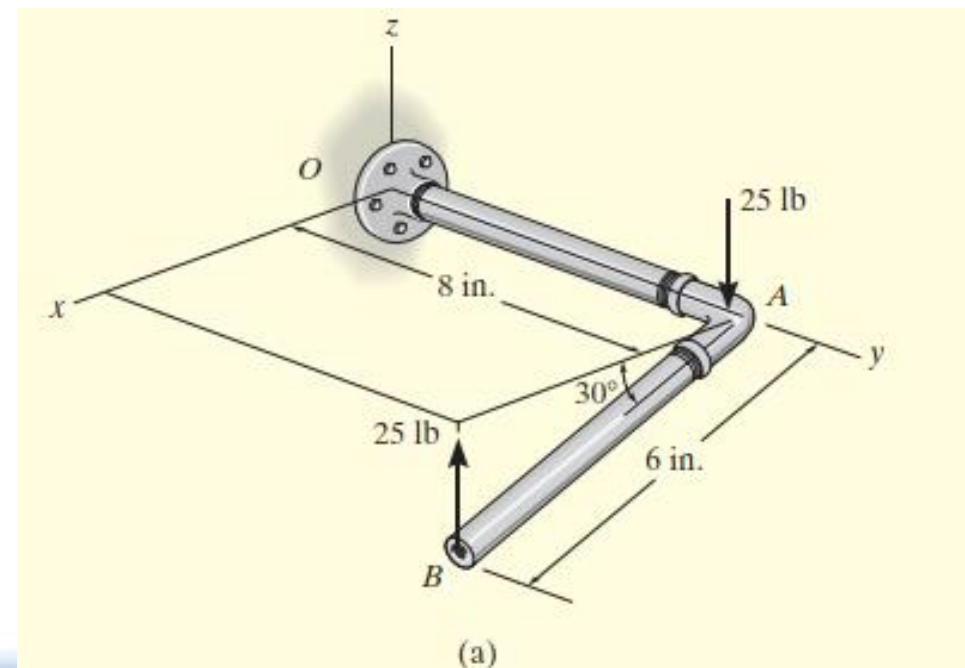
# MOMENT OF A COUPLE

## Example 4.11

### Question

- Determine the couple moment acting on the pipe shown in Fig. Segment AB is directed  $30^\circ$  below the x–y plane.

- **4 ways of solving this problem!**



# MOMENT OF A COUPLE

The moment of the two couple forces can be found about *any point*. If point  $O$  is considered, Fig. 4-32b, we have

$$\begin{aligned} \mathbf{M} &= \mathbf{r}_A \times (-25\mathbf{k}) + \mathbf{r}_B \times (25\mathbf{k}) \\ &= (8\mathbf{j}) \times (-25\mathbf{k}) + (6 \cos 30^\circ \mathbf{i} + 8\mathbf{j} - 6 \sin 30^\circ \mathbf{k}) \times (25\mathbf{k}) \\ &= -200\mathbf{i} - 129.9\mathbf{j} + 200\mathbf{i} \\ &= \{-130\mathbf{j}\} \text{ lb} \cdot \text{in.} \end{aligned}$$

*Ans.*

It is *easier* to take moments of the couple forces about a point lying on the line of action of one of the forces, e.g., point  $A$ , Fig. 4-32c. In this case the moment of the force at  $A$  is zero, so that

$$\begin{aligned} \mathbf{M} &= \mathbf{r}_{AB} \times (25\mathbf{k}) \\ &= (6 \cos 30^\circ \mathbf{i} - 6 \sin 30^\circ \mathbf{k}) \times (25\mathbf{k}) \\ &= \{-130\mathbf{j}\} \text{ lb} \cdot \text{in.} \end{aligned}$$

*Ans.*

## SOLUTION II (SCALAR ANALYSIS)

Although this problem is shown in three dimensions, the geometry is simple enough to use the scalar equation  $M = Fd$ . The perpendicular distance between the lines of action of the couple forces is  $d = 6 \cos 30^\circ = 5.196$  in., Fig. 4-32d. Hence, taking moments of the forces about either point  $A$  or point  $B$  yields

$$M = Fd = 25 \text{ lb} (5.196 \text{ in.}) = 129.9 \text{ lb} \cdot \text{in.}$$

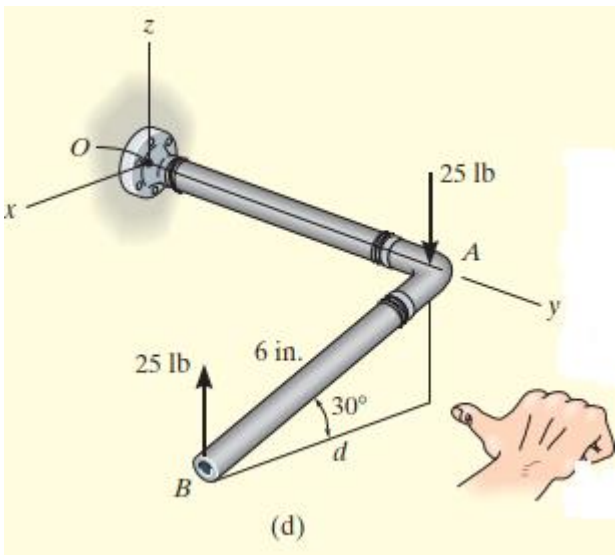
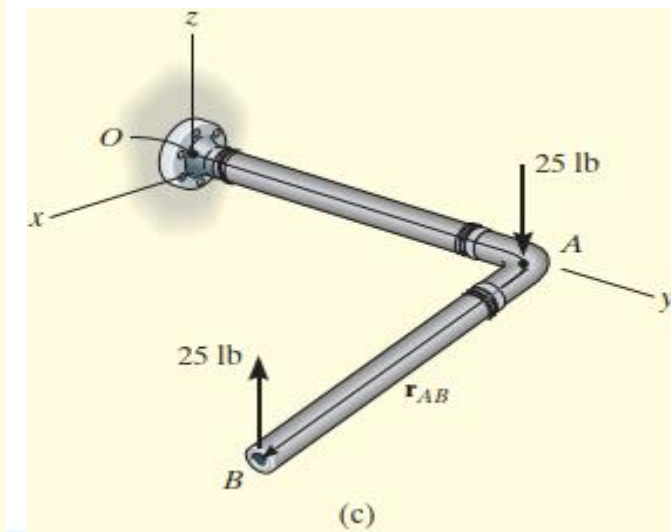
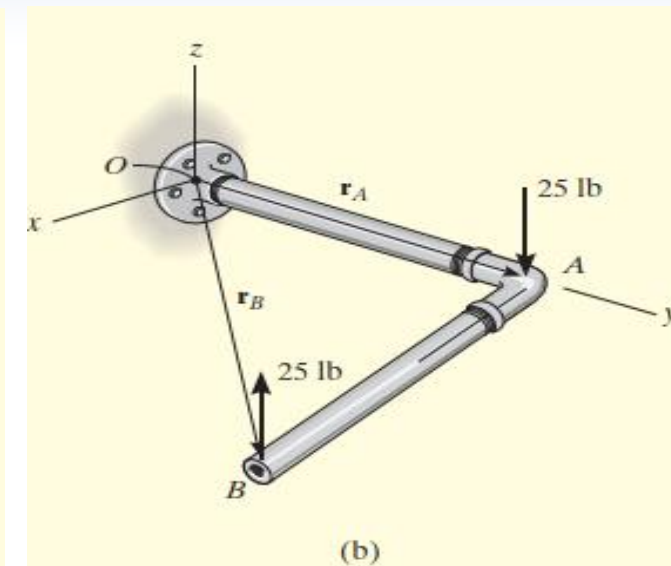
Applying the right-hand rule,  $\mathbf{M}$  acts in the  $-\mathbf{j}$  direction. Thus,

$$\mathbf{M} = \{-130\mathbf{j}\} \text{ lb} \cdot \text{in.}$$

*Ans.*

Solution

**4 different solutions!**



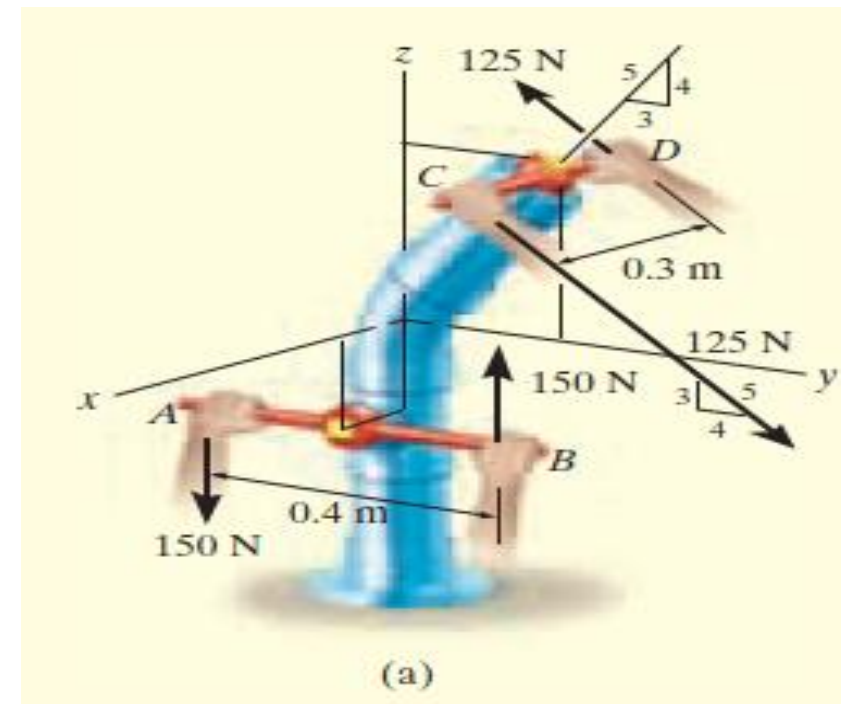
# MOMENT OF A COUPLE

## Example 4.12

### Question

- Replace the two couples acting on the pipe column in Fig. by a resultant couple moment

- **2 ways of solving this problem!**



# MOMENT OF A COUPLE

The couple moment  $\mathbf{M}_1$ , developed by the forces at  $A$  and  $B$ , can easily be determined from a scalar formulation.

$$M_1 = Fd = 150 \text{ N}(0.4 \text{ m}) = 60 \text{ N} \cdot \text{m}$$

Solution

By the right-hand rule,  $\mathbf{M}_1$  acts in the  $+\mathbf{i}$  direction, Fig. 4-33*b*. Hence,

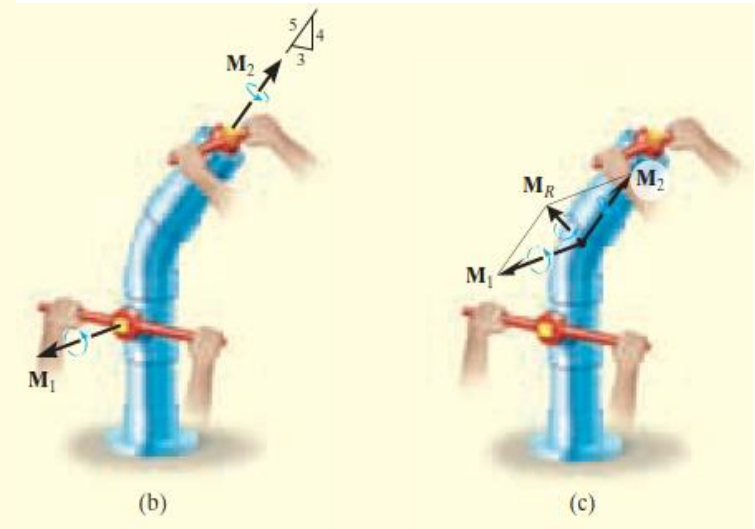
$$\mathbf{M}_1 = \{60\mathbf{i}\} \text{ N} \cdot \text{m}$$

Vector analysis will be used to determine  $\mathbf{M}_2$ , caused by forces at  $C$  and  $D$ . If moments are calculated about point  $D$ , Fig. 4-33*a*,  $\mathbf{M}_2 = \mathbf{r}_{DC} \times \mathbf{F}_C$ , then

$$\begin{aligned} \mathbf{M}_2 &= \mathbf{r}_{DC} \times \mathbf{F}_C = (0.3\mathbf{i}) \times \left[ 125\left(\frac{4}{5}\right)\mathbf{j} - 125\left(\frac{3}{5}\right)\mathbf{k} \right] \\ &= (0.3\mathbf{i}) \times [100\mathbf{j} - 75\mathbf{k}] = 30(\mathbf{i} \times \mathbf{j}) - 22.5(\mathbf{i} \times \mathbf{k}) \\ &= \{22.5\mathbf{j} + 30\mathbf{k}\} \text{ N} \cdot \text{m} \end{aligned}$$

Since  $\mathbf{M}_1$  and  $\mathbf{M}_2$  are free vectors, they may be moved to some arbitrary point and added vectorially, Fig. 4-33*c*. The resultant couple moment becomes

$$\mathbf{M}_R = \mathbf{M}_1 + \mathbf{M}_2 = \{60\mathbf{i} + 22.5\mathbf{j} + 30\mathbf{k}\} \text{ N} \cdot \text{m} \quad \text{Ans.}$$

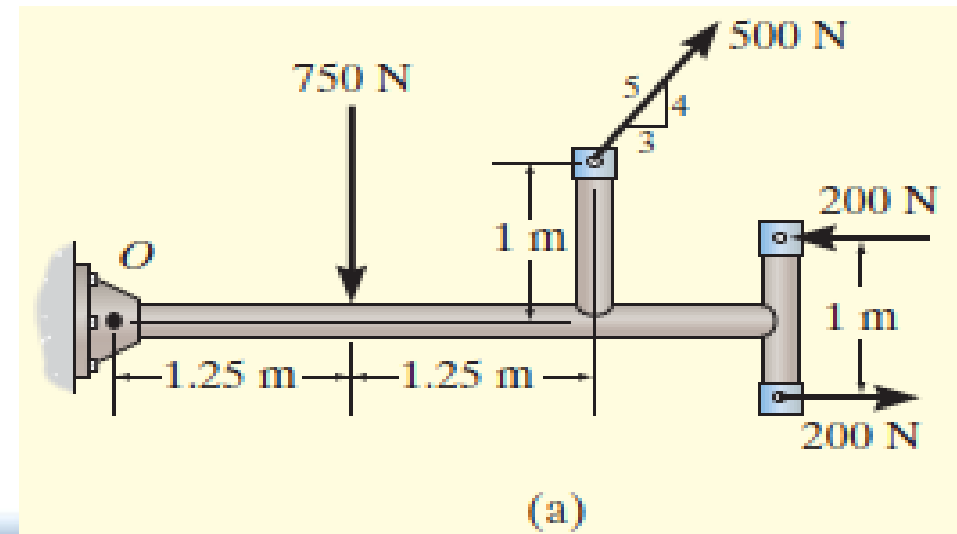


# MOMENT OF A COUPLE

## Example 4.13

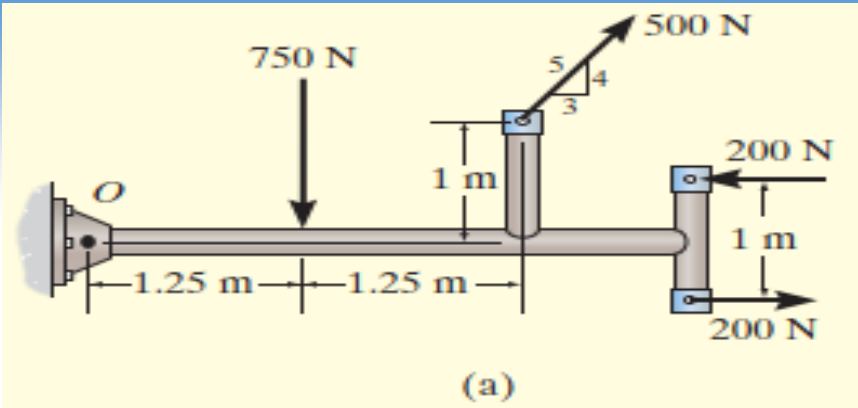
### Question

- Replace the force and couple system acting on the member in Fig. 4–38a by an equivalent resultant force and couple moment acting at point O.
- In Cartesian notation or as a vector
- **2 ways of solving this problem!**



# MOMENT OF A COUPLE

## Example 4.13



### Solution

**Force Summation.** Since the couple forces of 200 N are equal but opposite, they produce a zero resultant force, and so it is not necessary to consider them in the force summation. The 500-N force is resolved into its  $x$  and  $y$  components, thus,

$$\rightarrow (F_R)_x = \Sigma F_x; (F_R)_x = \left(\frac{3}{5}\right)(500 \text{ N}) = 300 \text{ N} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y; (F_R)_y = (500 \text{ N})\left(\frac{4}{5}\right) - 750 \text{ N} = -350 \text{ N} = 350 \text{ N} \downarrow$$

From Fig. 4-15b, the magnitude of  $\mathbf{F}_R$  is

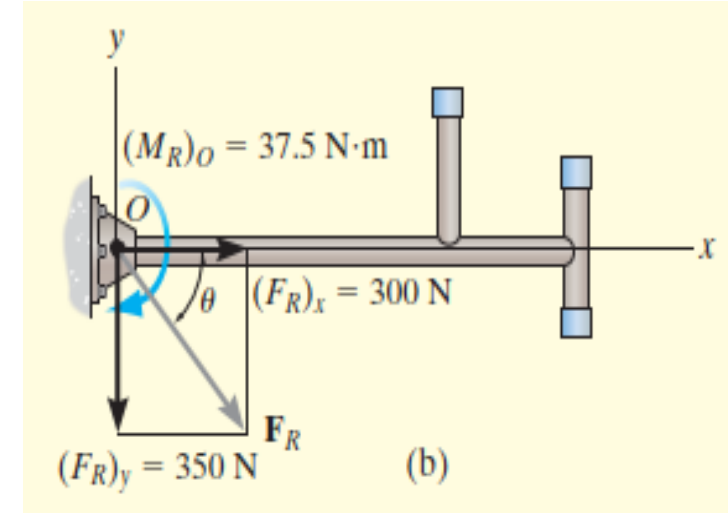
$$\begin{aligned} F_R &= \sqrt{(F_R)_x^2 + (F_R)_y^2} \\ &= \sqrt{(300 \text{ N})^2 + (350 \text{ N})^2} = 461 \text{ N} \end{aligned}$$

And the angle  $\theta$  is

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right) = \tan^{-1}\left(\frac{350 \text{ N}}{300 \text{ N}}\right) = 49.4^\circ$$

*Ans.*

*Ans.*



**Moment Summation.** Since the couple moment is a free vector, it can act at any point on the member. Referring to Fig. 4-38a, we have

$$\zeta + (M_R)_O = \Sigma M_O + \Sigma M$$

$$\begin{aligned} (M_R)_O &= (500 \text{ N})\left(\frac{4}{5}\right)(2.5 \text{ m}) - (500 \text{ N})\left(\frac{3}{5}\right)(1 \text{ m}) \\ &\quad - (750 \text{ N})(1.25 \text{ m}) + 200 \text{ N} \cdot \text{m} \\ &= -37.5 \text{ N} \cdot \text{m} = 37.5 \text{ N} \cdot \text{m} \curvearrowright \end{aligned}$$

*Ans.*

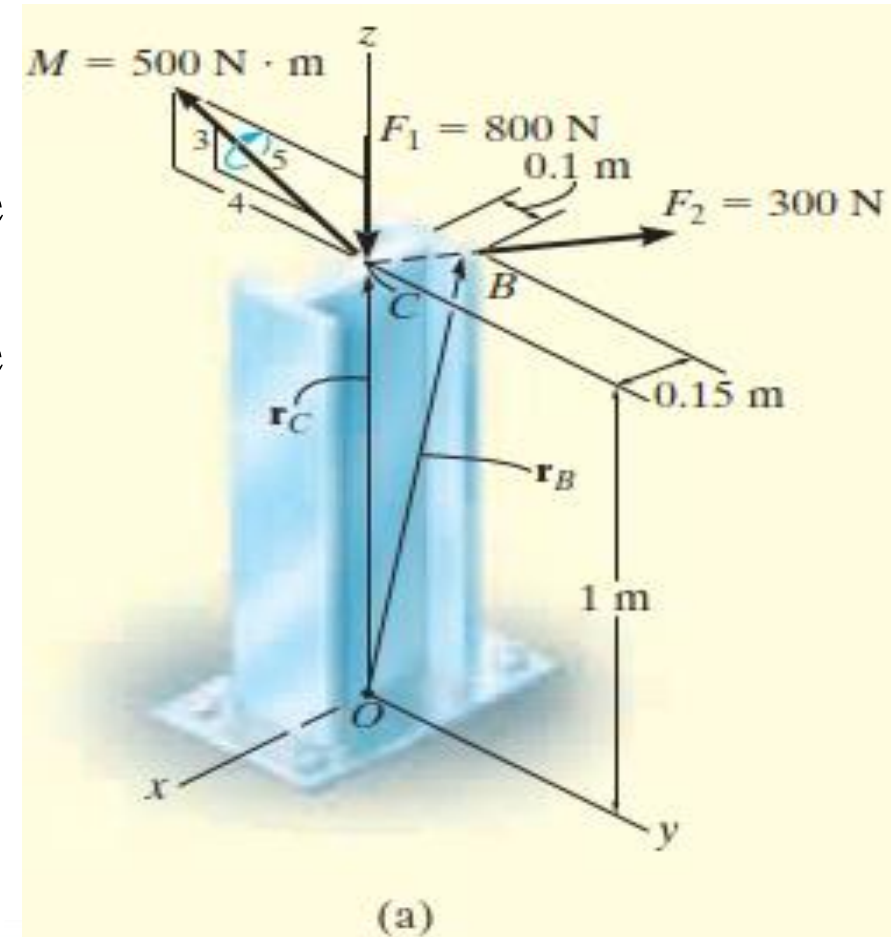
This clockwise moment is shown in Fig. 4-38b.

# MOMENT OF A COUPLE

## Example 4.14

### Question for Critical Visual & Analysis

- The structural member is subjected to a couple moment  $M$  and forces  $F_1$  and  $F_2$  in Fig. 4–39a. Replace this system by an equivalent resultant force and couple moment acting at its base, point  $O$ .



# MOMENT OF A COUPLE

## Example 4.14

### Solution

The three-dimensional aspects of the problem can be simplified by using a Cartesian vector analysis. Expressing the forces and couple moment as Cartesian vectors, we have

$$\mathbf{F}_1 = \{-800\mathbf{k}\} \text{ N}$$

$$\mathbf{F}_2 = (300 \text{ N})\mathbf{u}_{CB}$$

$$= (300 \text{ N})\left(\frac{\mathbf{r}_{CB}}{r_{CB}}\right)$$

$$= 300 \text{ N} \left[ \frac{\{-0.15\mathbf{i} + 0.1\mathbf{j}\} \text{ m}}{\sqrt{(-0.15 \text{ m})^2 + (0.1 \text{ m})^2}} \right] = \{-249.6\mathbf{i} + 166.4\mathbf{j}\} \text{ N}$$

$$\mathbf{M} = -500\left(\frac{4}{5}\right)\mathbf{j} + 500\left(\frac{3}{5}\right)\mathbf{k} = \{-400\mathbf{j} + 300\mathbf{k}\} \text{ N}\cdot\text{m}$$

### Force Summation.

$$\begin{aligned} \mathbf{F}_R &= \Sigma \mathbf{F}; & \mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 = -800\mathbf{k} - 249.6\mathbf{i} + 166.4\mathbf{j} \\ & & &= \{-250\mathbf{i} + 166\mathbf{j} - 800\mathbf{k}\} \text{ N} \end{aligned}$$

### Moment Summation.

$$(\mathbf{M}_R)_O = \Sigma \mathbf{M} + \Sigma \mathbf{M}_O$$

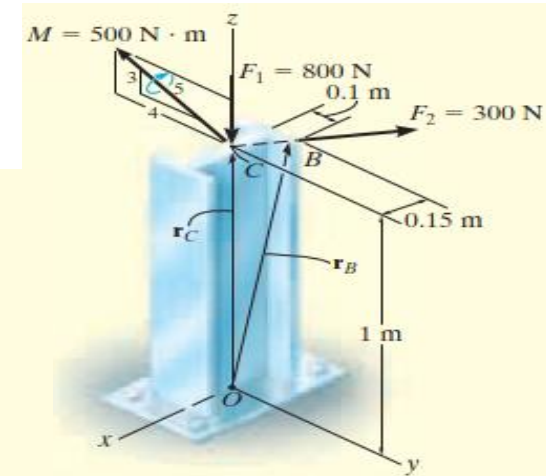
$$(\mathbf{M}_R)_O = \mathbf{M} + \mathbf{r}_C \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2$$

$$(\mathbf{M}_R)_O = (-400\mathbf{j} + 300\mathbf{k}) + (1\mathbf{k}) \times (-800\mathbf{k}) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.15 & 0.1 & 1 \\ -249.6 & 166.4 & 0 \end{vmatrix}$$

$$= (-400\mathbf{j} + 300\mathbf{k}) + (0) + (-166.4\mathbf{i} - 249.6\mathbf{j})$$

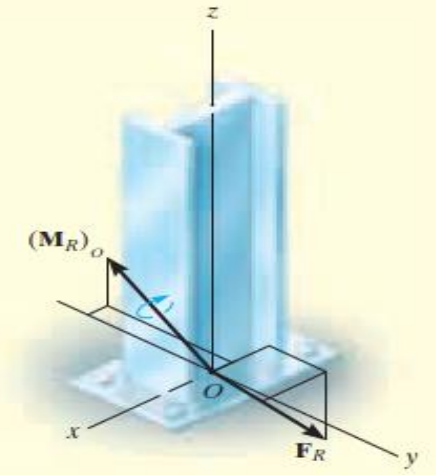
$$= \{-166\mathbf{i} - 650\mathbf{j} + 300\mathbf{k}\} \text{ N}\cdot\text{m}$$

The results are shown in Fig. 4-39b.



(a)

Ans.



(b)



# MOMENT OF A COUPLE

## HOME WORK EXERCISE

4-4, 4-12, 4-29, 4-34, 4-36, 4-38, 4-43, 4-46, 4-48, 4-50, 4-51, 4-54, 4-56,  
4-58, 4-66, 4-69, 4-72, 4-80, 4-83, 4-96, 4-97, 4-102, 4-108, 4-115 & 4-134.





**THE END**

*THANK YOU*

