MAT 1110:

Chapter 5

Calculus

2019/2020

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Differential Calculus

1.1. limits

Let $x = 1 - \frac{1}{n}$, where n is a positive integer. Then $0 < x < 1$ for all values of n. For example, if $n = 2$, then

$$x = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

If $n = 4$, then

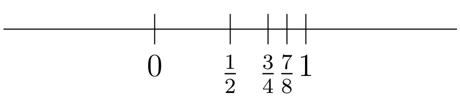
$$x = 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$

If $n = 8$ then

$$x = 1 - \tfrac{1}{8} = \tfrac{7}{8}.$$

and so on.

Note that as n increases the fraction $\frac{1}{n}$ decreases which results in the difference $1 - \frac{1}{n}$ to increase. Thus, as n increases, the values of x also increases but will not exceed 1 but will get closer and closer to the value 1.



We say that x approaches 1 from the left and write $x \to 1^-$

Similarly, if $x = 1 + \frac{1}{n}$ where n is a positive integer, then $1 < x < 2$ for all values of n. However, as n increases, the fraction $\frac{1}{n}$ becomes smaller, this means that we are adding a smaller number to 1. As a result, the values of $x = 1 + \frac{1}{n}$ decreases to 1. For example, when $n = 2$

$$x = 1 + \tfrac{1}{2} = \tfrac{3}{2}.$$

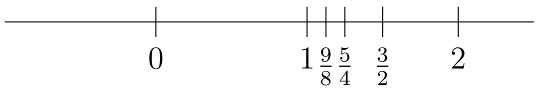
If $n = 4$, then

$$x = 1 + \tfrac{1}{4} = \tfrac{5}{4}.$$

If $n = 8$, then

$$x = 1 + \tfrac{1}{8} = \tfrac{9}{8},$$

and so on.



We say that x approaches 1 from the right and write $x \to 1^+$.

Let a be a real number. When we say 'as x approaches a' and write $x \to a$, we include both situations, that is, x approaches a from the left and from the right.

Let $y = f(x)$ be a function. We shall now investigate the behaviour of the function as x approaches a real number a.

Example 1.1.0.1 *Consider the function*

$$f(x) = x^2,$$

. We investigate its behaviour as x approaches 2 from the left and from the right.

Let first x approach 2 from the left.

As we approach 2 from the negative(left), we obtain the following:

x	*1.8*	1.9	1.99	1.999	1.9999
$y = x^2$	*3.24*	*3.61*	*3.9601*	*3.996001*	*3.99960001*

From the table above, we observe that as x approaches 2 from the left, the values of the function $f(x) = x^2$ approach the value 4. Thus, we say that the limit of the function $y = f(x)$ as x approaches 2 from the left is 4, and write it as

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$$

Suppose that now we approach 2 from the right.

If we now approach 2 from the positive (right), we obtain the following:

x	*2.2*	2.1	2.01	2.001	2.0001
$y = x^2$	*4.84*	*4.41*	*4.0401*	*4.004001*	*4.00040001*

From the table above, we observe that as x approaches 2 from the right, the values of the function $f(x) = x^2$ approach the value 4. Thus, we say that the limit of the function as x approaches 2 from the right is 4, and write it as

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 4$$

In this example we observe that the limit of the function $f(x) = x^2$ as x approaches 2 from the left is the same as its limit as x approach 2 from the right. That is,

$$\lim_{x \to 2^-} x^2 = 4 = \lim_{x \to 2^+} x^2 = f(2) = (2)^2 = 4.$$

In this case, we say that the limit of the function $f(x) = x^2$ at the point $x = 2$ exists and that limit is 4. That is

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 4$$

Definition 1.1.1 *Let a and L be real numbers. We say that the limit of the function $f(x)$ as x approaches a exists and the limit is L written as*

$$\lim_{x \to a} f(x) = L$$

if the limit from the left of the function is L and the limit from the right of the function is also L. That is, if

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

If either

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

or

$$\lim_{x\to a} f(x) = \infty,$$

then we say that the limit of the function $f(x)$ does not exist at $x = a$.

For any polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

the limit of such function exist at any real number $x = a$ and this limit is given by $f(a)$. For example, for the function $f(x) = x^2$, we have seen already that the limit of this function at $x = 2$ exists and this limit is $f(2) = (2)^2 = 4$.

Most of the rational functions

$$f(x) = \frac{ax+b}{cx+d}$$

also behave like polynomial functions at any real number where the denominator is not zero. That is, if p is a real number and $c(p) + d \neq 0$, then $\lim_{x\to p} f(x)$ exists and

$$\lim_{x\to p} f(x) = f(p) = \frac{a(p)+b}{c(p)+d}.$$

Example 1.1.0.2 *Evaluate the following limits*

(i) $\lim_{x\to 2} 3x^3 + 5x - 13$

(ii) $\lim_{x\to 1} \dfrac{x^2 - 2x + 7}{x+2}$

Solution:

(i) $f(x) = 3x^3 + 5x - 13$ is a polynomial function, therefore, its limit at $x = 2$ is just $f(2)$. Thus

$$\begin{aligned} \lim_{x\to 2}(3x^3 + 5x - 13) &= 3(2)^3 + 5(2) - 13 \\ &= 3(8) + 10 - 13 \\ &= 24 + 10 - 13 \\ &= 21 \end{aligned}$$

(ii) The function $f(x) = \frac{x^2-2x+7}{x+2}$ is a rational function in which the denominator is $x+2$. Substituting $x = 1$ into the denominator gives $1+2 = 3 \neq 0$. Since the denominator is non zero at $x = 1$, then the limit of this function at 1 is $f(1)$. Thus,

$$\begin{aligned}\lim_{x\to 1}\left(\frac{x^2-2x+7}{x+2}\right) &= \tfrac{1^2-2(1)+7}{1+2}\\ &= \tfrac{1-2+7}{3}\\ &= 2.\end{aligned}$$

If $f(x) = \frac{h(x)}{g(x)}$ is a rational function and a is a real number such that $g(a) = 0$ and $h(a) \neq 0$, then

$$\lim_{x\to a} f(x) = \infty \text{ (undefined)}.$$

Remember we have already mentioned that if the limit is infinity, then the limit does not exist.

Some points to note:

1. In general we do not evaluate the limit by actually substituting $x = a$ in $f(x)$ for a number of reasons. For example, the limit of the function may exist at $x = a$ and yet $f(a)$ may not be defined or may give a different value altogether.
2. The value of the limit can depend on which side it is approached, from left or right, i.e through the values of x less than a or through the values of x greater than a respectively. The two possible values may not be the same in which case the limit does not exist.
3. The limit may not exist at all and even if it does it may not be equal to $f(a)$.

Example 1.1.0.3 *(a) Let the function*

$$f(x) := \begin{cases} x^2-1 & \text{if } x \leq 4 \\ x-2 & \text{if } x > 4 \end{cases},$$

check whether the limit as x approaches 4 exist?

(b) Let the function

$$f(x) := \begin{cases} x^2+1 & \text{if } x \neq 2 \\ -4 & \text{if } x = 2 \end{cases},$$

(i) Find $f(2)$

(ii) Find limit as $\lim\limits_{x \to 2} f(x)$

Solution:

(a) For the function

$$f(x) := \begin{cases} x^2 - 1 & \text{if } x \leq 4 \\ x - 2 & \text{if } x > 4 \end{cases},$$

note that it is divided into two parts. For all values of x which are less than or equal to 4 the function is $f(x) = x^2 - 1$ while for all values of x greater than 4 the function is $f(x) = x - 2$. Therefore, we have

$$\begin{aligned} \lim_{x \to 4^-} f(x) &= \lim_{x \to 4^-} (x^2 - 1) \\ &= (4)^2 - 1 \\ &= 16 - 1 \\ &= 15 \end{aligned}$$

Thus, $\lim\limits_{x \to 4^-} f(x) = 15$.

On the other hand

$$\begin{aligned} \lim_{x \to 4^+} f(x) &= \lim_{x \to 4^+} (x - 2) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

Thus, $\lim\limits_{x \to 4^+} f(x) = 2$

Since $15 = \lim\limits_{x \to 4^-} f(x) \neq \lim\limits_{x \to 4^+} f(x) = 2$, then the limit of the function at $x = 4$ does not exist.

(b) For the function

$$f(x) := \begin{cases} x^2 + 1 & \text{if } x \neq 2 \\ -4 & \text{if } x = 2 \end{cases},$$

we have

(i) $f(2) = -4$

(ii) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2 + 1) = (2)^2 + 1 = 5 = \lim\limits_{x \to 2^+} f(x)$. Therefore, $\lim\limits_{x \to 2} f(x) = 5$.

Limits of the form $\frac{0}{0}$

Let $f(x) = \frac{h(x)}{g(x)}$ be a rational function. Let a be a real number such that $h(a) = 0$ and $g(a) = 0$. In this case, if you try to evaluate the limit of the function f by directly substituting $x = a$, you end up having the following expresion

$$\lim_{x \to a} f(x) = \frac{h(a)}{g(a)} = \frac{0}{0}.$$

This expression is meaningless. When this happens, it means that there is a common factor for $h(x)$ and $g(x)$. You need to find the common factor, cancel it out before making your substitution. For example, consider the limit

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}.$$

Direct substitution by $x = 1$ yields the expression $\frac{0}{0}$. However, to find the correct limit we first factorize the numerator as follows

$$\begin{aligned}
\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} &= \lim_{x \to 1} \frac{(x-1)(x+2)}{x - 1} \\
&= \lim_{x \to 1} (x + 2), \quad \text{after cancelling the common factor (x-1)} \\
&= 1 + 2 \\
&= 3
\end{aligned}$$

Thus, $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = 3$.

Example 1.1.0.4 *Evaluate each of the following limits.*

(a) $\lim_{x \to 0} \frac{3x}{x^2 + 2x}$

(b) $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 + x - 2}$

(c) $\lim_{x \to 2} \frac{4 - x^2}{3 - \sqrt{x^2 + 5}}$

Solution:

(a) We have

$$\begin{aligned}\lim_{x\to 0}\frac{3x}{x^2+2x} &= \lim_{x\to 0}\frac{3x}{x(x+2)}\\ &= \lim_{x\to 0}\frac{3}{x+2}\\ &= \tfrac{3}{0+2}\\ &= \tfrac{3}{2}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to -2}\frac{x^2+5x+6}{x^2+x-2} &= \lim_{x\to -2}\frac{(x+2)(x+3)}{(x+2)(x-1)}\\ &= \lim_{x\to -2}\frac{x+3}{x-1}\\ &= \tfrac{-2+3}{-2-1}\\ &= -\tfrac{1}{3}\end{aligned}$$

(c) We first rationalize the denominator

$$\begin{aligned}\lim_{x\to 2}\frac{4-x^2}{3-\sqrt{x^2+5}} &= \lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})(3+\sqrt{x^2+5})}\\ &= \lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{9-(x^2+5)}\\ &= \lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{9-x^2-5}\\ &= \lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{4-x^2}\\ &= \lim_{x\to 2}(3+\sqrt{x^2+5})\\ &= 3+\sqrt{(2)^2+5}\\ &= 3+\sqrt{9}\\ &= 6\end{aligned}$$

Properties of limits

The properties of limits are fairly well what we might expect. Thus if $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$, then

1. $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kb$ for any constant k
2. $\lim_{x\to a}(f(x)\pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = b\pm c$
3. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\lim_{x\to a} g(x) = bc$
4. $\lim_{x\to a}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{b}{c}$, $c\neq 0$.
5. $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n = b^n$.

Note

From the definition of limits, we observe the following:

1. $$\lim_{x\to\infty} \frac{1}{x} = 0$$

2. $$\lim_{x\to 0} \frac{1}{x} = \infty.$$

3. $$\lim_{x\to 0} x = 0.$$

4. $$\lim_{x\to\infty} x = \infty.$$

Limits of the form $\frac{\infty}{\infty}$

Consider the function $f(x) = \frac{7x^2-x}{3x^2+5}$. If we try to take the limit of the function as x goes to infinity we get the following expression

$$\lim_{x\to\infty} \frac{7x^2-x}{3x^2+5} = \frac{\infty}{\infty}$$

and this again is a meaningless expression. To deal with such limits, first divide both the numerator and the denominator by the highest power of x in the quotient before taking the limit. For example, in the function above we do the following

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^2(7-\frac{1}{x})}{x^2(3+\frac{5}{x^2})} \\
&= \lim_{x\to\infty} \frac{7-\frac{1}{x}}{3+\frac{5}{x^2}} \quad \text{after cancelling } x^2 \\
&= \frac{\lim_{x\to\infty}(7-\dfrac{1}{x})}{\lim_{x\to\infty}(3+\dfrac{5}{x^2})} \\
&= \frac{7-0}{3+0} \quad \text{by the properties of limits above} \\
&= \frac{7}{3}.
\end{aligned}$$

Example 1.1.0.5 *1.*

$$\lim_{x\to\infty} \frac{2x}{1+3x} = \lim_{x\to\infty} \frac{2}{\frac{1}{x}+3} = \frac{2}{3}.$$

2.

$$\lim_{x\to\infty} \frac{5-2x-9x^2}{2x^2+3} = \lim_{x\to\infty} \frac{\frac{5}{x^2}-\frac{2}{x}-9}{2+\frac{3}{x^2}} = -\frac{9}{2}.$$

3.

$$\lim_{x\to\infty} \frac{2x}{x^3+2x^2-1} = \lim_{x\to\infty} \frac{\frac{2}{x^2}}{1+\frac{2}{x}-\frac{1}{x^3}} = 0.$$

Other standard limits are given as follows:

(a) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(b) $\lim_{x\to 0} \frac{1-\cos x}{x} = 0.$

Example 1.1.0.6 *Evaluate each of the following limits.*

(a) $\lim_{x\to 0} \frac{\tan x}{x}$

(b) $\lim_{x\to 0} \frac{\sin 3x}{x}.$

Solution:

(a)

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x}{x} &= \lim_{x\to 0} \frac{\sin x}{x\cos x} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1}{\cos x} \\
&= 1 \times 1 \\
&= 1.
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 3x}{x} &= \lim_{x\to 0} \frac{3\sin 3x}{3x} \\
&= 3\lim_{x\to 0} \frac{\sin 3x}{3x} \\
&= 3 \times 1 \\
&= 3.
\end{aligned}$$

1.2. Continuity

Definition 1.2.1 *A function $f(x)$ is said to be continuous at $x = a$ if and only if the following are satisfied*

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exist
3. $\lim_{x\to a} f(x) = f(a)$

Example 1.2.0.7 *Investigate the continuity of each of the following:*

(a)

$$f(x) = \frac{x^2 - 1}{x + 1} \;\; at \;\; x = -1$$

(b)

$$f(x) := \begin{cases} 2x + 1 & \text{if } x \le -1 \\ x^2 - 2 & \text{if } x > -1 \end{cases}, \quad at \;\; x = -1$$

Solution:

(a) We have

$$\begin{aligned} \lim_{x\to -1} f(x) &= \lim_{x\to -1} \frac{x^2 - 1}{x + 1} \\ &= \lim_{x\to -1} \frac{(x-1)(x+1)}{x+1} \\ &= \lim_{x\to -1} (x - 1) \;\; \text{after cancelling common factor} \\ &= -1 - 1 = -2. \end{aligned}$$

However, $f(-1)$ is undefined. Therefore, since $f(-1) \neq \lim_{x\to -1} f(x)$ we conclude that the function is not continuous at $x = -1$.

(b) Note that this function is divided into two parts. For all values of x less or equal to -1, the function is $f(x) = 2x + 1$. Therefore, to find the left limit we use this linear part of the function. Thus

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (2x + 1) = 2(-1) + 1 = -2 + 1 = -1$$

For all values of x greater that -1, the function is $f(x) = x^2 - 2$. Therefore, to find the left limit we use this quadratic part. Thus

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2 - 2) = (-1)^2 - 2 = 1 - 2 = -1$$

Since the left limit is the same as the right limit, we have $\lim_{x \to -1} f(x) = -1$. On the other hand, $f(-1) = 2(-1) + 1 = -1$. Since $f(-1) = -1 = \lim_{x \to -1} f(x)$, we conclude that the function is continuous at $x = -1$.

Example 1.2.0.8 *If the function*

$$f(x) := \begin{cases} \frac{x^2-16}{x-4} & \text{if } x \neq 4 \\ C & \text{if } x = 4 \end{cases}$$

is continuous at $x = 4$, *what is the value of* C.

Solution:

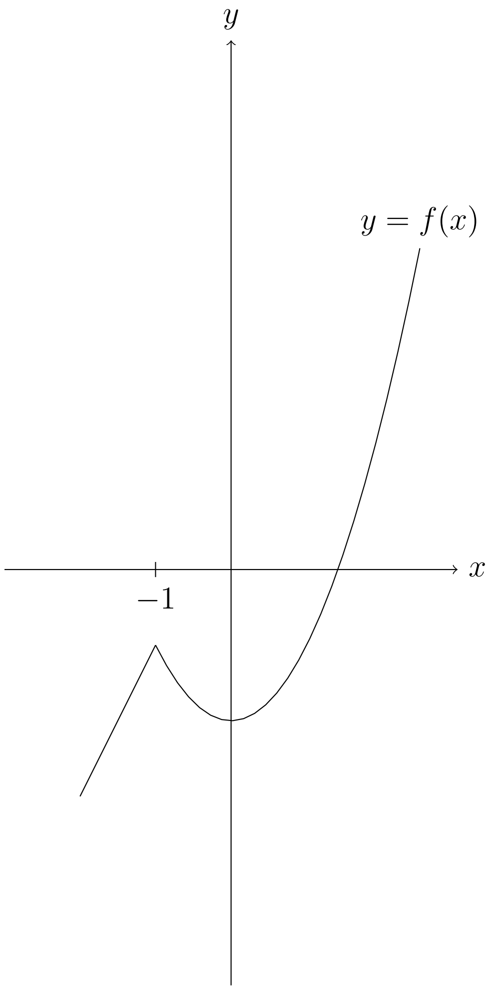
We have $\lim_{x \to 4} f(x) = \lim_{x \to 4} \dfrac{x^2 - 16}{x - 4} = \lim_{x \to 4} (x + 4) = 8$.

If $f(x)$ is continuous at $x = 4$ then $C = f(4) = \lim_{x \to 4} f(x) = 8$. Thus, $C = 8$.

Example 1.2.0.9 *Sketch the graph of*

$$f(x) := \begin{cases} 2x + 1 & \text{if } x \leq -1 \\ x^2 - 2 & \text{if } x > -1 \end{cases}.$$

Solution: We have already discussed the continuity of this function at the point $x = -1$. The graph of this function is given below:



1.3. Differentiation

1.3.1 Derivative

Definition 1.3.1 *Let $y = f(x)$ be a continuous function on a given interval and let x be a number in its domain. Then we define the derivative of f at x to be the function $f'(x)$ given by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{1.1}$$

provided this limit exists.

Note that this limit is of the form $\frac{0}{0}$ if you substitute $h = 0$ directly.

Other notations for the derivative of the function $y = f(x)$ are $\frac{dy}{dx}$, $\frac{df}{dx}$ or y'.

Differentiating a function $y = f(x)$ by taking the limit of the quotient $\frac{f(x+h)-f(x)}{h}$ as h approaches 0 is called differentiating the function from the first principle.

Definition 1.3.2 *Given a function $y = f(x)$, the first principle states that the gradient function of $f(x)$ denoted by $f'(x)$ is given by*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 1.3.1.1 *Use the first principle to find $f'(x)$ in each of the following:*

(a) $f(x) = x^2$

(b) $y = \frac{1}{x}$

(c) $y = \sin x$

Solution: Using the formula $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ we have

(a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} \frac{(2x+h)h}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x. \end{aligned}$$

(b) Since $y = f(x)$ we have

$$\begin{aligned} y' = f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{x - (x+h)}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\ &= -\lim_{h\to 0} \frac{1}{x(x+h)} \\ &= -\tfrac{1}{x^2}. \end{aligned}$$

(c)

$$\begin{aligned} y' = f'(x) &= \lim_{h\to 0} \frac{\sin x + h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h - \sin x + \sin h \cos x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\ &= 0 + \cos x \\ &= \cos x \end{aligned}$$

Thus, the derivative of $\sin x$ is $\cos x$.

Example 1.3.1.2 *Differentiate each of the following functions from the first principle*

(a) $f(x) = \sqrt{x}$

(b) $y = \frac{1}{\sqrt{x}}$

Solution: Again we use the formula $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, we have

(a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \quad \text{rationalizing the numerator} \\ &= \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \tfrac{1}{2(\sqrt{x})}. \end{aligned}$$

(b) Do it as an exercise.

By the other notations stated above, if $y = f(x)$, then we can also write the derivative as

$$\frac{dy}{dx} = f'(x)$$

Note that if $f(x) = c$ a constant function, then $f(x+h) = c$ so that $f(x+h) - f(x) = 0$. This says that the derivative of the constant function is zero.

We now want to outline general rules for differentiating complex functions. First we give standard derivatives of elementary functions.

Here are some of the standard derivatives:

1. If $y = x^n$ for any real number n, then $\frac{dy}{dx} = nx^{n-1}$
2. If $y = \sin x$, then $\frac{dy}{dx} = \cos x$
3. If $y = \cos x$, then $\frac{dy}{dx} = -\sin x$
4. If $y = e^x$, then $\frac{dy}{dx} = e^x$
5. If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

Let $f(x)$ and $g(x)$ be two functions, and let C be any real number, then

1. $y = Cf(x)$ implies that $\frac{dy}{dx} = Cf'(x)$
2. $y = f(x) + g(x)$ implies that $\frac{dy}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx} = f'(x) + g'(x)$
3. $y = f(x) - g(x)$ implies that $\frac{dy}{dx} = f'(x) - g'(x)$

Example 1.3.1.3 *Find the derivative of each of the following functions:*

(a) $y = x^3 - 6x + \frac{5}{x^2} + 17$

(b) $f(x) = 3\sin x - 5\ln(x+1)$

(c) $g(x) = x^{-7} + 2e^x - 5$

Solution:

(a) For $y = x^3 - 6x + \frac{5}{x^2} + 17$, we differentiate term by term as follows

(i) $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

(ii) $g(x) = -6x \Rightarrow g'(x) = -6$

(iii) $h(x) = \frac{5}{x^2} = 5x^{-2} \Rightarrow h'(x) = -10x^{-3} = -\frac{10}{x^3}$

(iv) $k(x) = 17 \Rightarrow k'(x) = 0$

Combining (i), (ii), (iii) and (iv) we have

$$\frac{dy}{dx} = 3x^2 - 6 - \frac{10}{x^3}$$

(b) We again differentiate $f(x) = 3\sin x - 5\ln(x+1)$ term by term to get

$$f'x = 3\cos x - \frac{5}{x+1}$$

(c) Differentiate $g(x) = x^{-7} + 2e^x - 5$ term by term to get

$$g'(x) = -7x^{-8} + 2e^x$$

Interpretation of the derivative

One interpretation of the derivative of a function is that the derivative of a function gives the gradient (slope) of a function at any given point. Note also that the gradient of a function at any point is equal to the gradient of the tangent line to the graph of the function at that given point.

Example 1.3.1.4 *Find the gradient of the tangent to the graph of the function at a given point in each of the following:*

(a) $y = x^3 - 2x - 5$ *at* $(2, -1)$

(b) $f(x) = \cos x + \frac{1}{4}$ *at* $(\frac{\pi}{3}, \frac{3}{4})$

(c) $f(x) = \frac{1}{x}$ *at* $(\frac{1}{2}, 2)$

Solution:

(a) Differentiating the function $y = x^3 - 2x - 5$ we get

$$\frac{dy}{dx} = 3x^2 - 2$$

Since the derivative is a function of x only, we substitute $x = 2$ to get the gradient of the tangent. Thus

$$m = 3(2)^2 - 2 = 12 - 2 = 10$$

(b) The derivative of $f(x) = \cos x + \frac{1}{4}$ is

$$f'(x) = -\sin x$$

So the gradient of the tangent at $(\frac{\pi}{3}, \frac{1}{4})$ is

$$m = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

(c) We have

$$f'(x) = -\frac{1}{x^2}$$

so that

$$m = -\frac{1}{(\frac{1}{2})^2} = -4$$

is the gradient of the tangent.

1.4. Methods of differentiation

We now consider some methods of differentiation which will enable us differentiate some complicated functions in which we can not apply above methods directly.

1.4.1 Chain Rule

The Chain Rule is a method of differentiating composition of functions. For example $f(x) = (x^5 - 2x^2 + 3)^8$. We can see that if we let $g(x) = x^5 - 2x^2 + 3$ and $h(x) = x^8$, the $f(x)$ is the composition of functions $f(x) = (h \circ g)(x)$. To differentiate such functions we use the following:

Theorem 1.4.1 *Let $u = g(x)$ and $y = f(u)$. Then y is a function of x and*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 1.4.1.1 *Use chain rule to differentiate each of the following functions:*

(a) $y = (3x+1)^2$

(b) $y = \ln(x^2 - 3x)$

(c) $y = e^{-3x^2}$

(d) $y = \cos(4x^3 + 2)$

Solution:

(a) Let $u = 3x + 1$, then $y = u^2$. We have

$$\begin{aligned} \tfrac{dy}{du} &= 2u \\ \tfrac{du}{dx} &= 3 \end{aligned}$$

Now by the chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ we get

$$\frac{dy}{dx} = 2u(3) = 6(3x+1) = 18x + 6$$

(b) Let $u = x^2 - 3x$. Then $y = \ln u$. We have

$$\begin{aligned} \tfrac{dy}{du} &= \tfrac{1}{u} \\ \tfrac{du}{dx} &= 2x - 3. \end{aligned}$$

$$\begin{aligned} \tfrac{dy}{dx} &= \tfrac{dy}{du}\tfrac{du}{dx} \\ &= \tfrac{1}{u} \times (2x-3) \\ &= \tfrac{2x-3}{x^2-3x}, \quad \text{since } u = x^2 - 3x. \end{aligned}$$

(c) Let $u = -3x^2$. Then $y = e^u$. We have

$$\begin{aligned} \tfrac{dy}{du} &= e^u \\ \tfrac{du}{dx} &= -6x \end{aligned}$$

Then

$$\tfrac{dy}{dx} = -6xe^{-3x^2}.$$

(d) Exercise.

1.4.2 Product Rule

Theorem 1.4.2 *Let $u = f(x)$ and $v = g(x)$ be two functions of x. Let $y = uv = f(x)g(x)$, then*

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = g(x)f'(x) + f(x)g'(x)$$

Example 1.4.2.1 *Use product rule to differentiate each of the following functions*

(a) $y = x^5 \ln x$

(b) $e^{2x} \sin x$

Solution:

(a) Let $u = x^5$ and $v = \ln x$, then

$$\begin{aligned} \tfrac{du}{dx} &= 5x^4 \\ \tfrac{dv}{dx} &= \tfrac{1}{x}. \end{aligned}$$

It follows that

$$\begin{aligned} \tfrac{dy}{dx} &= v\tfrac{du}{dx} + u\tfrac{dv}{dx} \\ &= \ln x(5x^4) + x^5(\tfrac{1}{x}) \\ &= 5x^4 \ln x + x^4 \\ &= x^4(5\ln x + 1) \end{aligned}$$

(b) Let $u = e^{2x}$ and $v = \sin x$, then

$$\begin{aligned} \tfrac{du}{dx} &= 2e^{2x} \text{ by chain rule} \\ \tfrac{dv}{dx} &= \cos x \end{aligned}$$

It follows that

$$\begin{aligned}\frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\ &= \sin x(2e^{2x}) + e^{2x}\cos x \\ &= 2e^{2x}\sin x + e^{2x}\cos x \\ &= e^{2x}(2\sin x + \cos x).\end{aligned}$$

1.4.3 Quotient Rule

Theorem 1.4.3 *Let $u = f(x)$ and $v = g(x)$ be two functions of x. If $y = \frac{u}{v} = \frac{f(x)}{g(x)}$, then*

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Example 1.4.3.1 *Find the derivative of each of the following functions*

(a) $y = \frac{x^3-2x}{x^2+1}$

(b) $y = \tan x$

Solution:

(a) Let $u = x^3 - 2x$ and $v = x^2 + 1$. We have

$$\begin{aligned}\frac{du}{dx} &= 3x^2 - 2 \\ \frac{dv}{dx} &= 2x\end{aligned}$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(x^2+1)(3x^2-2)-(x^3-2x)(2x)}{(x^2+1)^2} \\ &= \frac{3x^4-2x^2+3x^2-2-(2x^4-4x^2)}{(x^2+1)^2} \\ &= \frac{x^4+5x^2-2}{(x^2+1)^2}\end{aligned}$$

(b) Write $y = \tan x = \frac{\sin x}{\cos x}$ and let $u = \sin x$ and $v = \cos x$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{\cos x(\cos x)-\sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} \\ &= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \\ &= 1 + \tan^2 x\end{aligned}$$

Note also that $\frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$. This gives the identity

$$\sec^2 x = 1 + \tan^2 x$$

1.4.4 Implicit Differentiation

Consider the equation $y + xy + y^2 = 2$. We may sometimes be able to solve the equation for y and express it as $y = f(x)$, as a function of x. In such circumstances it would be easy to find the derivative $\frac{dy}{dx} = f'(x)$. Unfortunately it is not always possible to do so. In such a situation, it may still be possible under certain conditions to work out the derivative $\frac{dy}{dx}$ by implicit differentiation. To see how this can be done let us use again the equation $y + xy + y^2 = 2$. Write this equation as

$$f(x) + xf(x) + (f(x))^2 = 2 \tag{1.2}$$

where $y = f(x)$ and $\frac{dy}{dx} = f'(x)$

Differentiating equation (1.2) term by term we get the following

(i) $\frac{df(x)}{dx} = f'(x) = \frac{dy}{dx}$

(ii) $xf(x)$ is a product of functions and by product rule we have

$$\frac{d(xf(x))}{dx} = (1)f(x) + xf'(x) = y + x\frac{dy}{dx}$$

That is,

$$\frac{d(xy)}{dx} = y + x\frac{dy}{dx}$$

(iii) To differentiate $[f(x)]^2$ we use chain rule

$$\frac{d[f(x)]^2}{dx} = (2f(x))f'(x) = 2y\frac{dy}{dx}$$

That is,

$$\frac{d(y^2}{dx} = 2y\frac{dy}{dx}$$

(iv) $\frac{d(2)}{dx} = 0$

Combining these we see that differentiating the equation $y + xy + y^2 = 2$ with respect to x gives

$$\frac{dy}{dx} + (y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

or

$$(1 + x + 2y)\frac{dy}{dx} + y = 0 \tag{1.3}$$

If we now solve equation (1.3) for $\frac{dy}{dx}$ we get

$$\frac{dy}{dx} = -\frac{y}{1 + x + 2y}$$

1.5. Some Applications of Differentiation

In this chapter, we present some applications of differentiation in our day to day life. These include finding the slope or gradient of a curve at a given point,sketching curves,optimization problems, i.e. problems of determining the maximum or minimum point of a function.

1.5.1 The Gradient

We recall that the gradient function of a curve $y = f(x)$ at a point (a, b), denoted by $f'(a)$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Example 1.5.1.1 *Find the gradient of the curve*

$$y = \frac{1}{4}x^3 - 2x$$

at $P(1, -\frac{7}{4})$ and hence write down the equation of the tangent to the curve at the point P.

Solution:

The derivative of the function is $\frac{dy}{dx} = \frac{3}{4}x^2 - 2$. Now the gradient of a curve at a point is the derivative of the function evaluated at the given point. Thus, the gradient of the curve at P is

$$\frac{dy}{dx}|_{x=1} = \frac{3}{4}(1)^2 - 2 = \frac{3}{4} - 2 = -\frac{5}{4}$$

or

$$m = -\frac{5}{4}$$

This is also the gradient of the tangent to the curve at P. Now, the tangent to the curve at a given point is a straight line and is of the form

$$y = mx + c.$$

We already have that $m = -\frac{5}{4}$ so that the equation of the tangent has the form

$$y = -\frac{5}{4}x + c.$$

To find c we use the values of x and y at the point P. When $x = 1$ then $y = -\frac{7}{4}$. Substituting these values in the above equation and then solving for c we have

$$\begin{aligned} -\tfrac{7}{4} &= -\tfrac{5}{4}(1) + c \\ c &= -\tfrac{7}{4} + \tfrac{5}{4} \\ &= -\tfrac{1}{2}. \end{aligned}$$

The equation of the tangent then is

$$y = -\frac{5}{4}x - \frac{1}{2}.$$

1.5.2 Increasing and Decreasing Functions

Let $y = f(x)$ be a function continuous on a given interval and let x_0 be a point in the interval. The function $y = f(x)$ is said to be increasing at x_0 if for all x in the neighborhood of x_0 we have

$$f(x) < f(x_0) \text{ for } x < x_0 \text{ and}$$
$$f(x) > f(x_0) \text{ for } x > x_0.$$

The function $y = f(x)$ is decreasing at x_0 if for all x in some neighborhood of x_0 we have

$$f(x) > f(x_0) \text{ for } x < x_0 \text{ and}$$
$$f(x) < f(x_0) \text{ for } x > x_0.$$

Theorem 1.5.1 *Let $y = f(x)$ be a function defined in a neighborhood of a point x_0.*

(i) If $\frac{dy}{dx}|_{x_0} = f'(x_0) > 0$, then the function $y = f(x)$ is increasing at x_0.

(ii) If $\frac{dy}{dx}|_{x_0} = f'(x_0) < 0$, then the function $y = f(x)$ is decreasing at x_0.

Example 1.5.2.1 *Let $y = 2x^3 - 3x^2 - 12x$ be a function. Determine the intervals where the function is increasing and where it is decreasing.*

Solution:

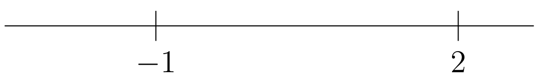
We now know that the function is increasing in the interval where the derivative $\frac{dy}{dx}$ is positive, and that it is decreasing in the interval where the derivative $\frac{dy}{dx}$ is negative.

Now the derivative of the function $y = 2x^3 - 3x^2 - 12x$ is

$$\frac{dy}{dx} = 6x^2 - 6x - 12 = 6(x - 2)(x + 1).$$

To get the intervals first we get the critical values of the derivative, that is, values of x which make the derivative zero. The critical values of the above derivative are $x = 2$ and $x = -1$

We now use the critical values to form intervals as follows



So we have the following intervals

$$\begin{aligned} I_1 &= (-\infty, -1) \\ I_2 &= (-1, 2) \\ I_3 &= (2, \infty). \end{aligned}$$

We now test the sign of the gradient of the function $\frac{dy}{dx} = 6(x - 2)(x + 1)$ in each of the intervals I_1, I_2 and I_3.

Take any value in I_1, say $x = -3$ as a test value. Then we have

$$\frac{dy}{dx}|_{x=-3} = 6(-3-2)(-3+1) = 60 > 0$$

Thus, the derivative is positive in the interval $I_1 = (-\infty, -1)$ meaning that the gradient of the function in this interval is positive. This is true for any value you take in this interval. Therefore, the function is increasing in the interval I_1

In the interval I_2 we take any value, say $x = 0$ in this interval. Then we have

$$\frac{dy}{dx}|_{x=0} = 6(0-2)(0+1) = -12 < 0$$

Since the derivative of the function is negative at $x = 0$, then the gradient of the function is negative there. This is true for any value of x in the interval $(-1, 2)$. Therefore, the function is decreasing in the interval I_2.

We do the same to the interval $I_3 = (2, \infty)$. Take any value say $x = 3$. Then

$$\frac{dy}{dx}|_{x=3} = 6(3-2)(3+1) = 24 > 0$$

Thus, the function is also increasing in the interval I_3.

We conclude that the function $y = 2x^3 - 3x^2 - 12x$ is increasing in the interval

$$(-\infty, -1) \cup (2, \infty)$$

and it is decreasing in the interval

$$(-1, 2).$$

1.5.3 Critical or Stationary Points

We have used the gradient of the curve $\frac{dy}{dx}$ to determine where the function is increasing ($\frac{dy}{dx} > 0$) and to determine where the function is decreasing ($\frac{dy}{dx} < 0$). We now look at a case where the gradient of the function is zero. That is, a case when $\frac{dy}{dx} = 0$.

The points where $\frac{dy}{dx} = 0$ are called **critical points**, and the corresponding values of x as we have already seen, are called **critical values**.

Example 1.5.3.1 *Find all the critical points of the function*

$$f(x) = \frac{1}{2}x^4 - \frac{1}{3}x^3 - \frac{5}{2}x^2 - 2x + 1.$$

Solution:

Differentiating the function gives

$\frac{dy}{dx} = 2x^3 - x^2 - 5x - 2$. To find the critical values we solve the equation $\frac{dy}{dx} = 0$, That is, we solve the equation

$$2x^3 - x^2 - 5x - 2 = 0$$

By factor theorem we see that $x + 1$ is a factor of $2x^3 - x^2 - 5x - 2$. We now use either long division or synthetic division to find other factors. Factorising the left hand side of the equation we get

$$(x + 1)(x - 2)(2x + 1) = 0$$

This gives the critical values to be $x = -1, -\frac{1}{2}, 2$.

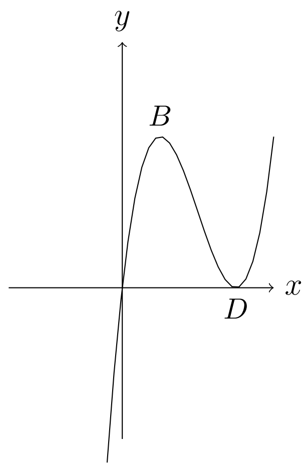
To find the critical points, we substitute each of the critical values into the function to find the corresponding values of y. We have

$$f(-1) = \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{5}{2} + 2 + 1 = \tfrac{4}{3}$$
$$f(-\tfrac{1}{2} = \tfrac{1}{32} + \tfrac{1}{24} - \tfrac{5}{8} + 1 + 1 = \tfrac{137}{96}$$
$$f(2) = 8 - \tfrac{8}{3} - 10 - 4 + 1 = -\tfrac{23}{3}.$$

Therefore, the critical points are $(-1, \frac{4}{3})$, $(-\frac{1}{2}, \frac{137}{96})$ and $(2, -\frac{23}{3})$.

Note that when $\frac{dy}{dx} = 0$, the function is neither increasing nor decreasing. It is like the function is stationary. Hence the term used to describe points where $\frac{dy}{dx} = 0$ is **stationary**

points.



The points such as B in the diagram where the function turns downward are called **maximum points**.

The points such as D in the diagram where the function turns upward are called **minimum points**.

The minimum and the maximum points of a function are also called the **extreme points**.

Note that the extreme points are also critical points.

To determine the extreme points of a function we have the following criteria.

Lemma 1.5.1 *Let $y = f(x)$ be a function and let p be a point in domain of f. Let the derivative at p be zero. That is $\frac{dy}{dx}|_{x=p} = f'(p) = 0$.*

(i) If $\frac{dy}{dx}$ changes sign from positive when $x < p$ to negative when $x > p$, then $y = f(x)$ has a maximum at $x = p$.

(ii) If $\frac{dy}{dx}$ changes sign from negative when $x < p$ to positive when $x > p$, then $y = f(x)$ has a minimum at $x = p$.

(iii) If $\frac{dy}{dx}$ does not change sign as x increases, then $y = f(x)$ neither has a maximum nor a minimum.

There is however, a much practical way of determining the extreme points of a function using the second derivative. If $y = f(x)$ is a function, then the second derivative of the function is obtained by differentiating the derivative function. That if $\frac{dy}{dx} = f'(x)$ is the derivative of f, then the second derivative is $\frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2} = f''(x)$.

Example 1.5.3.2 *Let $y = \frac{1}{5}x^5 + \frac{1}{12}x^4 - x^2 + 2$ be a function. Find the second derivative $\frac{d^2y}{dx^2}$ of the function.*

Solution:

We have

$$\frac{dy}{dx} = x^4 + \frac{1}{3}x^3 - 2x$$

Then, the second derivative is

$$\frac{d^2y}{dx^2} = 4x^3 + x^2 - 2.$$

We now state the Second Derivative Test.

Lemma 1.5.2 *Let $y = f(x)$ be a function, and let p be a point in the domain of f such that $\frac{dy}{dx}|_{x=p} = f'(p) = 0$.*

(i) If $\frac{d^2y}{dx^2} < 0$ at p, then $y = f(x)$ has a maximum there.

(ii) If $\frac{d^2y}{dx^2} > 0$ at p, then $y = f(x)$ has a minimum there.

(iii) If $\frac{d^2y}{dx^2} = 0$ at p, then the test fails.

A point where $\frac{d^2y}{dx^2} = 0$ is called a **point of inflection**.

1.5.4 Curve Sketching

To sketch the curve of a polynomial function we need to do the following:

1. Find the points where the curve crosses the axes. Remember that where the graph crosses the x-axis the value of y is zero, and where the graph crosses the y-axis $x = 0$.

2. Determine the behavior of the function $y = f(x)$ for large values of x, that is as $x \to \infty$ and as $x \to -\infty$.

3. Locate all the stationary points where $\frac{dy}{dx} = 0$.
4. Determine the extreme points, that is, maximum and minimum points.
5. Locate all points of inflection, where $\frac{d^2y}{dx^2} = 0$.
6. If necessary plot a few additional values.

Example 1.5.4.1 *Sketch the graph of the function* $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 2$

Solution:

1. When $x = 0$, $y = 2$. The points where the graph crosses the x-axis are irrational and so cannot be easily found.
2. When the values of x become large, the function is dominated by the term x^3. Therefore, when x becomes large and negative, y also becomes large and negative. When x becomes large and positive, y also becomes large and positive.
3. To find the stationary points we have $\frac{dy}{dx} = x^2 - x - 2$. Setting the derivative to zero and solving for x we have

   $x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0$. This gives the critical values to be $x = -1$ and $x = 2$.

   When $x = -1$ then $y = \frac{1}{3}(-1)^3 - \frac{1}{2}(-1)^2 - 2(-1) + 2 = \frac{19}{6}$

   When $x = 2$ then $y = \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 - 2(2) + 2 = -\frac{4}{3}$.

   Thus, the stationary points are $(-1, \frac{19}{6})$ and $(2, \frac{4}{3})$.
4. To classify the stationary points we have $\frac{d^2y}{dx^2} = 2x - 1$.

   At $x = -1$ we have $\frac{d^2y}{dx^2} = 2(-1) - 1 = -3 < 0$. This means that the point $(-1, \frac{19}{6})$ is a maximum point.

   At $x = 2$ we have $\frac{d^2y}{dx^2} = 2(2) - 1 = 3 > 0$. This means that the point $(2, \frac{4}{3})$ is a minimum point.

5. The point of inflection is when $\frac{d^2y}{dx^2} = 0 \Rightarrow 2x - 1 = 0$, or $x = \frac{1}{2}$. When $x = \frac{1}{2}$ then $y = \frac{11}{12}$. Thus, the point $(\frac{1}{2}, \frac{11}{12})$ is the point of inflection