INTRODUCTORY MATHEMATICS

An Introduction to Mathematical Methods

by

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Set Theory

1.1. Introduction

Sets are very cardinal to the comprehensive and logical study of mathematics. They are the building blocks of the whole structure of mathematics. We study the common numbers, space, area, volume e.t.c. using sets. It is therefore, important that we start our study of mathematics by looking at the theory of sets.

Definition 1.1.1. *A set is any well-defined collection, list or class of objects. The objects in a set are called elements or members of the set.*

1.1.1 Notations and Symbols

Having defined what a set is, we now look at the notations adopted in our discussion of set theory. The following are the symbols and notations used in the discussion of set theory in this publication

- sets are denoted by uppercase (capital) letters. e.g A, B, X, Y e.t.c
- elements are denoted by lowercase(small) letters. e.g a, b, c, x, etc
- the symbols $\subset$ and $\subseteq$ denote set inclusion. For example, if A and B are twos sets such that *all* elements in A are found in B, we say set A is included in set B. Mathematically, we write

  $$A \subset B$$

  This is read as 'set A is a subset of set B'. Note that $A \subset B$ is used when A is a proper subset of set B, otherwise, $A \subseteq B$ can be used.
- the symbols $\in$ and $\notin$ denote element inclusion and element exclusion, respectively. For example, if Y is a set given as

  $$Y = \{1, 2, 3, 4\},$$

  then we write $2 \in Y$ which is read as '2 is an element of set Y' or simply '2 is in set Y'. If an element is not in set Y, for example 7, we write $7 \notin Y$ which is read as '7 is not an element of set Y' or simply, '7 is not in set Y.'
- the symbols $\emptyset$ and $\{\}$ denote an empty set, which is a set without any elements.

- if the number of elements in a set, say A, can be counted and is finite, we write $n(A)$ to denote the total number of elements in set A. For example, if

$$A = \{1, 3, 5\} \text{ and } X = \{a, e, i, o, u\}$$

then $n(A) = 3$ and $n(X) = 5$. We say the cardinalities of sets A and X are 3 and 5, respectively. The cardinality of a set is the total number of elements in that set.

- the set containing all elements under discussion at any given time is called the Universal set. Quite often we will denote the universal set by the uppercase letter E, but any other uppercase letter can also be used.

1.1.2 Set Representation

Sets can be represented in a number of different ways. Some of the common representations are listing and set builder notations. This section discusses some of the important ways of representing the various types of sets

1. **Listing:**

   This approach simply lists all the elements of a set provided its elements are known explicitly. Listing is used to represent sets that are countable, whether finite or infinite.

   Example 1.1.1. $X = \{2, 4, 6, 8\}$ is a list of members of set X. Similarly, $Y = \{0, 1, 2, 3, \cdots\}$ is a list of the members of set Y. Set X is finitely countable while set Y is infinitely countable.

   Note that listing uses the curly brackets "{" and "}". The elements are then listed in between the two brackets while being separated by a comma. Listing is an effective and simple approach of describing a set. However, its main limitation is that it can only be applied to sets that are countable. In mathematics and science in general, we encounter a lot of sets that are not countable yet very useful. Since listing can not be used for such sets, other methods must be used describe such sets.

2. **Set-Builder Notation:**

   A more general approach in describing a set is the use of set-builder notation. Like listing, it also uses the curly brackets "{" and "}" but adds more fluidity to the approach. It is applied to both countable and non-countable sets. The examples below demonstrate the use of the set-builder notation.

   Example 1.1.2. If sets $A = \{4, 5, 6, 7, \cdots\}$, $X = \{1, 2, 3, 4, 5\}$ and $W = \{a, e, i, o, u\}$, use the set builder notation to describe sets A, X and W.

   Soln:

   (a) $A = \{x|\ x > 3, x \in \mathbb{N}\}$

   (b) $X = \{x|\ 1 \leq x \leq 5, x \in \mathbb{Z}\}$

   (c) $W = \{x|\ x \text{ is a vowel }\}$

The expression $A = \{x|\ x > 3, x \in \mathbb{N}\}$ is read as "A is a set of elements x such that x is greater than 3, and x is a natural number"

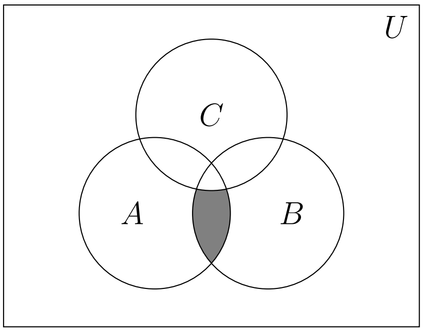
Similarly, $X = \{x|\ 1 \leq x \leq 5, x \in \mathbb{Z}\}$ is read as "X is a set of elements x such that 1 is less or equal to x yet x is less or equal to 5, and x belongs to the set of integers.

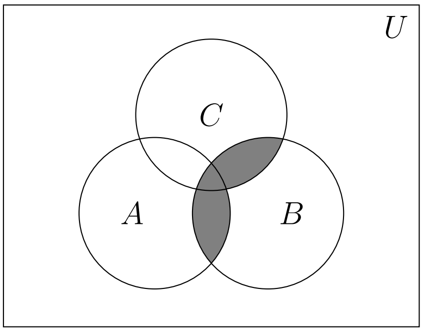
$W = \{x|\ x \text{ is a vowel } \}$ reads "W is a set of elements x such that x is a vowel.

The symbols $\mathbb{N}$ and $\mathbb{Z}$ denote sets of natural numbers and integers respectively. The letter x was used arbitrarily, any letter such as t, y e.t.c could have been used. Also, note that set-builder notation is not necessarily unique for some cases. We will see how set-builder notation can be used to describe uncountable sets.

3. **Diagrammatic Representation:**

   We can also use diagrams to represent sets. One of the most important diagrams used is the Venn Diagram which was introduced by the British Mathematician, John Venn (1834-1923). It can be used to represent sets as well as showing the results of respective set operations such as union, intersection e,t.c. However, like the listing approach, it is also limited and can not be used for certain cases, as we shall see. The diagrams below show examples of venn Diagrams.





4. **Number line Representation:** Another method used to describe sets is the use of a number line. Like the set-builder notation, this approach can be used to describe both the countable and uncountable sets. We will consider this approach when we look at intervals in $\mathbb{R}$

1.2. Sets of Numbers

At this point, we discuss the various types of sets of numbers. Numbers fall into different classes or groups depending on the properties they posses. We will discus two basic classes of numbers, the real numbers and the complex numbers.

1.2.1 Real Numbers

The concept of a real number is very important to the understanding of sets and mathematics in general. We understand that natural numbers are just positive whole numbers while integers are simply positive and negative whole numbers with zero inclusive. Now let us consider the following numbers:

$$0.5, \qquad \pi, \qquad \sqrt{2}, \qquad -\frac{1}{1000}, \qquad e, \qquad 0.0000000001, \qquad -1273.0001543$$

What type of numbers are they? We can clearly see that these numbers are neither part of the integers nor natural numbers. These are not whole numbers, yet they are *real* numbers. Infact, the type of numbers we normally use such as integers, natural numbers and non-integers (e.g those listed above), are all *real* numbers. Positive or negative, large or small, whole numbers or decimal numbers, these are all Real Numbers. Thus, a real number is a value that represents a quantity along a line. The following is a very basic definition of a real number. We will denote the set of real numbers by the symbol $\mathbb{R}$

Definition 1.2.1. *A real number is a number which can be represented by a point on a real number line that runs from negative infinite* $(-\infty)$ *to positive infinite* (∞)*. The collection of all real numbers denoted* $\mathbb{R}$*, is called the set of real numbers.*

Thus, from the definition above, we write

$$\mathbb{R} = (-\infty, \infty)$$

Note 1.2.1. Whole numbers, Natural numbers, integers, rational and irrational numbers are all real numbers. As matter of fact, any number YOU can think of at this moment, is nothing but a real number.

Real numbers can further be divided into several categories. The following are the different types of real numbers.

1. **Natural Numbers:** These are real numbers that have no decimal and are bigger than zero. The counting numbers from 1 to infinite are called natural numbers. We use $\mathbb{N}$ to denote the set of natural numbers. Thus,

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \cdots\}$$

2. **Whole Numbers:** These are positive real numbers that have no decimals, and also zero. Natural numbers are also whole numbers. The counting numbers from zero to infinite are called whole numbers. We use $\mathbb{W}$ to denote the set of whole numbers, i.e

$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots\}$$

Note that we can write

$$\mathbb{N} = \{x|\ x \geq 1, x \in \mathbb{W}\} \qquad \text{since } \mathbb{N} \subset \mathbb{W}$$

3. **Integers:** A collection of positive and negative whole numbers with zero inclusive, is called the set of integers and is denoted by $\mathbb{Z}$. Thus,

$$\mathbb{Z} = \{\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots\}$$

4. **Prime Numbers:** A prime number is a natural number that has only two factors, namely 1 and itself. In other words, a number that is divisible by 1 and itself. Lets denote the primes by $\mathbb{P}$, then
$$\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, \cdots\}$$

5. **Even Numbers:** An even number is an integer which is "evenly divisible" by two. This means that if the integer is divided by 2, it yields no remainder. Zero is an even number because zero divided by two equals zero. Even numbers can be either positive or negative. The collection of all even numbers constitute the set of even numbers.

6. **Odd Numbers:** An odd number is an integer which is not a multiple of two. If it is divided by two the result is a fraction. One is the first odd positive number. The next four bigger odd numbers are three, five, seven, and nine. So some sequential odd numbers are:
$$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25...\}$$

7. **Rational Numbers:** These are real numbers that can be written down as fractions of integers. Mathematically, they can be written in the form $\frac{a}{b}$, where a and b are integers with $b \neq 0$. Note that all Integers are also rational numbers. We will denote the set of all rational numbers by $\mathbb{Q}$. We can not display the set of rational numbers since it is infinitely uncountable. The following are some examples of rational numbers:
$$0.5, \qquad 17.312, \qquad 2.\overline{251}, \qquad 0.\overline{3}, \qquad -1, \qquad 0, \qquad \frac{3}{7}, \qquad -\frac{13}{15}$$

Example 1.2.1. Show that every integer is a rational number

Soln:

Let $\mathbb{Z}$ be the set of all integers. Also, let z be any integer chosen from $\mathbb{Z}$. Then z can be written as
$$z = \frac{z}{1}$$
Since both z and 1 are integers, z is rational. Since z was chosen arbitrary, every integer must be rational.

Example 1.2.2. Show that $0.\overline{3}$ is a rational number by writing it in the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$

Soln:

We let
$$x = 0.\overline{3}$$
Then multiplying both sides of this equation by 10 gives us
$$10x = 3.\overline{3}$$

$$\begin{aligned} 10x - x =& 3.\overline{3} - 0.\overline{3} \\ 9x =& 3 \\ x =& \frac{1}{3} \end{aligned}$$

Therefore, we conclude that $0.\overline{3} = \frac{1}{3}$. Since both 1 and 3 are integers, $0.\overline{3}$ is a rational number.

Example 1.2.3. Express $-3.5\overline{32}$ in the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$

Soln:

Let $y = -3.5\overline{32}$.

Then multiplying both sides of this equation by 10 gives us: $10y = -35.\overline{32}$ (i)

Further multiplication of $y = -3.5\overline{32}$ by 1000 gives us: $1000y = -3532.\overline{32}$ (ii)

subtracting the two equations, we have:

$$\begin{aligned} 1000y - 10y &= -\ 3532.\overline{32} - (-35.\overline{32}) \\ 990y &= -\ 3497 \\ y &= -\ \frac{3497}{990} \end{aligned}$$

Therefore, $-3.5\overline{32} = -\frac{3497}{990}$. Since 3497 and 990 are integers, $-3.5\overline{32}$ is rational.

8. **Irrational Numbers:** An irrational number is a real number whose decimal part does not repeat nor terminate. These are real numbers that can not be written as a fraction of integers. Irrational numbers can not be written in the form $\frac{a}{b}$, where a and b are integers with $b \neq 0$. If a real number is not rational, then it must be irrational. We denote the set of irrational numbers by $\mathbb{Q}'$. Like the rational numbers, irrational numbers are infinitely uncountable. The following are some examples of irrational numbers:

$$\sqrt{2}, \qquad \sqrt{3}, \qquad \sqrt{2}+4, \qquad \pi, \qquad e, \qquad \sqrt{3}-1, \qquad -3-\sqrt{2}, \qquad -\frac{\sqrt{2}}{3}$$

Theorem 1.2.2. *If $k \in \mathbb{Z}$ and k^2 is divisible by 2, then k is also divisible by 2.*

Example 1.2.4. Prove that $\sqrt{2}$ is irrational.

Proof: We prove this by contradiction. Suppose that $\sqrt{2}$ is rational. Then by definition of a rational number, we can write $\sqrt{2}$ as:

$$\sqrt{2} = \frac{a}{b}, \qquad \text{where} \quad a, b \in \mathbb{Z} \quad \text{with} \quad b \neq 0$$

and we assume that the fraction $\frac{a}{b}$ is in its lowest form, ie, there are no common factors of a and b. Then,

$$\sqrt{2} = \frac{a}{b} \Longrightarrow 2b^2 = a^2 \qquad\qquad *$$

This simply means that a^2 is divisible by 2. Since a is an integer, by the theorem above, we conclude that a is also divisible by 2. Hence, we can write $a = 2m$ for some $m \in \mathbb{Z}$. From $*$, we have

$$2b^2 = a^2 = (2m)^2 = 4m^2 \quad \text{so that} \quad b^2 = 2m^2$$

This means that b^2 is divisible by 2. By the theorem above, b is also divisible by 2 since $b \in \mathbb{Z}$.

Since both a and b are now divisible by 2, this contradicts the statement that $\frac{a}{b}$ is in its lowest form. Hence, $\sqrt{2}$ can not be written in the form $\frac{a}{b}$ with $a, b \in \mathbb{Z}$ such that $\frac{a}{b}$ is in its lowest form. Therefore $\sqrt{2}$ must be irrational.

Example 1.2.5. Prove that $5+\sqrt{2}$ is irrational.

Proof: Prove by contradiction. Suppose that $5+\sqrt{2}$ is rational. Then we can write $5+\sqrt{2}$ as a ratio of two integers:

$$5+\sqrt{2}=\frac{p}{q} \quad \text{where} \quad p,q\in\mathbb{Z} \quad \text{with} \quad q\neq 0$$

and assume that the fraction $\frac{p}{q}$ is in its lowest form. Making $\sqrt{2}$ the subject of the formula, we have

$$\sqrt{2}=\frac{p-5q}{q} \qquad\qquad *$$

From $*$, we can see that L.H.S is irrational while R.H.S is rational. This is not possible. Thus our assumption must be wrong. We conclude therefore, that $5+\sqrt{2}$ is irrational.

Surds and Rationalisation

Note that some of the examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{7}$. The square root symbol $\sqrt{\dots}$ is called the radical sign. We use it to indicate a positive square root of a number. For example, $\sqrt{16}$ indicates the positive square root of 16 which is 4. The symbol $\sqrt[n]{\dots}$ is used to indicate the n^{th} root of a number. For example, the 5^{th} root of 32 is 2, hence, we write $\sqrt[5]{32}=2$. The numbers of the form $\sqrt[n]{x}$ are called radicals or surds. For any two positive integers a and b, the following hold;

- $\sqrt{ab}=\sqrt{a}\sqrt{b}$ e.g. i) $\sqrt{6}=\sqrt{(2)(3)}=\sqrt{2}\sqrt{3}$ ii) $\sqrt{80}=\sqrt{(16)(5)}=\sqrt{16}\sqrt{5}=4\sqrt{5}$
- $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$ e.g. i) $\sqrt{\frac{16}{81}}=\frac{\sqrt{16}}{\sqrt{81}}=\frac{4}{9}$ ii) $\sqrt{\frac{2}{3}}=\frac{\sqrt{2}}{\sqrt{3}}$
- $a\sqrt{b}=\sqrt{a^2b}$ e.g. $4\sqrt{3}=\sqrt{4^2\times 3}=\sqrt{(16)(3)}=\sqrt{48}$
- $\sqrt{a^2}=a$ e.g. $\sqrt{7^2}=7$ Similarly, we have $\sqrt{(53)^2}=53$

Example 1.2.6. Simplify: i) $\sqrt{45}$ ii) $\sqrt{450}$ iii) $\sqrt{50}+\sqrt{2}-2\sqrt{18}+\sqrt{8}$

Sol:

i) $\sqrt{45}=\sqrt{(9)(5)}=\sqrt{9}\sqrt{5}=3\sqrt{5}$

ii) $\sqrt{450}=\sqrt{(9)(25)(2)}=\sqrt{9}\sqrt{25}\sqrt{2}=(3)(5)\sqrt{2}=15\sqrt{2}$

iii) we simplify and add up the surds

$$\begin{aligned}\sqrt{50}+\sqrt{2}-2\sqrt{18}+\sqrt{8} &=\sqrt{(25)(2)}+\sqrt{2}-2\sqrt{(9)(2)}+\sqrt{(4)(2)}\\ &=\sqrt{25}\sqrt{2}+\sqrt{2}-2\sqrt{9}\sqrt{2}+\sqrt{4}\sqrt{2}\\ &=5\sqrt{2}+\sqrt{2}-2(3)\sqrt{2}+2\sqrt{2}\\ &=5\sqrt{2}+\sqrt{2}-6\sqrt{2}+2\sqrt{2}\\ &=(5+1-6+2)\sqrt{2}\\ &=2\sqrt{2}\end{aligned}$$

Certain radicals can be made simple by transforming the denominator into a rational number. This process is called rationalization of the denominator. The method makes use of the difference of two squares,

$$a^2 - b^2 = (a-b)(a+b)$$

The following examples help demonstrate the approach

Example 1.2.7. For each of the following, rationalize the denominator and simplify.

i) $\frac{1}{\sqrt{2}}$ ii) $\frac{1}{1+\sqrt{3}}$ iii) $\frac{1}{\sqrt{2}} + \sqrt{8}$ iv) $\frac{1+\sqrt{x}}{3-\sqrt{x}}$

Sol:

i) We have $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times 1 = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

ii) We have $\frac{1}{1+\sqrt{3}} \times 1 = \frac{1}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{1-\sqrt{3}}{1-3} = \frac{1-\sqrt{3}}{-2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

iii) We have $\frac{1}{\sqrt{2}} + \sqrt{8} = \frac{\sqrt{2}}{2} + \sqrt{(4)(2)} = \frac{\sqrt{2}}{2} + 2\sqrt{2} = \frac{\sqrt{2}+4\sqrt{2}}{2} = \frac{5\sqrt{2}}{2}$

iv) We have $\frac{1+\sqrt{x}}{3-\sqrt{x}} = \frac{1+\sqrt{x}}{3-\sqrt{x}} \times 1 = \frac{1+\sqrt{x}}{3-\sqrt{x}} \times \frac{3+\sqrt{x}}{3+\sqrt{x}} = \frac{(1+\sqrt{x})(3+\sqrt{x})}{9-x} = \frac{(x+3)+4\sqrt{x}}{9-x}$

Note 1.2.2. It is important to take note of the following

- Any real number is either rational or irrational.
- Any number with a terminating or recurring decimal part is rational
- Any number with a non terminating yet not recurring decimal part is irrational
- $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$
- $\mathbb{Q} \cap \mathbb{Q}' = \emptyset$
- $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Intervals in $\mathbb{R}$

We now discuss the important concept of intervals of real numbers. These are basically subsets of $\mathbb{R}$. Let $a, b \in \mathbb{R}$ such that $a < b$. Then, there is an infinite number of real numbers that lie between the two real numbers a and b. The set of *all* real numbers between a and b constitute an interval in $\mathbb{R}$ from point a to point b. The following are the types of intervals from a to b:

- $I_1 = (a, b)$ is the set of all real numbers between a and b, excluding a and b. In set builder notation form, we have

  $$(a, b) = \{x \mid a < x < b, x \in \mathbb{R}\}$$

  Here, the interval $I_1 = (a, b)$ consists of all real numbers **between** a and b but not including a nor b.

- $I_2 = (a, b]$ is the set of all real numbers between a and b including b, but excluding a. In set builder notation form, we have

$$(a, b] = \{x|\ a < x \leq b, x \in \mathbb{R}\}$$

Here, the interval $I_2 = (a, b]$ consists of all real numbers between a and b with b inclusive, yet a excluded

- $I_3 = [a, b)$ is the set of all real numbers between a and b including a, but excluding b. In set builder notation form, we have

$$[a, b) = \{x|\ a \leq x < b, x \in \mathbb{R}\}$$

Here, the interval $I_3 = [a, b)$ consists of all real numbers between a and b with a inclusive, yet b excluded

- $I_4 = [a, b]$ is the set of all real numbers between a and b, including both a and b. In set builder notation form, we have

$$[a, b] = \{x|\ a \leq x \leq b, x \in \mathbb{R}\}$$

Here, the interval $I_4 = [a, b]$ consists of all real numbers **between** a and b, with both a and b included in the set.

1.2.2 Set Operations

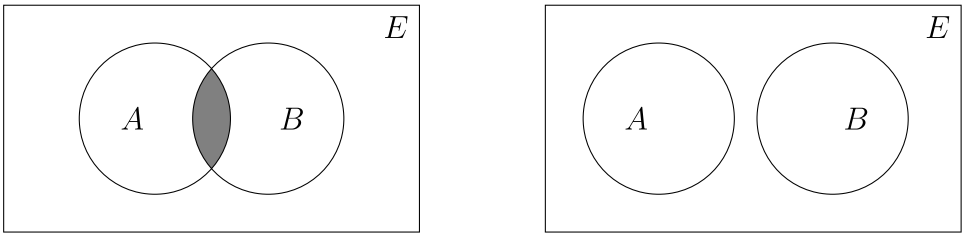
We are familiar with operations of addition, subtraction, multiplication and division. These are number operations since they are performed on numbers. Sets behave like numbers. They have their own operations which include union, intersection, complementation and product. This section discuses some of the set operations as well as the laws that govern them.

1. **Intersection:**

   The intersection of two sets, say A and B, is denoted by $A \cap B$ and is defined as the set that contains elements common to both A and B. In symbols, we can write

$$A \cap B = \{x|\ x \in A \text{ and } \ x \in B\}$$

   Pictorially, we have



Example 1.2.8. If $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$ and $C = \{4, 5, 6\}$, then we can see that:

$$A \cap B = \{3\}, \qquad A \cap C = \{\}, \qquad \text{and} \qquad B \cap C = \{4, 5\}$$

Example 1.2.9. If $A = (1, 3)$, $B = [3, 5]$ and $C = (0, 4]$, find the follwing:

i) $A \cap B$ ii) $A \cap C$ and iii) $B \cap C$

Soln:

i)
$$\begin{aligned} A \cap B &= (1, 3) \cap [3, 5] \\ &= \{\} \\ &= \emptyset \end{aligned}$$

ii)
$$\begin{aligned} A \cap C &= (1, 3) \cap (0, 4] \\ &= (1, 3) \end{aligned}$$

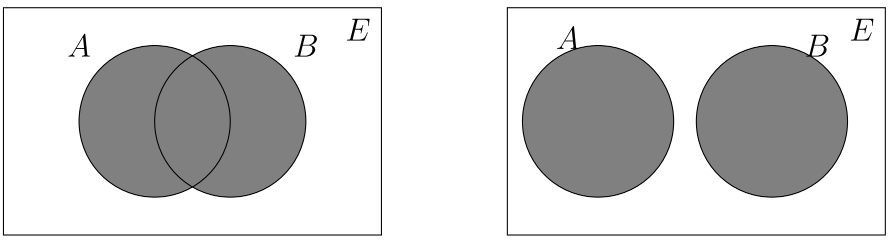
iii)
$$\begin{aligned} B \cap C &= [3, 5] \cap (0, 4] \\ &= [3, 4] \end{aligned}$$

2. **Union:**

   The union of two sets, say A and B, is denoted by $A \cup B$ and is defined as the set that contains elements found either in A or in B. In symbols, we can write

   $$A \cup B = \{x |\ x \in A \text{ or } \ x \in B\}$$

   Pictorially, we have



Example 1.2.10. If $X = \{1, 2, 3\}$, $Y = \{3, 4, 5\}$ and $Z = \{4, 5, 6\}$, then, clearly:

$$X \cup Y = \{1, 2, 3, 4, 5\}, \qquad X \cup Z = \{1, 2, 3, 4, 5, 6\}, \qquad \text{and} \qquad Y \cup Z = \{3, 4, 5, 6\}$$

Example 1.2.11. If $A = (1, 3)$, $B = [3, 5]$ and $C = (0, 4]$, find the follwing:

i) $A \cup B$ ii) $A \cup C$ and iii) $B \cup C$

Soln:

i)
$$\begin{aligned} A \cup B &= (1, 3) \cup [3, 5] \\ &= (1, 5] \end{aligned}$$

ii)
$$\begin{aligned} A \cup C &= (1, 3) \cup (0, 4] \\ &= (0, 4] \end{aligned}$$

iii)
$$\begin{aligned} B \cup C &= [3, 5] \cup (0, 4] \\ &= (0, 5] \end{aligned}$$

Example 1.2.12. If $A = (0, 1)$, $B = [0, 1]$ and $C = [1, \infty)$, find the following:

i) $A \cup B$ ii) $A \cup C$ and iii) $B \cup C$

Soln:

i)
$$\begin{aligned} A \cup B &= (0, 1) \cup [0, 1] \\ &= [0, 1] \\ &= B \qquad \text{Since } A \subset B. \text{ Similarly } A \cap B = A \end{aligned}$$

ii)
$$\begin{aligned} A \cup C &= (0, 1) \cup [1, \infty) \\ &= (0, \infty) \end{aligned}$$

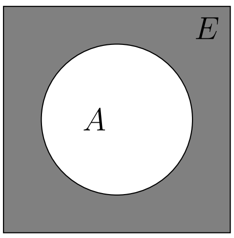
iii)
$$\begin{aligned} B \cup C &= [0, 1] \cup [1, \infty) \\ &= [0, \infty) \end{aligned}$$

3. **Complementation:**

Let E be the universal set and $A \subset E$. The set of all the elements in the universal set E but not in set A is called the complement of A in E. It is denoted by A' or A^c. In symbols, we write

$$A' = \{x | \ x \notin A \text{ but } \ x \in E\}$$

Pictorally



We sometimes treat the complement as subtraction, i.e

$$A' = E - A = E \cap A'$$

Example 1.2.13. Let $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set. Further, let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = [5, 6]$

Soln:

First we notice that $A \subset E$ and $B \subset E$. However C is not a subset of E.

Then, $A' = \{6, 7, 8, 9, 10\}$ and $B' = \{1, 3, 5, 7, 9\}$. However, No C'

Example 1.2.14. Let $\mathbb{R} = (-\infty, \infty)$ be the universal set. Further, suppose that $X = [1, 4]$, $Y = [-3, 2)$, $A = (-2, \infty)$ and $B = [-\infty, 0]$ are subsets of $\mathbb{R} = (-\infty, \infty)$. Find the complements of X, Y, A and B and display the results on the real number line.

Soln:

i)
$$\begin{aligned} X' &= \mathbb{R} - X \\ &= (-\infty, \infty) - [1, 4] \\ &= (-\infty, 1) \cup (4, \infty) \end{aligned}$$

ii)
$$\begin{aligned} Y' &= \mathbb{R} - Y \\ &= (-\infty, \infty) - [-3, 2) \\ &= (-\infty, -3) \cup [2, \infty) \end{aligned}$$

iii)
$$\begin{aligned} A' &= \mathbb{R} - A \\ &= (-\infty, \infty) - (-2, \infty) \\ &= (-\infty, -2] \end{aligned}$$

iv)
$$\begin{aligned} B' &= \mathbb{R} - B \\ &= (-\infty, \infty) - [-\infty, 0] \\ &= (0, \infty) \end{aligned}$$

Example 1.2.15. Let $E = [-10, 10]$ be the universal set. Further, let $X = [-2, 8)$, $Y = (-10, 3)$ and $Z = (-1, 10]$ be subsets of E. Find the following sets and display the results on the number line, where possible.

a) $X - Y$ b) $Y - X$ c) $X - (Y - Z)$ d) $X \cup (Y \cap Z)$ e) $X' \cup Z$

Sol:

a) $X - Y$ denotes the elements in X but not in Y. i.e, $X - Y = X \cap Y'$. Hence,

$$X - Y = [-2, 8) - (-10, 3) = [3, 8)$$

b) $Y - X$ denotes the elements in Y but not in X. i.e, $Y - X = Y \cap X'$. Hence,

$$Y - X = (-10, 3) - [-2, 8) = (-10, -2)$$

c) $X - (Y - Z) = [-2, 8) - \{(-10, 3) - (-1, 10]\} = [-2, 8) - (-10, -1] = (-1, 8)$

d) $X \cup (Y \cap Z) = [-2, 8) \cup ((-10, 3) \cap (-1, 10]) = [-2, 8) \cup (-1, 3) = [-2, 8)$

e) $X' \cup Z = [-10, -2) \cup [8, 10] \cup (-1, 10] = [-10, -2) \cup (-1, 10]$

Laws On Set Operations

The following are the laws obeyed by sets. We call **The Boolean Laws**. Suppose that X, Y and Z are subsets of the universal set E. Then the following hold:

- Idempotence:
  a) $X \cup X = X$ b) $X \cap X = X$
- Associativity:
  a) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$ b) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$
- Commutativity:
  a) $X \cup y = Y \cup X$ b) $X \cap Y = Y \cap X$
- Distributivity:
  a) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ b) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
- De Morgan's:
  a) $(X \cup Y)' = Y' \cap X'$ b) $(X \cap Y)' = Y' \cup X'$
- Properties of the Complement:
  a) $X' \cup X = E$ b) $X' \cap X = \emptyset$ c) $(X')' = X$
- Properties of the universal set:
  a) $X \cup E = E$ b) $X \cap E = X$ c) $E' = \emptyset$
- Properties of the empty set:
  a) $X \cup \emptyset = X$ b) $X \cap \emptyset = \emptyset$ c) $\emptyset' = E$

We will use these laws to simplify some complex set expressions

Example 1.2.16. Let $E = [0, 1]$ be the universal set. If $X = (0.5, 1)$ and $Y = (0.8, 1]$ Verify the De Morgan's Law shown below

$$(X \cap Y)' = X' \cup Y'$$

Soln: We can see that $X' = [0, 0.5] \cup \{1\}$ and $Y = [0, 0.8]$. Therefore,

$$\begin{aligned} LHS =& (X \cap Y)' \\ =& ((0.5, 1) \cap (0.8, 1])' \\ =& (0.8, 1)' \\ =& [0, 0.8] \cup \{1\} \end{aligned}$$

Thus $(X \cap Y)' = [0, 0.8] \cup \{1\}$

$$\begin{aligned} RHS =& X' \cup Y' \\ =& ((0.5, 1) \cap (0.8, 1])' \\ =& (0.8, 1)' \\ =& [0, 0.8] \cup \{1\} \end{aligned}$$

Thus $X' \cup Y' = [0, 0.8] \cup \{1\}$

Therefore, $(X \cap Y)' = X' \cup Y'$, proving the De Morgan's Law.

Example 1.2.17. Let E denote the universal set. Further, let X and Y be two sets in E. simplify as far as possible the expression

$$[(X \cap Y)' \cup (X - Y)]'$$

Sol: we use the laws on set operation

$$\begin{aligned} [(X \cap Y)' \cup (X - Y)]' =& [(X \cap Y)' \cup (X \cap Y')]' \\ =& [(X \cap Y)' \cup (X \cap Y')]' \\ =& (X \cap Y) \cap (X \cap Y')' \\ =& (X \cap Y) \cap (X' \cup Y) \\ =& [(X \cap Y) \cap X'] \cup [(X \cap Y) \cap Y] \\ =& \emptyset \cup (X \cap Y) \\ =& X \cap Y \end{aligned}$$

Example 1.2.18. Prove the DeMorgan's Law: $(A \cup B)' = A' \cap B'$

Sol: Suppose that $x \in (A \cup B)'$. Then, $x \in A \cup B$. This is only possible if $x \in A$ and $x \in B$. This means that $x \in A'$ and $x \in B'$. Hence, $x \in A' \cap B'$. Therefore, we conclude that $(A \cup B)' \subset A' \cap B'$

Conversely, suppose $x \in A' \cap B'$. Then $x \in A'$ and $x \in B'$. This means that $x \in A'$ and $x \in B'$. Hence, $x \in (A \cup B)'$ implying that $x \in (A \cup B)'$. Therefor, $A' \cap B' \subset (A \cup B)'$

1.2.3 Complex Numbers

So far, we have discussed the real numbers and some of their properties. We now turn our attention to another type of numbers called complex numbers. The collection of all complex numbers is called the set of complex numbers and is usually denoted as $\mathbb{C}$. To better understand the concept of a complex number, let us consider the following radicals;

$\sqrt{-2} = \sqrt{(2)(-1)} = \sqrt{2}\sqrt{-1}$

$\sqrt{-3} = \sqrt{(3)(-1)} = \sqrt{3}\sqrt{-1}$

$\sqrt{-4} = \sqrt{(4)(-1)} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1}$

$\sqrt{-5} = \sqrt{(5)(-1)} = \sqrt{5}\sqrt{-1}$

It is convenient to introduce the symbol;

$$i = \sqrt{-1}$$

This is called the imaginary unit. The above surds can now be written as shown below;

$\sqrt{-2} = \sqrt{(2)(-1)} = \sqrt{2}\sqrt{-1} = i\sqrt{2}$

$\sqrt{-3} = \sqrt{(3)(-1)} = \sqrt{3}\sqrt{-1} = i\sqrt{3}$

$\sqrt{-4} = \sqrt{(4)(-1)} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1} = 2i$

$\sqrt{-5} = \sqrt{(5)(-1)} = \sqrt{5}\sqrt{-1} = i\sqrt{5}$

Thus, we can write $\sqrt{-25} = 5i$, $\sqrt{-16} = 4i$, $\sqrt{-100} = 10i$ and so on.....

Properties of the imaginary unit i

- $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} = -1$. Hence, $\quad i^2 = -1$
- $i^3 = i \times i = \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} = -1 \times i = -i$. Hence, $\quad i^3 = -i$
- $i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$. Hence, $\quad i^4 = 1$
- Generally, $i^{2n} = (-1)^n$
- Generally, $i^{2n+1} = (-1)^n\ i$

Definition 1.2.3. *Let z be a complex number. The standard form or cartesian form of a complex number z is given as*

$$z = x + yi \text{ or } z = x + iy$$

where $x, y \in \mathbb{R}$. The real number x is called the real part of z while the real number y is called the imaginary part of z

From this definition, we see that every complex number z is written in terms of its real and imaginary parts. The imaginary part is the part that is multiplied by the imaginary unit i. Hence, if $z = x + iy$, then

$x = Re(z)$ which is read as "x is the real part of z"

$y = Im(z)$ which is read as "y is the imaginary part of z"

Example 1.2.19. For each of the following complex numbers, determine the real part and the imaginary part.

i) $3+2i$ ii) $7-4i$ iii) $1-i$ iv) $4i$ v) -2 vi) -10i

Sol: Exercise

Definition 1.2.4. *A complex number $z = x+yi$ is said to be pure real if $Im(z) = 0$, i.e $y = 0$.*

Definition 1.2.5. *A complex number $z = x+yi$ is said to be pure imaginary if $Re(z) = 0$, i.e $x = 0$.*

Example 1.2.20. The number $4 = 4+0i$. Hence 4 is pure real. Similarly, the number $-\frac{2}{7} = -\frac{2}{7}+0i$ is pure real.

On the other hand, the number $-3i = 0-3i$ is pure imaginary. similarly, $0.023i = 0+0.023i$ is pure imaginary.

Definition 1.2.6. *The collection of all complex numbers constitute the set of complex numbers denoted by $\mathbb{C}$. Therefore*

$$\mathbb{C} = \left\{x+yi|\ x, y \in \mathbb{R}, \quad i = \sqrt{-1}\right\}$$

Note 1.2.3. The following follow from the definitions above.

- Any real number $a \in \mathbb{R}$ can be written as $a = a+0i$. Hence all real numbers are also complex numbers
- Therefore the set of real numbers is a subset of the set of complex numbers
- $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Definition 1.2.7. *Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are also equal.*

From the definition, if $z_1 = x_1+y_1i$ and $z_2 = x_2+y_2i$ are two complex numbers, $z_1 = z_2$ implies that $x_1 = x_2$ and $y_1 = y_2$

Example 1.2.21. Find the values of x and y given that $2x+4i = 6+i(x+y)$

Sol: Using the definition for equality of complex numbers, we equate the real parts and the imaginary parts. Thus,

$$2x = 6 \text{ and } 4 = x+y$$

Solving these two simultaneously gives $x = 3$ and $y = 1$

Addition and Subtraction of Complex Numbers

To add two complex numbers, add the real part of one complex number to the real part of the other complex number and add the imaginary of one complex number to the imaginary of the other complex number. Similarly, to subtract two complex numbers, subtract the real parts and subtract the imaginary parts.

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ be two complex numbers. Then

a) $z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$ and

b) $z_1 - z_2 = (x_1 + y_1 i) - (x_2 + y_2 i) = (x_1 - x_2) + (y_1 - y_2)i$

Example 1.2.22. Given that $z_1 = -2 + 3i$ and $z_2 = 4i$, find i) $z_1 + z_2$ ii) $z_2 - z_1$

Sol: Note that $z_2 = 4i$ can be written as $z_2 = 0 + 4i$. Thus,

i) $z_1 + z_2 = (-2 + 3i) + (0 + 4i) = (-2 + 0) + (3 + 4)i = -2 + 7i$

ii) $z_2 - z_1 = (0 + 4i) - (-2 + 3i) = (0 - (-2)) + (4 - 3)i = 2 + 1i = 2 + i$

Example 1.2.23. If $z = 3 - 11i$ and $w = -1 + i$, find i) $z + w$ ii) $z - w$

Sol:

i) $z + w = (3 - 11i) + (-1 + i) = (3 + (-1)) + (-11 + 1)i = 2 - 10i$

ii) $z - w = (3 - 11i) - (-1 + i) = (3 - (-1)) + (-11 - 1)i = 4 - 12i$

Multiplication of Complex Numbers

To multiply complex numbers, we multiply as we usually do in algebra, but remembering that $i^2 = -1$ Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ be two complex numbers. Then

$$\begin{aligned} z_1 z_2 =& (x_1 + y_1 i)(x_2 + y_2 i) \\ =& x_1 x_2 + x_1 y_2 i + y_1 i x_2 + y_1 y_2 i^2 \\ =& x_1 x_2 + x_1 y_2 i + y_1 i x_2 - y_1 y_2 \\ =& (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i \end{aligned}$$

Hence, if $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are two complex numbers, then

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$$

This is the multiplication of two complex numbers.

Example 1.2.24. Let $z_1 = 3 + 2i$ and $z_2 = 4 + 5i$. Determine $z_1 z_2$

Sol: using $z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$, we have

$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i = [(3)(4) - (2)(5)] + [(3)(5) + (4)(2)]i = 2 + 23i$

Example 1.2.25. Evaluate the following:

i) $(2+i\sqrt{2})(1+i)$ ii) $(2+i\sqrt{2})^2$ iii) $(1-i)(i)$ iv) $(3-i)(1+2i)$

Sol:

i) $(2+i\sqrt{2})(1+i) = 2+2i+i\sqrt{2}+i^2\sqrt{2} = 2-\sqrt{2}+(2+\sqrt{2})i$

ii) $(2+i\sqrt{2})^2 = 4+4i\sqrt{2}+2i^2 = 4-2+4i\sqrt{2} = 2+i4\sqrt{2}$

iii) $(1-i)(i) = i-i^2 = i-(-1) = 1-i$

iv) $(3-i)(1+2i) = 3+6i-i-2i^2 = 5+5i$

We have discussed the addition, subtraction and multiplication of complex numbers. Division however is not as straight forward. We require a preliminary discussion. Thus, before we discuss division, we need to discuss the concept of a conjugate.

Conjugate of a Complex number

Definition 1.2.8. *Let $z = x+yi$ be a complex number. Then the conjugate of z denoted by $\overline{z}$ is defined as*

$$\overline{z} = x - yi$$

Example 1.2.26. Find the conjugate of the following complex numbers;

i) $3+4i$ ii) $1-3i$ iii) $7i$ iv) 5 v) $1+i$ vi) $-5-i\sqrt{2}$

Sol:

i) If $z = 3+4i$, then $\overline{z} = 3-4i$

ii) If $z = 1-3i$, then $\overline{z} = 1+3i$

iii) If $z = 0+7i$, then $\overline{z} = 0-7i = -7i$

iv) If $z = 5+0i$, then $\overline{z} = 5-0i = 5$

v) If $z = 1+i$, then $\overline{z} = 1-i$

vi) If $z = -5-i\sqrt{2}$, then $\overline{z} = -5+i\sqrt{2}$

Note from iv) that the conjugate of a pure real number is just itself.

Properties of the Conjugate

Let $z = x+yi$ be a complex number and $\overline{z} = x-yi$ be its conjugate. Then the following properties hold:

- $z\overline{z} = x^2+y^2$
- $|z| = \sqrt{x^2+y^2}$. This is called the modulus or absolute value of z.
- Hence, it is easy to see that $z\overline{z} = |z|^2$
- $z+\overline{z} = 2Re(z)$. In this case, $z+\overline{z} = 2x$
- $z-\overline{z} = 2Im(z)$. In this case, $z-\overline{z} = 2y$
- If z_1 and z_2 are two complex numbers, then $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

- If z_1 and z_2 are two complex numbers, then $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- If z_1 and z_2 are two complex numbers, then $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
- If z_1 and z_2 are two complex numbers, then $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Example 1.2.27. Find the modulus for each of the following complex numbers.

i) $z = 1 - 6i$ ii) $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ iii) $z = -4$

Sol: Recall that the modulus of a complex number is given by $|z| = \sqrt{x^2 + y^2}$

i) If $z = 1 - 6i$, then $|z| = \sqrt{(1)^2 + (-6)^2} = \sqrt{37}$

ii) If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$, then $|z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

iii) If $z = -4 + 0i$, then $|z| = \sqrt{(-4)^2 + 0^2} = \sqrt{4} = 2$

Example 1.2.28. Let $z = x + yi$. Given that $z\overline{z} + 2iz = 12 + 16i$, find z.

Sol: Exercise

Division of Complex Numbers

Let $z_1 = x_1 + y_2$ and $z_2 = x_2 + y_2$ be two complex numbers. To divide any two complex numbers, say z_1 and z_2, we write $\frac{z_1}{z_2}$ and the multiply both the numerator and the denominator by the conjugate of z_2. Then we simplify the expression. i.e

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

Example 1.2.29. Evaluate the following and express the result in standard form:

i) $\frac{2-4i}{4+3i}$ ii) $\frac{2}{2-i}$ iii) $\frac{1}{(1-i)^2}$ iv) $\frac{1}{(2+i)(1+i)}$ v) $\frac{1-2i}{(1+i)^3}$

Sol: Multiply by the conjugate of the denominator

i)

$$\begin{aligned}\frac{2-4i}{4+3i} &= \frac{(2-4i)(4-3i))}{(4+3i)(4-3i)} \\ &= \frac{8-6i-16i+12i^2}{4^2+3^2} \\ &= \frac{8-12-22i}{25} \\ &= -\frac{4}{12} - \frac{22}{25}i\end{aligned}$$

ii)

$$
\begin{aligned}
\frac{2}{2-i} &= \frac{2(2+i)}{(2-i)(2+i)} \\
&= \frac{4+2i}{4+1} \\
&= \frac{4+2i}{5} \\
&= \frac{4}{5} + \frac{2}{5}i
\end{aligned}
$$

iii)

$$
\begin{aligned}
\frac{1}{(1-i)^2} &= \frac{1}{(1-i)(1-i)} \\
&= \frac{1}{1-2i-1} \\
&= \frac{1}{-2i} \\
&= \frac{1}{0-2i} \\
&= \frac{1(0+2i)}{(0-2i)(0+2i)} \\
&= \frac{2i}{-4i^2} \\
&= \frac{2i}{4} \\
&= \frac{i}{2}
\end{aligned}
$$

iv)

$$
\begin{aligned}
\frac{1}{(2+i)(1+i)} &= \frac{1}{2+2i+i+i^2} \\
&= \frac{1}{2-1+3i} \\
&= \frac{1}{1+3i} \\
&= \frac{1(1-3i)}{(1+3i)(1-3i)} \\
&= \frac{1-3i}{1+9} \\
&= \frac{1-3i}{10} \\
&= \frac{1}{10} - \frac{3}{10}i
\end{aligned}
$$

iv) Exercise

1.3. Binary Operations

We conclude our discussion of set theory by looking at the binary operations on a set.

Definition 1.3.1. *Let X denote a set. A binary operation on set X, denoted by $*$, is an operation which assigns to each pair of elements $a, b \in X$, a unique element $a * b \in X$.*

Note 1.3.1. Two points are worth remembering when considering whether an operation $*$ is binary or not

- the operation $*$ must act on **every** pair $a, b \in X$ to produce an output $a * b$.
- the output $a * b$ **must** be **unique** and an element of X.

Example 1.3.1. Let $\mathbb{N}$ denote the set of all natural numbers. Show that $* = +$ is a binary operation on $\mathbb{N}$.

Sol: Let $a, b \in \mathbb{N}$. Then $a * b = a + b \in \mathbb{N}$. Also, for any two natural numbers, we know that $a + b$ is unique and is also a natural number. Hence, $* = +$ is a binary operation on $\mathbb{N}$.

Example 1.3.2. Let $\mathbb{N}$ denote the set of all natural numbers. Show that $* = \times$ is a binary operation on $\mathbb{N}$.

Sol: Let $a, b \in \mathbb{N}$. Then $a * b = a \times b \in \mathbb{N}$. Also, for any two natural numbers, $a \times b$ is unique. Hence, $* = \times$ is indeed a binary operation on $\mathbb{N}$.

The two examples above, show that the operations of addition and multiplication are each a binary operation on the set of natural numbers. We can say that the set of natural numbers is closed under addition or multiplication.

Example 1.3.3. Let $\mathbb{N}$ denote the set of all natural numbers. Show that $* = -$ is NOT a binary operation on $\mathbb{N}$.

Sol: Let $a, b \in \mathbb{N}$. Then $a * b = a - b$ is not in $\mathbb{N}$ if $a < b$. Hence, $* = -$ is not a binary operation on $\mathbb{N}$. To see this, suppose $a = 2$ and $b = 7$. Then $a * b = 2 * 7 = 2 - 7 = -5$, and we can clearly see that -5 is not in $\mathbb{N}$

Example 1.3.4. Let $\mathbb{Z}$ denote the set of all integers, show that $* = \div$ is NOT a binary operation on $\mathbb{Z}$

Sol: Exercise

1.3.1 Characteristics of Binary Operations

Let X be a set. Further, let $a, b, c \in X$ and $*$ be a binary operation on set X

1. **Closure:** if $a * b \in X$ for any pair $a, b \in X$, then set X is said to be closed under the binary operation $*$
2. **Commutative:** if $a * b = b * a$ for every pair $a, b \in X$, then the binary operation $*$ is said to be commutative.
3. **Associativity:** if $(a * c) * c = a * (b * c)$ for any elements $a, b, c \in X$, then the binary operation $*$ is said to be associative.
4. **Identity:** If there is an element $I \in X$ such that $I * a = a * I = a$ for any element $a \in X$, then I is called the identity element with respect to $*$
5. **Inverse:** if for every element $a \in X$, there exists another element $a^{-1} \in X$ such that $a * a^{-1} = a^{-1} * a = I$, then a^{-1} is called an inverse of a with respect to $*$

Example 1.3.5. Let $X = \{x | \ x > 0, \quad x \in \mathbb{R}\}$. For any pair $a, b \in X$, define an operation $*$ by $a * b = 2a + b$.

i) determine whether $*$ is a binary operation on X.

ii) determine whether $*$ is commutative

iii) evaluate $2 * 3$ and $\frac{2}{7} * 10$

iv) determine the value $(a * b) * c$ and $a * (b * c)$

Sol: Let $a, b, c \in X$. Then $a > 0$, $b > 0$ and $c > 0$

i) For any elements $a, b \in X$, $a * b = 2a + b > 0$. Hence, $2a + b \in X$ and is unique.

Therefore, $*$ is a binary operation on X

ii) $a * b = 2a + b$ while $b * a = 2b + a$. We can see that $2a + b \neq 2b + a$ implying that $a * b \neq b * a$. Hence, $*$ is not commutative.

iii) $2 * 3 = 2(2) + 3 = 7$. and $\frac{2}{7} * 10 = 2(\frac{2}{7}) + 10 = \frac{74}{7}$

iv) $(a * b) * c = (2a + b) * c = 2(2a + b) + c = 4a + 2b + c$ while

$a * (b * c) = 2a + (b * c) = 2a + (2b + c) = 2a + 2b + c$. Thus

$(a * b) * c \neq a * (b * c)$. Therefore, $*$ is not associative

Example 1.3.6. Let $*$ be a binary operation defined by $a * b = (b - a)^3 + 2ab$, where $a, b \in \mathbb{Z}$.

i) determine whether $*$ is commutative on $\mathbb{Z}$

ii) evaluate $-10 * (3 * 10)$ and $(-10 * 3) * 10$

iii) Is $*$ associative on $\mathbb{Z}$?

Sol: Exercise

Exercise 1

1. Let $E = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ be the universal set. If $A = \{-2, -1, 0, 1, 2\}$, $B = \{2, 4, 6\}$ and $C = \{-5, -4\}$ are subsets of U;

   a) Find: i) $C \cap (A - B)$ ii) $(U - A) \cap (B - C)$ iii) $A - (C - B)$

   b) Verify that: i) $A \cap (B \cap C) = (A \cap B) \cap C$ and ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   c) Verify the De Morgan's laws: i) $(A \cap B)' = A' \cup B'$ (ii) $(A \cup B)' = A' \cap B'$

2. Given that X and Y are subsets of the universal set E, show that:
   i) $(X \cap Y) \cup (X \cap Y') = X$ (ii) $X \cup Y = X \cup (X' \cap Y)$

3. Let X and Y denote two sets. If $X \subset Y$ simplify each of the following as far as possible and where necessary, show the results in a Venn diagram.
   i) $X \cap Y$ ii) $X \cup Y$ iii)$X' \cap Y'$ iv)$X \cap Y'$ v)$X' \cup Y'$

4. Let E denote the universal set. If X, Y, Z are subsets of E such that X, Y, Z all intersect, simplify and shade the parts described by the following sets in separate Venn diagrams:
   i) $[(X' \cup Z)' \cup Y']'$ ii) $(Y - X') - X'$ iii) $[X' \cup (Y - X)]'$ iv) $Y \cup (X \cap Y)$

5. Let $X = [-10, 10]$ be the universal set. Let $A = (-2, 6]$, B=[-4,7], $C = [-1, 8]$ and $D = (3, 5]$. Find each of the following sets and display it on the real number line.
   i) A' ii) $X - A$ iii) $(A \cup C)'$ iv) $(B - A) \cup C$ v) $X - (C - D)$

6. Let $\mathbb{R} = (-\infty, \infty)$ be the universal set. Further, let $A = (-8, 6]$, $B = [4, \infty)$, $C = [0, 1)$, $D = [1, \infty)$, $E = (-\infty, 1)$ and $F = [5, \infty)$ be subsets of the universal set $\mathbb{R}$.

   a) Find each of the following and represent the results on the real number line:
   i) $A \cup B$ ii) $(A \cap B)'$. iii) C' iv) $B \cap D$ v) $C \cap E'$ vi) $B \cup F$ vii) $D \cap E$

   b) Verify De morgan's laws: i) $(A \cap B)' = A' \cup B'$ and ii) $(A \cup B)' = A' \cap B'$

   c) Verify that: i) $A \cap (B \cap C) = (A \cap B) \cap C$ and ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

7. Let $\mathbb{R} = (-\infty, \infty)$ be the universal set. If $A = (-5, 8)$, $X = (0, 1)$, $B = [7, \infty)$ and $Y = [0, 1]$, find:
   i) X' ii) Y' iii) $B - A$ iv) $A - B$ v) $Y \cap X$ vi) $\mathbb{R} - X$ vii) $\mathbb{R} - Y$ viii) $X - \mathbb{R}$

8. Express the following numbers in the form $\frac{a}{b}$ where a and b are integers, with $b \neq 0$.
   i) $0.\overline{1}$ ii) $12.\overline{13}$ iii) 3.375 iv) $-3.5\overline{32}$ v) $0.\overline{714285}$ v) $-0.\overline{7}$ vi) $21.3211\overline{3}$

9. Prove that the following are irrational numbers
   i) $\sqrt{2}$ ii) $\sqrt{3}+1$ iii) $\sqrt{3}$ iv) $-1-\sqrt{2}$ v) $6+\sqrt{2}$ vi) $-\sqrt{3}$ vii)$\sqrt{2}-2$

10. a) Simplify the following leaving your answer in surd form where necessary

   i) $\sqrt{48}\sqrt{6}$ ii) $\sqrt{2x^2y^2}\sqrt{2x}$ vi)$(\sqrt{3}-\sqrt{7})(\sqrt{3}+\sqrt{7})$ iii)$\frac{\sqrt{84}}{\sqrt{6}\sqrt{14}}$ iv)$\sqrt{12}+\sqrt{147}-\sqrt{27}$

   b) Rationalize the denominator of each of the following:

   i) $\frac{2\sqrt{3}-\sqrt{2}}{4\sqrt{3}}$ ii) $\frac{x}{x+\sqrt{y}}$ iii) $\frac{2\sqrt{7}+\sqrt{3}}{3\sqrt{7}-\sqrt{3}}$ iv) $\frac{x-\sqrt{x^2-9}}{x+\sqrt{x^2-9}}$ v) $\frac{1}{(\sqrt{2}+1)(\sqrt{3}-1)}$ vi) $\frac{h\sqrt{x}\sqrt{x+h}}{\sqrt{x}-\sqrt{x+h}}$ vii) $\frac{6+2\sqrt{7}}{3-\sqrt{7}}$

   c) Rationalize the numerator of each of the following:

   i) $\frac{2\sqrt{3}-\sqrt{2}}{4\sqrt{3}}$ (ii) $\frac{x-\sqrt{y}}{x}$ (iii) $\frac{2\sqrt{7}+\sqrt{3}}{3\sqrt{7}-\sqrt{3}}$ (iv) $\frac{x-\sqrt{x^2-9}}{x+\sqrt{x^2-9}}$ (v) $\frac{(\sqrt{2}+1)(\sqrt{3}-1)}{6}$ (vi) $\frac{h\sqrt{x}\sqrt{x+h}}{\sqrt{x}-\sqrt{x+h}}$ (vii) $\frac{6+2\sqrt{7}}{3-\sqrt{7}}$

11. Express $\sqrt{108}$ in the form $k\sqrt{3}$ where k is an integer.

12. a) Let $\mathbb{Z}$ be the set of integers with $a, b \in \mathbb{Z}$. State which of the following is a binary operation on $\mathbb{Z}$.

   i) $a*b = b^a$ (ii) $a*b = a+b$ (iii) $a*b = (a+b)(a-b)$ (iv) $a*b = (ab)^2$ (v) $a*b = a+2b$

   b) which operations in (a) are commutative? Which ones are associative?

13. Let $*$ be a binary operation on $\mathbb{R} = (-\infty, \infty)$ defined as $a * b = a + b - ab$ for $a, b, c \in \mathbb{R}$.

   i) Is * commutative? (ii) Calculate $4*(0.5*6)$ and $(4*0.5)*6$ (iii) Calculate $10*(-10)$

14. Let $\mathbb{Z}$ denote the set of all integers. Define an operation $*$ on $\mathbb{Z}$ by $a * b = a + b - ab$

   a) show that $*$ is a binary operation on $\mathbb{Z}$

   b) determine whether $*$ is associative on $\mathbb{Z}$

   c) evaluate $-3 * 2 * 5$

15. Let $\mathbb{N}$ be the set of all natural numbers. Define an operation $*$ on $\mathbb{N}$ by $a*b = (a-b)^2+2ab$

   a) show that $*$ is a binary operation on $\mathbb{N}$

   b) determine whether $*$ is associative on $\mathbb{N}$

   c) determine whether $*$ is commutative on $\mathbb{N}$

16. Express $z = \frac{(4-2i)(1-2i)}{(1+2i)^2}$ in the form $x + yi$ where $x, y \in \mathbb{R}$

17. Express $z = \frac{(4+22i}{(1-2i)^3}$ in the form $x + yi$ where $x, y \in \mathbb{R}$

18. Let * be a binary operation on $\mathbb{R} = (-\infty, \infty)$ defined as $a * b = a + b - ab$ for $a, b, c \in \mathbb{R}$.

   i) Is * commutative? ii) Calculate $4 * (0.5 * 6)$ and $(4 * 0.5) * 6$ (iii) Is * associative?

19. Let $*$ be a binary operation defined by $a * b = (a - b)^2 + 2ab$ where $a, b \in \mathbb{Z}$

   (a) Determine whether $*$ is commutative on $\mathbb{Z}$

   (b) Compute $0 * (1 * 2)$ and $(0 * 1) * 2$

   (c) Determine whether $*$ is associative on $\mathbb{Z}$

20. Let $*$ be a binary operation defined by $a * b = 2^{ab} - 1$ where $a, b \in \mathbb{R}$

   (a) Determine whether $*$ is commutative on $\mathbb{R}$

   (b) Compute $0 * (\frac{2}{3} * 3)$ and $(0 * \frac{2}{3}) * 3$

   (c) Determine whether $*$ is associative on $\mathbb{R}$

2
Relations and Functions

2.1. Introduction

We now study special type of sets called relations. The concept of a relation as a set, helps us comprehend the very important topic of functions in mathematics. We start with some basic, yet cardinal definitions.

2.1.1 Product Set

Definition 2.1.1. *Let X and Y be two non-empty sets. The cartesian product of the two sets, X and Y, is denoted as $X \times Y$ and defined as*

$$X \times Y = \{(x, y) | x \in X \text{ and } y \in Y\}$$

.

Here, (x, y) denotes an ordered pair, and x is called the first coordinate and y is the second coordinate of the pair (x, y). The cartesian product of X and Y is also referred to as the product set. As defined above, it is the collection of ***all*** ordered pairs (x, y) where the first entry x is from set X and the second entry y, is from set Y. The following examples help us to understand the cartesian product much better.

Example 2.1.1. Let $X = \{1, 2, 3\}$ and $Y = \{3, 4\}$. Find the cartesian product from set X to set Y.

Soln: Let $X \times Y$ denote the cartesian product from X to Y. Then we have

$$X \times Y = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$$

Note 2.1.1. In general, $X \times Y \neq Y \times X$. From our example , we see that $Y \times X = \{(3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3)\}$ which confirms that $X \times Y \neq Y \times X$.

Always remember that $X \times Y$ is a SET of ordered pairs.

Example 2.1.2. Let $A = [1, 4]$ and $B = [3, 4]$ be two sets in $\mathbb{R}$. Find the cartesian product from set A to set B.

Soln: Let $A \times B$ denote the cartesian product from set A to set B. Then we have

$$A \times B = \{(x, y)|1 \leq x \leq 4, 3 \leq y \leq 4\}$$

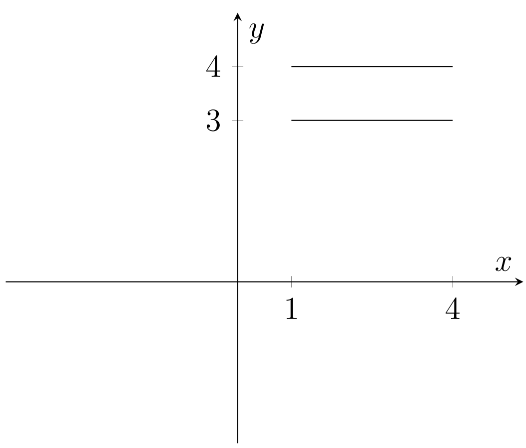
Example 2.1.3. Let $X = (-\infty, \infty)$ and $Y = [0, 1]$ be two sets in $\mathbb{R}$. Find the cartesian product from set X to set Y.

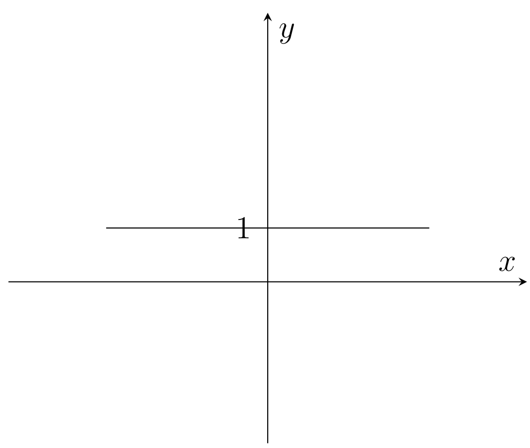
Soln: Let $X \times Y$ denote the cartesian product from set X to set Y. Then we have

$$X \times Y = \{(x, y)| -\infty < x < \infty, 0 \leq y \leq 1\}$$

Pictorial Representation of Product Set

We can use a pictorial representation to clearly show the cartesian product of two sets. The example above can be represented as shown below





Note 2.1.2. If $X \times Y$ is the cartesian product of X and Y and if (a, b) is in $X \times Y$,

- then $a \in X$ and $b \in Y$.
- (a, b) simply means a is mapped to b
- if r is a relation from set X to set Y, then r is a subset of $X \times Y$
- if (a, b) is one of the elements in r, we write $(a, b) \in r$
- also $(a, b) \in r$ can be written as $r(a) = b$, which means "r maps x to y".
- $r(a) = b$ is simply read as "r maps x to y".

2.2. Relations

The concept of the Cartesian Product of two sets is very important to the understanding of relationships that may exist between any two or more sets. This in turn helps us understand the concept of a function. We start our discussion of relations with the following definition of a relation.

Definition 2.2.1. *Let X and Y be two non-empty sets. A relation denoted r, from X to Y is a subset of the set $X \times Y$.*

From this definition, we see that any subset of the cartesian product $X \times Y$, qualifies to be a relation from X to Y. Since any set is a subset of itself, $X \times Y$ is also a relation from X to Y.

Example 2.2.1. Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$. Further, let

$r_1 = \{(1, 3), (1, 4), (3, 4)\}$, $r_2 = \{(1, 3), (3, 3), (2, 4)\}$, $r_3 = \{(1, 4)\}$ and $r_4 = \{(4, 3), (2, 4), (3, 3)\}$

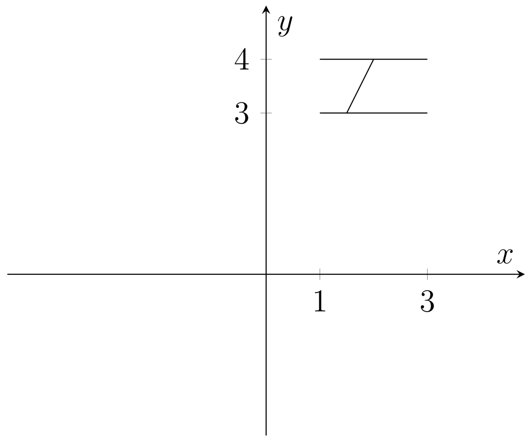
Then we can see that r_1 is a subset of the cartesian product $A \times B$. Therefore, we say that r_1, is a relation from A to B.

Similarly, r_2 and r_3 are relations from A to B since each of them is a subset of $A \times B$. However, notice that r_4 is not a subset of $A \times B$. Hence, r_4 is not a relation from A to B. Pictorially, we can represent the above relations as:

Example 2.2.2. Let $A = [1, 3]$ and $B = [3, 4]$ be two intervals in $\mathbb{R}$. Further, define

$$r = \{(x, y)| \ \ 1 \leq x \leq 3, \quad 3 \leq y \leq 4, \quad y = 2x\}$$

Then we see that r is a relation from set A to set B, since r is a subset of $A \times B$

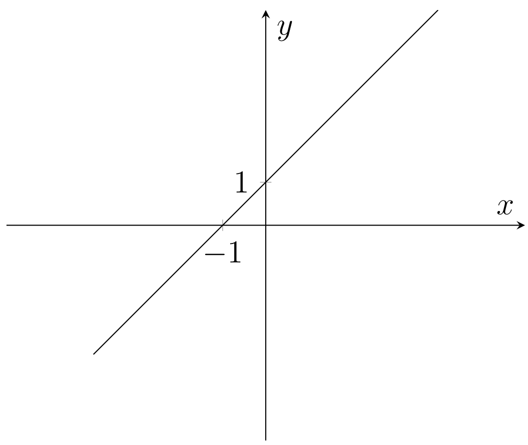


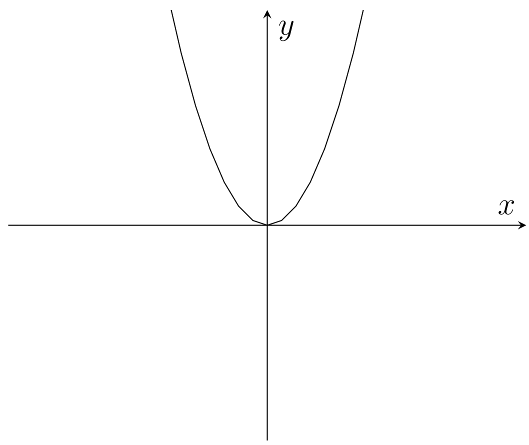
Example 2.2.3. Let $X = (-\infty, \infty)$ and $Y = (-\infty, \infty)$. The we see that

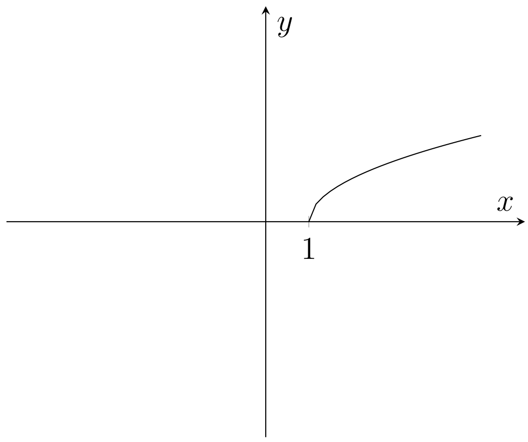
$$X \times Y = \{(x, y)| \ x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$$

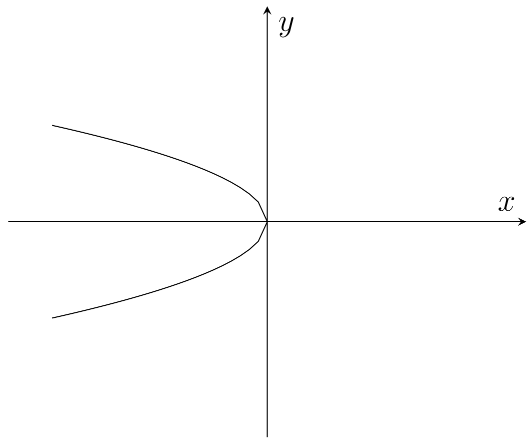
Further, define: $r_1 = \{(x, y)| \ y = x + 1, \ \ x, y \in \mathbb{R}\}$, $\quad r_2 = \{(x, y)| \ y = x^2, \ \ x, y \in \mathbb{R}\}$.
$r_3 = \{(x, y)| \ y = \sqrt{x - 1}, \quad x, y \in \mathbb{R}\}$ and $r_4 = \{(x, y)| \ y^2 = x, \quad x, y \in \mathbb{R}\}$

Since $r_1, \quad r_2, \quad r_3$ and r_4 are all subsets of $X \times Y$, they form relations from X to Y. See the diagrams below and identify each relation







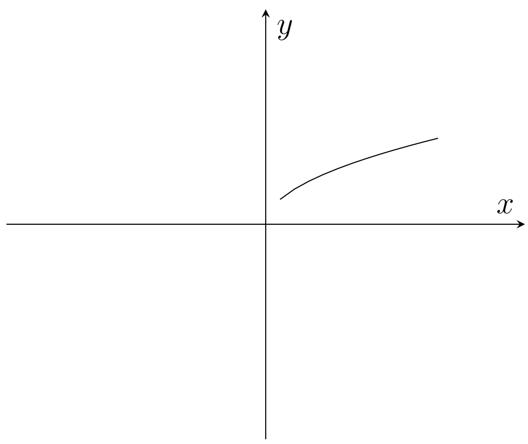


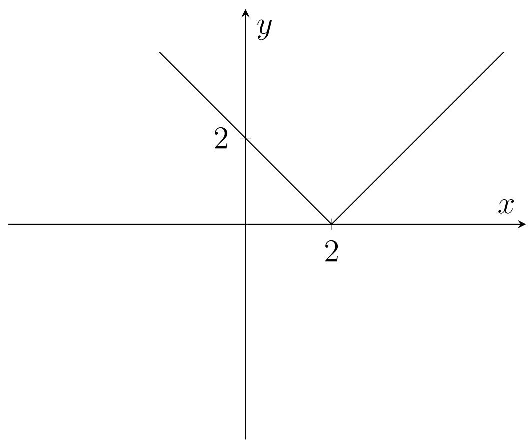
Example 2.2.4. Let $X = (-\infty, \infty)$ and $Y = [0, \infty)$. Then the cartesian product is given as:

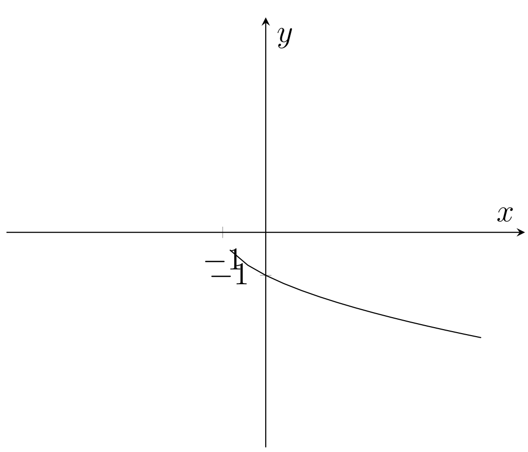
$$X \times Y = \{(x, y)| \ \ x \in \mathbb{R}, \quad y \geq 0 \text{ and } y \in \mathbb{R}\}$$

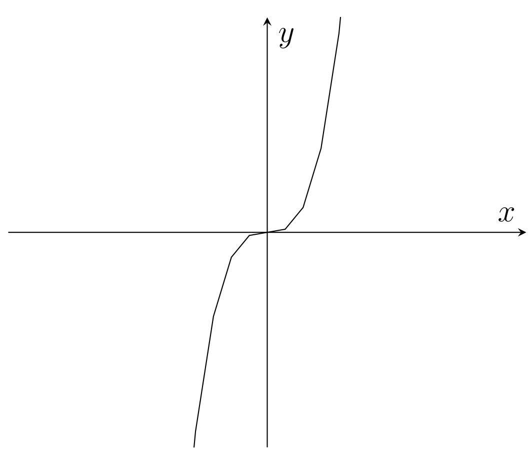
Further, define: $r_1 = \{(x, y)| \ y = \sqrt{x}, \ \ x, y \in \mathbb{R}\}$, $\qquad r_2 = \{(x, y)| \ y = |x - 2|, \ \ x, y \in \mathbb{R}\}$.
$r_3 = \{(x, y)| \ y = -\sqrt{x+1}, \quad x, y \in \mathbb{R}\}$ and $r_4 = \{(x, y)| \ y = x^3, \ \ 1 < y \leq 5, \ \ x, y \in \mathbb{R}\}$

Since r_1, and r_2 are subsets of $X \times Y$, they form relations from X to Y. However, note that r_3 and r_4 are not subsets of $X \times Y$. Hence, they are not relations from X to Y.









Definition 2.2.2. *Let X and Y be two non-empty sets. The domain of a relation r from X to Y denoted D_r, is defined as*

$$D_r = \{x \in X| \ \ (x, y) \in r \text{ for some } y \in Y\}$$

Definition 2.2.3. *Let X and Y be two non-empty sets. The range of a relation r from X to Y denoted R_r, is defined as*

$$R_r = \{y \in Y| \ \ (x, y) \in r \text{ for some } x \in X\}$$

Example 2.2.5. Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$. Further, let

$$r_1 = \{(1, 3), (1, 4), (3, 4)\}, \quad r_2 = \{(1, 3), (3, 3), (2, 4)\} \quad \text{and } r_3 = \{(1, 4)\}$$

a) Find the domain of:

(i) r_1 (ii) r_2 (iii) r_3

b)Find the range of:

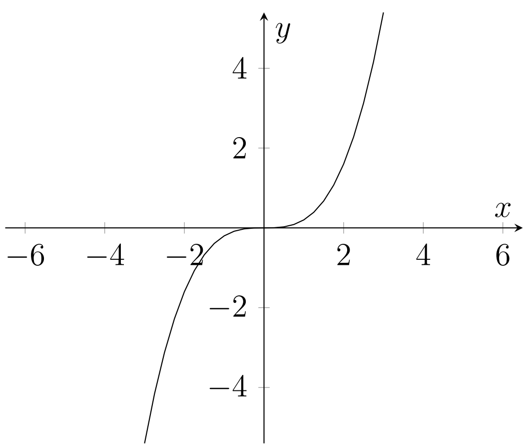
(i) r_1 (ii) r_2 (iii) r_3

Soln: Exercise

2.2.1 Characteristics of Relations

1. We have looked at a number of relations. We now discuss some of the important characteristics of these relations.

   (a) **One to One (1-1):** A relation r from set A to set B is said to be one to one if $r(x_1) = r(x_2)$ implies that $x_1 = x_2$ for any points $x_1, x_2 \in A$.
   This definition implies that a relation is one to one if no element is mapped to more than one element. See the diagrams below;

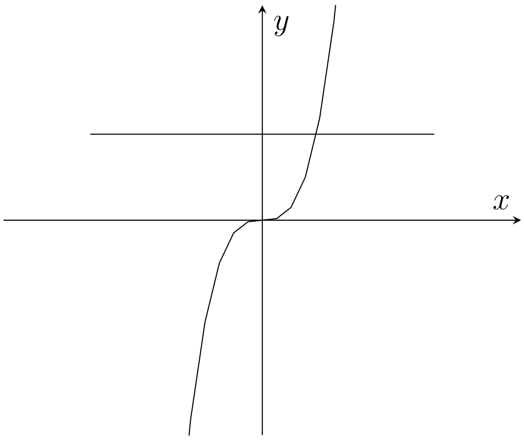


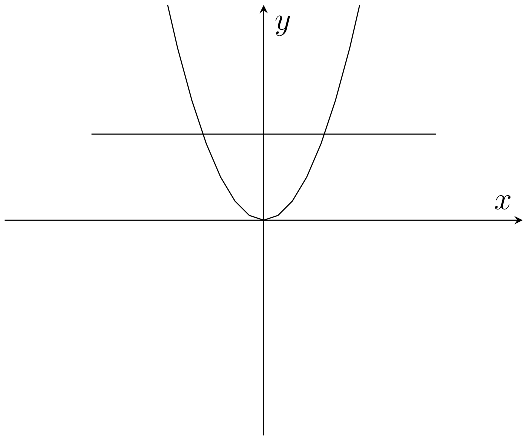
Example 2.2.6. Let $X = [1, 4]$ and $Y = [0, 6]$ be two intervals in $\mathbb{R}$. Further, let

$$r = \{(x, y)|\, x \in X,\ \ y \in Y,\ \ y = x^3\} \text{ and } g = \{(x, y)|\, 1 \leq x \leq 4,\ \ 0 \leq y \leq 6,\ \ y = x^2\}$$

be two relations from X to Y. Show that r is one-one and that g is not one-one.

Soln: The sketch below shows that f is indeed one-one and g is not one-one. Note that a relation is one-one if either a vertical or horizontal line cuts the graph of the relation at exactly one point.





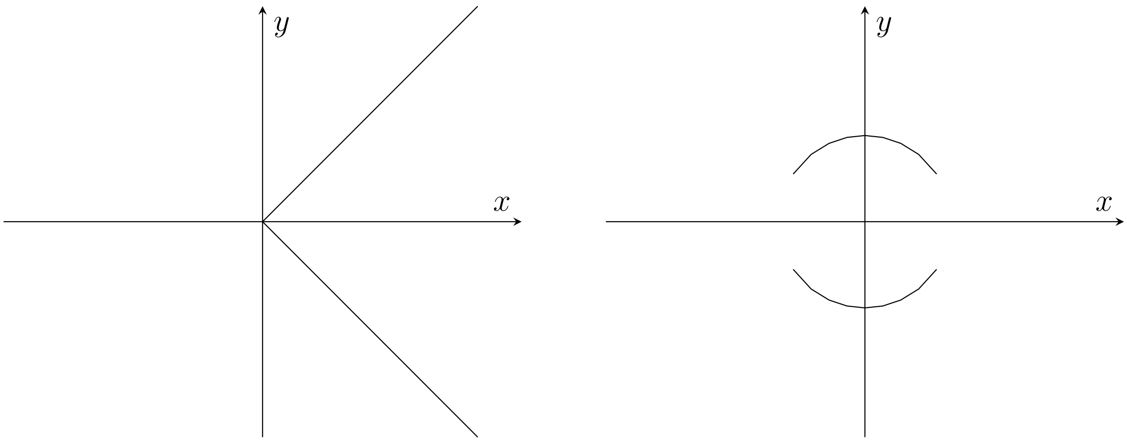
(b) **One to Many** A relation r from set A to set B is said to be one to many if a single value $x \in A$ may be mapped to more than one image in B. See the diagrams below

Example 2.2.7. Let $X = (-\infty, \infty)$ and $Y = (-\infty, \infty)$ be two intervals. Further, let

$$r = \{(x, y)|\, -\infty < x < 4,\ \ y \in Y,\ \ x = |y|\} \text{ and } g = \{(x, y)|\, x \in \mathbb{R},\ \ -\infty < y < \infty,\ \ y^2 + x^2 = 1\}$$

be two relations from X to Y. By means of a sketch, show that both r and g are one to many relations from set X to set Y.

Soln: For a relation from set X to set Y to be one to many, a vertical line must cut the graph of the relation at more than one point. See the graph below



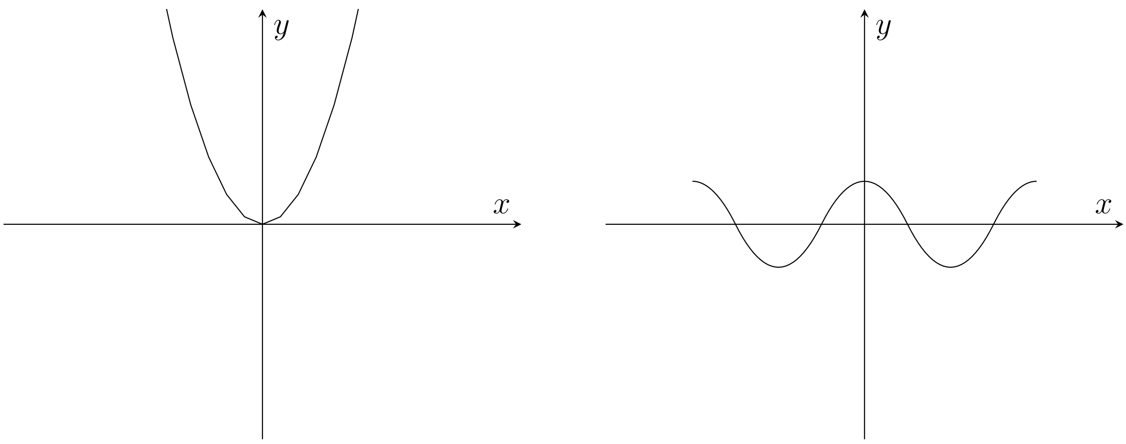
(c) **Many to One:** A relation r from set A to set B is said to be many to one if at least two elements from set A can be mapped to a single element in set B.

Example 2.2.8. Let $X = (-\infty, \infty)$ and $Y = (-\infty, \infty)$ be two intervals. Further, let

$$r = \{(x, y)|\, x \in X,\ -\infty < y < \infty,\ y = \cos x\} \text{ and } g = \{(x, y)|\, x \in X,\ y \in Y,\ y = x^2\}$$

be two relations from X to Y. Show that r is many-one and that g is not many-one.

Soln: The sketch below shows that r is indeed many-one and g is not many-one. Note that a relation is many-one if a horizontal line cuts the graph of the relation at more than one point.



(d) **onto:**A relation r from set A to set B is said to be onto if the entire A is mapped to the entire B.

We can also define relations within a set. In other words, we can define relations from one element of a set to another element of the same set. Such a relation is simply a binary operation defined on the same set.

Definition 2.2.4. *Let r be a binary relation on a set X. r is called a reflexive relation on X if for all $x \in X$, $(x, x) \in r$*

Definition 2.2.5. *Let r be a binary relation on a set X. r is called a symmetric relation on X if for all $x, y \in X$, $(x, y) \in r$ implies that $(y, x) \in r$*

Definition 2.2.6. *Let r be a binary relation on a set X. r is called a transitive relation on X if for all $x, y, z \in X$, $(x, y) \in r$ and $(x, y) \in r$ implies that $(x, z) \in r$*

Example 2.2.9. Let $X = \{1, 2, 3\}$ be a set of the first three natural numbers.

Then, a relation $r_1 = \{(1, 1), (1, 2), (2, 2), (3, 3), (3, 2)\}$ is reflexive on set X since all pairs of the form $(x, x) \in r_1$. That is, $(1, 1) \in r_1$, $(2, 2) \in r_1$ and $(3, 3) \in r_1$.

The relation $r_2 = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)\}$ is symmetric on X since $(x, y) \in r_2$ implies that $(y, x) \in r_2$. It is a two way path.

The relation $r_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (1, 3)\}$ is transitive on X since $(x, y) \in r_3$ and $(y, z) \in r_3$ implies that $(x, z) \in r_3$.

Example 2.2.10. Determine whether the following relations are reflexive, transitive or symmetric on $Y = \{1, 2, 3, 4\}$

i) $r = \{(1, 2), (2, 1), (3, 1), (4, 2)\}$

ii) $r = \{(1, 1), (2, 1), (3, 1), (4, 2), (2, 2), (3, 4), (3, 3), (4, 4)\}$

iii) $r = \{(1, 2), (2, 3), (1, 3), (2, 1)\}$

iv) $r = \{(1, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$

Sol: Exercise

2.3. Functions

Before proceeding to this section, make sure you understand set theory, product set (Cartesian product) as well as relations. Functions are simply relations between two sets, that satisfy certain conditions. Since relations are just sets and functions are indeed relations, we can see that functions are just special type of sets. Let us give some definitions

Definition 2.3.1. *Let X and Y be two non-empty sets. A relation f, from X to Y is called a function if for **every** element $x \in X$, there exists a **unique** element $y \in Y$ such that $(x, y) \in f$ i.e $f(x) = y$.*

Note 2.3.1. If f is a function from set X to set Y, then the following hold:

- f is a relation from set X to set Y
- $(x, y) \in f$ means that f maps the input value $x \in X$ to a unique output value $y \in Y$. This is written as $f(x) = y$ which we read as "f maps x to y".
- if $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$. This means that an element x from the domain, can not be mapped to more than one element in the range.
- Every element in the domain of f **must** be mapped to a unique element in the range set of f

Example 2.3.1. Let $A = \{1, 2, 3, 4\}$ and $B = \{7, 9, 11\}$. Define the relations f_1, f_2, f_3, f_4, f_5 and f_6 as shown below. Determine which of these relations are functions

i) $f_1 = \{(1, 7), (2, 7), (3, 7), (4, 7)\}$

ii) $f_2 = \{(1, 7), (2, 11), (3, 11), (4, 11), (2, 9), (3, 9), (3, 7), (4, 9)\}$

iii) $f_3 = \{(1, 7), (1, 9), (1, 11)\}$

iv) $f_4 = \{(1, 11), (2, 9), (3, 7)\}$

iv) $f_5 = \{(1, 11), (2, 9), (3, 11), (4, 9)\}$

iv) $f_6 = \{(1, 7), (2, 9), (3, 11), (4, 7)\}$

Represent each relation using an arrow diagram

Sol: Exercise

2.3.1 General Characteristics of Functions

The following are the general characteristics of functions:

1. **One to One (Injective):** A function $f : X \longrightarrow Y$ is one to one (1-1) if for any two elements $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. A one to one function is also called an injective function. The graph of a one to one function cuts any horizontal line at exactly one point.

   Example 2.3.2. Show that the function $f(x) = 2x - 7$ is a one to one function.

   Soln: Let $x_1, x_2 \in D_f$ such that $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Thus,

$$\begin{aligned} f(x_1) &= f(x_2) \\ 2x_1 - 7 &= 2x_2 - 7 \\ 2x_1 &= 2x_2 \\ x_1 &= x_2 \end{aligned}$$

   Hence, f is a 1-1 function since $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

Example 2.3.3. Show that the function $f(x) = 3x^2 - 4$ is not a one to one function.

Soln: Let $x_1, x_2 \in D_f$ such that $f(x_1) = f(x_2)$. We need to show that $x_1 \neq x_2$. Thus,

$$\begin{aligned} f(x_1) &= f(x_2) \\ 3(x_1)^2 - 4 &= 3(x_2)^2 - 4 \\ 3x_1^2 &= 3x_2^2 \\ x_1^2 &= x_2^2 \\ x_1^2 - x_2^2 &= 0 \\ (x_1 - x_2)(x_1 + x_2) &= 0 \end{aligned}$$

This shows that either $x_1 = x_2$ or that $x_1 = -x_2$. Hence, f is not a 1-1 function since $f(x_1) = f(x_2)$ does not necessarily imply that $x_1 = x_2$. The graph of a 1-1 function does not cut any horizontal line more than once.

2. **Many to One:** A function $f : X \longrightarrow Y$ is said to be many to one if at least two elements from set X can be mapped to a single element in set Y. The graph of a many to one function may cut a horizontal line at more than one point.

3. **Surjective (ONTO):** A function $f : X \longrightarrow Y$ is said to be surjective if the range of f is the entire Y. A surjective function is also called an onto function.

4. **INTO:** A function $f : X \longrightarrow Y$ is said to be an into function if the range of f is a proper subset of Y.

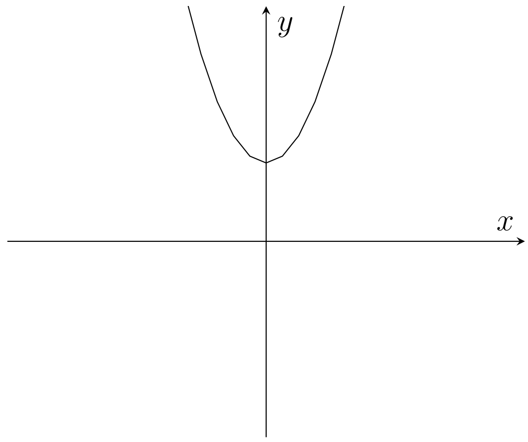
5. **Bijective (1-1 and ONTO):** A function $f : X \longrightarrow Y$ is said to be bijective if it is both one to one and onto. Thus, bijective means that a function is both injective and surjective.

6. **Even:** A function $f : X \longrightarrow Y$ is said to be an even function if $f(-x) = f(x)$ for all elements $x \in X$.

   Example 2.3.4. Show that the function $f(x) = 3x^2 + 11$ is an even function. Sketch its graph and comment.

   Soln: Let $x \in D_f$ be an arbitrary element. Then

$$\begin{aligned} f(-x) &= 3(-x)^2 + 11 \\ &= 3(-1)^2(x)^2 + 11 \\ &= 3(1)(x)^2 + 11 \\ &= 3x^2 + 11 \\ &= f(x) \end{aligned}$$



The graph of an even function is symmetric about the y-axis.

7. **Odd:** A function $f : X \longrightarrow Y$ is said to be an odd function if $f(-x) = -f(x)$ for all elements $x \in X$.

   Example 2.3.5. Show that the function $f(x) = 7x^3 + 2x$ is an odd function. Sketch its graph and comment.

   Soln: Let $x \in D_f$ be an arbitrary element. Then

$$\begin{aligned} f(-x) =& 7(-x)^3 + 2(-x) \\ =& 7(-1)^3(x)^3 + 2(-1)x \\ =& 7(-1)(-1)(-1)(x)^3 - 2x \\ =& -7x^3 - 2x \\ =& -(7x^3 + 2x) \\ =& -f(x) \end{aligned}$$

   The graph of an odd function is symmetric about the origin. As an exercise, sketch the graph of this function.

2.3.2 The Domain and Range of a Function

Recall that a function is a relation from one set to another set. Also recall that every relation has a domain which is the set of all first entries of an ordered pair, and a range which is the set of all second entries of an ordered pair. Let us discuss further the domain and range of a function.

Definition 2.3.2. *Let $f : X \longrightarrow Y$ be a function. The domain of f, denoted D_f, is defined as*

$$D_f = \{x|\ f(x) = y,\ \text{for some unique } y \in Y\}$$

Definition 2.3.3. *Let $f : X \longrightarrow Y$ be a function. The range of f, denoted R_f, is defined as*

$$R_f = \{y|\ y = f(x),\ \text{for some } x \in X\}$$

NOTE:

- In certain situations, the function and the domain are given but the range is not given. We will have to work out the range.
- If the function is given and the domain and range are not, we need to find the domain and the range from $\mathbb{R}$, by eliminating those real values that make the function undefined. In other words, if the domain is not given, first assume that the whole of $\mathbb{R}$ is the domain. secondly, from $\mathbb{R}$, remove values which can not apply to the given function.
- The range is usually not given. We may work out the range by:
    1. making x the subject of the formula and excluding values of y that can not apply to the function.
    2. sketching the graph of the function
    3. inspection

Example 2.3.6. Given that f is a function defined as $f(x) = 3x - 1$ for any $x \in D_f$ and that $D_f = \{-2, -1, 0, 1, 2\}$, find the range of f.

Soln:

$$\begin{aligned} f(-2) =& 3(-2) - 1 \\ =& -6 - 1 \\ =& -7 \end{aligned}$$

$$\begin{aligned} f(-1) =& 3(-1) - 1 \\ =& -3 - 1 \\ =& -4 \end{aligned}$$

$$\begin{aligned} f(0) =& 3(0) - 1 \\ =& -1 \end{aligned}$$

$$\begin{aligned} f(1) =& 3(1) - 1 \\ =& 2 \end{aligned}$$

$$\begin{aligned} f(2) =& 3(2) - 1 \\ =& 5 \end{aligned}$$

Hence, f maps: -2 to -7, -1 to -4, 0 to -1, 1 to 2 and 2 to 5

Therefore, $D_f = \{-7, -4, -1, 2, 5\}$

Example 2.3.7. Find the domain and the range for each of the following functions:

(i) $f(x) = k$ where $k \in \mathbb{R}$ (ii) $g(x) = -5$ (iii) $h(x) = 2x - 5$ (iv) $r(x) = \frac{3}{5}x - 1$

Soln:

(i) For $f(x) = k$, we have

$D_f = \{x|\ x \in \mathbb{R}\}$ since the function f is defined for all real numbers.

$R_f = \{k|\ k \in \mathbb{R}\}$

(ii) For $g(x) = -5$, we have

$D_g = \{x|\ x \in \mathbb{R}\}$ since the function g is defined for all $x \in \mathbb{R}$.

$R_g = \{-5\}$ by inspection

(iii) For $h(x) = 2x - 5$ we have

$D_h = \{x|\ x \in \mathbb{R}\}$ since the function h is defined for all $x \in \mathbb{R}$.

R_h : We can make x the subject;

$$\begin{aligned} y &= 2x - 5 \\ 2x &= y + 5 \\ x &= \frac{y+5}{2} \end{aligned}$$

Therefore, $R_h = \{y|\ y \in \mathbb{R}\}$ since $\frac{y+5}{2}$ is defined for all $y \in \mathbb{R}$

(iv) For $r(x) = \frac{3}{5}x - 1$, we have

$D_r = \{x|\ x \in \mathbb{R}\}$ since the function r is defined for all real numbers.

R_h : We can make x the subject;

$$\begin{aligned} y &= \frac{3}{5}x - 1 \\ \frac{3}{5}x &= y + 1 \\ x &= \frac{5y+5}{3} \end{aligned}$$

Therefore, $R_r = \{y|\ y \in \mathbb{R}\}$ since $\frac{5y+5}{3}$ is defined for all $y \in \mathbb{R}$

Example 2.3.8. Find the domain and the range for each of the following functions:

(i) $f(x) = \frac{1}{x}$ (ii) $g(x) = \frac{3}{x+1}$ (iii) $h(x) = \frac{x-2}{5x+1}$ (iv) $r(x) = \frac{1-2x}{x+1}$ (v) $s(x) = \frac{1}{x^2+1}$

Soln:

(i) For $f(x) = \frac{1}{x}$, we have

$D_f = \{x|\ x \neq 0, x \in \mathbb{R}\}$ since the function f is defined for all real numbers except 0.

R_h : We can make x the subject;

$$
\begin{aligned}
y =& \frac{1}{x} \\
xy =& 1 \\
x =& \frac{1}{y}
\end{aligned}
$$

Therefore, $R_f = \{y|\ y \neq 0,\ y \in \mathbb{R}\}$ since $\frac{1}{y}$ is defined for all $y \in \mathbb{R}$ provided $y \neq 0$

(ii) For $g(x) = \frac{3}{x+1}$, denominator must not be zero as division by zero is undefined. Thus

$$
\begin{aligned}
x + 1 \neq& 0 \\
x \neq& -1
\end{aligned}
$$

$D_g = \{x|\ x \neq -1, x \in \mathbb{R}\}$ since $\frac{3}{x+1}$ is defined for all real numbers except -1.

R_g : We make x the subject;

$$
\begin{aligned}
y =& \frac{3}{x+1} \\
y(x+1) =& 3 \\
xy + y =& 3 \\
xy =& 3 - y \\
x =& \frac{3-y}{y}
\end{aligned}
$$

Therefore, $R_g = \{y|\ y \neq 0,\ y \in \mathbb{R}\}$ since $\frac{3-y}{y}$ is defined for all $y \in \mathbb{R}$ provided $y \neq 0$

(iii) For $h(x) = \frac{x-2}{5x+1}$, denominator must not be zero as division by zero is undefined. Thus

$$
\begin{aligned}
5x + 1 \neq& 0 \\
5x \neq& -1 \\
x \neq& -\frac{1}{5}
\end{aligned}
$$

$D_h = \{x|\ x \neq -\frac{1}{5}, x \in \mathbb{R}\}$ since $\frac{x-2}{5x+1}$ is defined for all real numbers except $-\frac{1}{5}$.

$R_:$ We make x the subject;

$$
\begin{aligned}
y =& \frac{x-2}{5x+1} \\
y(5x+1) =& x - 2 \\
5xy + y =& x - 2 \\
x - 5xy =& y + 2 \\
x(1 - 5y) =& y + 2 \\
x =& \frac{y+2}{1-5y}
\end{aligned}
$$

Therefore, $R_h = \{y|\ y \neq \frac{1}{5},\ y \in \mathbb{R}\}$ since $\frac{y+2}{1-5y}$ is defined for all $y \in \mathbb{R}$ provided $y \neq \frac{1}{5}$

(iv) For $r(x) = \frac{1-2x}{x+1}$, denominator must not be zero as division by zero is undefined. Thus

$$\begin{aligned} x + 1 &\neq 0 \\ x &\neq -1 \end{aligned}$$

$D_r = \{x|\ x \neq -1, x \in \mathbb{R}\}$ since $\frac{1-2x}{x+1}$ is defined for all real numbers except -1.

R_r : We make x the subject;

$$\begin{aligned} y =& \frac{1-2x}{x+1} \\ y(x+1) =& 1 - 2x \\ xy + 2x =& 1 - y \\ x(y+2) =& 1 - y \\ x =& \frac{1-y}{y+2} \end{aligned}$$

Therefore, $R_r = \{y|\ y \neq -2,\ y \in \mathbb{R}\}$ since $\frac{1-y}{y+2}$ is defined for all $y \in \mathbb{R}$ provided $y \neq -2$

(v) For $s(x) = \frac{1}{x^2+1}$, denominator can not be zero for any real number x. We see that,

$$x^2 + 1 > 0$$

for all real values $x \in \mathbb{R}$. whether x is negative, positive or zero, $\frac{1}{x^2+1}$ makes sense. Hence,

$$\begin{aligned} D_s =& \{x|\ x \in \mathbb{R}\} \\ =& (-\infty, \infty) \end{aligned}$$

since $\frac{1}{x^2+1}$ is defined for all real numbers.

R_s : We make x the subject;

$$\begin{aligned} y =& \frac{1}{x^2+1} \\ y(x^2+1) =& 1 \\ x^2y + y =& 1 \\ x^2y =& 1 - y \\ x^2 =& \frac{1-y}{y} \\ x =& \pm\sqrt{\frac{1-y}{y}} \end{aligned}$$

Three cases arise here

- $y \neq 0$

- $1 - y \geq 0$ implying that $y \leq 1$
- $y > 0$

Therefore, $R_s = \{y|\ 0 < y \leq 1,\ y \in \mathbb{R}\}$ since $\pm\sqrt{\frac{1-y}{y}}$ is defined for all $y \in (0, 1]$.

Example 2.3.9. Find the domain and the range for each of the following functions:
(i) $f(x) = \sqrt{x+1}$ (ii) $g(x) = \sqrt{-x-1}$ (iii) $h(x) = \sqrt{2x+7}$ (v) $r(x) = \frac{\sqrt{x-3}}{\sqrt{4-x}}$

Soln:

(i) For $f(x) = \sqrt{x+1}$, we need $x + 1 \geq 0$ as the square root of a negative number is not *real.* Hence;

$$\begin{aligned} x + 1 \geq& 0 \\ x \geq& -1 \end{aligned}$$

$D_f = \{x|\ x \geq -1, x \in \mathbb{R}\}$ since $f(x)$ is defined for all real numbers in the interval $[-1, \infty)$.

R_f : The range for this function can be determined as follows;

$$\begin{aligned} y =& \sqrt{x+1} \\ y^2 =& x + 1 \\ x =& y^2 - 1 \end{aligned}$$

Therefore, $R_f = \{y|\ y \geq 0\ y \in \mathbb{R}\} = [0, \infty)$ since $y^2 - 1$ is defined for all $y \in [0, \infty)$

ii) For $g(x) = \sqrt{-x-1}$, we need $-x-1 \geq 0$ as the square root of negative numbers is not *real.* Hence;

$$\begin{aligned} -x - 1 \geq& 0 \\ x + 1 \leq& 0 \\ x \leq& -1 \end{aligned}$$

$D_g = \{x|\ x \leq -1, x \in \mathbb{R}\} = (-\infty, -1]$ since $g(x)$ is defined for all real numbers in the interval $(-\infty, -1]$.

R_g : The range for this function can be determined as follows;

$$\begin{aligned} y =& \sqrt{-x-1} \\ y^2 =& -x - 1 \\ x =& -y^2 - 1 \end{aligned}$$

Therefore, $R_g = \{y|\ y \geq 0\ y \in \mathbb{R}\} = [0, \infty)$ since $-y^2 - 1$ is defined for all $y \in [0, \infty)$

iii) For $h(x) = \sqrt{2x+7}$ we need $2x + 7 \geq 0$ as the square root of negative numbers is not *real*. Hence;

$$\begin{aligned} 2x + 7 \geq& 0 \\ 2x + \geq& -7 \\ x \geq& \frac{-7}{2} \end{aligned}$$

$D_h = \{x|\ x \geq \frac{-7}{2}, x \in \mathbb{R}\}$ since $h(x)$ is defined for all real numbers in the interval $[\frac{-7}{2}, \infty)$.

Similarly, we see that, $R_h = \{y|\ y \geq 0\ y \in \mathbb{R}\} = [0, \infty)$

iv) For $r(x) = \frac{\sqrt{x-3}}{\sqrt{4-x}}$, we need $x - 3 \geq 0$ for the numerator, and $4 - x > 0$ for the denominator. Hence we have

$$x \geq 3 \qquad \text{and} \qquad x < 4$$

.

Therefore, $D_r = \{x|\ \ 3 \leq x < 4, \ \ x \in \mathbb{R}\} = [3, 4)$ since $r(x)$ is defined only for all $x \in [3, 4)$.

Similarly, we see that, $R_r = \{y|\ y \geq 0\ y \in \mathbb{R}\} = [0, \infty)$

Example 2.3.10. Let $g(x) = \begin{cases} 1 - 2x & \text{if } x \leq -1; \\ x^2 - 2 & \text{if } x > -1. \end{cases}$

a) Find: i) $g(-3)$ ii) $g(-1)$ iii) $g(1)$ b) Find the values of a for which $g(a) = 14$.

Sol: a) This function has a partitioned domain at $x = -1$

i) Since $-3 < -1$, we use $g(x) = 1 - 2x$ so that we have $g(-3) = 1 - 2(-3) = 7$

ii) Similarly, $g(-1) = 1 - 3(-1) = 4$

iii) Since $1 > -1$, we use $g(x) = x^2 - 2$ so that $g(1) = (1)^2 - 2 = -1$

b) From $g(x) = 1 - 2x$, we have $1 - 2a = 14 \Longrightarrow a = -\frac{13}{2}$. From $g(x) = x^2 - 2$, we have $a^2 - 2 = 14 \Longrightarrow x^2 = 16 \Longrightarrow x = \pm 4$. we discard -4 since $-4 < -1$ and say $x = 4$. Hence, the required values of a are $-\frac{13}{2}$ and $x = 4$.

2.3.3 Composite Functions

Definition 2.3.4. *Let f and g be functions. The composition of f with g, denoted by fog is given by $(fog)(x) = f[g(x)]$. Similarly, the composition of g with f denoted by gof is given as $(gof)(x) = g[f(x)]$*

Note that in general, $(fog)(x) \neq (gof)(x)$.

Example 2.3.11. Given that $f(x) = \frac{1}{2x-1}$ and $g(x) = x + 3$, find:

(i) $(fog)(x)$ (ii) $(gof)(x)$

Soln:

(i)

$$\begin{aligned}(fog)(x) =& f[g(x)]\\ =& \frac{1}{2[g(x)]-1}\\ =& \frac{1}{2[x+3]-1}\\ =& \frac{1}{2x+2}\end{aligned}$$

(ii)

$$\begin{aligned}(gof)(x) =& g[f(x)]\\ =& [f(x)]+3\\ =& [\frac{1}{2x-1}]+3\\ =& \frac{1}{2x-1}+3\\ =& \frac{1+3(2x-1)}{2x-1}\\ =& \frac{6x-2}{2x-1}\end{aligned}$$

Example 2.3.12. Given that $f(x) = \sqrt{2-x}$ and $g(x) = \frac{2}{x+1}$, find:

(i) the domain of $(gof)(x)$ (ii) $(gof)(-3)$ (iii) $(gof)(2)$ (iv) $(gof)(0)$

Soln:

(i) We first find the composition $(gof)(x)$ as we did above.

$$\begin{aligned}(gof)(x) =& g[f(x)]\\ =& \frac{2}{[f(x)]+1}\\ =& \frac{2}{[\sqrt{2-x}]+1}\\ =& \frac{2}{\sqrt{2-x}+1}\\ =& \frac{2}{\sqrt{2-x}+1}\end{aligned}$$

For the domain, we need $2-x \geq 0$ implying that $x \leq 2$. Hence domain, $D_{gof} = (-\infty, 2]$

(ii) Simply substitute x for -3.

$$
\begin{aligned}
(gof)(-3) =& \frac{2}{\sqrt{2-(-3)}+1} \\
=& \frac{2}{\sqrt{5}+1}
\end{aligned}
$$

(iii) Similarly,

$$
\begin{aligned}
(gof)(-2) =& \frac{2}{\sqrt{2-(2)}+1} \\
=& \frac{2}{\sqrt{0}+1} \\
=& 2
\end{aligned}
$$

(iv) Similarly,

$$
\begin{aligned}
(gof)(k) =& \frac{2}{\sqrt{2-(k)}+1} \\
=& \frac{2}{\sqrt{2-k}+1}
\end{aligned}
$$

2.3.4 Inverse Functions

Definition 2.3.5. *Let $f : X \mapsto Y$ be a bijective function from set X to set Y. An inverse function of f denoted f^{-1} is another function that maps elements from Y to X.*

Note that both f and f^{-1} must be bijective. The graphs of the two functions f and f^{-1} are symmetric about the line $y = x$

Example 2.3.13. Find the inverse of each of the following functions:

(i) $f(x) = x + 1$ (ii) $g(x) = \frac{1}{2x-1}$ (iii) $h(x) = \sqrt{-1-x}$

Soln:

(i) To find the inverse of a function, it is customary to make x the subject of the formula

$$
\begin{aligned}
y =& f(x) \\
y =& x + 1 \\
x =& y - 1
\end{aligned}
$$

Therefore, $f^{-1}(x) = x - 1$ is the inverse function of $f(x) = x + 1$

(ii) Similarly,

$$
\begin{aligned}
y =& g(x) \\
y =& \frac{1}{2x-1} \\
(2x-1)y =& 1 \\
2xy =& 1+y \\
x =& \frac{1+y}{2y}
\end{aligned}
$$

Therefore, $g^{-1}(x) = \frac{1+x}{2x}$ is the inverse function of $g(x) = \frac{1}{2x-1}$

(iii) Similarly

$$
\begin{aligned}
y =& h(x) \\
y =& \sqrt{-1-x} \\
y^2 =& -1-x \\
x =& -1-y^2
\end{aligned}
$$

Therefore, $h^{-1}(x) = x-1$ is the inverse function of $h(x) = x+1$

Example 2.3.14. Given the function $f(x) = \frac{4}{3x+2}$ and $g(x) = 2x-3$, find $(g \circ f)^{-1}(x)$ and state its domain.

Sol:

$$
\begin{aligned}
(g \circ f)(x) =& g[f(x)] \\
=& 2[f(x)] - 3 \\
=& 2\left[\frac{4}{3x+2}\right] - 3 \\
=& \frac{8-3(3x+2)}{3x+2} \\
=& \frac{2-9x}{3x+2} \qquad \text{make } x \text{ the subject to get } (g \circ f)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
y =& \frac{2-9x}{3x+2} \\
y(3x+2) =& 2-9x \\
3xy+9x =& 2-2y \\
x(3y-9) =& 2-2y \\
x =& \frac{2-2y}{3y-9} \qquad \text{Hence, } (g \circ f)^{-1}(x) = \frac{2-2x}{3x-9}
\end{aligned}
$$

For the domain, we need $3x-9 \neq 0 \Longrightarrow x \neq 3$. Hence, $D_{(g \circ f)^{-1}} = \{x| \quad x \neq 3, \quad x \in \mathbb{R}\}$

3
Linear Functions

3.1. Introduction

We have looked at functions in general. We now turn our attention to a specific type of functions called the linear functions. These are some of the most basic functions, yet very useful. They are used extensively in applied science to show the relationship between two quantities that are linearly related.

Definition 3.1.1. *A linear function in the variable x with real constants m and c, is a function of the form*

$$f(x) = mx + c$$

Example 3.1.1. The following are some examples of linear functions;

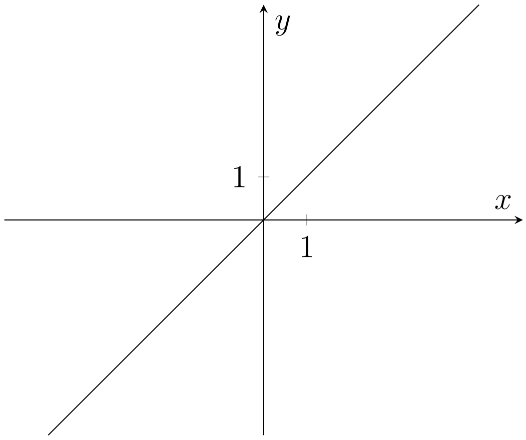
i) $f(x) = 2x + 7$ ii) $g(x) = x$ iii) $h(x) = 2 - 5x$ iv) $k(x) = \frac{3}{11}x - 1$

3.2. Graphs of Linear Functions

The term linear functions implies that the graph of any Linear function is a straight line. Quite often, the domain and the range will comprise of all real numbers unless some specific restrictions are imposed on the function. If $f(x) = mx + c$ is a linear function, the graph of this function, denoted $y = mx + c$ is a straight line whose gradient or slope is m and the $Y-$intercept is c. We will examine various techniques for sketching the graphs of these linear functions.

Example 3.2.1. Sketch the graph of $f(x) = x$. Hence, state the domain and the range of this function.

Sol: The graph of this function passes through all points such that $y = x$. Hence,



From the graph, we can see that

$$D_f = \{x|\ \ x \in \mathbb{R}\} = (-\infty, \infty) \quad \text{and} \quad R_f = \{y|\ \ y \in \mathbb{R}\} = (-\infty, \infty)$$

Example 3.2.2. Sketch the graph of $f(x) = -x$. Hence, state the domain and the range for this function.

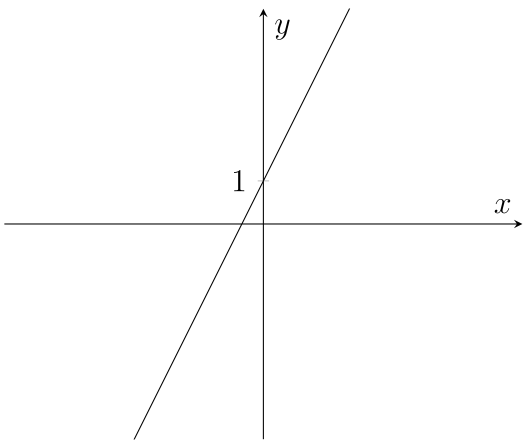
Sol: Exercise

Example 3.2.3. Given the function $f(x) = 2x + 1$, sketch the graph. Hence or otherwise, state the domain and the range for this function.

Sol: The graph of this function is given by $y = 2x + 1$. Further, to sketch the graph of any linear function, we just need two points through which the graph passes.

when $x = 0$, then $y = 2(0) + 1 = 1$. Hence, the graph passes through the point $(0, 1)$

when $y = 0$, then $0 = 2x + 1 \Longrightarrow x = -\frac{1}{2}$. Hence, the graph passes through the point $(-\frac{1}{2}, 0)$



From the graph, we can see that

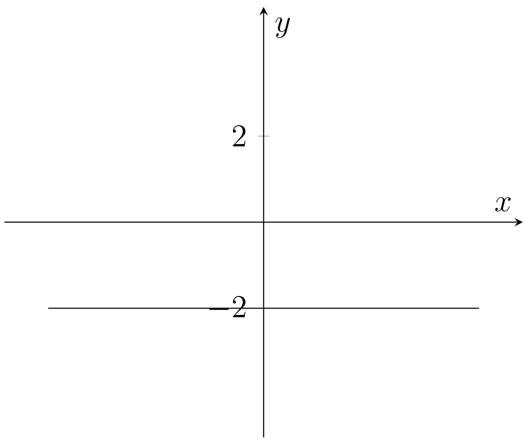
$$D_f = \{x|\ \ x \in \mathbb{R}\} = (-\infty, \infty) \quad \text{and} \quad R_f = \{y|\ \ y \in \mathbb{R}\} = (-\infty, \infty)$$

Example 3.2.4. Sketch the graph of $f(x) = 7 - 3x$. Hence, state the domain and the range for this function.

Sol: Exercise

Example 3.2.5. Given the function $f(x) = -2$, sketch the graph. Hence or otherwise, state the domain and the range for this function.

Sol: Note that this is just a straight line through $(x, -2)$ for all values of x and is parallel to the $X-$axis.



we can see that

$$D_f = \{x|\ \ x \in \mathbb{R}\} = (-\infty, \infty) \text{ and } R_f = \{-2\}$$

Note 3.2.1. The following points apply to lines that are either parallel to the $X-$axis or parallel to the $Y-$axis.

- The graph of a line $y = k$, where $k \in \mathbb{R}$ is parallel to the $X-$axis. It represents the graph of a many to one function, $f(x) = k$. It has a gradient of zero.
- The graph of a line $x = k$, where $k \in \mathbb{R}$ is parallel to the $Y-$axis. It is NOT a graph of a function. The gradient is undefined.

Example 3.2.6. Sketch the graphs of the following relations, on the same axes.

i) $y = \frac{7}{2}$ ii) $y = -5$ iii) $x = -\frac{5}{2}$ iv) $x = 1$ v) $2x - 5y = 11$

Sol: Exercise

3.2.1 Gradient of a Straight Line

The gradient of straight line is a measure of the steepness of that line relative to the $X-$axis.

Definition 3.2.1. *The gradient m, of a straight line L, that passes through two points (x_1, y_1) and (x_2, y_2) is given as*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 3.2.7. Find the gradient of a straight line through the points $(-5, 3)$ and $(4, 7)$

Sol: Let $(x_1, y_1) = (-5, 3)$ and $(x_2, y_2) = (4, 7)$. Then $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-3}{4-(-5)} = \frac{4}{9}$

Example 3.2.8. Find the gradient of the line through the points:
i) $(-2, -3)$ and $(3, 5)$ ii) $(-2, 5)$ and $(4, -2)$ iii) $(2, -3)$ and $(2, 11)$ iv) $(-3, -1)$ and $(7, -1)$

Sol: Exercise

Note 3.2.2. We take note of the following:

- Parallel lines have equal gradients. Conversely, if two lines have equal gradients, then they are parallel
- If m_1 and m_2 are the gradients of two parallel lines, then $m_1 m_2 = -1$
- Lines parallel to the $X-$axis have gradients equal to zero
- Lines parallel to the $Y-$axis have undefined gradients.

3.2.2 Equation of a Straight Line

We can determine the equation of a straight line using the concept of the gradient. The equation of a strait line can be determined depending on the presented information:

a) **Given one point $p(x_1, y_1)$ on the line and its gradient m:** Choosing an arbitrary point (x, y) on the line, we can show that the equation is given by

$$y - y_1 = m(x - x_1)$$

b) **Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the line:** Choosing an arbitrary point (x, y) on this line and using the concept of a straight line, we can show that the equation of the line is given by

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

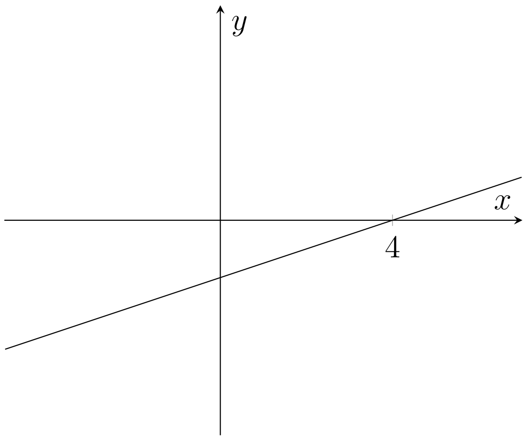
Example 3.2.9. Determine the equation of the line through the point $(1, -1)$ with gradient $\frac{1}{3}$. Sketch the graph of this line.

Sol: Let $(x_1, y_1) = (1, -1)$ and $m = \frac{1}{3}$. Then using $y - y_1 = m(x - x_1)$, we have

$y - (-1) = \frac{1}{3}(x - 1) \Longrightarrow 3y + 3 = x - 1$ which simplifies to $3y = x - 4$

To sketch the graph we need to know the intercepts. Thus;

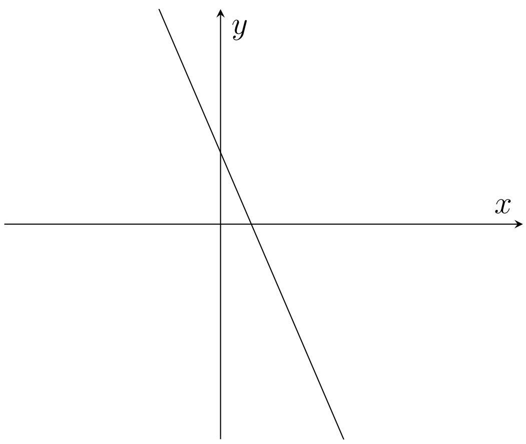
Verify that the graph passes through the point $(0, -\frac{4}{3})$ and $(4, 0)$



Example 3.2.10. Determine the equation of the line through $(2, -3)$ and $(-1, 4)$. Hence or otherwise, sketch the graph of this line.

Sol: Let $(x_1, y_1) = (2, -3)$ and $(x_2, y_2) = (-1, 4)$. Then, using $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$, we have

$\frac{y-(-3)}{4-(-3)} = \frac{x-2}{-1-2} \Longrightarrow \frac{y+3}{7} = \frac{x-2}{-3}$ which simplifies to $3y + 7x = 5$. The sketch is shown below.



From the graphs above we can see that the domain of a linear function is $\mathbb{R}$ and the range is the same, $\mathbb{R}$

3.3. Equations and Inequalities

We now look at some basic equations and inequalities involving linear terms.

Example 3.3.1. Solve the equation $-2x + 7 = x - 11$

Sol: Collect like terms. $-2x + 7 = x - 11 \Longrightarrow -3x = -18 \Longrightarrow x = 6.$ Hence,

$$SS = \{6\}$$

Example 3.3.2. Solve the inequality $-x - 3 \geq 2x + 7$

Sol: Collect like terms. $-x - 3 \geq 2x + 7 \Longrightarrow -3x \geq 10 \Longrightarrow x \leq -\frac{10}{3}$. Hence, the solution set is given by;

$$SS = \left\{x|\ x \leq -\frac{10}{3}\right\} = \left(-\infty, -\frac{10}{3}\right]$$

Example 3.3.3. Solve the following pair of simultaneous equations

$$2x + 3y = -1 \qquad (i)$$

$$x(x - y) = 2 \qquad (ii)$$

Sol: Use substitution.

From (i), we have $y = \frac{-1-2x}{3}$. Substituting this into equation (ii), we have

$x(x - \frac{-1-2x}{3}) = 2$ so that $x(\frac{5x+1}{3}) = 2$, which simplifies to a quadratic equation $5x^2 + x - 6 = 0$.

$5x^2 + x - 6 = 0 \Longrightarrow (5x + 6)(x - 1) = 0$. This gives $x = -\frac{6}{5}$ and $x = 1$

When $x = -\frac{6}{5}$, from (i), we get $y = \frac{-1-2(-\frac{6}{5})}{3} = \frac{7}{15}$

When $x = 1$, from (i), we get $\frac{-1-2(1)}{3} = -1$

Hence, $x = -\frac{6}{5}$ when $y = \frac{7}{15}$ and $x = 1$ when $y = -1$

Example 3.3.4. Find the coordinates of the points where the line $x + 2y = 7$ meets the curve $x^2 - 4x + y^2 = 1$

Sol: We solve the two equations simultaneously.

$$x + 2y = 7 \qquad \text{(i)}$$

$$x^2 - 4x + y^2 = 1 \qquad \text{(ii)}$$

From (i), we have $x = 7 - 2y$. Substituting for x in (ii), we have

$(7 - 2y)^2 - 4(7 - 2y) + y^2 = 1$ so that $49 - 28y + 4y^2 - 28 + 8y + y^2 - 1 = 0$.

This reduces to a quadratic $y^2 - 4y + 4 = 0 \Longrightarrow (x - 2)^2 = 0$. This gives $x = 2$

From (i), $x = 7 - 2(2) = 3$.

Therefore, the line and the curve meet at the point $(2, 3)$

Example 3.3.5. Solve the following system of linear equations

$$x + 2y - 3z = 12 \qquad \text{(i)}$$

$$3x - y - 2z = 1 \qquad \text{(ii)}$$

$$2x + 5y + 4z = 18 \qquad \text{(iii)}$$

Sol: For now, we will use substitution.

From (i), we have $x = 12 + 3z - 2y$ substituting this for x in (ii) and (iii), we have

$3(12 + 3z - 2y) - y - 2z = 1$ which simplifies to $z - y = -5$. Similarly,

$2(12 + 3z - 2y) + 5y + 4z = 18$ which simplifies to $10z + y = -6$. We now solve the following simultaneously.

$$z - y = -5 \qquad \text{(iv)}$$

$$10z + y = -6 \qquad \text{(v)}$$

From (iv), we have $z = y - 5$ and substituting this into (v) gives $10(y - 5) + y = -6$ so that we have $11y = 44 \Longrightarrow y = 4$.

Now we can use (iv), i.e $z - 4 = -5 \Longrightarrow z = -1$. Further, using (i), we have

$x = 12 + 3z - 2y \Longrightarrow x = 12 + 3(-1) - 2(4) = 1$

Therefore, $x = 1$, $y = 4$ and $z = -1$. The solution set,

$$SS = \{(1, 4, -1)\}$$

Example 3.3.6. Solve the following system of linear equations

$$21x + 35y - 7z = -125 \qquad \text{(i)}$$

$$7x + 7y + 7z = -8 \qquad \text{(ii)}$$

$$2x - 2y + 7z = 21 \qquad \text{(iii)}$$

Example 3.3.7. Solve the following system of linear equations

$$2x - 3y + z = -10 \qquad \text{(i)}$$

$$3x + 7y - 2z = 8 \qquad \text{(ii)}$$

$$6x + 5y - 4z = -1 \qquad \text{(iii)}$$

Example 3.3.8. Solve the following system of linear equations

$$2x + 5y - 4z = 2 \qquad \text{(i)}$$

$$3x + 7y - 8z = 0 \qquad \text{(ii)}$$

$$12z - 13y + 2x = 4 \qquad \text{(iii)}$$

Sol: Exercise

4
Quadratic Functions

4.1. Introduction

We now study one of the most important functions in mathematics, the quadratic function. It is a polynomial of degree two. This function is used to model a wide variety of random phenomenon. Its applications ranges from the fields of economics, social sciences and natural sciences.

Definition 4.1.1. *A quadratic function in x variable, with real constants a, b and c, is a function of the form*

$$f(x) = ax^2 + bx + c$$

Note:

- when $c = 0$, then we have $f(x) = ax^2 + bx$, which is still a quadratic function.
- when $c = 0$ and $b = 0$ then we have $f(x) = ax^2$, which is also still a quadratic function.
- when $a = 0$ the its no longer a quadratic function.

Definition 4.1.2. *An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation in terms of the random variable x.*

We should distinguish the quadratic function $f(x) = ax^2 + bx$ and the quadratic equation $ax^2 + bx + c = 0$. Before we study the quadratic function further, we need to explore some nice features of the quadratic equation.

4.2. Roots of a Quadratic Equation

The values of x which satisfy the quadratic equation $ax^2 + bx + c = 0$ are called roots of the quadratic equation. The roots are also referred to as the solutions or zeros of the quadratic equation. Roots maybe real numbers or indeed complex numbers. The concept of roots is vital to the understanding of the quadratic function. To determine the roots of the quadratic equation, we need to solve the equation $ax^2 + bx + c = 0$. Thus, it is very important that we develop that necessary skill of solving quadratic equations

Methods of Solving Quadratic Equations

We will consider three methods of solving the quadratic equation. Note that when we are solving the quadratic equation $ax^2 + bx + c = 0$, we are simply finding its roots.

1. **Factorisation:**

   This is arguably the simplest method when the roots are rational. The following steps are involved:

   i) obtain the product: $P = ac$

   ii) obtain the sum: $S = b$

   iii) find two factors whose product $P = ac$ and whose sum is $S = b$.

   iv) substitute b from the quadratic equation with the two numbers, then factorise.

   Example 4.2.1. Use the factorization method to solve the following quadratic equations:
   (i) $2x^2+7x-15 = 0$ (ii) $-5x^2-3x+2 = 0$ (iii) $x^2-x-2 = 0$ (iv) $5x^2-6x-2 = 0$

   Soln:

   (i) For $2x^2 + 7x - 15 = 0$, we have $a = 2$, $b = 7$, $c = -15$.

   $P:\ ac = (2)(-15) = -30$

   $S:\ b = 7.$

   factors: $+10$ and -3

   We now factorise as follows:

$$\begin{aligned} 2x^2 + 7x - 15 &= 0 \\ 2x^2 + 10x - 3x - 15 &= 0 \\ 2x(x+5) - 3x(x+5) &= 0 \\ (2x-3)(x+5) &= 0 \end{aligned}$$

   So either $(2x - 3) = 0$ or $(x + 5) = 0$ implying that either $x = \frac{3}{2}$ or $x = -5$

   (ii) For $-5x^2 - 3x + 2 = 0$, we have $a = -5$, $b = -3$, $c = 2$.

   $P:\ ac = -10$

   $S:\ b = -3.$

   factors: -5 and 2

$$\begin{aligned} -5x^2 - 3x + 2 &= 0 \\ -5x^2 - 5x + 2x + 2 &= 0 \\ -5x(x+1) + 2(x+1) &= 0 \\ (-5x+2)(x+1) &= 0 \end{aligned}$$

   So either $(-5x + 2) = 0$ or $(x + 1) = 0$.

   Therefore, $x = \frac{2}{5}$ or $x = -1$

(iii) For $x^2 - x - 2 = 0$, we have $a = 1$, $b = -1$, $c = -2$.

$P:\ ac = -2$

$S:\ b = -1.$

factors: 1 and -2

$$\begin{aligned} x^2 - x - 2 &= 0 \\ x^2 + x - 2x - 2 &= 0 \\ x(x+1) - 2(x+1) &= 0 \\ (x-2)(x+1) &= 0 \end{aligned}$$

So either $(x-2) = 0$ or $(x+1) = 0$ implying that either $x = 2$ or $x = -1$

Hence the roots are $x = 2$ and $x = -1$

(iv) For $5x^2 - 6x - 2 = 0$, the factorization can not work (**Verify**). This is because the roots are not rational numbers. In parts (i)-(iii) of our example we see that all obtained roots were rational.

2. **Completing the Square:**

This is a more general approach of solving quadratic equations. It can be used whether the roots are real or complex.The following steps are involved:

i) write the terms with the unknown variable on one side.

ii) divide through by the coefficient of x^2

iii) add the square of half the coefficient of x

iv) factorise the side with the variables and then solve

Example 4.2.2. Given $ax^2 + bx + c = 0$, complete the square.

Soln:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \\ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

Example 4.2.3. Given $2x^2 + 7x + 5 = 0$, complete the square.

Soln:

$$\begin{aligned}
2x^2 + 7x + 5 =& 0\\
x^2 + \frac{7}{2}x + \frac{5}{2} =& 0\\
x^2 + \frac{7}{2}x =& -\frac{5}{2}\\
x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 =& -\frac{5}{2} + \left(\frac{7}{4}\right)^2\\
\left(x + \frac{7}{4}\right)^2 =& \frac{9}{16}
\end{aligned}$$

Example 4.2.4. By completing the square, solve the quadratic equation $5x^2-6x-2 = 0$.

Soln:

$$\begin{aligned}
5x^2 - 6x - 2 =& 0\\
5x^2 - 6x =& 2\\
x^2 - \frac{6}{5}x =& \frac{2}{5}\\
x^2 - \frac{6}{5}x + \left(-\frac{3}{5}\right)^2 =& \frac{2}{5} + \left(-\frac{3}{5}\right)^2\\
\left(x - \frac{3}{5}\right)^2 =& \frac{19}{25}\\
x - \frac{3}{5} =& \pm\sqrt{\frac{19}{25}}\\
x =& \frac{3}{5} \pm \sqrt{\frac{19}{25}}
\end{aligned}$$

Hence, $x = \frac{3+\sqrt{19}}{5}$ or $x = \frac{3-\sqrt{19}}{5}$

Example 4.2.5. Given $9 - 2x - 5x^2 = 0$, complete the square. Hence, find the roots of the quadratic equation.

Soln: Exercise

3. **Quadratic Formula**

   This formula is just a consequence of the method of completing the square. If $f(x) = ax^2 + bx + c$, by completing the square, we get:

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Hence for any quadratic $f(x) = ax^2 + bx + c$, the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

can be used to determine the roots

Example 4.2.6. Solve the equation $3x^2 - 7x - 11 = 0$

Soln: $a = 3$, $b = -7$ and $c = -11$. Using the formula, we have;

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-11)}}{2(3)} \\
&= \frac{7 \pm \sqrt{49 + 132}}{2(3)} \\
&= \frac{7 \pm \sqrt{181}}{6}
\end{aligned}
$$

Hence, $x = \frac{7+\sqrt{181}}{6}$ or $x = \frac{7-\sqrt{181}}{6}$

4.2.1 Nature of Roots and The Discriminant

We now turn to our attention to the nature of the roots of a quadratic equation.

Definition 4.2.1. *The discriminant D of a quadratic equation $ax^2 + bx + c = 0$ is given by*

$$D = b^2 - 4ac$$

The concept of a discriminant is very important to the understanding of the nature of roots. There are basically three types of roots, depending on the discriminant.

1. If the discriminant is positive, i.e $b^2-4ac > 0$, then the quadratic equation $ax^2+bx+c = 0$ has two distinct real roots.
2. If the discriminant is zero, i.e $b^2 - 4ac = 0$, then the quadratic equation $ax^2 + bx + c = 0$ has two equal real roots. Having two equal roots is the same as having only one root.
3. If the discriminant is negative, i.e $b^2-4ac < 0$, then the quadratic equation $ax^2+bx+c = 0$ has no real roots. It has complex roots

Example 4.2.7. Determine the nature of the roots of the equation $4x^2 - 7x + 3 = 0$

Sol: $a = 4$, $b = -7$ and $c = 3$

Then $b^2 - 4ac = (-7)^2 - 4(4)(3) = 1 > 0$.

Since $b^2 - 4ac > 0$, the equation $4x^2 - 7x + 3 = 0$ has two distinct real roots.

Example 4.2.8. Determine the nature of the roots of the equation $x^2 + 6x + 9 = 0$

Sol: $a = 1$, $b = 6$ and $c = 9$

Then $b^2 - 4ac = (6)^2 - 4(1)(9) = 0$.

Since $b^2 - 4ac = 0$, the equation $x^6x + 9 = 0$ has two equal real roots.(one root)

Example 4.2.9. Determine the nature of the roots of the equation $5x^2 - x + 9 = 0$

Sol: $a = 5$, $b = -1$ and $c = 9$

Then $b^2 - 4ac = (-1)^2 - 4(5)(9) = -179 < 0$.

Since $b^2 - 4ac < 0$, the equation $5x^2 - x + 9 = 0$ has no real roots. (it has complex roots)

Example 4.2.10. For what values of k does the equation $4x^2 + kx + 9 = 0$ have equal roots?

Sol: $a = 4$, $b = k$ and $c = 9$. For equal roots, we need $b^2 - 4ac = 0$

Hence,

$$\begin{aligned} b^2 - 4ac &= 0 \\ (k)^2 - 4(4)(9) &= 0 \\ k^2 - 144 &= 0 \\ (k + 12)(k - 12) &= 0 \end{aligned}$$

Hence, $k = -12$ and $k = 12$

4.2.2 Sum and Product of the Roots

Let α and β be the roots of the quadratic $ax^2 + bx + c = 0$. We will examine the relationship that exists between the roots and the coefficients. Since α and β are roots of the quadratic $ax^2 + bx + c = 0$, we can assume that

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Now, for $ax^2 + bx + c = 0$, we can divide through by a to get;

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$————————————————————(i)

Also, since Since α and β are roots of the quadratic $ax^2 + bx + c = 0$, we can write

$(x-\alpha)(x-\beta) = 0$—————————————————————-(ii)

equating (i) and (ii), we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x-\alpha)(x-\beta)$$

.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (\alpha+\beta)x + \alpha\beta$$

Comparing the terms, we get

$$\alpha + \beta = -\frac{b}{a}$$

and

$$\alpha\beta = \frac{c}{a}$$

which denote the sum and the product of the roots of the quadratic equation respectively.

Example 4.2.11. Find the sum and the product of the roots of $2x^2 - 3x + 1 = 0$

Sol: $a = 2$, $b = -3$ and $c = 1$.

Let α and β be the roots of $2x^2 - 3x + 1 = 0$.

Then $\alpha + \beta = -\frac{b}{a} = -\frac{(-3)}{2} = \frac{3}{2}$

and $\alpha\beta = \frac{c}{a} = \frac{1}{2}$

Some Important Identities:

- $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$
- $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$
- $\alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$

Example 4.2.12. If the roots of the quadratic $3x^2 - 5x + 1 = 0$ are α and β, find the value of

i) $\alpha^2 + \beta^2$ ii) $\frac{1}{\alpha} + \frac{1}{\beta}$ iii) $\alpha^3 + \beta^3$

Sol: $a = 3$, $b = -5$ and $c = 1$.

We have $\alpha + \beta = -\frac{b}{a} = -\frac{(-5)}{3} = \frac{5}{3}$ and $\alpha\beta = \frac{c}{a} = \frac{1}{3}$

Therefore, (i)

$$\begin{aligned}\alpha^2 + \beta^2 =& (\alpha + \beta)^2 - 2\alpha\beta \\ =& \left(\frac{5}{3}\right)^2 - 2\left(\frac{1}{3}\right) \\ =& \frac{25}{9} - \frac{2}{3} \\ =& \frac{19}{9}\end{aligned}$$

(ii)

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} =& \frac{\beta + \alpha}{\alpha\beta} \\ =& \frac{\frac{5}{3}}{\frac{1}{3}} \\ =& \frac{5}{3} \times \frac{3}{1} \\ =& 5\end{aligned}$$

(iii)

$$\begin{aligned}\alpha^3 + \beta^3 =& (\alpha + \beta)(\alpha^2 + \beta^2 - 2\alpha\beta) \\ =& \left(\frac{5}{3}\right)\left(\frac{19}{9} - \frac{1}{3}\right) \\ =& \frac{5}{3} \times \frac{16}{9} \\ =& \frac{80}{27}\end{aligned}$$

Example 4.2.13. The roots of $x^2 - 2x + 3 = 0$ are α and β. Find the equation whose roots are $\alpha + 2$ and $\beta + 2$

Sol: Exercise

Example 4.2.14. Find the value of k if the roots of $3x^2 + 5x - k = 0$ differ by 2.

Sol: Exercise

4.3. Graphs of Quadratic Functions

Let $f(x) = x^2 + bx + c$ be any quadratic function. The graph $y = x^2 + bx + c$ of any quadratic function is a parabola. If $a > 0$, the parabola opens upwards (cup-shaped). If $a < 0$, the parabola opens downwards (cap-shaped). To sketch the graph of a quadratic function, the following must be determined:

- **orientation:** To determine the orientation of the parabola, the constant a is used. If $a > 0$, the parabola opens upwards (cup-shaped). If $a < 0$, the parabola opens downwards (cap-shaped).

- **y-intercept:** To determine the $Y-$intercept, we let $x = 0$ and evaluate the corresponding y value. It is easy to see that the graph cuts the $Y-$axis at the point $(0, c)$.

- **turning point:** The turning point occurs when $x = -\frac{b}{2a}$, i.e at the point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

- **x-intercept:** To determine the $X-$intercept, we let $y = 0$ and evaluate the corresponding x value(s). This simply means we solve the equation $x^2 + bx + c = 0$ to obtain the roots. The graph cuts the $X-$axis at the points α and β, where α and β are just the usual roots we have discussed.

- **Sketch:** Once the above quantities are determined, we are ready to sketch the graph $y = x^2 + bx + c$.

Example 4.3.1. Sketch the graph of $f(x) = 2x^2 - 7x + 5$

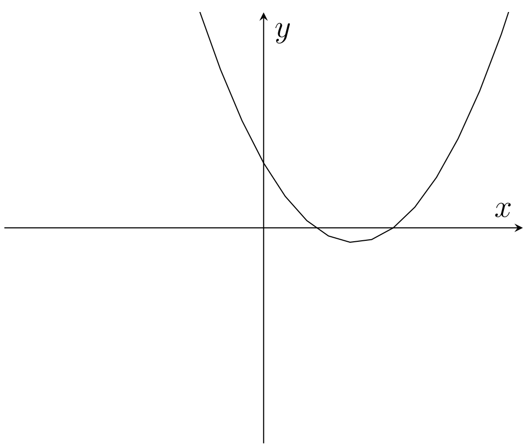
Sol: $a = 2$, $b = -7$ and $c = 5$.

i) Since $a > 0$, the orientation of the graph is cup-shaped, i.e it opens upwards.

ii) $f(0) = 2(0)^2 - 7(0) + 5 = 5$. Hence, the graph cuts the $y-$axis at $(0, 5)$.

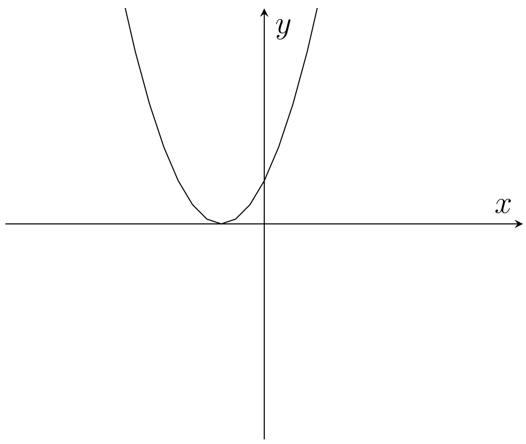
iii) The turning point occurs at $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right) = \left(-\frac{(-7)}{2(2)}, \frac{4(2)(5)-(-7)^2}{4(2)}\right) = \left(\frac{7}{4}, -\frac{9}{8}\right)$

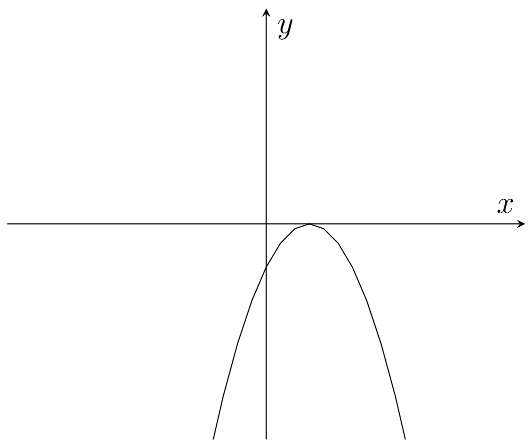
iv) $2x^2 - 7x + 5 = 0 \Longrightarrow (x-1)(2x-5) = 0 \Longrightarrow x = 1$ and $x = \frac{5}{2}$. Hence $\alpha = 1$ and $\beta = \frac{5}{2}$ are the roots. This means that the graph cuts the $x-$axis at $(1, 0)$ and $\left(\frac{5}{2}, 0\right)$ We have what we need to sketch the graph of the quadratic function $f(x) = 2x^2 - 7x + 5$. See below

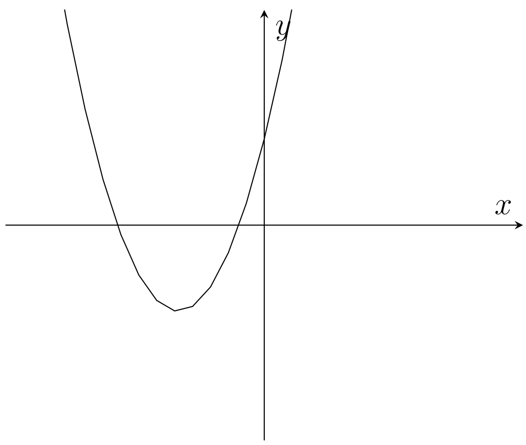


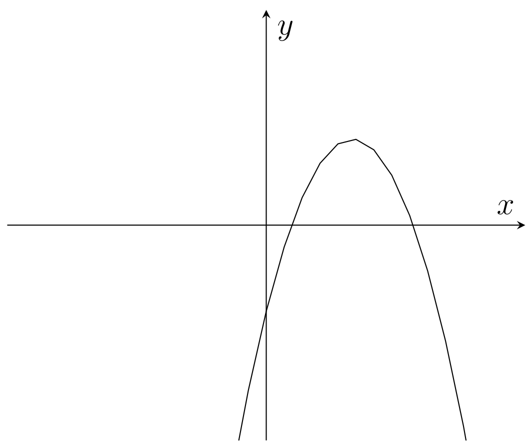
4.3.1 Discriminant and The Graph of a Quadratic Function

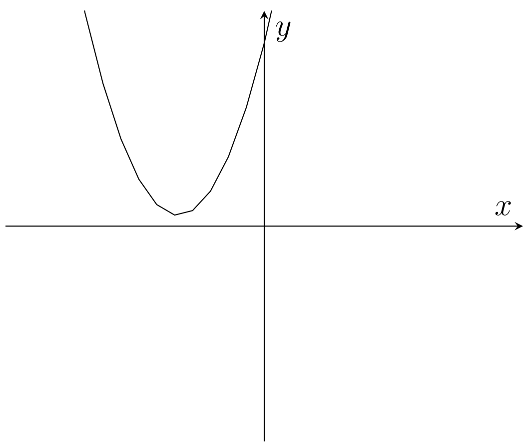
Let us examine the relationship between the discriminant and the graph of $f(x) = x^2 + bx + c$. Using the discriminant $b^2 - 4ac$ and the value of a, the graph of a quadratic falls into three categories, shown below.

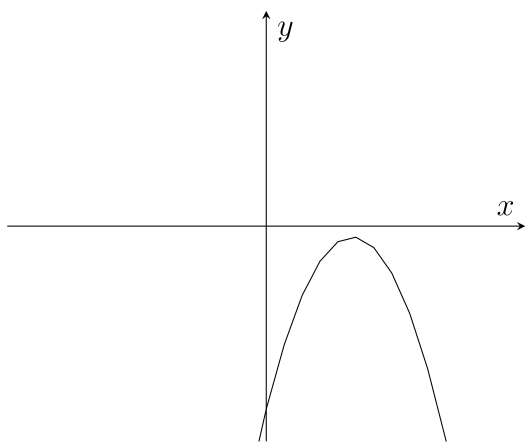












4.3.2 Maximum and Minimum Values of a Quadratic Function

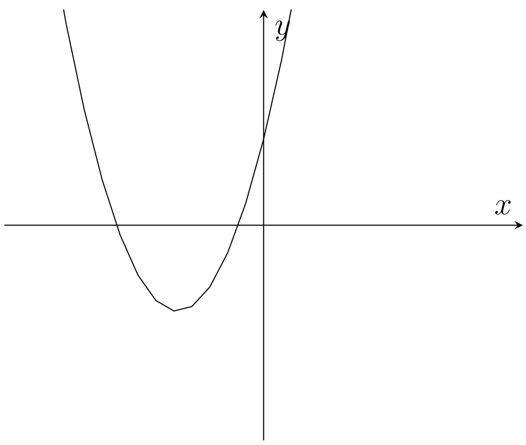
Let $f(x) = x^2 + bx + c$ be a quadratic function. If $a < 0$, the quadratic has a maximum point. If $a > 0$ the quadratic has a minimum point. We need to determine the coordinates of the Minimum/Maximum points. To do this, recall the concept of completing the square:

Completing the square for a quadratic function $f(x) = x^2+bx+c$ yields the following important result

$$\begin{aligned} f(x) =& ax^2 + bx + c \\ =& a\left(x^2 + \frac{b}{a}\right) + c \\ =& a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + c \\ =& a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \end{aligned}$$

Minimum Value

The graph below shows the minimum point of a quadratic curve.

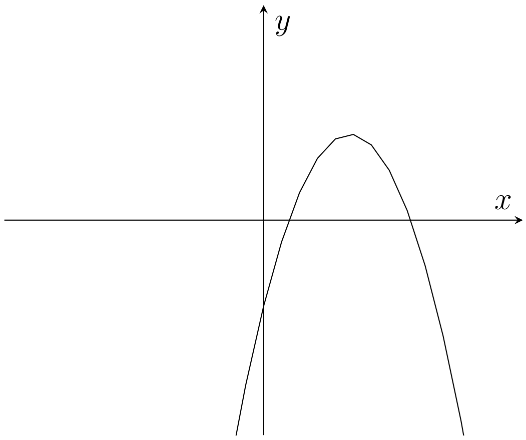


This occurs when $a > 0$. Further, from the results of completing the square, we see that;

- the value $x = -\frac{b}{2a}$ attains the minimum value for the quadratic function
- the minimum value for the quadratic function is $f\left(-\frac{b}{2a}\right) = \frac{4ac-b^2}{4a}$
- hence, the minimum point is $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$
- the range of $f(x)$ is $[\frac{4ac-b^2}{4a}, \infty)$

Maximum Value

The graph below shows the maximum point of a quadratic curve.



This occurs when $a < 0$. Further, form the results of completing the square, we see that;

- the value $x = -\frac{b}{2a}$ attains the maximum value for the quadratic function
- the maximum value for the quadratic function is $f\left(-\frac{b}{2a}\right) = \frac{4ac-b^2}{4a}$
- hence, the maximum point is $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$
- the range of $f(x)$ is $(-\infty, \frac{4ac-b^2}{4a}]$

Example 4.3.2. Complete the square of the quadratic function $f(x) = 2x^2 - 10x + 22$. Hence, find the maximum/minimum value of $f(x)$ and the value of x at which it occurs, and sketch the graph.

Sol: $a = 2$, $b = -10$ and $c = 22$. Since we have $a > 0$, we have a minimum.

$$\begin{aligned}
f(x) =& 2x^2 - 10x + 22 \\
=& 2\left(x^2 - \frac{10}{2}\right) + 22 \\
=& 2\left(x^2 - 5\right) + 22 \\
=& 2\left(x - \frac{5}{2}\right)^2 - 2\left(-\frac{5}{2}\right)^2 + 22 \\
=& 2\left(x - \frac{5}{2}\right)^2 + \frac{19}{2}
\end{aligned}$$

From this, the minimum value occurs at $x = \frac{5}{2}$

The minimum value of the function is $f\left(\frac{5}{2}\right) = \frac{19}{2}$

Hence, the minimum point is $\left(\frac{5}{2}, \frac{19}{2}\right)$

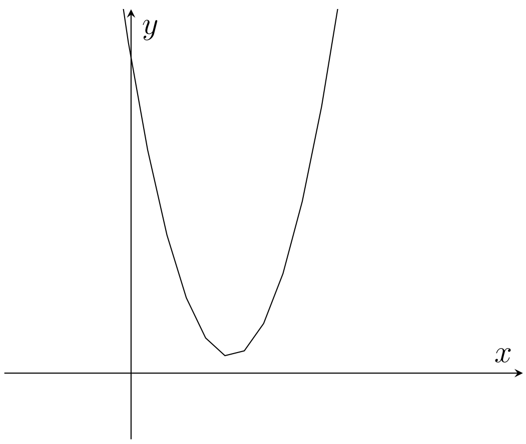
Also, the range is $R_f = [\frac{19}{2}, \infty)$

To sketch the graph, note that $f(0) = 22$. Hence the graph cuts the $Y-$axis at the point $(0, 22)$.

To find the roots, we set $y = o$ and solve;

$$\begin{aligned} 2\left(x - \frac{5}{2}\right)^2 + \frac{19}{2} =& 0 \\ 2\left(x - \frac{5}{2}\right)^2 =& -\frac{19}{2} \\ \left(x - \frac{5}{2}\right)^2 =& -\frac{19}{4} \\ x - \frac{5}{2} =& \pm\sqrt{-\frac{19}{4}} \\ x =& \frac{5}{2} \pm \sqrt{-\frac{19}{4}} \\ x =& \frac{5 \pm \sqrt{-19}}{2} \end{aligned}$$

Hence, $f(x) = 2x^2 - 10x + 22$ has no real roots. Therefore, it does not cut the $X-$axis. The sketch is shown below.



Example 4.3.3. Given the quadratic function $f(x) = 2 - 6x - x^2$;

i) complete the square of $f(x)$.

ii) Hence, find the maximum value of the function $f(x)$, and the value of x at which it occurs.

iii) sketch the graph $y = 2 - 6x - x^2$ and state the range.

Example 4.3.4. Given the quadratic function $f(x) = x^2 + 7x + 6$;

i) complete the square of $f(x)$.

ii) Hence, find the minimum value of the function $f(x)$, and the value of x at which it occurs.

iii) sketch the graph $y = f(x)$ and state the range.

Sol: Exercise

4.4. Quadratic Inequalities

We have looked at ways of solving the quadratic equations. We conclude our discussion of quadratic functions by looking at the quadratic inequalities. To solve a quadratic inequality such as $ax^2 + bx + c \leq 0$, we use any of the two methods:

1. Factorise completely, then use the table of signs to determine the solution set.

   OR

2. Sketch the graph of the quadratic and determine the solution set from the graph.

Example 4.4.1. Find the solution set to the inequality $x^2 + 5x + 8 \geq 2$

Sol: We factorise the expression to obtain critical values

$$\begin{aligned} x^2 + 5x + 8 &\geq 2 \\ x^2 + 5x + 6 &\geq 0 \\ x^2 + 2x + 3x + 6 &\geq 0 \\ x(x+2) + 3(x+2) &\geq 0 \\ (x+2)(x+3) &\geq 0 \end{aligned}$$

Critical values: $x + 2 = 0 \Longrightarrow x = -2$ and $x + 3 = 0 \Longrightarrow x = -3$

Critical values are $x = -2$ and $x = -3$

	$-\infty < x < -3$	$-3 < x < -2$	$-2 < x < \infty$
$x+3$	$-$	$+$	$+$
$x+2$	$-$	$-$	$+$
$(x+3)(x+2)$	$+$	$-$	$+$

From the table, the solution set denoted SS is given by

$SS = \{x \mid -3 \leq x \leq -2 \text{ or } x \geq -2, x \in \mathbb{R}\}$ OR $SS = [-3, -2] \cup [-2, \infty)$

Example 4.4.2. Solve the inequality $2x^2 + 7x - 15 < 0$

Sol: We factorise the expression to obtain critical values

$$\begin{aligned} 2x^2 + 7x - 15 &< 0 \\ 2x^2 + 10x - 3x - 15 &< 0 \\ 2x(x+5) - 3x(x+5) &< 0 \\ (2x-3)(x+5) &< 0 \end{aligned}$$

Critical values are $x = \frac{3}{2}$ and $x = -5$

	$-\infty < x < -5$	$-5 < x < \frac{3}{2}$	$\frac{3}{2} < x < \infty$
$2x-3$	$-$	$-$	$+$
$x+5$	$-$	$+$	$+$
$(2x-3)(x+5)$	$+$	$-$	$+$

The solution set is given by

$$SS = \{x| \quad -5 < x < \frac{3}{2}, x \in \mathbb{R}\}$$

.

$$SS = \left(-5, \frac{3}{2}\right)$$

Example 4.4.3. Find the solution set to the inequality $x^2 + x + 2 \leq 0$

Sol: Since $b^2 - 4ac = 1 - 4(1)(2) = -7$, the quadratic has no real roots. Hence it does not cut the $X-$axis. Further, note that this quadratic is above the $x-$axis, it is never negative for all values of x. Therefore,$x^2 + x + 2 \leq 0$ has no solutions.

$$SS = \emptyset$$

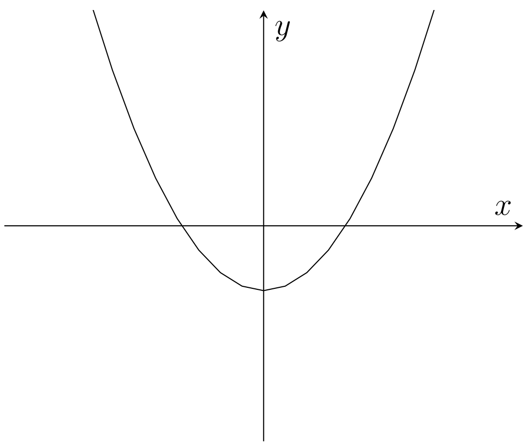
Example 4.4.4. Find the solution set to the inequality $x^2 + 1 < 0$

Example 4.4.5. Find the solution set to the inequality $x^2 - 1 \leq 0$

Sol: Exercise

Example 4.4.6. Sketch the graph of $f(x) = x^2 - 4$. Hence, use the graph to find the solution set to the inequality $x^2 - 4 \geq 0$

Sol: The sketch is shown below



Shaded interval is the solution set. ie $SS = (-\infty, -2] \cup [2, \infty)$

5
Polynomial Functions

5.1. Introduction

So far, we have looked at functions of the form $f(x) = ax + b$, the linear functions and functions of the form $f(x) = ax^2 + bx + c$, the quadratic functions. Linear functions are polynomials of degree 1 while quadratics are polynomials of degree 2. We now study polynomials of higher degree.

Definition 5.1.1. *A polynomial of degree n, is a function of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \cdots, a_0$ are constants and x is the variable. n is usually an integer.

Note that the highest power of x determines the degree of the polynomial. This is usually denoted by n.

Example 5.1.1. The following are examples of polynomials and their respective degrees:

$f(x) = x^3 - 5x^2 + x - 7$ is a polynomial of degree 3

$g(x) = x^5 - 5x^4 + x^3 - 7x^2 - 4x + 1$ is a polynomial of degree 5

$h(x) = x^7 + 1$ is a polynomial of degree 7

$k(x) = x^{30} - 2x + 5$ is a polynomial of degree 30

5.2. Division of Polynomials

Let $f(x)$ be a polynomial of degree n. Then the polynomial $f(x)$ can be decomposed into polynomials of lower degrees as follows:

$$f(x) = d(x)q(x) + r(x)$$

OR

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

Where $d(x)$, $q(x)$ and $r(x)$ are polynomials of lower degree than $f(x)$. From the expression above, $d(x)$ is a polynomial called the divisor, $q(x)$ is called the quotient and $r(x)$ is called the remainder.

Any polynomial $f(x)$ can be expressed in the form

$$f(x) = d(x)q(x) + r(x)$$

where $q(x)$ is the quotient, $d(x)$ is the divisor and $r(x)$ is the remainder. We will look at 2 methods used in dividing polynomials.

Long Division

This method can be used to divide a polynomial by a linear factor. The example below demonstrates the use of long division.

Example 5.2.1. Determine whether $x-1$ is a factor of the polynomial $f(x) = x^3+3x^2+3x+1$

Sol: We use long division to determine the remainder.

$$\begin{array}{r|l} & x^2 + 4x + 7 \\ \hline x-1) & x^3 + 3x^2 + 3x + 1 \\ & \underline{x^3 - x^2} \\ & \quad 4x^2 + 3x + 1 \\ & \quad \underline{4x^2 - 4x + 0} \\ & \qquad\quad 7x + 1 \\ & \qquad\quad \underline{7x - 1} \\ & \qquad\qquad\quad 8 \end{array}$$

Hence, the quotient $q(x) = x^2 + 4x + 7$ and the remainder $r(x) = 8$

Since the remainder is not 0, we conclude that $x-1$ is not a factor of $f(x) = x^3+3x^2+3x+1$.

Note: In this example,

- $x - 1$ is the divisor, $d(x)$
- $x^2 + 4x + 7$ is the quotient, $q(x)$
- 8 is the remainder, $r(x)$

Example 5.2.2. Determine whether $x-1$ is a factor of the polynomial $g(x) = x^3 - x^2 - x + 1$

Sol: We use long division to determine the remainder.

$$\begin{array}{r|l} & x^2 - 1 \\ \hline x-1) & x^3 - x^2 - x + 1 \\ & \underline{x^3 - x^2} \\ & \qquad\quad -x + 1 \\ & \qquad\quad \underline{-x + 1} \\ & \qquad\qquad\quad 0 \end{array}$$

Hence, the quotient $q(x) = x^2 - 1$ and the remainder $r(x) = 0$

We see that the remainder is 0. Hence, we conclude that $x-1$ is a factor of $g(x) = x^3-x^2-x+1$.

Synthetic Division

This method, like long division is used to divide polynomials. arguably, it simplifies the process involved in long division. The example below demonstrates the use of synthetic division.

Example 5.2.3. Determine the quotient and the remainder when $f(x) = x^3 + 3x^2 + 3x + 1$ is divided by $x - 1$.

Sol: We use synthetic division to determine the remainder. We let $x - 1 = 0 \Longrightarrow x = 1$. Then we have

$$\begin{array}{r|rrrrl} & 1 & 3 & 3 & 1 & \text{all coeficints of } f(x) \\ & 0 & 1 & 4 & 7 & \\ \hline 1 & 1 & 4 & 7 & 8 = r(x) & \text{remainder} \end{array}$$

Hence, quotient is $q(x) = x^2 + 4x + 7$ and remainder is $r(x) = 8$.

Example 5.2.4. Determine the quotient and the remainder when $f(x) = 2x^3 - 3x^2 + 2x$ is divided by $2x + 1$.

Sol: Let $2x + 1 = 0 \Longrightarrow x = -\frac{1}{2}$. Using synthetic division, we have

$$\begin{array}{r|rrrr} & 2 & -3 & 2 & 0 \\ & 0 & -1 & 2 & -2 \\ \hline -\frac{1}{2} & 2 & -4 & 4 & -2 = r(x) \end{array}$$

Hence, quotient is $q(x) = 2x^2 - 4x + 4$ and remainder is $r(x) = -2$.

Example 5.2.5. Use synthetic division to determine whether $x + 2$ is a factor of $4x^4 + x^2 - 1$

Sol: Let $x + 2 = 0 \Longrightarrow x = -2$. Using synthetic division, we have

$$\begin{array}{r|rrrrr} & 4 & 0 & 1 & 0 & -1 \\ & 0 & -8 & 16 & -34 & 68 \\ \hline -2 & 4 & -8 & 17 & -34 & 67 = r(x) \end{array}$$

Hence, quotient is $q(x) = 4x^2 - 8x^3 + 17x - 34$ and remainder is $r(x) = 67$.

5.2.1 The Remainder Theorem

Let us look at an important theorem used in the evaluation of remainders when dividing polynomials by linear terms. This is known as the remainder theorem.

Theorem 5.2.1. *Let $f(x)$ be a polynomial of degree $n \geq 2$. If $f(x)$ is divided by a linear term $px + q$, the remainder is $f\left(-\frac{q}{p}\right)$.*

Example 5.2.6. Find the remainder when $f(x) = x^3 - x^2 + 3x - 2$ is divided by $x + 2$

Sol: let $x + 2 = 0$ so that $x = -2$.

Then, $f(-2) = (-2)^3 - (-2)^2 + 3(-2) - 2 = -20$.

The remainder is -20

Example 5.2.7. Find the remainder when $f(x) = x^3 - x^2 + 3x - 2$ is divided by $2x - 1$

Sol: let $2x - 1 = 0$ so that $x = \frac{1}{2}$.

Then, $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 + 3(\frac{1}{2}) - 2 = -\frac{5}{8}$.

The remainder is $-\frac{5}{8}$

Example 5.2.8. Let $f(x) = x^3 + rx^2 + tx - 3$ be a polynomial. When $f(x)$ is divided by $x - 1$ and $x + 1$, the remainders are 1 and -9 respectively. Find the values of r and t.

Sol:

Dividing by $x-1$ gives remainder$=f(1) = (1)^3 + r(1)^2 + t(1) - 3 = r + t - 2$. Hence, $r + t - 2 = 1$

Dividing by $x + 1$ gives remainder$=f(-1) = (-1)^3 + r(-1)^2 + t(-1) - 3 = r - t - 4$. Hence, $r - t - 4 = -9$

Solving $r + t - 2 = 1$ and $r - t - 4 = -9$ simultaneously for r and t gives, $r = -1$ and $t = 4$

5.2.2 Factor Theorem

We now turn our attention to another important theorem, the factor theorem. This theorem basically adds on to the remainder theorem and aids us in the determination of factors of polynomials. This is important in factorization of polynomials.

Theorem 5.2.2. *Let f(x) be a polynomial of degree n and px+q be a linear term. If $f\left(-\frac{q}{p}\right) = 0$, then $px + q$ is a factor of $f(x)$. Conversely, if $px + q$ is a factor of $f(x)$, then $f\left(-\frac{q}{p}\right) = 0$.*

Example 5.2.9. Show that $x - 1$ is a factor of $f(x) = x^3 - 6x^2 - x + 6$

Sol: let $x - 1 = 0$ so that $x = 1$.

Then, $f(1) = (1)^3 - 6(1)^2 - (1) + 6 = 0$.

Since the remainder is 0, $x - 1$ is a factor of $f(x) = x^3 - 6x^2 - x + 6$

Example 5.2.10. The expression $f(x) = 2x^3 + ux^2 + vx - 2$ is exactly divisible by $x - 2$ and $2x + 1$. Find the values of u and v.

Sol:

Dividing $f(x)$ by $x - 2$ gives remainder$=f(2) = 2(2)^3 + u(2)^2 + v(2) - 2 = 16 + 4u + 2v - 2$

Since $x - 2$ is a factor, $16 + 4u + 2v - 2 = 0$ so that $2u + v = -7$

Similarly,

Dividing $f(x)$ by $2x + 1$ gives remainder$=f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + u\left(-\frac{1}{2}\right)^2 + v\left(-\frac{1}{2}\right) - 2$

This gives $-\frac{1}{4} + \frac{u}{4} - \frac{v}{2} - 2 = 0$ so that $u - 2v = 9$

Solving $2u + v = -7$ and $u - 2v = 9$ simultaneously gives $u = -1$ and $v = -5$.

5.3. Roots of Polynomials

To determine the roots of a polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$, we first need to factorise and then equate each of the factors to zero. However, factorization is not straight forward as it involves a trial and error approach. To simplify our search, we may use the following approach:

Suppose we have a polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, and we wish to determine the roots of the equation $f(x) = 0$ i.e $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$. Then;

- obtain the factors of the constant term a_0
- obtain factors of a_n (the coefficient of the highest power of x)
- divide factors of a_0 by factors of a_n. This gives **possible** rational roots.
- use the factor and remainder theorems, or synthetic division to obtain the **actual** roots of $f(x)$

Example 5.3.1. Solve the equation $2x^3 + 3x^2 - 3x - 2 = 0$

Sol: Here, $a_0 = -2$ and $a_n = 2$

coefficients of -2: ±1, ±2

coefficients of 2: ±1, ±2

Possible factors: ±1, ±2, $\pm\frac{1}{2}$

To find the actual factors, we use the Remainder and Factor Theorems.

$f(1) = 2(1)^3 + 3(1)^2 - 3(1) - 2 = 0$. Hence $x - 1$ is a factor. Letting $x - 1 = 0$, gives $x = 1$, which is one of the roots. To find the remaining roots, we divide $2x^3 + 3x^2 - 3x - 2$ by the found linear factor $x - 1$. You can use long division or synthetic division.

We use synthetic division:

	2	3	-3	-2
	0	2	5	2
1	2	5	2	$0 = r(x)$

This gives quotient $q(x) = 2x^2 + 5x + 2$ and remainder as $r(x) = 0$.

Hence, $2x^3 + 3x^2 - 3x - 2 = (x - 1)(2x^2 + 5x + 2) = (x - 1)(2x + 1)(x + 2)$

Since $2x^3 + 3x^2 - 3x - 2 = 0$, we have $(x - 1)(2x + 1)(x + 2) = 0$ so that $x = 1$ or $x = -\frac{1}{2}$ or $x = -2$

Therefore, the solution set is $\{1, -\frac{1}{2}, -2\}$

Example 5.3.2. Given that $f(x) = x^3 - 3x^2 + x + 2$, solve $f(x) = 0$

Sol: Here, $a_0 = 2$ and $a_n = 1$

coefficients of 2: ± 1, ± 2

coefficients of 1: ± 1

Possible factors: ± 1, ± 2

To find the actual factors, we use the Remainder and Factor Theorems.

$f(1) = (1)^3 - 3(1)^2 + (1) + 2 = 1 - 3 + 1 + 2 = 1 \neq 0$. Hence $x - 1$ is not a factor.

$f(-1) = (-1)^3 - 3(-1)^2 + (-1) + 2 = -1 - 3 - 1 + 2 \neq 0$. Hence $x + 1$ is not a factor.

$f(2) = (2)^3 - 3(2)^2 + (2) + 2 = 8 - 12 + 2 + 2 = 0$. Hence $x - 2$ is a factor.

We can now apply the synthetic division:

	1	-3	1	2
	0	2	-2	-2
2	1	-1	-1	$0 = r(x)$

This gives quotient $q(x) = x^2 - x - 1$ and remainder as $r(x) = 0$ so that

$x^3 - 3x^2 + x + 2 = (x - 2)(x^2 - x - 1)$. Verify that $x^2 - x - 1$ can not be factorised any further.

Since $x^3 - 3x^2 + x + 2 = 0$, we have $(x - 2)(x^2 - x - 1) = 0$ so that $x = 2$

Therefore, the solution set is $\{2\}$

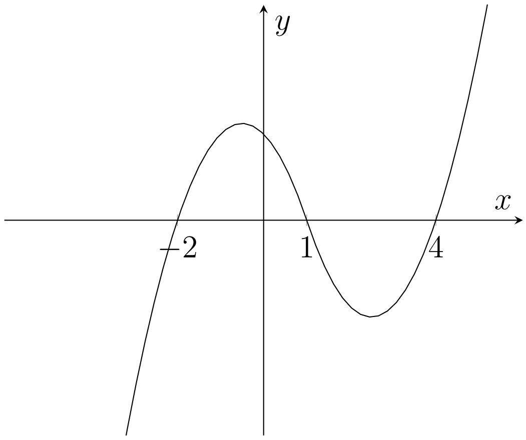
5.4. Graphs of Polynomial Functions

We have looked at ways of sketching the graphs of linear and quadratic functions. We extend our discussion to the sketching of graphs of polynomials of higher degrees.

Example 5.4.1. Sketch the graph of the polynomial function $f(x) = (x-1)(x+2)(x-4)$

Sol: The $y-$intercept is determined by letting $x = 0$. Thus $f(0) = (0-1)(0+2)(0-4) = 8$ To determine the $x-$intercepts, we let $f(x) = 0$ and solve the polynomial equation. Thus, we have

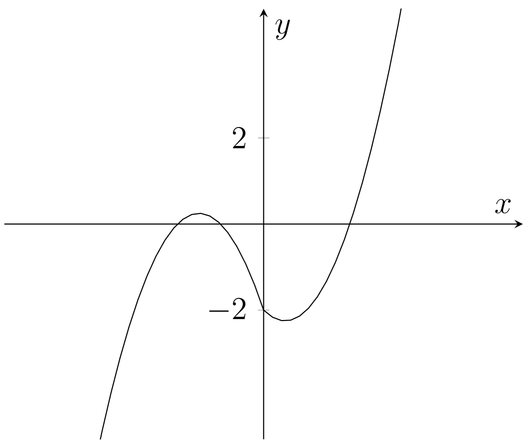
$$\begin{aligned} (x-1)(x+2)(x-4) &= 0 \\ \text{so that} \quad (x-1) = 0, \quad (x+2) = 0 \quad &\text{or} \quad (x-4) = 0 \\ \text{Hence,} \quad x = 1, \quad x = -2 \quad &\text{and} \quad x = 4 \end{aligned}$$



Example 5.4.2. Sketch the graph of the function $f(x) = 2x^3 + 3x^2 - 3x - 2$

Sol: The $y-$intercept is determined by letting $x = 0$. Thus $f(0) = 2(0)^3+3(0)^2-3(0)-2 = -2$ To determine the $x-$intercepts, we let $f(x) = 0$ and solve the polynomial equation. Thus, we have

$$\begin{aligned} 2x^3 + 3x^2 - 3x - 2 &= 0 \\ (x-1)(x+2)(x+\frac{1}{2}) &= 0 \\ \text{so that} \quad (x-1) = 0, \quad (x+2) = 0 \quad &\text{or} \quad (x+\frac{1}{2}) = 0 \\ \text{Hence,} \quad x = 1, \quad x = -2 \quad &\text{and} \quad x = -\frac{1}{2} \end{aligned}$$



The technique in sketching the graphs of polynomial function involves determining the intercept. Further technique of determining the turning points will be discussed after covering calculus.

5.5. Polynomial Inequalities

We have looked at polynomial equations of the form $f(x) = 0$, whose results are just the roots of the polynomial or values of x for which the equation holds. We now look at the polynomial inequalities whose solutions are infinite sets of real numbers.

Example 5.5.1. Solve the inequality $(x+3)(x+1)(x-2) \geq 0$

Sol: First, obtain the critical values: $(x+3)(x+1)(x-2) = 0$ so that $x = -3, \quad x = -1$ and $x = 2$ are the critical points. Constructing the table of signs, we have;

factors	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 2$	$2 < x < \infty$
$x+3$	$-$	$+$	$+$	$+$
$x+1$	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$(x+3)(x+1)(x-2)$	$-$	$+$	$-$	$+$

From the table,

$$SS = \{x| \quad -3 \leq x \leq -1 \text{ or } \quad x \geq 2, \quad x \in \mathbb{R}\} \qquad = \qquad [-3,-1] \cup [2, \infty)$$

Example 5.5.2. Solve the inequality $(x+5)(x+1)(x-5) < 0$

Sol: First, obtain the critical values: $(x+5)(x+1)(x-5) = 0$ so that $x = -5, \quad x = -1$ and $x = 5$ are the critical points. Constructing the table of signs, we have;

factors	$-\infty < x < -5$	$-5 < x < -1$	$-1 < x < 5$	$5 < x < \infty$
$x+5$	$-$	$+$	$+$	$+$
$x+1$	$-$	$-$	$+$	$+$
$x-5$	$-$	$-$	$-$	$+$
$(x+5)(x+1)(x-5)$	$-$	$+$	$-$	$+$

From the table,

$$SS = \{x| \ x < -5 \text{ or } \quad -1 < x < 5, \ x \in \mathbb{R}\} \qquad = \qquad (-\infty, -5) \cup (-1, 5)$$

Example 5.5.3. Solve the polynomial inequality $2x^3 + 3x^2 - 3x - 2 \leq 0$

Sol: We first determine the critical values by factorisation:

$$\begin{aligned} 2x^3 + 3x^2 - 3x - 2 &\leq 0 \\ (x-1)(2x^2 + 5x + 2) &\leq 0 \\ (x-1)(2x+1)(x+2) &\leq 0 \end{aligned}$$

Therefore, the critical values are: $x = 1, \ x = -\frac{1}{2}$ and $x = -2$. We now construct the table of signs as shown below:

factors	$-\infty < x < -2$	$-2 < x < -\frac{1}{2}$	$-\frac{1}{2} < x < 1$	$1 < x < \infty$
$x-1$	$-$	$-$	$-$	$+$
$2x+1$	$-$	$-$	$+$	$+$
$x+2$	$-$	$+$	$+$	$+$
$(x-1)(2x+1)(x+2)$	$-$	$+$	$-$	$+$

From the table,

$$SS = \{x| \ \ x \leq -2 \text{ or } \ -\frac{1}{2} \leq x \leq 1, \ x \in \mathbb{R}\} \qquad = \qquad (-\infty, -2] \cup [-\frac{1}{2}, 1]$$

Example 5.5.4. Solve the inequality $6x^3 - 5x^2 \geq 7x - 4$

Sol: Factorise to obtain the critical values:

$$\begin{aligned} 6x^3 - 5x^2 &\geq 7x - 4 \\ 6x^3 - 5x^2 - 7x + 4 &\geq 0 \\ (3x-4)(2x-1)(x+1) &\geq 0 \end{aligned}$$

Critical values are: $x = \frac{4}{3}$, $x = \frac{1}{2}$, and $x = -1$. We now construct the table of signs as shown below:

factors	$-\infty < x < -1$	$-1 < x < \frac{1}{2}$	$\frac{1}{2} < x < \frac{4}{3}$	$\frac{4}{3} < x < \infty$
$3x-4$	$-$	$-$	$-$	$+$
$2x-1$	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$
$(3x-4)(2x-1)(x+1)$	$-$	$+$	$-$	$+$

From the table,

$$SS = \{x| \ \ -1 \leq x \leq \frac{1}{2} \text{ or } \ x \geq \frac{4}{3}, \ x \in \mathbb{R}\} \qquad = \qquad [-1, \frac{1}{2}] \cup [\frac{4}{3}, \infty)$$

Example 5.5.5. Solve the inequality $x(2-x)(3+x) < 0$

Sol: Critical values are: $x = 0$, $x = 2$ and $x = -3$

factors	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 2$	$2 < x < \infty$
x	$-$	$-$	$+$	$+$
$2-x$	$+$	$+$	$+$	$-$
$3+x$	$-$	$+$	$+$	$+$
$x(2-x)(3+x)$	$+$	$-$	$+$	$-$

From the table,

$$SS = \{x| \ \ -3 < x < 0 \text{ or } \ x > 2, \ x \in \mathbb{R}\} \qquad = \qquad (-3, 0) \cup (2, \infty)$$

5.6. Partial Fractions

In order to resolve an algebraic rational fraction into its partial fractions, we must factorise the denominator completely. The numerator must be at least one degree less than the denominator, otherwise, we divide. Consider the addition of two fractions below;

$$\begin{aligned}\frac{2}{x+1}+\frac{1}{x-2} &= \frac{2(x-2)+(x+1))}{(x+1)(x-2)}\\ &= \frac{2x-4+x+1}{(x+1)(x-2)}\\ &= \frac{3x-3}{(x+1)(x-2)}\end{aligned}$$

We say, the partial fraction decomposition of $\frac{3x-3}{(x+1)(x-2)}$ gives the two fractions $\frac{2}{x+1}$ and $\frac{1}{x-2}$. Partial fraction decomposition involves the reverse process of the above process, ie it involves "breaking down" a seemingly complex algebraic fraction into its simpler partial fractions. There are basically three types of partial fraction and the form of partial fraction used is summarized in the table below. The following are the three types we will consider:

1. The first type involves a rational function whose denominator can be factorised completely into linear terms.
2. The second type involves a rational function whose denominator can factorised, but some terms are repeating.
3. The third involves a rational function whose denominator contains an irreducible quadratic function, ie a quadratic that can not be factorised into linear terms

Type	Denominator containing	Expression	Form of Partial Fraction
1	Linear Factors	$\frac{f(x)}{(x-a)(x+b)(x+c)}$	$\frac{A}{x-a}+\frac{B}{x+b}+\frac{C}{x+c}$
2	Repeating Factors	$\frac{f(x)}{(x+a)(x+b)^3}$	$\frac{A}{(x+a)}+\frac{B}{(x+b)}+\frac{C}{(x+b)^2}+\frac{D}{(x+b)^3}$
3	Quadratic Factors	$\frac{f(x)}{(ax^2+bx+c)(x-d)}$	$\frac{Ax+B}{ax^2+bx+c}+\frac{D}{x-d}$

Example 5.6.1. Resolve the fraction $\frac{11-3x}{x^2+2x-3}$ into its partial fractions

Sol: Factorise the denominator completely and use the type 1 decomposition:
This gives $x^2+2x-3=(x+3)(x-1)$ so that $\frac{11-3x}{x^2+2x-3}=\frac{11-3x}{(x+3)(x-1)}$. Therefore,

$$\begin{aligned}\frac{11-3x}{(x+3)(x-1)} &= \frac{A}{x+3}+\frac{B}{x-1}\\ &= \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}\end{aligned}$$

Equating the numerators, we have: $11-3x=A(x-1)+B(x+3)$ (i)

Our task is to determine the values of A and B using (i). When $x = 1$, $11 - 3(1) = A(1-1) + B(1+3)$ which gives $B = 2$

When $x = -3$, we have $11 - 3(-3) = A(-3-1) + B(-3+3)$ which gives us $A = -5$

$$\text{Therefore, } \frac{11-3x}{x^2+2x-3} = \frac{2}{x-1} - \frac{5}{x+3}$$

Example 5.6.2. Decompose $\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$ into partial fractions.

Sol: The denominator is already factorised completely.

Let $\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)} = \frac{A}{(x+1)} + \frac{B}{x-2} + \frac{C}{x+3} = \frac{A(x-2)(x+3)+B(x+1)(x+3)+C(x-1)(x+2)}{(x+1)(x-2)(x+3)}$

Equating the numerators, we have:

$$2x^2 - 9x - 35 = A(x-2)(x+3) + B(x+1)(x+3) + C(x+1)(x-2)$$

When $x = 2$, we have

$$\begin{aligned} 2(2)^2 - 9(2) - 35 =& A(2-2)(2+3) + B(2+1)(2+3) + C(2+1)(2-2) \\ -45 =& 15B \\ B =& -3 \end{aligned}$$

When $x = -3$, we have

$$\begin{aligned} 2(-3)^2 - 9(-3) - 35 =& A(-3-2)(-3+3) + B(-3+1)(-3+3) + C(-3+1)(-3-2) \\ 10 =& 5C \\ C =& 2 \end{aligned}$$

When $x = -1$, we have

$$\begin{aligned} 2(-1)^2 - 9(-1) - 35 =& A(-1-2)(-1+3) + B(-1+1)(-1+3) + C(-1+1)(-1-2) \\ -24 =& -6C \\ C =& 4 \end{aligned}$$

So that

$$\frac{11-3x}{x^2+2x-3} = \frac{4}{(x+1)} - \frac{3}{x-2} + \frac{2}{x+3}$$

Example 5.6.3. Resolve $\frac{5x^2-2x-19}{(x+3)(x-1)^2}$ into partial fractions

Sol: This involves a denominator with a repeating factor. Thus,

let $\frac{5x^2-2x-19}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2+B(x+3)(x-1)+C(x+3)}{(x+3)(x-1)^2}$

Hence, by algebraic addition,

$$5x^2 - 2x - 19 = A(x-1)^2 + B(x+3)(x-1) + C(x+3)$$

When $x = 1$, we have

$$\begin{aligned} 5(1)^2 - 2(1) - 19 =& A(1-1)^2 + B(1+3)(1-1) + C(1+3) \\ -16 =& 4C \\ C =& -4 \end{aligned}$$

When $x = -3$

$$\begin{aligned} 5(-3)^2 - 2(-3) - 19 =& A(-3-1)^2 + B(-3+3)(-3-1) + C(-3+3) \\ 32 =& 16A \\ A =& 2 \end{aligned}$$

When $x = 0$

$$\begin{aligned} 5(0)^2 - 2(0) - 19 =& A(0-1)^2 + B(0+3)(0-1) + C(0+3) \\ -19 =& A - 3B + 3C \\ 3B =& 19 + A + 3C \\ 3B =& 19 + (2) + 3(-4) \\ 3B =& 9 \\ B =& 3 \end{aligned}$$

Hence,

$$\frac{5x^2 - 2x - 19}{(x+3)(x-1)^2} = \frac{2}{x+3} + \frac{3}{x-1} - \frac{4}{(x-1)^2}$$

Example 5.6.4. Decompose the rational function $\frac{7x^2+5x+13}{(x^2+2)(x+1)}$ into partial fractions.

Sol: We have an irreducible quadratic in the denominator.

Let $\frac{7x^2+5x+13}{(x^2+2)(x+1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x+1} = \frac{(Ax+B)(x+1)+C(x^2+2)}{(x^2+2)(x+1)}$

This gives us:

$$7x^2 + 5x + 13 = (Ax+B)(x+1) + C(x^2+2)$$

When $x = -1$, we have

$$\begin{aligned} 7(-1)^2 + 5(-1) + 13 =& (A(-1) + B)(-1+1) + C((-1)^2 + 2) \\ 15 =& 3C \\ C =& 5 \end{aligned}$$

When $x = 0$, we have

$$\begin{aligned} 7(0)^2 + 5(0) + 13 =& (A(0) + B)(0+1) + C((0)^2 + 2) \\ 13 =& B + 2C \\ B =& 13 - 2C \\ B =& 13 - 2(5) \\ B =& 3 \end{aligned}$$

When $x = 1$, we have

$$\begin{aligned} 7(1)^2 + 5(1) + 13 =& (A(1) + B)(1 + 1) + C((1)^2 + 2) \\ 25 =& 2(A + 3) + 5(3) \\ 2A =& 25 - 6 - 15 \\ 2A =& 4 \\ A =& 2 \end{aligned}$$

Hence, decomposition is given by

$$\frac{7x^2 + 5x + 13}{(x^2 + 2)(x + 1)} \quad = \quad \frac{2x + 3}{x^2 + 2} + \frac{5}{x + 1}$$

Example 5.6.5. Resolve $\frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}$ into partial fractions

Sol: The denominator contains a repeating term and an irreducible quadratic.

Let $\quad \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)} \quad = \quad \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3} \quad = \quad \frac{Ax(x^2+3)+B(x^2+3)+x^2(Cx+D)}{x^2(x^2+3)}$

$$\begin{aligned} 3 + 6x + 4x^2 - 2x^3 =& Ax(x^2 + 3) + B(x^2 + 3) + x^2(Cx + D) \\ =& Ax^3 + 3Ax + Bx^2 + 3B + Cx^3 + Dx^2 \\ =& (A + C)x^3 + (B + D)x^2 + 3Ax + 3B \end{aligned}$$

$$\text{Hence,} \quad 3 + 6x + 4x^2 - 2x^3 \quad = \quad (A + C)x^3 + (B + D)x^2 + 3Ax + 3B$$

Using the equality of polynomials and equating the like terms, we have:

$$(A + C)x^3 = -2x^3 \quad \Longrightarrow \quad A + C = -2 \qquad \text{(i)}$$

$$(B + D)x^2 = 4x^2 \quad \Longrightarrow \quad B + D = 4 \qquad \text{(ii)}$$

$$3Ax = 6x \quad \Longrightarrow \quad 3A = 6 \qquad \text{(iii)}$$

$$3B = 3 \quad \Longrightarrow \quad 3B = 3 \qquad \text{(iv)}$$

From (iv) we get $B = 1$, and from (ii), $D = 4 - B = 4 - 1 = 3$

From (iii) we get $A = 2$, and from (i), $C = -2 - A = -2 - 2 = -4$

Therefore, we get

$$\frac{3 + 6x + 4x^2 - 2x^3}{x^2(x^2 + 3)} \quad = \quad \frac{2}{x} + \frac{1}{x^2} + \frac{3 - 4x}{x^2 + 3}$$

6
Rational Functions

6.1. Introduction

Recall that a rational number is a number that can be written in the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$. In this section, we will now consider rational functions.

Definition 6.1.1. *A rational function $f(x)$, is a function of the form*

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are functions of x with $q(x) \neq 0$ for all values of $x \in D_f$

Example 6.1.1. Examples of rational functions are: $f(x) = \frac{x+2}{x-5}$ and $g(x) = \frac{2x}{x^2+x-7}$

We will be concerned with the sketching of the graphs, finding domains and ranges of these functions. We will also look at the equations and inequalities involving rational functions.

6.2. Graphs of Rational Functions

To sketch the graph of a rational function, we need to determine the domain, the range, intercepts and asymptotes if they exists. We have already discussed methods of determining the domain and range of a rational function. The $X-$intercept is the value of x where the graph cuts the $X-$axis. It can be found by letting $y = 0$ in the equation $y = f(x)$. The $Y-$intercept is the value of y where the graph cuts the $Y-$axis. It can be found by letting $x = 0$ in the equation $y = f(x)$.

An asymptote of a graph is a line graph such that the graph does not touch nor cut it. We have vertical, horizontal and slant asymptotes. Vertical asymptotes are determined by values of x for which the function is not defined, while horizontal asymptotes are determined by values of y for which the function is not defined.

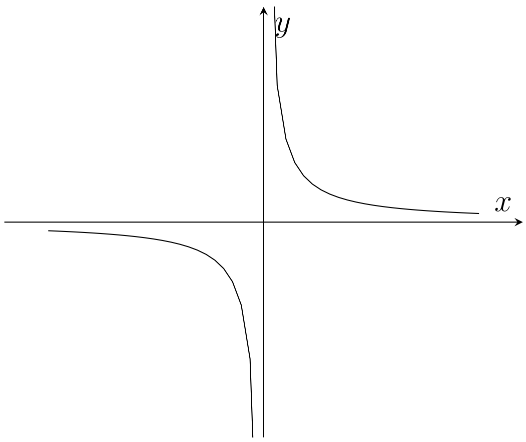
Example 6.2.1. Sketch the graph of $f(x) = \frac{1}{x}$

Sol: For this function, verify that $D_f = \{x|\ x \neq 0,\ x \in \mathbb{R}\}$ and $R_f = \{y|\ y \neq 0,\ y \in \mathbb{R}\}$

From the domain, since $x \neq 0$, we conclude that the line $x = 0$ is a vertical asymptote.

From the range, since $y \neq 0$, we conclude that the line $y = 0$ is a horizontal asymptote

We have no intercepts. The sketch is shown below



Example 6.2.2. Given $f(x) = \frac{2}{x+1}$,

i) find the domain of f

ii) find the range of f

iii) sketch the graph of f

iv) Write down the equation of the asymptotes

Sol: This is an example of a rational function

i) Domain: we need $x + 1 \neq 0 \Longrightarrow x \neq -1$. Hence, $D_f = \{x | \ x \neq -1, \quad x \in \mathbb{R}\}$

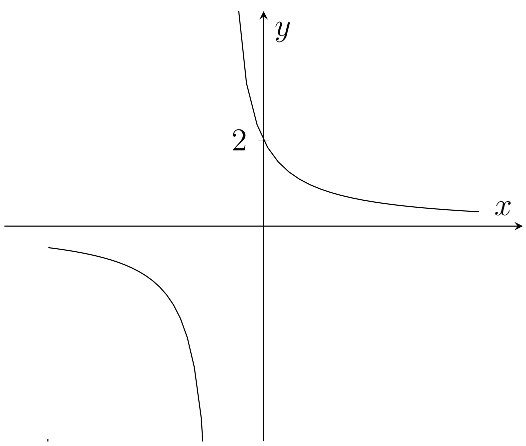
ii) Range: we make x the subject of the formula. Thus, $y = \frac{2}{x+1} \Longrightarrow x = \frac{2-y}{y}$

Hence, $R_f = \{y | \ y \neq 0, \quad y \in \mathbb{R}\}$

iii) Sketch: verify that the $Y-$intercept is $y = 2$ and the $X-$intercept does not exists

From the domain, the vertical asymptote is $x = -1$.

From the range, the horizontal asymptote is $y = 0$ The graph is shown below:



iv) vertical asymptote: $x = -1$ and horizontal asymptote: $y = 0$

Example 6.2.3. Given the function $f(x) = \frac{2x+3}{5x-4}$

i) find the domain of f

ii) find the range of f

iii) sketch the graph of f

Sol: This is a rational function

i) Domain: we need $5x - 4 \neq 0 \Longrightarrow x \neq \frac{4}{5}$. Hence, $D_f = \{x| \ x \neq \frac{4}{5}, \quad x \in \mathbb{R}\}$

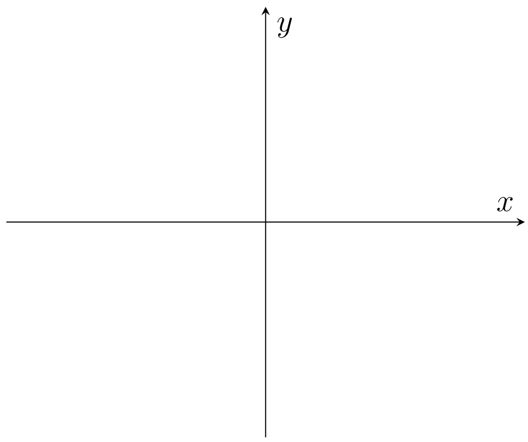
ii) Range: we make x the subject of the formula. Thus, $y = \frac{2x+3}{5x-4} \Longrightarrow x = \frac{3+4y}{5y-2}$

Hence, $R_f = \{y| \ y \neq \frac{2}{5}, \quad y \in \mathbb{R}\}$

iii) Sketch: verify that the $Y-$intercept is $y = -\frac{3}{4}$ and the $X-$intercept is $-\frac{3}{2}$

From the domain, the vertical asymptote is $x = \frac{4}{5}$.

From the range, the horizontal asymptote is $y = \frac{2}{5}$ The graph is shown below:



Example 6.2.4. Given the function $f(x) = \frac{1}{x^2+5x-6}$

i) find the domain of $f(x)$

ii) find the range of $f(x)$

iii) sketch the graph of $f(x)$

Sol: This is also a rational function

i) Domain: we need $x^2 + 5x - 6 \neq 0$. The solution set is the required domain.

$$\begin{aligned} x^2 + 5x - 6 &\neq 0 \\ x^2 - x + 6x - 6 &\neq 0 \\ x(x-1) + 6(x-1) &\neq 0 \\ (x-1)(x+6) &\neq 0 \end{aligned}$$

We have $x \neq 1$ and $x \neq -6$. Therefore,

$$D_f = \{x |\ x \neq 1 \text{ and } x \neq -6, \quad x \in \mathbb{R}\}$$

ii) Range: Let $y = \frac{1}{x^2+5x-6}$ so that $y(x^2+5x-6) = 1 \Longrightarrow yx^2 + 5yx - 6y - 1 = 0$

Note that this is a quadratic with $a = y$, $b = 5y$ and $c = -6y - 1$.

we need $b^2 - 4ac \geq 0$

$$\begin{aligned}(5y)^2 - 4(y)(-6y-1) \geq& 0 \\ 25y^2 + 24y^2 + 4y \geq& 0 \\ 49y^2 + 4y \geq& 0 \\ y(49y+4) \geq& 0\end{aligned}$$

Critical values: we have $y = 0$ and $y = -\frac{4}{49}$

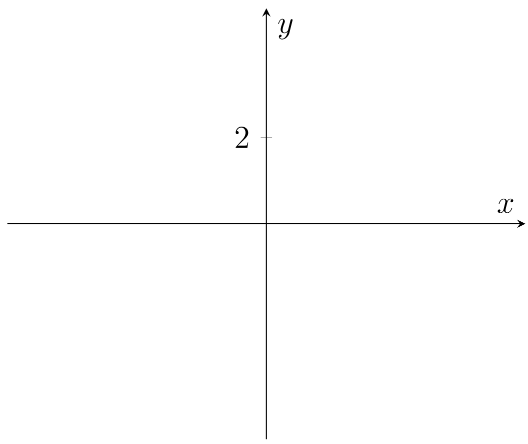
	$-\infty < y < -\frac{4}{49}$	$-\frac{4}{49} < y < 0$	$0 < y < \infty$
y	$-$	$-$	$+$
$49y + 4$	$-$	$+$	$+$
$y(49y + 4)$	$+$	$-$	$+$

From the table, $R_f = \left(-\infty, -\frac{4}{49}\right) \cup (0, \infty)$

iii) Sketch: verify that the $Y-$intercept is $y = -\frac{1}{6}$ and the $X-$intercepts do not exist.

From the domain, the vertical asymptote is $x = 1$ and $x = -6$.

From the range, the horizontal asymptote is $y = 0$ and $y = -\frac{4}{49}$. Graph shown below:



Example 6.2.5. Given the function $f(x) = \frac{3x-9}{x^2-x-2}$,

i) find the domain

ii) find the range

iii) sketch the graph

Sol: Exercise

6.3. Equations and Inequalities

To solve the equations involving rational function, the basic thing is to cross multiply. The following examples demonstrate the approach.

Example 6.3.1. Solve the equation $\frac{x+1}{2x-5} = 1$

Sol: We cross multiply;

$$\begin{aligned} \frac{x+1}{2x-5} &= 1 \\ 2x-5 &= x+1 \\ 2x-x &= 5+1 \\ x &= 6 \end{aligned}$$

Hence, $SS = \{6\}$ is the solution set.

Example 6.3.2. Solve the equation $\frac{2x-7}{x+3} = 0$

Sol: If $\frac{2x-7}{x+3} = 0$, then the numerator is zero. Hence, $2x - 7 = 0 \Longrightarrow x = \frac{7}{2}$

Example 6.3.3. Solve the equation $\frac{1}{x^2+9x+7} = \frac{1}{x}$

Sol: We can cross multiply

$$\begin{aligned} \frac{1}{x^2+9x+7} &= \frac{1}{x} \\ x^2+9x+7 &= x \\ x^2+9x+7-x &= 0 \\ x^2+8x+7 &= 0 \\ (x+1)(x+7) &= 0 \end{aligned}$$

Hence, $x + 1 = 0 \Longrightarrow x = -1$ and $x + 7 = 0 \Longrightarrow x = -7$. Thus, we have $SS = \{-1, -7\}$

To solve the rational inequality, do NOT CROSS MULTIPLY. The following examples demonstrate the technique to be used.

Example 6.3.4. Solve the inequality $\frac{2x+3}{x-4} < 3$

Sol: We do not cross multiply.

$$\begin{aligned} \frac{2x+3}{x-4} &< 3 \\ \frac{2x+3}{x-4} - 3 &< 0 \\ \frac{2x+3-3(x-4)}{x-4} &< 0 \\ \frac{2x-3x+3+12}{x-4} &< 0 \\ \frac{-x+15}{x-4} &< 0 \end{aligned}$$

At this point, we get the critical points:

from the numerator, $-x + 15 = 0 \Longrightarrow x = 15$ is a critical point

from the denominator, $x - 4 = 0 \Longrightarrow x = 4$ is a critical point

	$-\infty < x < 4$	$4 < x < 15$	$15 < x < \infty$
$-x + 15$	$+$	$+$	$-$
$x - 4$	$-$	$+$	$+$
$\frac{-x+15}{x-4}$	$-$	$+$	$-$

From the table,

$$SS = (-\infty, 4) \cup (15, \infty)$$

Example 6.3.5. Find the solution set to the inequality $\frac{x-1}{x^2+6x+5} \leq 0$

Sol: We need to find the critical values;

$$\begin{aligned}
\frac{x-1}{x^2+6x+5} &\leq 0 \\
\frac{x-1}{x^2+x+5x+5} &\leq 0 \\
\frac{x-1}{x(x+1)+5(x+1)} &\leq 0 \\
\frac{x-1}{(x+1)(x+5)} &\leq 0
\end{aligned}$$

At this point, we get the critical points:

from the numerator, $x - 1 = 0 \Longrightarrow x = 1$ is a critical point

from the denominator, $(x + 1)(x + 5) = 0 \Longrightarrow x = -1$ and $x = -5$ are critical points

factors	$-\infty < x < -5$	$-5 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
$x - 1$	$-$	$-$	$-$	$+$
$x + 1$	$-$	$-$	$+$	$+$
$x + 5$	$-$	$+$	$+$	$+$
$\frac{x-1}{(x+1)(x+5)}$	$-$	$+$	$-$	$+$

From the table,

$$SS = (-\infty, -5) \cup (-1, 1]$$

7

Radical Functions

7.1. Introduction

In this section, we discuss another type of functions called radical functions.

Definition 7.1.1. *A radical function in x variable, is a function of the form*

$$f(x) = k\sqrt{q(x)} + h$$

where $q(x)$ is some function of x. k and h are constants

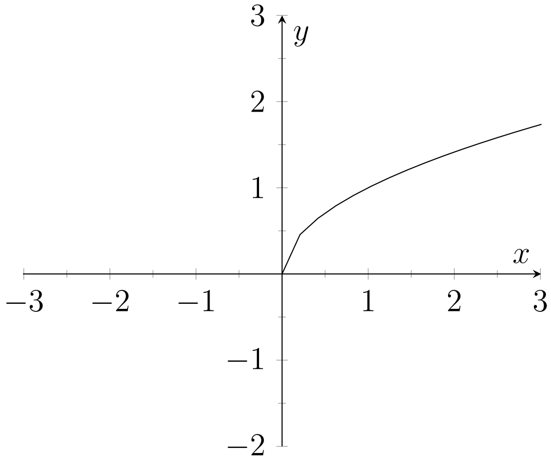
We will examine some standard radical functions through examples. Here, we will restrict our discussion to cases where $q(x)$ is either linear or quadratic.

Radicals of the form $k\sqrt{ax+b}+h$

Example 7.1.1. Sketch the graph of the function $f(x) = \sqrt{x}$.

Sol: This is one of the standard radical functions. Before we sketch, let us determine the domain. Recall that the square root of a negative number is not real. Hence, we only need positive values of x for this to hold as a function. Therefore

$$D_f = \{x|\ x \geq 0, x \in \mathbb{R}\} \quad = \quad [0, \infty)$$



From the graph, we can see that

$$R_f = \{y| \ \ y \geq 0, x \in \mathbb{R}\} = [0, \infty)$$

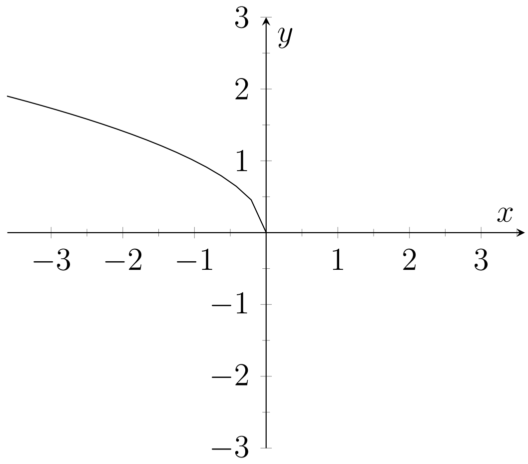
Example 7.1.2. Sketch the graph of the function $f(x) = \sqrt{-x}$.

Sol: This is another standard radical function. To determine the domain, recall that the square root of a negative number is not real. Hence, we need $-x \geq 0 \Longrightarrow x \leq 0$. Thus, we need only the negative values of x for this to hold as a function. Therefore

$$D_f = \{x| \ x \leq 0, x \in \mathbb{R}\}$$

OR

$$D_f = (-\infty, 0]$$



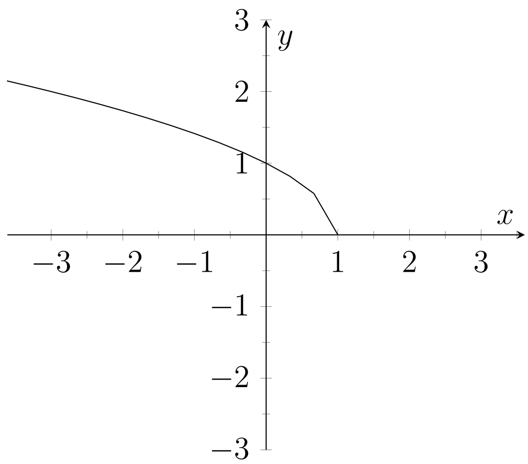
From the graph, we can see that

$$R_f = \{y| \ \ y \geq 0, x \in \mathbb{R}\} = [0, \infty)$$

Example 7.1.3. Sketch the graph of the function $f(x) = \sqrt{1-x}$.

Sol: For the domain, we need $1 - x \geq 0 \Longrightarrow x \leq 1$. Hence,

$$D_f = \{x| \ x \leq 1, \ x \in \mathbb{R}\} = (-\infty, 1]$$



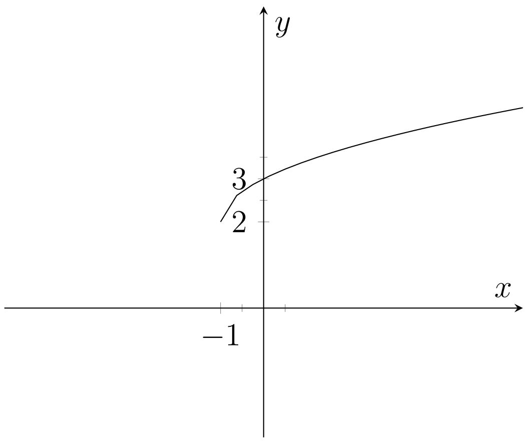
From the graph, we can see that

$$R_f = \{y| \ \ y \geq 0, x \in \mathbb{R}\} = [0, \infty)$$

Example 7.1.4. Sketch the graph of the function $f(x) = \sqrt{x+1} + 2$.

Sol: For the domain, we need $x + 1 \geq 0 \Longrightarrow x \geq -1$. Hence,

$$D_f = \{x| \ x \geq -1, \ x \in \mathbb{R}\} = [-1, \infty)$$



From the graph, we can see that

$$R_f = \{y| \ \ y \geq 2, x \in \mathbb{R}\} = [1, \infty)$$

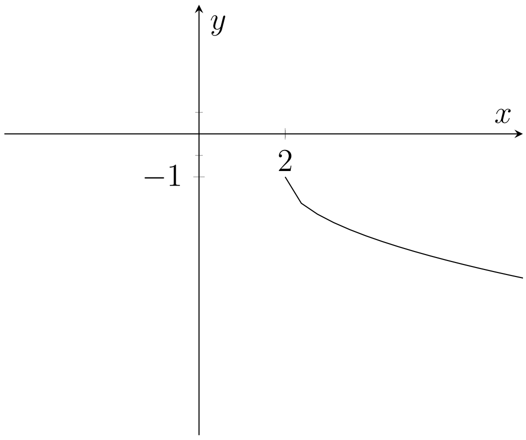
Note 7.1.1. If $f(x) = k\sqrt{ax+b} + h$. Then

- k determines whether the function is above or below the $x-$axis. If k is positive, then the graph is above $x-$axis, if k is negative, then the graph is below the $x-$axis.
- h is the vertical shift. If h is positive, then the graph shifts h units upwards. If h is negative, then the graph shifts h units downward.

Example 7.1.5. Find the domain and sketch the graph of $f(x) = -\sqrt{x-2} - 1$

Sol: For the domain, we need $x - 2 \geq 0 \Longrightarrow x \geq 2$. Hence,

$$D_f = \{x| \ x \geq 2, \ x \in \mathbb{R}\} = [2, \infty)$$



From the graph, we can see that

$$R_f = \{y|\ \ y \leq -1, x \in \mathbb{R}\} = (-\infty, -1]$$

Radicals of the form $k\sqrt{ax^2+bx+c}+h$

Here, the domain is all values of x such that $ax^2+bx+c \geq 0$. It faces up if $k > 0$ and faces down if $k < 0$.

Example 7.1.6. Find the domain, range and sketch the graph of the function $f(x) = \sqrt{x^2+6x+5}$

Sol: For the domain, we need $x^2+6x+5 \geq 0$. Hence, the domain is just the solution set of the inequality $x^2+6x+5 \geq 0$. Thus,

$$\begin{aligned} x^2+6x+5 &\geq 0 \\ x^2+x+5x+5 &\geq 0 \\ x(x+1)+5(x+1) &\geq 0 \\ (x+5)(x+1) &\geq 0 \end{aligned}$$

Critical values: $x = -5$ and $x = -1$

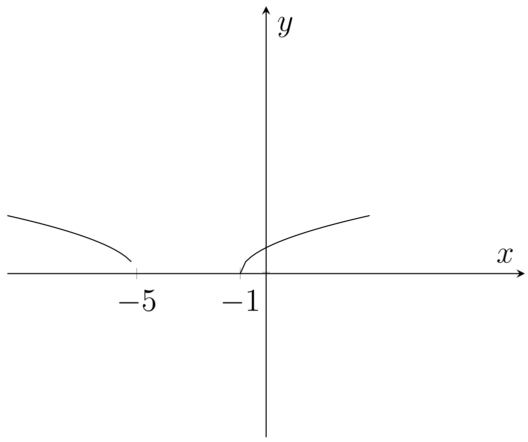
	$-\infty < x < -5$	$-5 < x < -1$	$-1 < x < \infty$
$x+5$	$-$	$+$	$+$
$x+1$	$-$	$-$	$+$
$(x+5)(x+1)$	$+$	$-$	$+$

The solution set is

$$SS = \{x|\ \ x \leq -5 \text{ or } x \geq -1,\quad x \in \mathbb{R}\}$$

. Therefore,

$$D_f = \{x|\ \ x \leq -5 \text{ or } x \geq -1,\quad x \in \mathbb{R}\} = (-\infty, -5] \cup [-1, \infty)$$



From the graph, we can see that

$$R_f = \{y| \ \ y \geq 2, x \in \mathbb{R}\} = [1, \infty)$$

7.2. Equations and Inequalities

We need to remember that the square root symbol without a negative, denotes a positive square root. To remove the square root symbol requires squaring on both sides. At times, this process may be repeated.

Example 7.2.1. Solve the equation $\sqrt{3x+7} - x = 1$

Sol: This equation involves a radical.

$$\begin{aligned} \sqrt{3x+7} - x =& 1 \\ \sqrt{3x+7} =& x+1 \\ \left(\sqrt{3x+7}\right)^2 =& (x+1)^2 \\ 3x+7 =& x^2+2x+1 \\ x^2 - x - 6 =& 0 \\ (x+2)(x-3) =& 0 \end{aligned}$$

Hence, we have $x + 2 = 0 \implies x = -2$ and $x - 3 = 0 \implies x = 3$. We discard -2 since it does not hold when we substitute it in the equation. Hence,

$$SS = \{3\}$$

Example 7.2.2. Solve the expression $\sqrt{6x+7} - \sqrt{3x+3} = 1$

Sol: This is an equation involving radicals.

$$\begin{aligned}\sqrt{6x+7}-\sqrt{3x+3} =&1\\ \sqrt{6x+7} =&1+\sqrt{3x+3}\\ \left(\sqrt{6x+7}\right)^2 =&\left(1+\sqrt{3x+3}\right)^2\\ 6x+7 =&1+2\sqrt{3x+3}+3x+3\\ 3x+3 =&2\sqrt{3x+3}\\ (3x+3)^2 =&\left(2\sqrt{3x+3}\right)^2\\ 9x^2+18x+9 =&4(3x+3)\\ 9x^2+6x-3 =&0\\ 3x^2+2x-1 =&0\\ 3x^2+3x-x-1 =&0\\ 3x(x+1)-1(x+1) =&0\\ (3x-1)(x+1) =&0\end{aligned}$$

Hence, $x = 1-$ or $x = \frac{1}{3}$ implying that $SS = \{-1, \frac{1}{3}\}$

Example 7.2.3. Given $\frac{x}{\sqrt{x+1}} + \frac{2x}{\sqrt{x+5}} = 0$ find the solution set to the equation.

Sol: This equation involves positive radical or positive square root.

$$\begin{aligned}\frac{x}{\sqrt{x+1}}+\frac{2x}{\sqrt{x+5}} =&0\\ \frac{x}{\sqrt{x+1}} =&-\frac{2x}{\sqrt{x+5}}\\ \left(\frac{x}{\sqrt{x+1}}\right)^2 =&\left(-\frac{2x}{\sqrt{x+5}}\right)^2\\ \frac{x^2}{x+1} =&\frac{4x^2}{x+5}\\ \frac{x^2}{x+1}-\frac{4x^2}{x+5} =&0\\ x^2\left(\frac{1}{x+1}-\frac{4}{x+5}\right) =&0\\ \frac{x^2(1-3x)}{(x+1)(x+5)} =&0\end{aligned}$$

Either $x^2 = 0 \Longrightarrow x = 0$ or $1 - 3x = 0 \Longrightarrow x = \frac{1}{3}$. Hence, we have

$$SS = \{0, \frac{1}{3}\}$$

Example 7.2.4. Given the inequality $\frac{1}{\sqrt{x}} < \frac{2}{\sqrt{x+27}}$, find the solution set

Sol: This is an inequality involving positive square root.

$$
\begin{aligned}
\frac{1}{\sqrt{x}} &< \frac{2}{\sqrt{x+27}}\\
\left(\frac{1}{\sqrt{x}}\right)^2 &< \left(\frac{2}{\sqrt{x+27}}\right)^2\\
\frac{1}{x} &< \frac{4}{x+27}\\
\frac{1}{x}-\frac{4}{x+27} &< 0\\
\frac{27-3x}{x(x+27)} &< 0\\
\frac{3(9-x)}{(x-0)(x+27)} &< 0
\end{aligned}
$$

Critical Values: From the denominator, $x = 0$ and $x = -27$. From the numerator, $x = 9$

factors	$-\infty < x < -27$	$-27 < x < 0$	$0 < x < 9$	$9 < x < \infty$
x	$-$	$-$	$+$	$+$
$9-x$	$+$	$+$	$+$	$-$
$x+27$	$-$	$+$	$+$	$+$
$x(9-x)(x+27)$	$+$	$-$	$+$	$-$

From the table, $x \in (-27, 0) \cup (9, \infty)$, from $\sqrt{x}$, we need $x \geq 0 \Longrightarrow x \in [0, \infty)$ and from $\sqrt{x+27}$, we need $x + 27 \geq 0 \Longrightarrow x \geq -27 \Longrightarrow x \in [-27, \infty)$. Hence, $SS = (9, \infty)$

Example 7.2.5. Solve the inequality $\sqrt{2} - \sqrt{x+6} \leq -\sqrt{x}$

Sol: We rewrite the expression so that we have only positive terms, then square.

$$
\begin{aligned}
\sqrt{2}-\sqrt{x+6} &\leq -\sqrt{x}\\
\sqrt{2}+\sqrt{x} &\leq \sqrt{x+6}\\
\left(\sqrt{2}+\sqrt{x}\right)^2 &\leq \left(\sqrt{x+6}\right)^2\\
2+2\sqrt{2}\sqrt{x}+x &\leq x+6\\
2\sqrt{2x} &\leq 4\\
\sqrt{2x} &\leq 2\\
\left(\sqrt{2x}\right)^2 &\leq 2^2\\
2x &\leq 4\\
x &\leq 2
\end{aligned}
$$

From this expression, we need $x \leq 2$ meaning that $x \in (-\infty, 2]$

From $\sqrt{x+6}$, we need $x + 6 \geq 0 \Longrightarrow x \geq -6$ meaning that $x \in [-6, \infty)$

From $\sqrt{x}$, we need $x \geq 0$ meaning that $x \in [0, \infty)$

Therefore, $SS = (-\infty, 2] \cap [-6, \infty) \cap [0, \infty) = [0, 2]$

8
Modular Functions

8.1. Introduction

We now look at another type of function called the modular function. Let us have some basic defination

Definition 8.1.1. *The modulus of a real number $x \in \mathbb{R}$ denoted $|x|$ is defined as*

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

The modulus of a number is just the absolute value of that number. The following examples show the meaning of a modulus.

Example 8.1.1. The modulus of 2 is denoted by $|2|$. By definition, $|2| = 2$ since $2 > 0$

The modulus of -2 is denoted by $|-2|$. By definition, $|-2| = -(-2) = 2$ since $-2 < 0$

Example 8.1.2. Determine the following;

i) $|-10|$ ii) $\left|-\frac{2}{7}\right|$ iii) $|0.356|$ iv) $\left|\sqrt{2}\right|$ v) $\left|-\sqrt{2}\right|$ vi) $|0|$ vii) $|10|$

Sol:

i) $|-10| = -(-10) = 10$ ii) $\left|-\frac{2}{7}\right| = -(-\frac{2}{7}) = \frac{2}{7}$ iii) $|0.356| = 0.356$

iv) $\left|\sqrt{2}\right| = \sqrt{2}$ v) $\left|-\sqrt{2}\right| = -\left(-\sqrt{2}\right) = \sqrt{2}$ vi) $|0| = 0$

vi) $|10| = 10$

NOTE: For any real number $x \in \mathbb{R}$, $|-x| = |x|$

Example 8.1.3. Evaluate the following expressions

i) $|2-7|+|3-1|$ ii) $||-3|-|-9||$ iii) $||2-6|-|1-9||$ iv) $|3-6|-|-2+4|+|-2-3|$

Sol: Exercise

Having defined the modulus of a real number, we can extend the definition to the real valued functions.

Definition 8.1.2. *Let $f(x)$ be a real valued function. The modulus of $f(x)$ denoted by $|f(x)|$ is defined as*

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0; \\ -f(x) & \text{if } f(x) < 0. \end{cases}$$

Example 8.1.4. If $f(x) = x + 2$, then $|f(x)| = |x + 2|$. Similarly, if $f(x) = \tan x$, then $|f(x)| = |\tan x|$.

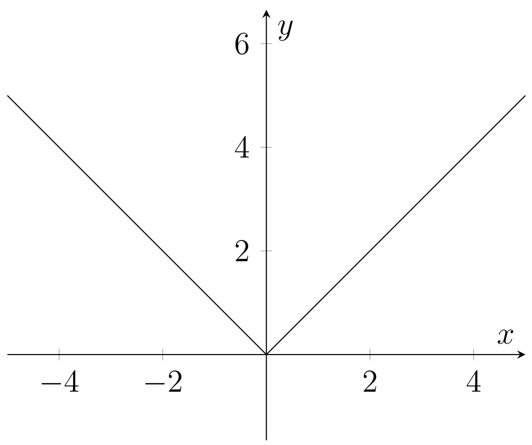
8.2. Graphs of Modular Functions

We now look at graphs of the Modular functions. To sketch the graph of $y = |f(x)| + d$, the following steps can be taken:

- sketch the graph of $y = f(x)$
- reflect the negative part of $y = f(x)$ in the $X-$axis.
- outline the positive part only, to get the graph of $y = |f(x)|$
- shift the graph of $y = |f(x)|$, d units up if $d > 0$ to obtain $y = |f(x)| + d$.
- shift the graph of $y = |f(x)|$, d units down if $d < 0$ to obtain $y = |f(x)| + d$.

Example 8.2.1. Sketch the graph of $y = |x|$. Hence or otherwise, state the domain and the range of this function.

Sol: This graph is obtained by reflecting the negative part of $y = x$ in the $x-$axis. The graph is shown below.



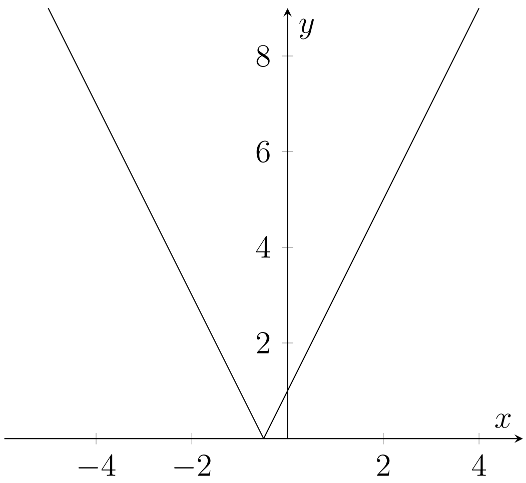
From the graph, $D_f = (-\infty, \infty)$ and $R_f = [0, \infty)$

Example 8.2.2. Sketch the graph of $y = |2x + 1|$. Hence or otherwise, state the domain and the range of this function.

Sol: We need to determine the points where the graph cuts the $x-$axis and the $y-$axis.

When $x = 0$, we get $y = 1$ so that the graph cuts the $y-$axis at $y = 1$.

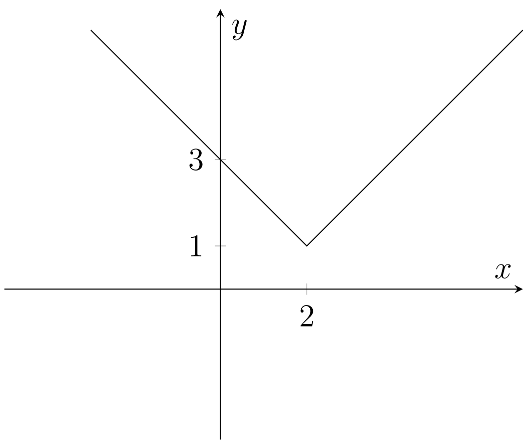
When $y = 0$, then $x = -\frac{1}{2}$. Therefore, the graph is given by.



From the graph, $D_f = (-\infty, \infty)$ and $R_f = [0, \infty)$

Example 8.2.3. Sketch the graph of $y = |x - 2| + 1$. Hence or otherwise, state the domain and the range of this function.

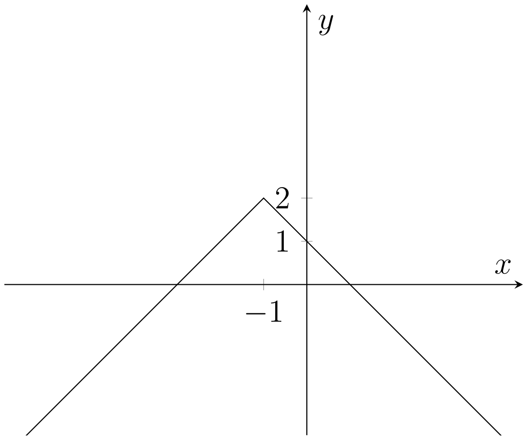
Sol: The graph is shown below.



From the graph, $D_f = (-\infty, \infty)$ and $R_f = [1, \infty)$

Example 8.2.4. Sketch the graph of $y = 2 - |x + 1|$. Hence or otherwise, state the domain and the range of this function.

Sol: The graph is shown below. To determine the $x-$intercepts, solve the equation $2-|x+1| = 0$, and verify that $x = 1$ and $x = -3$



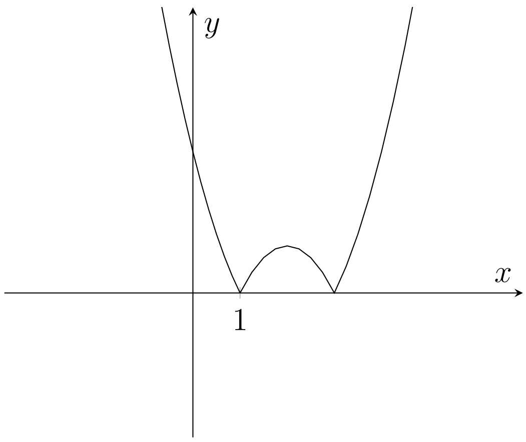
From the graph, $D_f = (-\infty, \infty)$ and $R_f = (-\infty, 2]$

Example 8.2.5. Sketch the graph of $y = |2x^2 - 7x + 5|$. Hence, state the domain and the range of the function.

Sol: We start with the graph of $y = 2x^2 - 7x + 5$, then reflect the negative part in the $X-$axis.

$a = 2$, $b = -7$ and $c = 5$. Since $a > 0$, it is cup-shaped. It cuts the y-axis at $y = 5$. The minimum point is $\left(\frac{7}{4}, -\frac{9}{8}\right)$

$2x^2 - 7x + 5 = 0 \Longrightarrow (2x - 5)(x - 1) = 0 \Longrightarrow x = 1$ and $x = \frac{5}{2}$ are the roots.



From the graph, we have $D_f = (-\infty, \infty)$ and $R_f = [0, \infty)$

Example 8.2.6. Given $f(x) = \left|\frac{3}{x-1}\right|$,

i) find the domain of f

ii) find the range of f

iii) sketch the graph of f

iv) Write down the equation of the asymptotes, if any.

Sol: Exercise

8.3. Equations and Inequality

Let us now discuss equations involving Modulus. The following examples show the technique involved

Example 8.3.1. Solve the equation $|2x-1| = 4$

Sol: we square both sides to get rid of the modulus sign.

$$\begin{aligned}|2x-1| &= 4\\(2x-1)^2 &= 4^2\\(2x-1)^2-4^2 &= 0\\(2x-1-4)(2x-1+4) &= 0\\(2x-5)(2x+3) &= 0\end{aligned}$$

Hence, $2x-5=0 \Longrightarrow x=\frac{5}{2}$ or $2x+3=0 \Longrightarrow x=-\frac{3}{2}$

$$SS = \left\{-\frac{3}{2}, \frac{5}{2}\right\}$$

Example 8.3.2. Solve the following equation $|x+3| = |x-1|$

Sol: Again, we square both sides to get rid of the modulus sign.

$$\begin{aligned}|x+3| &= |x-1|\\(x+3)^2 &= (x-1)^2\\x^2+6x+9 &= x^2-2x+1\\8x &= -8\\x &= -1\end{aligned}$$

Hence, $x=-1$ so that

$$SS = \{-1\}$$

Example 8.3.3. Solve the modular equation $|x+1| = 4$

Sol: $|x+1| = 4$ implies that $x+1 = \pm 4$. Hence, $x+1=4 \Longrightarrow x=3$ or $x+1=-4 \Rightarrow x=-5$. Hence,

$$SS = \{-5, 3\}$$

Example 8.3.4. Solve the equation $|x^3-2x^2+5x-11| = -2$

Sol: Exercise

Example 8.3.5. Solve the modular equation $|x+1| = |x| - |x-1|$.

Sol: we square more than once to get rid of the modulus sign.

$$\begin{aligned}
|x+1| =& |x| - |x-1| \\
|x+1| + |x-1| =& |x| \\
(|x+1| + |x-1|)^2 =& |x|^2 \\
(x+1)^2 + 2|x+1||x-1| + (x-1)^2 =& x^2 \\
x^2 + 2x + 1 + 2|x+1||x-1| + x^2 - 2x + 1 =& x^2 \\
2|(x+1)(x-1)| =& -x^2 - 2 \\
(2|(x+1)(x-1)|)^2 =& (-x^2-2)^2 \\
4\left(x^2-1\right)^2 =& (-1)^4\left(x^2+2\right)^2 \\
4\left(x^2-1\right)^2 =& \left(x^2+2\right)^2 \\
4x^4 - 8x^2 + 4 =& x^4 + 4x^2 + 4 \\
3x^4 - 12x^2 =& 0 \\
x^4 - 4x^2 =& 0 \\
x^2(x^2-4) =& 0 \\
x^2(x-2)(x+2) =& 0
\end{aligned}$$

This means that either $x^2 = 0 \implies x = 0$ or $x - 2 = 0 \implies x = 2$ or $x + 2 = 0 \implies x = -2$
Check that substituting any of these values into the equation, does not make the equation valid. Hence,

$$SS = \emptyset$$

Example 8.3.6. Solve the inequality $|2x-1| < 2$

Sol: we square both sides to get rid of the modulus sign.

$$\begin{aligned}
|2x-1| <& 2 \\
(2x-1)^2 <& 2^2 \\
(2x-1)^2 - 2^2 <& 0 \\
(2x-1-2)(2x-1+2) <& 0 \\
(2x-3)(2x+1) <& 0
\end{aligned}$$

Critical values: $x = \frac{3}{2}$ and $x = -\frac{1}{2}$

	$-\infty < x < -\frac{1}{2}$	$-\frac{1}{2} < x < \frac{3}{2}$	$\frac{3}{2} < x < \infty$
$2x-3$	$-$	$-$	$+$
$2x+1$	$-$	$+$	$+$
$(2x-3)(2x+1)$	$+$	$-$	$+$

$$SS = \left\{x \mid -\frac{1}{2} < x < \frac{3}{2},\ x \in \mathbb{R}\right\} = \left(-\frac{1}{2}, \frac{3}{2}\right)$$

Example 8.3.7. Solve the inequality $|2x + 1| < -4$

Sol: Recall that a modulus is never negative. Therefore, the solution set to $|2x + 1| < -4$ is empty. Hence

$$SS = \emptyset$$

Example 8.3.8. Solve the inequality $|2x + 1| \geq |x - 1|$

Sol: We square both sides to get rid of the modulus sign.

$$\begin{aligned} |2x + 1| &\geq |x - 1| \\ (2x + 1)^2 &\geq (x - 1)^2 \\ 4x^2 + 4x + 1 &\geq x^2 - 2x + 1 \\ 3x^2 + 6x &\geq 0 \\ x^2 + 2x &\geq 0 \\ x(x + 2) &\geq 0 \end{aligned}$$

Critical Values: the critical values are $x = 0$ and $x = -2$

	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
x	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
$x(x+2)$	$+$	$-$	$+$

$$SS = (-\infty, -2] \cup [0, \infty)$$

Exercise

1. Define the following:

   a) function b) Domain of a function c) Range of a function d) Composite function

   v) An inverse of a function (vi) An odd function (vii) A bijective function

2. a) For each of the following functions, state the domain and where possible, the range:

   i) $f(x) = \frac{1}{x}$ (ii) $f(x) = \frac{-2}{x+1}$ (iii) $f(x) = \frac{x+4}{x-5}$ (iv) $f(x) = \frac{4x+1}{x}$ (v) $f(x) = \frac{1}{x+2} + 2$

   vi) $f(x) = \frac{5}{x^2-5x-6}$ (vii) $f(x) = \frac{x}{x^2-x-2}$ (viii) $f(x) = \frac{x^2}{x^2-4}$ (ix) $f(x) = \frac{2x^2+1}{x}$

   b) For each function in 2(a), sketch the graphs indicating the intercepts, vertical asymptotes, horizontal asymptotes and slant asymptotes if they exist.

3. If $f(x) = 2x + 3$ and $g(x) = 3x - 5$, find

   a) $(f \circ g)^{-1}(x)$ *b)* $(f^{-1} \circ g^{-1})(x)$ *c)* $(g^{-1} \circ g^{-1})(x)$ *d)* $(g^{-1} \circ f^{-1})(x)$

4. Find the values of k for which $f(k) = f(1)$, where $f(x)$ is given by $f(x) = \frac{x+1}{x^2-x+1}$

5. Let $f(x) = x^2 + x - 3$. Find $f(x+h)$, h is a constant. Hence express $\frac{f(x+h)-f(x)}{h}$ in its simplest form.

6. Sketch the graphs of the following functions and state the range in each case:

   a) $y = \begin{cases} x, & x \leq 0; \\ x^2, & x > 0. \end{cases}$ b) $y = \begin{cases} 7 - x & \text{if } x \leq 2; \\ 4x - 3 & \text{if } x > 2. \end{cases}$ c) $f(x) = \begin{cases} -x - 2 & \text{if } x < -\frac{1}{3}; \\ 5x & \text{if } -\frac{1}{3} \leq x \leq \frac{1}{2}; \\ x + 2 & \text{if } x > \frac{1}{2}. \end{cases}$

7. Let $f : x \to \frac{3}{x^2-4}$ and $h : x \to 3x - 1$. State the domains of f and h. Find:

   a) $f^{-1}(x)$ b) $h^{-1}(x)$ c) $foh(x)$ d) $(hof)^{-1}(x)$ e) domain and range of $foh(x)$

8. Let $g(x) = \begin{cases} 1 - 2x & \text{if } x \leq -1; \\ x^2 - 2 & \text{if } x > -1. \end{cases}$

   i) Find $g(-3)$, $g(-1)$ and $g(1)$ (ii) Find the values of a for which $g(a) = 14$.

   iii) Sketch the graph of $g(x)$ and state its range

9. For each of the following functions, state whether the function is odd, even or neither.

   a) $f(x) = x^2 + 2$ b) $f(x) = x^3$ b) $f(x) = x^2 + 10x + 9$ c) $f(x) = |x|$ d) $f(x) = \frac{1}{x^2}$

10. The function f is defined as $f(x) = \frac{1}{x+2}$, $x \in \mathbb{R}, x \neq -2$. Further, g is another function defined as $g(x) = x - 3$ $x \in \mathbb{R}$. Find:

    i)$f^{-1}(x)$ (ii) $g^{-1}(x)$ (iii) $fog(x)$ (iv) $(gof)^{-1}(x)$ (v) domain and range of $fog(x)$

11. Sketch the graphs of the following modulus functions and state the range in each case:

    a)$f(x) = -|x| + 2$ b)$f(x) = |(x+1)(2-x)|$ c)$f(x) = |x^2 + 6x + 5|$ d)$f(x) = |x - 2|$

    e)$f(x) = -|4 - x| + 2$ f)$f(x) = |x^3|$ g)$f(x) = |\frac{1}{x}|$ h)$f(x) = |x - 2|$

12. State the domain and sketch the graphs of the following functions involving radicals. State the range:

    a)$f(x) = \sqrt{x-1}$ b)$f(x) = 3 + \sqrt{2-x}$ c)$f(x) = -1 + \sqrt{-x-2}$ d)$f(x) = 2 - \sqrt{1+x}$

    e)$f(x) = \sqrt{x^2 - x - 2}$ f)$f(x) = \sqrt{x^2 - 4}$ g)$f(x) = -2 - \sqrt{-x}$ h)$f(x) = \sqrt{x+5}$

13. Solve each of the following equations:

    i) $|x - 2| = 6$ (ii) $|-2x - 1| = 6$ (iii) $|5x + 4| = 3 - 7x$ (iv) $|2x + 1| = |4x - 3|$

    v) $|x^2 - 4x| = 8 - 4x$ (vi) $|\frac{-2}{x+3}| = 5$ (vii) $|\frac{x+1}{x-2}| = 3$ (viii) $f(x) = |x^2 - 4| = 2$

14. Solve each of the following equations:

    a) $\sqrt{2x-1} + 2 = x$ (b) $\sqrt{2x-1} - \sqrt{x+3} = 1$ (c) $\sqrt{6x+7} - \sqrt{3x+3} = 1$

    d) $\frac{x}{\sqrt{x+1}} + \frac{2x}{\sqrt{x+5}} = 0$ (e) $\sqrt{x-2} - \sqrt{2x-11} = \sqrt{x-5}$ (f) $\sqrt{1+2\sqrt{x}} = \sqrt{x+1}$

15. Solve each of the following inequalities:

    a) $|6x - 11| < 2$ (b) $|2x+3| \geq 4$ (c) $|\frac{x+1}{x-2}| < 5$ (d) $|2x+1| > |x-1|$ (e) $|x+6| < |x-2|$

    f) $|3x+1| < -3$ (g) $|-3x^2 + x - 5| < 0$ (h) $\sqrt{x-3} > \sqrt{x+4} - 1$ (i) $\sqrt{x+1} \leq -1$

16. Solve each of the following inequalities involving radicals:

a) $10-\sqrt{2x+7} \leq 3$ b) $\sqrt{2}-\sqrt{x+6} \leq -\sqrt{x}$ c) $\frac{1}{\sqrt{x}} < \frac{2}{\sqrt{x+27}}$ d) $\sqrt{2x+9}-\sqrt{9+x} > 0$

17. Find the partial fraction decomposition for the following fractions

a) $\frac{20x-3}{6x^2+7x-3}$ (b) $\frac{-9x^2+7x-4}{x^3-3x^2-4x}$ (c) $\frac{3x+7}{(x+1)(x+2)(x+3)}$ (d) $\frac{8x^2+15x+12}{(x^2+4)(3x-4)}$ (e) $\frac{x^2+1}{x^2(2x+1)}$ (f) $\frac{2}{(2-x)(x+1)^2}$

g) $\frac{x}{(x+1)(x^2+2x+2)}$ h) $\frac{x^3-3x^2-3x-5}{x^2-4}$ (i) $\frac{x^3}{x^3-1}$ (j) $\frac{x^2+1}{x^2-1}$ (k) $\frac{4x^2+6x-10}{(x+3)(x^2+x+2)}$ (l) $\frac{x^2}{(x-1)(x+2)^2}$

18. Find the roots of each quadratic equation subject to the given conditions:

(a) $(2k+2)^2x^2+(4-4k)x+k-2=0$ has roots which are reciprocals of each other

(b) $kx^2-(1+k)x+3k+2=0$ has the sum of its roots equal to twice the product of its roots

(c) $(x+k)^2 = 2-3k$ has equal roots

19. Find two numbers whose sum is 10 and whose product is as large as possible

20. The function f is defined as $f(x) = \frac{1}{x+2}$, $x \in \mathbb{R}, x \neq -2$. Further, g is another function defined as $g(x) = x-3$ $x \in \mathbb{R}$. Find:

(i) $f^{-1}(x)$ (ii) $g^{-1}(x)$ (iii) $fog(x)$ (iv) $(gof)^{-1}(x)$ (v) domain and range of $fog(x)$

21. a) Solve the following quadratic equations:

(i) $2x^2-3x-1=0$ (ii) $2x^2-3x+4=0$ (iii) $x^2+6x+5=0$ (iv) $x^2+1=0$

(v) $2x^2-3=0$ (vi) $x^2-9x-10=0$ (vii) $5x^2=7x-13=0$ (viii) $9-x^2=0$

b) For each quadratic in (a), determine the type of roots and sketch their graphs.

22. For what range of values of p, does the equation $x^2-(p+2)x+p^2+3p=3$ have real roots?

23. If the equation $x^2+3=k(x+1)$ has real roots, find the range of values of k

24. Find the solution sets to the following inequalities

(i) $2x^2-3x-1 \leq 0$ (ii) $2x^2+3x-5 \geq 0$ (iii) $2x^2-4x-3 < 0$ (iv) $x^2-x-6 > 0$

(v) $2x^2-3 \leq 0$ (vi) $x^2-9x-10 \geq 0$ (vii) $2x+1 \leq 13x-60$ (viii) $x^2+16 \leq 0$

25. Determine the remainder when x^3+2x^2-x-1 is divided by

(i) $x-1$ (ii) $2x+1$ (iii) $x+1$ (iv) $x+2$ (v) $3x+1$ (vi) $x+3$ (vii) $2x-1$

26. a) Factorise completely each of the following:

(i) x^3-2x^2-5x+6 (ii) x^3+x^2-4x-4 (iii) $6x^3-13x^2+9x-2$ (iv) x^4-1

(v) $4x^3-8x^2-x+2$ (vi) $3x^3+3x^2-3x-2$ (vii) x^3-2x^2+4x-1 (viii) x^3+2

27. a) Solve the following polynomial equations:

(i) $x^3 - 3x^2 - 11x + 15 = 0$ (ii) $x^4 - 2x^2 + 1 = 0$ (iii) $4x^3 - 8x^2 - x + 2 = 0$ (iv) $x^3 - 1 = 0$

(v) $2x^2 - 3 = 0$ (vi) $x^2 - 9x - 10 = 0$ (vii) $5x^2 = 7x - 13 = 0$ (viii) $9 - x^2 = 0$

28. a) Solve the following polynomial inequalities:

(i) $x^3 - 3x^2 - 11x + 15 \leq 0$ (ii) $x^4 - 2x^2 + 1 > 0$ (iii) $4x^3 - 8x^2 - x + 2 \leq 0$ (iv) $x^3 - 1 \leq 0$

(v) $x^2(2x+1) \leq 3x - 60$ (vi) $4x^3 - 12x^2 - 5x + 6 < 0$ (vii) $5x^2 = 7x - 13 \geq 0$

29. Sketch the graphs of the following:

(i) $f(x) = (x-1)(x+2)(x-3)$ (ii) $f(x) = x^4 - 2x^2 + 1$ (iii) $p(x) = 4x^3 - 8x^2 - x + 2$ (iv)$x^3 + 2$

(v) $h(x) = x^3 + 3x^2 + 2x$ (vi) $k(x) = 2x^3 + 3x^2 - 3x - 2$ (vii) $g(x) = (x-2)(2x-1)(2x+1)$

30. Find in terms of p, the remainder when $3x^3 - 2x^2 + px - 6$ is divided by $x + 2$. Hence, write down the value of p for which the expression is exactly divisible by $x + 2$

31. Given that the expression $x^3 + ax^2 + bx + c$ leaves the same remainder when divided by $x - 1$ or $x + 2$, show that $a - b = 3$

32. If $4x^3 - 11x^2 - 6x + 7 = (Ax + B)(x + 1)(x - 3) + C$ for all values of x, evaluate A, B and C.

33. The expression $f(x) = 3x^3 + 2x^2 - px + q$ is divisible by $x - 1$, but leaves a remainder of 10 when divided by $x + 1$. Find the values of p and q.

34. The polynomial $f(x) = A(x-1)^2 + (x+2)^2$ is divided by $x+1$ and $x-2$. The remainders are 3 and -15 respectively. Find the values of A and B.

9
Trigonometric Functions

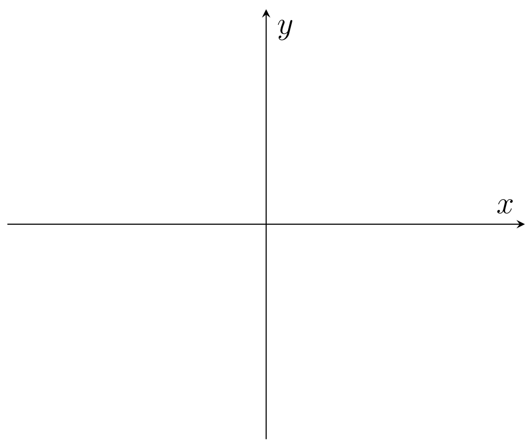
9.1. Introduction

Trigonometry is the branch of mathematics which deals with the measurement of sides and angles of triangles, and their relationship with each other. We will be interested in the relationship that exists between the sides and the angle of a triangle.

Definition 9.1.1. *An angle is a measure of rotation which is measured either in degrees or in radians.*

Note 9.1.1. The following angles are worth noting

- A quarter turn is called a right angle and is $90°$
- A half turn is an angle on a straight line and is $180°$
- A complete turn is a full circle and is $360°$
- An angle read in anticlockwise is positive and an angle read in clockwise is negative



Angles and Quadrants

A complete circle is divided into four quadrants:

1. **First Quadrant:** Angles in the first quadrant are between $0°$ and $90°$ moving in the positive direction. Thus, if x is an angle in the first quadrant, then $0° \leq x \leq 90°$
2. **Second Quadrant:** Angles in the second quadrant are between $90°$ and $180°$ moving in the positive direction. Thus, if x is an angle in the second quadrant, then $90° < x \leq 180°$
3. **Third Quadrant:** Angles in the third quadrant are between $180°$ and $270°$ moving in the positive direction. Thus, if x is an angle in the third quadrant, then $180° < x \leq 270°$
4. **Fourth Quadrant:** Angles in the fourth quadrant are between $270°$ and $360°$ moving in the positive direction. Thus, if x is an angle in the fourth quadrant, then $270° < x \leq 360°$

If x is a positive angle and we add multiples of $360°$ to x, the resulting new angle will be in the same quadrant as x. For example, if $x = 36°$ and $y = 36° + (360k)°$, then y is an angle in the first quadrant.

Similarly, if x is a negative angle and we subtract multiples of $360°$ from x, the resulting angle will be in the same quadrant as x. For example, if $x = -36°$ and $y = -36° - (360k)°$, then y is an angle in the fourth quadrant.

Note 9.1.2. Suppose we want to know the quadrant in which a given angle, say $t°$ is. Then, we could proceed as follows:

- determine $k = \frac{t}{360}$, where k is the value of the integer part.
- determine $x = t - 360k$, if t is either positive or negative.
- the evaluated angle x will be between $0°$ and $360°$. ie, $0° \leq x \leq 360°$ if t is positive, and x will be between $-360°$ and $0°$. ie, $-360° \leq x \leq 0°$ if t is negative
- Finally, the required quadrant of $t°$ is just the quadrant of x.

Example 9.1.1. Determine the quadrant of angle a) $711°$ b) $8220°$ c) $-420°$ iv) $-1070°$

Sol:

i) $\frac{711}{360} = 1.975$. Hence, $k = 1$. Since 711 is positive, we take $x = 711 - 360(1) = 351$ which is in the fourth quadrant. Hence, we conclude that $711°$ is also in the fourth quadrant.

ii) $\frac{820}{360} = 2.278$. Hence, $k = 2$. Since 820 is positive, we take $x = 820 - 360(2) = 100$ which is in the second quadrant. Hence, we conclude that $820°$ is also in the second quadrant.

iii) $\frac{-420}{360} = -1.167$. Hence, $k = -1$. Since -420 is negative, we take $x = -420 - 360(-1) = -60$ which is in the fourth quadrant. Hence, we conclude that $-420°$ is also in the fourth quadrant.

iv) $\frac{-1070}{360} = -2.972$. Hence, $k = -2$. Since -1070 is negative, we take $x = -1070 - 360(-1) = -350$ which is in the first quadrant. Hence, we conclude that $-1070°$ is also in the first quadrant.

Radian Measure

A radian is a measure of the angle at the centre of the circle making an arc of length r on the circumference of the circle with radius r.

- 1 Radian $= \frac{180^\circ}{\pi}$
- $1^\circ = \frac{\pi}{180}$ Radians

Example 9.1.2. Convert the following degrees into radians a) 60° b) -120° c) 180°

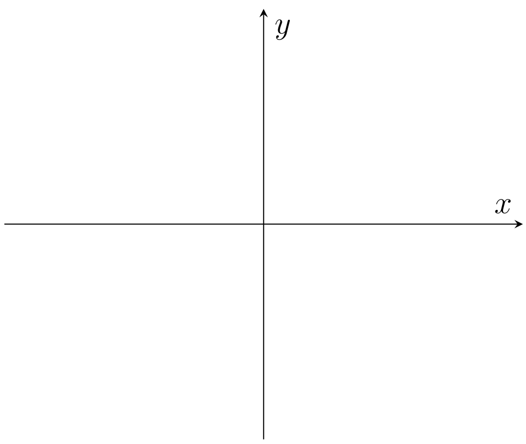
Sol: Using the formula 1 Radian $= \frac{180^\circ}{\pi}$, we have
$60^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$ radians
$-120^\circ = -120 \times \frac{\pi}{180} = -\frac{2\pi}{3}$ radians
$180^\circ = 180 \times \frac{\pi}{180} = \pi$ radians

Example 9.1.3. Convert the following radians to degrees a) 2π b) $-\frac{\pi}{4}$ c) $\frac{2\pi}{3}$

Sol: Using the formula $1^\circ = \frac{\pi}{180}$ Radians, we have
$2\pi = 2\pi \times \frac{180}{\pi} = 360^\circ$
$-\frac{\pi}{4} = -\frac{\pi}{4} \times \frac{180}{\pi} = -45^\circ$
$\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180}{\pi} = 120^\circ$

9.2. Trigonometric Functions

Consider the triangle in the first quadrant below with point (x, y) on the circumference of a circle with radius r.



From the diagram above, $|OA| = x$, $|OB| = r$, $|AB| = y$ and $A\hat{O}B = \theta$ Then we have the following:

1. **The Sine Function:** The sine of an angle θ is the ratio of the length of the opposite side, to that of the hypotenuse side. Thus, from our figure above,

$$\sin\theta = \frac{y}{r}$$

2. **The Cosine Function:** The cosine of an angle θ is the ratio of the length of the adjacent side, to that of the hypotenuse side. Thus, from our figure above,

$$\cos\theta = \frac{x}{r}$$

3. **The Tangent Function:** The tangent of an angle θ is the ratio of the length of the opposite side, to that of the adjacent side. Thus, from our figure above,

$$\tan\theta = \frac{y}{x}$$

4. **The Secant Function:** The secant of an angle θ is the ratio of the length of the hypotenuse , to that of the adjacent side. Thus, from our figure above,

$$\sec\theta = \frac{y}{x}$$

5. **The Cosecant Function:** The cosecant of an angle θ is the ratio of the length of the hypotenuse, to that of the opposite side. Thus, from our figure above,

$$\csc\theta = \frac{r}{y}$$

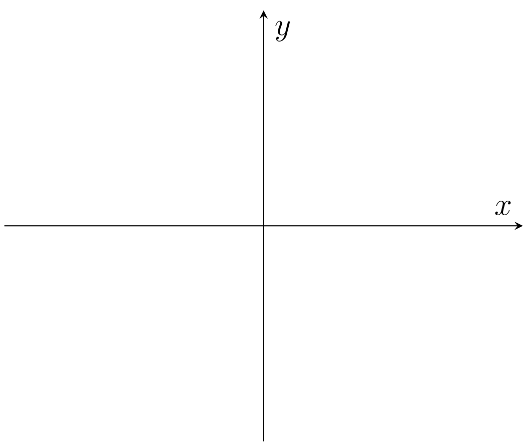
6. **The Cotangent Function:** The cotangent of an angle θ is the ratio of the length of the adjacent side, to that of the opposite side. Thus, from our figure above,

$$\cot\theta = \frac{x}{y}$$

Note 9.2.1. From the definitions above, we notice the following:

i) $\sec\theta = \frac{1}{\cos\theta}$ *ii)* $\csc\theta = \frac{1}{\sin\theta}$ *iii)* $\cot\theta = \frac{1}{\tan\theta}$ *iv)* $\tan\theta = \frac{\sin\theta}{\cos\theta}$ *v)* $\cot\theta = \frac{\cos\theta}{\sin\theta}$

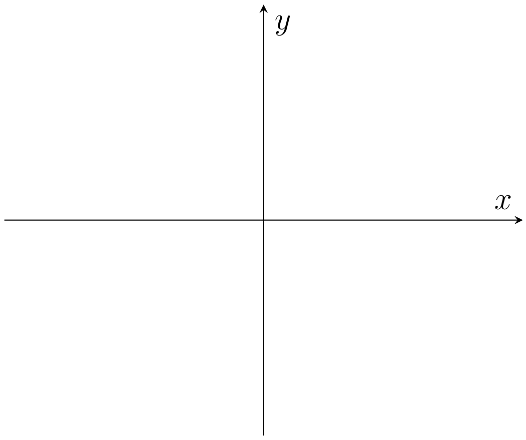
Example 9.2.1. Consider the triangle below. Find all the trigonometric functional values of θ



Sol: From the figure above, we can see that

$$\sin\theta = \frac{3}{5} \qquad \cos\theta = \frac{4}{5} \qquad \tan\theta = \frac{3}{4} \qquad \csc\theta = \frac{5}{3} \qquad \sec\theta = \frac{5}{4} \qquad \cot\theta = \frac{4}{3}$$

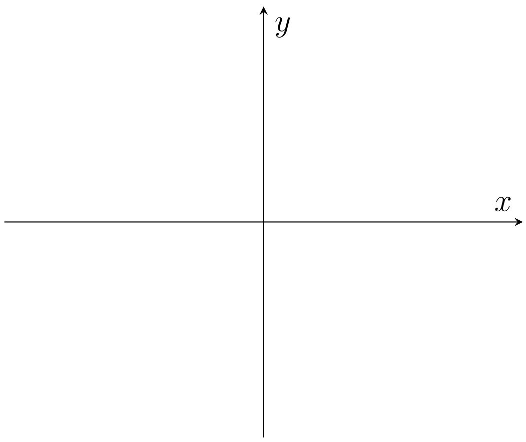
Example 9.2.2. Consider the triangle in the second quadrant. Find all the trigonometric functional values of θ



Sol: From the figure above, we can see that

$$\sin\theta = \frac{3}{5} \qquad \cos\theta = -\frac{4}{5} \qquad \tan\theta = -\frac{3}{4} \qquad \csc\theta = \frac{5}{3} \qquad -\sec\theta = \frac{5}{4} \qquad \cot\theta = -\frac{4}{3}$$

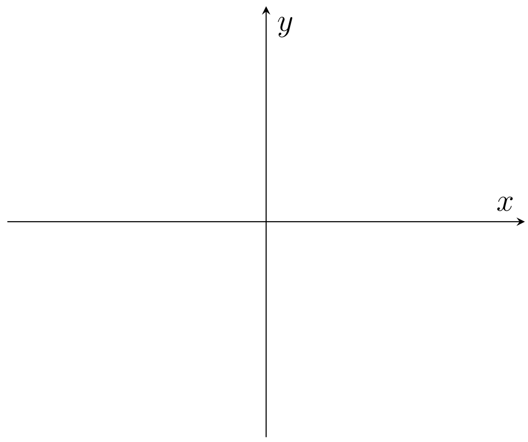
Example 9.2.3. Consider the triangle below, that is in the third quadrant. Find all the trigonometric functional values of θ



Sol: From the figure above, we can see that

$$\sin\theta = -\frac{3}{5} \qquad \cos\theta = -\frac{4}{5} \qquad \tan\theta = \frac{3}{4} \qquad \csc\theta = -\frac{5}{3} \qquad \sec\theta = -\frac{5}{4} \qquad \cot\theta = \frac{4}{3}$$

Example 9.2.4. Consider the triangle that is in the fourth quadrant. Find all the trigonometric functional values of θ



Sol: From the figure above, we can see that

$$\sin\theta = -\frac{3}{5} \qquad \cos\theta = \frac{4}{5} \qquad \tan\theta = -\frac{3}{4} \qquad \csc\theta = -\frac{5}{3} \qquad \sec\theta = \frac{5}{4} \qquad \cot\theta = -\frac{4}{3}$$

Trigonometric Functions and Quadrants

The above examples give us some interesting results, which we can now generalize:

- All the six trigonometric functions are positive in the first quadrant
- Only the sine and cosec functions are positive in the second quadrant
- Only the tangent and cotangent functions are positive in the third quadrant
- Only the cosine and sec functions are positive in the fourth quadrant

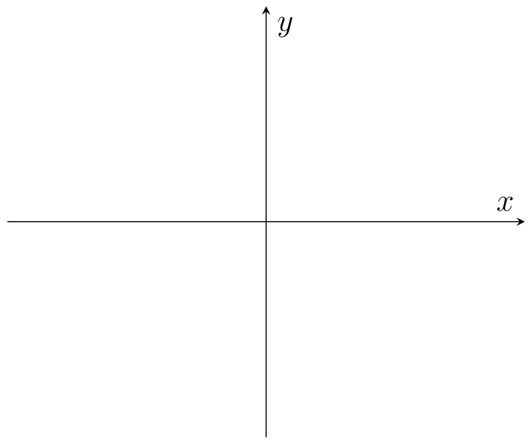
Therefore, each function is positive in the first and one other quadrant, and negative in the other two quadrants.

Example 9.2.5. Find the quadrant containing the terminal side of θ if $\sin\theta < 0$ and $\sec\theta > 0$

Sol: Since $\sin\theta < 0$, the quadrant is either the third or the fourth. Further, since $\sec\theta > 0$, then the quadrant must be the fourth. Hence, the terminal side of θ lies in the fourth quadrant

Example 9.2.6. If the terminal side of θ is in the second quadrant, find all the trigonometric functional values given that $\cos\theta = -\frac{1}{2}$:

Sol: First, we sketch the triangle in the second quadrant as shown below.



From pythagoras theorem, we can see that $y^2 = 2^2 - 1^1 = 3$. Hence, $y = \sqrt{3}$.

$$\sin\theta = \frac{\sqrt{3}}{2} \qquad \tan\theta = -\frac{\sqrt{3}}{1} = -\sqrt{3} \qquad \csc\theta = \frac{2}{\sqrt{3}} \qquad \sec\theta = -\frac{2}{1} = -2 \qquad \cot\theta = -\frac{1}{\sqrt{3}}$$

Common Angles

The angles $0°$, $30°$, $45°$, $60°$, $90°$, $180°$, $270°$ and $360°$ are called the common, standard or special angles. Their trigonometric functional values **must** be known as they are often applied. The table below summarises the trigonometric functional values of these special angles.

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	undefined
$180°$	π	0	-1	0
$270°$	$\frac{4\pi}{3}$	-1	0	undefined
$360°$	2π	0	1	0

Example 9.2.7. Calculate *i)* $\cos 300°$ *ii)* $\cot(-135°)$ *iii)* $\sin 405°$ *iv)* $\cos(-420°)$

Sol: We use the special angles

i) Angle 300° lies in the fourth quadrant. Hence $\cos 300°$ is positive. Thus,

$$\begin{aligned}\cos 300° &= \cos(360° - 300°)\\ &= \cos 60°\\ &= \frac{1}{2}\end{aligned}$$

ii) Angle $-135°$ lies in the third quadrant. Hence $\cot(-135°)$ is positive. Thus,

$$\begin{aligned}\cot -135° &= \cot(180° - 135°)\\ &= \cot 45°\\ &= 1\end{aligned}$$

iii) Angle 405° lies in the first quadrant. Hence $\sin 405°$ is positive. Thus,

$$\begin{aligned}\sin 405° &= \sin(405° - 360°)\\ &= \sin 45°\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

iv) Angle $-420°$ lies in the fourth quadrant. Hence $\cos(-420)°$ is positive. Thus,

$$\begin{aligned}\cos(-420)° &= \cos(420° - 360°)\\ &= \cos 60°\\ &= \frac{1}{2}\end{aligned}$$

9.2.1 Trigonometric Identities

We have seen that $\frac{x}{r} = \cos\theta$ and that $\frac{y}{r} = \sin\theta$. This gives $x = r\cos\theta$ and $y = r\sin\theta$. Using the Pythagoru's Theorem, we have

$$\begin{aligned}r^2 &= x^2 + y^2\\ &= (r\cos\theta)^2 + (r\sin\theta)^2\\ &= r^2\cos^2\theta + r^2\sin^2\theta\\ &= r^2(\cos^2\theta + \sin^2\theta)\end{aligned}$$

Dividing both sides by r^2 gives

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{(i)}$$

Dividing both sides of equation $\cos^2\theta + \sin^2\theta = 1$ by $\cos\theta$, we have

$$\begin{aligned}\cos^2\theta + \sin^2\theta &= 1\\ \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta}\\ 1 + \tan^2\theta &= \sec^2\theta\end{aligned}$$

Hence, we have another identity:

$$\sec^2\theta = 1 + \tan^2\theta \qquad \text{(ii)}$$

Dividing both sides of equation $\cos^2\theta + \sin^2\theta = 1$ by $\sin\theta$, we have

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta} \\ \cot^2\theta + 1 &= \csc^2\theta \end{aligned}$$

Hence, we have the identity:

$$\csc^2\theta = 1 + \cot^2\theta \qquad \text{(iii)}$$

Let α and β denote two angles, then the following identities hold:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \text{(iv)}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad \text{(v)}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \qquad \text{(vi)}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha \qquad \text{(vii)}$$

From identity (iv), we obtain

$$\begin{aligned} \cos(\alpha+\alpha) &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\ &= \cos^2\alpha - \sin^2\alpha \\ \text{Hence, } \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \qquad \text{(viii)} \end{aligned}$$

From identity (vi), we obtain

$$\begin{aligned} \sin(\alpha+\alpha) &= \sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ &= 2\sin\alpha\cos\alpha \\ \text{Hence, } \sin 2\alpha &= 2\sin\alpha\cos\alpha \qquad \text{(ix)} \end{aligned}$$

If α and β are two angles, then

$$\begin{aligned}
\tan(\alpha+\beta) =& \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\
=& \frac{\sin\alpha\cos\beta+\sin\beta\cos\alpha}{\cos\alpha\cos\alpha-\sin\alpha\sin\alpha}\\
=& \frac{\sin\alpha\cos\beta+\sin\beta\cos\alpha}{\cos\alpha\cos\alpha-\sin\alpha\sin\alpha}\times\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}\\
=& \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\
=& \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}
\end{aligned}$$

Therefore, we have the identity: $$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \text{(x)}$$

Similarly, we can show that: $$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \qquad \text{(xi)}$$

From the identity (x), we have $$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \qquad \text{(xii)}$$

Example 9.2.8. Evaluate *i)* $\cos 75°$ *ii)* $\sin 75°$ *iii)* $\tan 75°$ *ii)* $\sec 75°$

Sol:

i) Using the identity $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ with $\alpha = 30°$ and $\beta = 45°$, we have

$$\begin{aligned}
\cos(75°) =& \cos(30° + 45°)\\
=& \cos 30° \cos 45° - \sin 30° \sin 45°\\
=& \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\
=& \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\
=& \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

ii) Using the identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ with $\alpha = 30°$ and $\beta = 45°$, we have

$$\begin{aligned}
\sin(75°) &= \sin(30° + 45°) \\
&= \sin 30° \cos 45° + \cos 30° \sin 45° \\
&= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
&= \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}$$

iii) Using the identity (x) with $\alpha = 30°$ and $\beta = 45°$, we have

$$\begin{aligned}
\tan(75°) &= \tan(30° + 45°) \\
&= \frac{\tan 30° + \tan 45°}{1 - \tan 30° \tan 45°} \\
&= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \left(\frac{1}{\sqrt{3}}\right)(1)} \\
&= \frac{\frac{\sqrt{3}+3}{3}}{\frac{3-\sqrt{3}}{3}} \\
&= 2 + \sqrt{3}
\end{aligned}$$

Example 9.2.9. Evaluate and leave the answer in surd form: i) $\cos\frac{7\pi}{6}$ *ii)* $\tan\left(-\frac{5\pi}{3}\right)$

Sol:

i) Note that $\frac{7\pi}{6} = \frac{(3+4)\pi}{6} = \frac{\pi}{2} + \frac{2\pi}{3}$ so that

$$\begin{aligned}
\cos\frac{7\pi}{6} &= \cos\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) \\
&= \cos\frac{\pi}{2}\cos\frac{2\pi}{3} - \sin\frac{\pi}{2}\sin\frac{2\pi}{3} \\
&= (0)(\cos\frac{2\pi}{3}) - (1)(\sin\frac{2\pi}{3}) \\
&= -\sin\frac{2\pi}{3} \\
&= -\frac{\sqrt{3}}{2}
\end{aligned}$$

ii) Verify that $-\frac{5\pi}{3} = -\pi - \frac{2\pi}{3} = (-\pi) + (-\frac{2\pi}{3})$ so that

$$\begin{aligned}
\tan\left(-\frac{5\pi}{3}\right) =& \tan\left((-\pi) + (-\frac{2\pi}{3})\right) \\
=& \frac{\tan(-\pi) + \tan(-\frac{2\pi}{3})}{1 - \tan(-\pi)\tan(-\frac{2\pi}{3})} \\
=& \frac{0 - \tan\frac{2\pi}{3}}{1 - (0)\tan(-\frac{2\pi}{3})} \\
=& \tan -\frac{2\pi}{3} \\
=& \tan\left(\pi - \frac{2\pi}{3}\right) \\
=& \tan\frac{\pi}{3} \\
=& \sqrt{3}
\end{aligned}$$

Example 9.2.10. Verify that $\sin(\alpha + \pi) = -\sin\alpha$, where $\alpha \in \mathbb{R}$

Sol: Use the identity (vi)

$$\begin{aligned}
\sin(\alpha + \pi) =& \sin\alpha\cos\pi + \cos\alpha\sin\pi \\
=& (\sin\alpha)(-1) + (\cos\alpha)(0) \\
=& -\sin\alpha + 0 \\
=& -\sin\alpha
\end{aligned}$$

Hence, shown.

We can rewrite certain trigonometric functions using the above discussed identities

Example 9.2.11. Prove the identity $\sin x \cot x = \cos x$

Sol: Pick the side that is more complex than the other

$$\begin{aligned}
\text{L.H.S } =& \sin x \cot x \\
=& \sin x \frac{\cos x}{\sin x} \\
=& \cos x = \text{ R.H.S}
\end{aligned}$$

Example 9.2.12. Prove the identity $(1 + \tan^2 x)(1 - \sin^2 x) = 1$

Sol: Picking the left hand side,

$$\begin{aligned}
\text{L.H.S } =& (1 + \tan^2 x)(1 - \sin^2 x) \\
=& (\sec^2 x)(\cos^2 x) \\
=& (\frac{1}{\cos^2 x})(\cos^2 x) \\
=& 1 = \text{ R.H.S}
\end{aligned}$$

Example 9.2.13. Prove the identity $(\csc\theta - \cot\theta)^2 = \frac{1-\cos\theta}{1+\cos\theta}$

Sol: We start with the L.H.S

$$\begin{aligned}
\text{L.H.S } =&(\csc\theta - \cot\theta)^2 \\
=&\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2 \\
=&\left(\frac{1-\cos\theta}{\sin}\right)^2 \\
=&\frac{(1-\cos\theta)^2}{\sin^2\theta} \\
=&\frac{(1-\cos\theta)(1-\cos\theta)}{1-\cos^2\theta} \\
=&\frac{(1-\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} \\
=&\frac{1-\cos\theta}{1+\cos\theta} = \text{ R.H.S}
\end{aligned}$$

9.2.2 Trigonometric Equations

We can solve equations involving basic trigonometric functions, such as $\cos x = k$ where $x \in \mathbb{R}$, using the following steps:

- find the first quadrant angle α for which $\cos\alpha = |k|$.
- find the quadrants in which x will lie
- then determine the corresponding angles for those quadrants

A basic equation will usually have two solutions for $0 \leq x \leq 2\pi$. If the angles are in degrees, they should be read correct to one decimal place.

Example 9.2.14. Solve for x in the equation $\sqrt{2}\sin x = 1$ were $0 \leq x \leq 2\pi$

Sol: Write $\sqrt{2}\sin x = 1$ as $\sin x = \frac{1}{\sqrt{2}}$. Get α in the first quadrant such that $\sin\alpha = |\frac{1}{\sqrt{2}}| = \frac{1}{\sqrt{2}}$. Verify that $\alpha = \frac{\pi}{4}$ since $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Now, since $\sin x = \frac{1}{\sqrt{2}}$ means that the sine function is positive, the angle x must be in the first and second quadrants. Hence, $x = \frac{\pi}{4}$ and $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$$\text{Hence, the solution set is: } SS = \{\frac{\pi}{4}, \frac{3\pi}{4}\}$$

Example 9.2.15. solve the equation $\tan^2 x - 1 = 0$ where $0° \leq x \leq 360°$

Sol: We have $(\tan x - 1)(\tan x + 1) = 0$. Hence, $\tan x = 1$ or $\tan x = -1$

For $\tan x = 1$, we get $x = 45°,\ 225°$. For $\tan x = -1$, we have $x = 135°$ and $x = 315°$

$$\text{Hence, the solution set is: } SS = \{45°,\ 135°,\ 225°,\ 315°\}$$

Example 9.2.16. Solve the equation $2\sin x - \tan x = 0$ for $0 \leq x \leq 2\pi$

Sol: We have $2\sin x - \frac{\sin x}{\cos x} = 0$ so that $2\cos x \sin x - \sin x = 0$. Factorising gives $\sin x(2\cos x - 1) = 0$. Hence, $\sin x = 0$ or $\cos x = \frac{1}{2}$.

From $\sin x = 0$, we obtain $x = 0, \pi$, and 2π. From $\cos x = \frac{1}{2}$, we obtain $x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

$$\text{Hence, the solution set is: } SS = \left\{0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 2\pi\right\}$$

Example 9.2.17. solve for θ given the equation $\cos\theta = \sin 2\theta$, where $0 \leq \theta \leq 2\pi$:

Sol: We can rewrite as $\cos\theta = 2\sin\theta\cos\theta$ so that we obtain $\cos\theta(1 - 2\sin\theta) = 0$. Hence, $\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

From $\cos\theta = 0$, we obtain $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

From $\sin\theta = \frac{1}{2}$, we obtain $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$

$$\text{Hence, the solution set is: } SS = \left\{\frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{\pi}{2},\ \frac{3\pi}{2}\right\}$$

Example 9.2.18. Solve the equation $2\sin^2\theta - \cos\theta - 1 = 0$ for $0 \leq \theta \leq 2\pi$

Sol: We rewrite as $2(1 - \cos^2\theta) - \cos\theta - 1 = 0$ so that $2\cos^2\theta + \cos\theta - 1 = 0$. Let $\cos\theta = y$. Then we have a quadratic equation $2y^2 + y - 1 = 0$ verify that solving this quadratic gives

$(2y - 1)(y + 1) = 0$ so that $y = -1$ or $\frac{1}{2}$. Thus, $\cos\theta = -1$ and $\cos\theta = \frac{1}{2}$.

From $\cos\theta = -1$, we get $\theta = \pi$ only.

From $\cos\theta = \frac{1}{2}$, we get $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$

$$\text{Hence, the solution set is: } SS = \left\{\frac{\pi}{3},\ \pi,\ \frac{5\pi}{2}\right\}$$

Note 9.2.2. Expressions of the form $a\cos x + b\sin x$ where $a, b \in \mathbb{R}$ can be written as

$$r\cos(x + \alpha) \quad \text{or} \quad r\sin(x + \alpha)$$

where $\alpha \in \mathbb{R}$ and $r \in \mathbb{R}^+$

Example 9.2.19. Express $f(\theta) = \sin\theta - \cos\theta$ in the form $f(\theta) = r\cos(\theta + \alpha)$. Hence or otherwise, solve the equation $\sin\theta - \cos\theta = 1$

Sol: Let $\sin\theta - \cos\theta = r\cos(\theta + \alpha)$. Then we expand the R.H.S so that

$$\begin{aligned} \sin\theta - \cos\theta &= r\cos(\theta+\alpha) \\ \sin\theta - \cos\theta &= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ \text{Hence,} \qquad 1 &= -r\sin\alpha \qquad \text{(i)} \\ \text{and,} \qquad -1 &= r\cos\alpha \qquad \text{(ii)} \end{aligned}$$

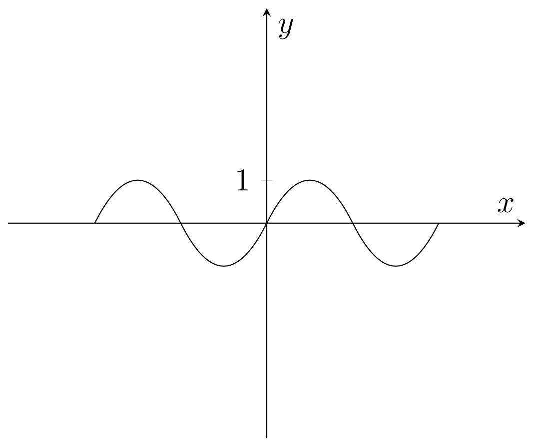
Solving (i) and (ii) simultaneously, gives $r = \sqrt{2}$ and $\alpha = \frac{5\pi}{4}$. Hence, $f(\theta) = \sqrt{2}\cos(\theta + \frac{5\pi}{4})$

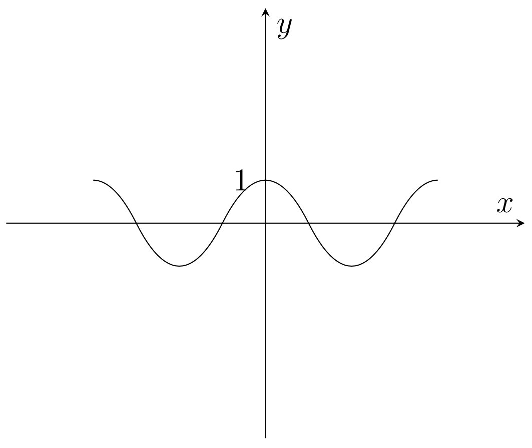
$$\begin{aligned} \sin\theta - \cos\theta &= 1 \\ \sqrt{2}\cos\left(\theta + \frac{5\pi}{4}\right) &= 1 \\ \cos\left(\theta + \frac{5\pi}{4}\right) &= \frac{1}{\sqrt{2}} \end{aligned}$$

Hence, $\theta + \frac{5\pi}{4} = \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, \frac{\pi}{4} + 2\pi, \cdots$. Hence, $\theta = \frac{\pi}{2}, \pi$ since we need $0 \leq \theta \leq 2\pi$

9.3. Graphs of Trigonometric Functions

Let x be a random measure, we examine the graphs of $\sin x$, $\cos x$ and $\tan x$. The graphs of these trigonometric functions are periodic functions. Below are the graphs of the basic trigonometric functions





Amplitude, Period and Phase Shift

Let $f(x) = A\sin(Bx + C)\ +\ D$ denote the standard trigonometric function. Then

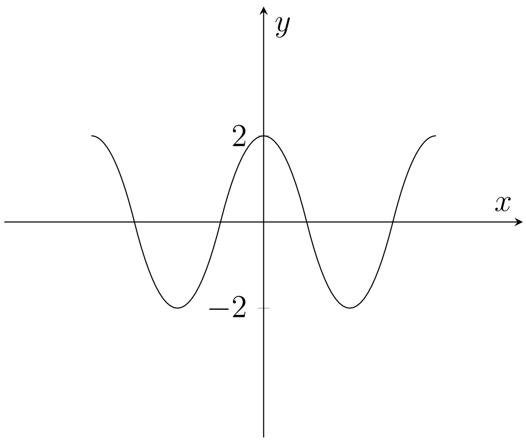
- $|A|$ is called the amplitude of the function. This is the maximum displacement from a given fixed line, usually the line $y = 0$
- The period of a periodic function is the interval required for one complete cycle. The period, $P = \frac{2\pi}{B}$

- The Phase Shift or horizontal shift is given as $-\frac{C}{B}$. It determines the units required for the horizontal shifting of the graph.
- The vertical shift is given by the value of D.

Example 9.3.1. Sketch the graph of $f(x) = 2\sin(x + \frac{\pi}{2})$ on the interval $[-2\pi, 2\pi]$

Sol: We have $A = 2$, $B = 1$, $C = \frac{\pi}{2}$ and $D = 0$.

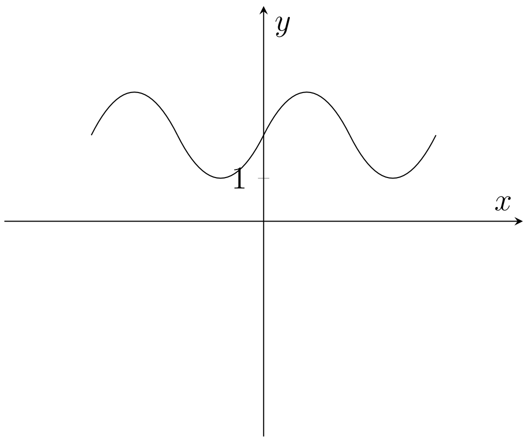
Therefore, amplitude $= |A| = |2| = 2$, phase shift $= -\frac{C}{B} = -\frac{\frac{\pi}{2}}{1} = -\frac{\pi}{2}$, period $= \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi$ and the vertical shift is 0. Since the phase shift is $-\frac{\pi}{2}$, we shift the graph of $2\sin x$ to the left $\frac{\pi}{2}$ units.



Example 9.3.2. Sketch the graph of $f(x) = 2 + \cos(x - \frac{\pi}{2})$ for $+2\pi \leq x \leq 2\pi$

Sol: We have $A = -1$, $B = 1$, $C = -\frac{\pi}{2}$ and $D = 2$.

Therefore, amplitude $= |A| = |-1| = 1$, phase shift $= -\frac{C}{B} = -\frac{-\frac{\pi}{2}}{1} = \frac{\pi}{2}$, period $= \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi$ and the vertical shift is 2.



Example 9.3.3. Sketch the graph of $f(x) = \tan(x - \frac{\pi}{2})$

Sol: Exercise

Double Angle Formulae

The following is a summary of the double angle formulae:

- $\sin 2x = 2\sin x \cos x$ $\qquad \cos 2x = 2\cos^2 x - 1$
- $\sin 4x = 2\sin 2x \cos 2x$ $\qquad \cos 4x = 2\cos^2 2x - 1$
- $\sin 6x = 2\sin 3x \cos 3x$ $\qquad \cos 6x = 2\cos^2 3x - 1$
- $\sin 8x = 2\sin 4x \cos 4x$ $\qquad \cos 8x = 2\cos^2 4x - 1$
- $\sin x = 2\sin \frac{1}{2}x \cos \frac{1}{2}x$ $\qquad \cos x = 2\cos^2 \frac{1}{2}x - 1$
- $\sin^2 x = \frac{1-\cos 2x}{2}$ $\qquad \cos^2 x = \frac{1+\cos 2x}{2}$
- $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$ $\qquad \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$ $\qquad \tan x = \frac{2\tan(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}$

Exercise 9

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1. Express each of the following in radians:

   i) $60°$ ii) $-420°$ iii) $135°$ iv) $-45°$ v) $570°$ vi) $-60°$ vii) $-780°$

2. Express each of the following in degrees:

   i) $\frac{5\pi}{6}$ ii) $\frac{17\pi}{4}$ iii) $\frac{-4\pi}{3}$ iv) $\frac{-37\pi}{24}$ v) $\frac{\pi}{36}$ vi) $\frac{-\pi}{3}$ vii) $\frac{7\pi}{5}$ viii) $\frac{11\pi}{3}$

3. Find the quadrant containing the terminal side of θ if the given conditions hold.

   i) $\sin\theta < 0$ and $\cos\theta > 0$ ii) $\tan\theta < 0$ and $\cos\theta > 0$ iii) $\csc\theta < 0$ and $\cot\theta > 0$

   iv) $\sec\theta < 0$ and $\tan\theta > 0$ v) $\csc\theta > 0$ and $\cot\theta < 0$ vi) $\sec\theta > 0$ and $\cot\theta > 0$

4. If the terminal side of θ is in the given quadrant, find all the trigonometric functional values:

   i) 1^{st} quadrant and $\sin\theta = \frac{4}{5}$ ii) 2^{nd} quadrant and $\cos\theta = -\frac{5}{13}$ iii) $\csc\theta = -\frac{5}{4}$ and $\sec\theta < 0$

   iv) 4^{th} quadrant and $\cot\theta = -\frac{5}{12}$ v) $\csc\theta > 0$ and $\tan\theta = -\frac{3}{4}$ vi) θ is obtuse and $\cot\theta = -\frac{4}{3}$

5. Find the exact values of the following, leaving your answer in surd form where necessary:

   i) $\sin 225°$ ii) $\cos 150°$ iii) $\tan 330°$ iv) $\csc(-240)°$ v) $\sec 420°$ vi) $\cot 135°$

   vii) $\sin\frac{2\pi}{3}$ ii) $\cos(-\frac{5\pi}{3})$ iii) $\tan\frac{4\pi}{3}$ iv) $\sec(-\frac{7\pi}{6})$ v) $\csc\frac{9\pi}{4}$ vi) $\cot(-\frac{\pi}{3})$

6. Given that $\theta = \frac{\pi}{3}$, find the following

   i) $\sin 2\theta$ ii) $2\sin\theta$ iii) $\sec\frac{1}{2}\theta$ iv) $\cos(-3\theta)$ v) $\tan^2\theta$ vi) $\cot\theta$

7. Prove the following identities:

   a) $\sec x - \cos x = \sin x \tan x$ b) $\sin x + \sin x\tan^2 x = \tan x \sec x$ c) $\tan^2 x + 1 = \sec^2 x$
   d) $\frac{1+\sec x}{\sin x+\tan x} = \csc x$ e) $(\sin x - \cos x)^2 = 1 - 2\sin x \cos x$ f) $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2\tan x \sec x$

   g) $\frac{(\tan x)(1+\cot^2 x)}{1+\tan x^2} = \cot x$ h) $\cot x \sin 2x = 1 + \cos 2x$ i) $\frac{1-\tan^2 x}{1+\tan^2 x} = \cos 2x$

8. Simplify the following to a single trigonometric function or a constant:

   a) $\sec\theta - \sin\theta\tan\theta$ b) $(\cos^2 x - 1)(\tan^2 x + 1)$ c) $\cos x + \tan x \sin x$ d) $\frac{\tan\theta\sin\theta}{\sec^2\theta - 1}$
   e) $\frac{\sec x - \cos x}{\tan x}$

9. Solve each of the following equations for θ where $0° \le \theta \le 360°$:

   i) $2\sin\theta + \sqrt{2} = 0$ ii) $\sin^2\theta - 1 = 0$ iii) $2\cos^2\theta = \cos\theta$ (iv) $2\tan\theta = 1$ v) $2\cos^3\theta = \cos\theta$

   vi) $2\sin^2\theta - \cos\theta - 1 = 0$ vii) $2\sin\theta = \tan\theta$ vii) $\tan\theta = \cot\theta$ viii) $4\cos^2\theta = 3$ iv) $\tan^2\theta = 1$

10. Solve each of the following equations for x where $0 \le x \le 2\pi$:

    a) $\tan x + 1 = \sec x$ b) $\sin x = 1 - \cos x$ c) $2\tan x \sec x - \tan x$ d) $\sin x = \csc x$

    e)$\sec^2 x - \sec x = 2$ f)$2\cos^2 x + 3\cos x + 1 = 0$ g)$3\tan^2 x - 1 = 0$ h)$\cos x \sin x + \sin x - \cos x = 1$

11. Find the period, amplitude and phase shift of the given functions and sketch their graphs for $-2\pi \le x \le 2\pi$:

    a) $f(x) = 2\sin(x - \frac{\pi}{2})$ b) $f(x) = 4\cos 2(x + \frac{\pi}{2})$ c) $f(x) = \frac{1}{2}\cos(2x + \frac{\pi}{2})$

12. Sketch the graphs of the following functions on the stated domains:

    i) $f(x) = -2\sin(x + \frac{\pi}{2})$, $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ ii) $f(x) = 3 + \cos(2x - \pi)$, $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$
    i) $f(x) = 2 - \sin x$, $-\pi \le x \le 2\pi$ (ii) $f(x) = -2 + \cos(x - \frac{\pi}{2})$, $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$

13. Sketch the graphs of the following functions:

    a) $y = 3\cot(x - \frac{\pi}{4})$ (b) $y = \tan(x + \frac{\pi}{4})$ (c) $y = -\csc(x + \frac{\pi}{4})$ (c) $y = 2\sec(x - \frac{\pi}{2})$

14. Verify each of the following identities:

    a) $\cos(\alpha + 90°) = -\sin\alpha$ (b) $\sin(\alpha + 90°) = \cos\alpha$ (c) $\sin(\alpha + \pi) = -\sin\alpha$ (d) $\cos(\alpha - \pi) = -\cos\alpha$

    f) $\tan(\alpha + \frac{\pi}{4}) = \frac{1+\tan\alpha}{1-\tan\alpha}$ (g) $\tan(\alpha - \frac{\pi}{4}) = \frac{\tan\alpha - 1}{\tan\alpha + 1}$ (h) $\frac{\sin 2\theta \sin\theta}{2\cos\theta} + \cos^2\theta = 1$ (i) $\sec 2\theta = \frac{\sec^2\theta}{2 - \sec^2\theta}$

15. Express: (a) $\sin 3\theta$ in terms of $\sin\theta$ (b) $\cos 3\theta$ in terms of $\cos\theta$ (c) $\cos 4\theta$ in terms of $\cos\theta$

16. solve each of the following for θ, where $0 \le \theta \le 2\theta$.

    (a)$\cos\theta = \sin 2\theta$ (b)$2 - \sin^2\theta = 2\cos^2\frac{\theta}{2}$ (c)$\sin\frac{\theta}{2} + \cos\theta = 1$ (d)$\cos 2\theta - 3\sin\theta = 2$ (e)$\tan 2\theta + \sec 2\theta = 1$

17. Express $f(x) = \sqrt{3}\cos x - \sin x$ in the form $f(x) = r\cos(x + \alpha)$. Hence, sketch its graph.

18. Find the general solution of: (a) $\cos\theta - \sqrt{3}\sin\theta = 1$ (b) $\cos x + \sin x = \sqrt{2}$.

10
Logarithmic and Exponential Functions

10.1. Introduction to exponentials

An exponential function is a function of the form $f(x) = b^x$. Here, the real number b is called the base and the power x is called the exponent. A function of the form $f(x) = b^{x+c}$ is also considered to be an exponential function, which can be written in the form $f(x) = ab^x$ where $a = b^c$. The base is often taken to be $e \approx 2.718$, where e is Euler's number, a number such that the function $f(x) = e^x$ is its own derivative. The exponential function is used to model a relationship in which a constant change in the independent variable gives the same proportional change in the dependent variable.

Basic Rules of Indices

The following are the basic laws of the indices.

- **Multiplication:** If the base is the same, then add the powers $x^m \times x^n = x^{m+n}$
- **Division:** If the base is the same, then subtract the powers $x^m \div x^n = x^{m-n}$
- **Power:** If an exponential is raised to another power, multiply the powers $(x^m)^n = x^{mn}$

Example 10.1.1. Evaluate i) $100^{\frac{3}{2}}$ ii) $32^{-\frac{2}{5}}$ iii) $\left(-\frac{8}{27}\right)^{-\frac{1}{3}}$ iv) $(-3)^0$

Sol: We apply the basic rules of indices

i) $100^{\frac{3}{2}} = \left(100^{\frac{1}{2}}\right)^3 = \left(\sqrt{100}\right)^3 = 10^3 = 1000$

ii) $32^{-\frac{2}{5}} = \frac{1}{32^{\frac{2}{5}}} = \frac{1}{\left(\sqrt[5]{32}\right)^2} = \frac{1}{(2)^2} = \frac{1}{4}$

iii) $\left(-\frac{8}{27}\right)^{-\frac{1}{3}} = \left(-\frac{27}{8}\right)^{\frac{1}{3}} = (-1)^{\frac{1}{3}}\left(\frac{27}{8}\right)^{\frac{1}{3}} = (-1)\frac{\sqrt[3]{27}}{\sqrt[3]{8}} = -\frac{3}{2}$

iv) $(-3)^0 = 1$

Example 10.1.2. Solve the equation $2^x = 64$

Sol: Note that $64 = 2^5$. Hence we have $2^x = 2^5 \implies x = 5$

Example 10.1.3. Solve the equation $2^{2x+1} - 15(2^x) = 8$

Sol: We rewrite $2^{2x+1} - 15(2^x) = 8$ as $2\left(2^x\right)^2 - 15(2^x) = 8$ and then let $y = 2^x$. Hence,

$$\begin{aligned} 2^{2x+1} - 15(2^x) =& 8 \\ 2\left(2^x\right)^2 - 15(2^x) =& 8 \\ 2y^2 - 15y - 8 =& 0 && \text{letting } 2^x = y \\ (2y+1)(y-8) =& 0 \end{aligned}$$

We get $x = -\frac{1}{2}$ and $x = 8$. Hence, $2^x = -\frac{1}{2}$ and $2^x = 8$. From $2^x = 8$, we get $x = 3$. However, $2^x = -\frac{1}{2}$ is not valid, hence it is discarded.

Example 10.1.4. Solve the following simultaneous equations

$$3^x \times 9^y = 1 \qquad \text{(i)}$$

$$2^{2x} \times 4^y = \frac{1}{8} \qquad \text{(ii)}$$

Sol: From equation (i), we have

$$\begin{aligned} 3^x \times 9^y =& 1 \\ 3^x \times 3^{2y} =& 3^0 \\ \text{Hence, } x + 2y =& 0 && \text{(iii)} \end{aligned}$$

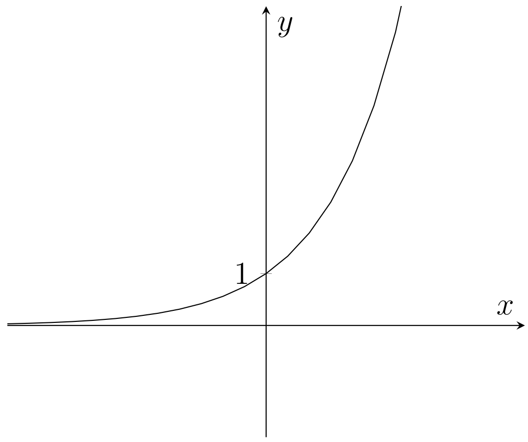
Similarly, from (ii), we get

$$\begin{aligned} 2^{2x} \times 4^y =& \frac{1}{8} \\ 2^{2x} \times 2^{2y} =& 2^{-3} \\ 2^{2x+2y} =& 2^{-3} \\ \text{Hence, } 2x + 2y =& -3 && \text{(iv)} \end{aligned}$$

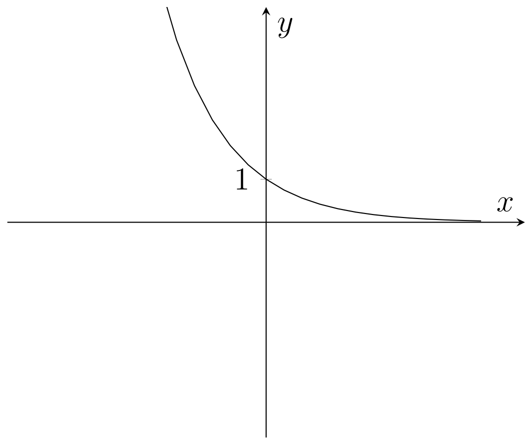
Solving equations (iii) and (iv) simultaneously gives $x = -3$ and $y = \frac{3}{2}$

10.1.1 Graphs of Exponential Functions

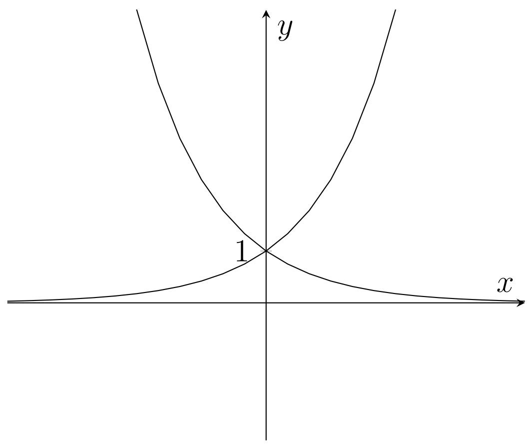
Let $f(x) = b^x$ be an exponential function with $b \in (1, \infty)$. Then the graph of $y = f(x)$ looks as shown below. Note that if there is no vertical shift, the graph of an exponential will pass through the point $(0, 1)$ and is asymptotic to the line $y = 0$. Below is a sketch of the graph when $b > 1$



When the value of the base b is a less than 1, the graph has the orientation shown below.



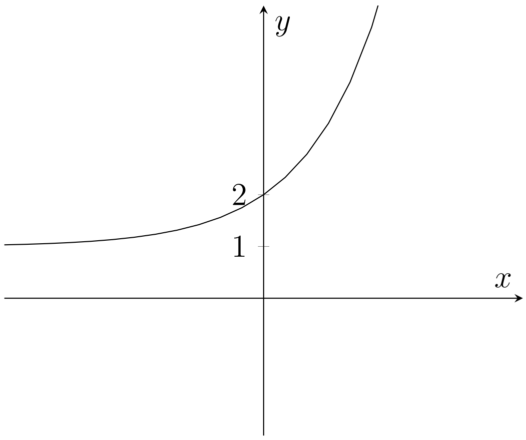
Example 10.1.5. Sketch the graphs of the function $f(x) = 3^x$ and $g(x) = \frac{1}{3}x^2$ on the same axis. State the domain and the range for each.



From the graph, we see that $D_f = (-\infty, \infty)$ and $R_f = (0, \infty)$. The line $y = 0$ is an asymptote.

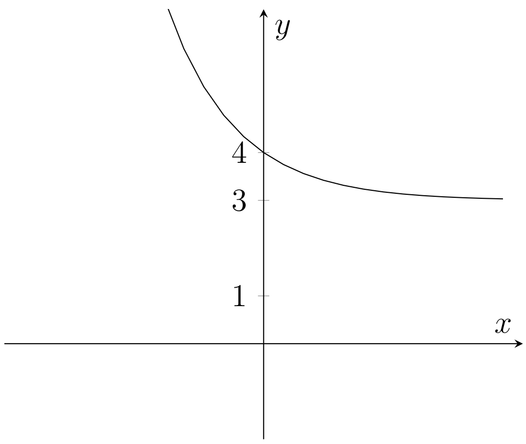
Example 10.1.6. Sketch the graph of the function $f(x) = e^x + 1$. State the domain, range and the equation of the asymptote.

Sol: The graph of this function is obtained by shifting the graph of e^x one unit upwards.



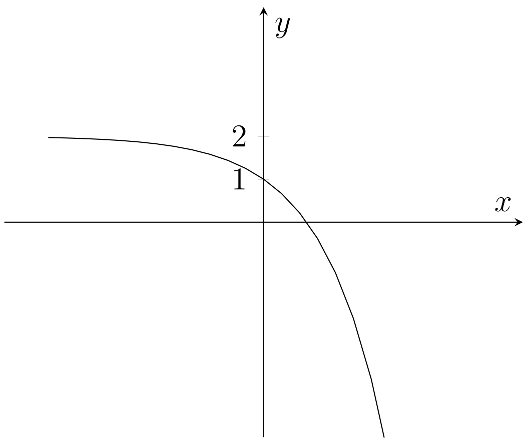
From the graph, we see that $D_f = (-\infty, \infty)$ and $R_f = (1, \infty)$. The line $y = 1$ is an asymptote.

Example 10.1.7. sketch the graph of the function $f(x) = e^{-x} + 3$. State the domain, range and the line of asymptote.



From the graph, we see that $D_f = (-\infty, \infty)$ and $R_f = (3, \infty)$. The line $y = 3$ is an asymptote.

Example 10.1.8. Sketch the graph of the function $f(x) = 2 - e^x$. State the domain and the range



From the graph, we see that $D_f = (-\infty, \infty)$ and $R_f = (-\infty, 2)$. The line $y = 2$ is an asymptote.

10.2. Introduction to Logarithms

The use of calculators has made the logarithmic tables to be rarely used for calculations. However, the theory of logarithms is important, for there are several scientific laws that involve the rules of logarithms.

Definition 10.2.1. *Let $y \in \mathbb{R}$. Further, let $b \in \mathbb{R}^+ - \{1\}$. Then the number x is called the logarithm of y to the base b if*

$$y = b^x$$

We can make a conversion from the exponential form to the logarithmic form using the following relationship

$$\text{If } y = b^x \text{ then } x = \log_b y$$

Example 10.2.1. Convert the following exponentials to logarithmic form.

$i)\ 10^5 = 100000 \qquad ii)\ 2^{-2} = \dfrac{1}{4} \qquad iii)\ \left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$

Sol:

i) If $10^5 = 100000$, then $\log_{10} 100000 = 5$

ii) If $2^{-2} = \frac{1}{4}$, then $\log_2 \frac{1}{4} = -2$

iii) If $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$, then $\log_{\frac{1}{2}} \frac{1}{8} = 3$

Example 10.2.2. Convert the following logarithms to exponential form.

$i)\ 4 = \log_3 x \qquad ii)\ x = \log_7 5 \qquad iii)\ 2 = \log_x 10$

Sol:

i) If $4 = \log_3 x$, then $3^4 = x$

ii) If $x = \log_7 5$, then $7^x = 5$

iii) If $2 = \log_x 10$, then $x^2 = 10$

Example 10.2.3. Solve for x in the following expressions

$i)\ x = \log_2 64 \qquad ii)\ \log_x 25 = 2 \qquad iii)\ \log_3 x = 4 \qquad iv)\ x = \log_3 \dfrac{1}{3}$

Sol: To solve logarithmic equations, we need to convert to exponential form.

i) If $x = \log_2 64$, then $2^x = 64$. Writing in the same base, we have $2^x = 2^6$ so that $x = 6$

ii) If $\log_x 25 = 2$, then $x^2 = 25$ so that $x = 5$

iii) If $\log_3 x = 4$, then $x = 3^4 = 81$. Hence, $x = 81$

iv) If $x = \log_3 \frac{1}{3}$, then $3^x = \frac{1}{3} = 3^{-1}$. Thus, $3^x = 3^{-1}$ so that $x = -1$

10.2.1 Laws of Logarithms

Let $b \in \mathbb{R}^+ - \{1\}$. Further, let x and y be real numbers. Then the following hold

1. Raising a number to a given power: $\log_b x^n = n \log_b x$
2. Changing from base N to base b: $\log_N x = \frac{\log_b x}{\log_b N}$
3. Multiplying two numbers: $\log_b(xy) = \log_b x + \log_b y$
4. Dividing two numbers: $\log_b(\frac{x}{y}) = \log_b x - \log_b y$
5. Log of 1 to any base: $\log_b 1 = 0$
6. log of the base: $\log_b b = 1$

Example 10.2.4. Simplify the following expressions

$i)\ \log_9 81 \qquad ii)\ \log_5\left(\frac{1}{25}\right) \qquad iii)\ \log_2 9 + \log_2 21 - \log_2 7$

Sol:

i) $\log_9 81 = \log_9 9^2 = 2\log_9 9 = (2)(1) = 2$

ii) $\log_5\left(\frac{1}{25}\right) = \log_5 5^{-2} = -2\log_5 5 = (-2)(1) = -2$

iii) $\log_2 9 + \log_2 21 - \log_2 7 = \log_2(9 \times 21 \div 7) = \log_2 27 = \log_2 3^3 = 3\log_2 3$

Note 10.2.1. Two special logarithms are worth noting:

- **Common Logarithms:** These are logarithms whose base is 10. We denote the common logarithms by lg. For example, $\log_{10} x = \lg x$. Similarly, $\log_{10} 7 = \lg 7$, $\log_{10} \frac{1}{2} = \lg \frac{1}{2}$
- **Natural Logarithms:** These are logarithms whose base is e. We denote the common logarithms by ln. For example, $\log_e x = \ln x$. Similarly, $\log_e 7 = \ln 7$, $\log_e \frac{1}{2} = \ln \frac{1}{2}$

Example 10.2.5. Write the expression $\ln\left(\frac{8\times\sqrt[4]{5}}{81}\right)$ in terms of $\ln 2$, $\ln 3$ and $\ln 5$

Sol: We apply the laws of logarithms.

$$\begin{aligned}\ln\left(\frac{8\times\sqrt[4]{5}}{81}\right) &= \ln 8 + \ln\sqrt[4]{5} - \ln 81\\ &= \ln 2^3 + \ln 5^{\frac{1}{4}} - \ln 3^4\\ &= 3\ln 2 + \frac{1}{4}\ln 5 - 4\ln 3\end{aligned}$$

10.2.2 Logarithmic Equations

Example 10.2.6. Find the possible values of x for which $\log 5 + \log x = 1$

Sol: Use the rules of logarithms and convert to exponential form.

$$\begin{aligned}
\log_2 5 + \log_2 x =& 1 \\
\log_2 5x =& 1 \\
5x2 & \\
x =& \frac{2}{5}
\end{aligned}$$

Example 10.2.7. Solve the equation $\ln(x-1) + \ln(x+1) = 2\ln(x+2)$

Sol: We use the rules of logarithms

$$\begin{aligned}
\ln(x-1) + \ln(x+1) =& 2\ln(x+2) \\
\ln(x-1)(x+1) =& \ln(x+2)^2 \\
(x-1)(x+1) =& (x+2)^2 \\
x^2 - 1 =& x^2 + 4x + 4 \\
4x =& -5 \\
x =& -\frac{5}{4}
\end{aligned}$$

Example 10.2.8. Solve the following equation $9\log_x 5 = \log_5 x$

Sol: Make sure the base is the same through out. Hence, note that $9\log_x 5 = 9\frac{\log_5 5}{\log_5 x} = \frac{9}{\log_5 x}$. Therefore,

$$\begin{aligned}
9\log_x 5 =& \log_5 x \\
\frac{9}{\log_5 x} =& \log_5 x
\end{aligned}$$

Let $\log_5 x = y$. Then substituting this above gives

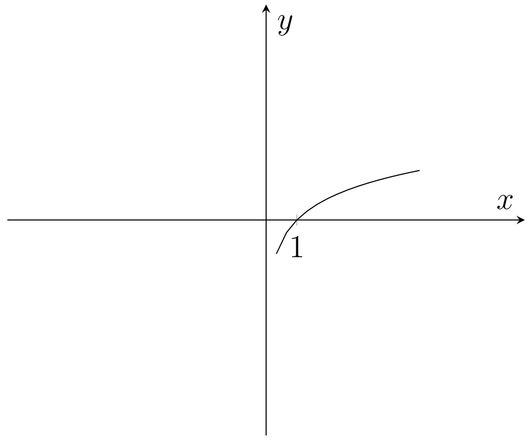
$$\begin{aligned}
\frac{9}{y} =& y \\
9 =& y^2 \\
y^2 - 9 =& 0 \\
(y-3)(y+3) =& 0
\end{aligned}$$

$$\text{Hence, } x = 3 \quad \text{and} \quad x = -3$$

Since we can not have a negative base, discard $x = -3$ and we say $x = 3$

10.2.3 Graphs of Logarithmic Functions

Since the logarithmic function is the inverse of the exponential function, we can obtain the graph of $y = \log_b x$ by reflecting the graph of $y = b^x$ in the line $y = x$. Below is the sketch of the graph of $y = \log_b x$ when $b > 1$.



Example 10.2.9. Sketch the graphs of the following functions clearly indicating the intercepts and asymptotes. Hence, state the domain and the range.

i) $f(x) = 2 - \ln x$ *ii)* $gx = 1 + \log_{\frac{1}{2}} x$ *iii)* $h(x) = 3 + \ln x$ *iv)* $2 - \log_5 x$

Sol: Exercise

10.3. Applications of Exponential and Logarithmic Functions

The exponential function is widely used in physics, chemistry, engineering, mathematical biology, economics and mathematics. The exponential function is used to model a number of random and natural phenomenon.

Example 10.3.1. A highly infectious disease is introduced into a small isolated village of population 200. The number of individuals y who have contracted the disease t days after the outbreak begins is modelled by the equation

$$y = \frac{200}{1 + 199e^{-0.2t}}$$

Determine:

i) the number of individuals infected initially

ii) the number of individuals infected after 10 days

iii) determine the time when half the population will be infected.

Sol:

i) Initially implies that $t = 0$. Therefore,

$$\begin{aligned} y =& \frac{200}{1+199e^{-0.2t}} \\ =& \frac{200}{1+199e^{-0.2(0)}} \\ =& \frac{200}{1+199e^{0}} \\ =& \frac{200}{1+199(1)} && \text{since } e^0 = 1 \\ =& \frac{200}{200} \\ =& 1 && \text{Hence, one individual was infected initially} \end{aligned}$$

ii) Taking $t = 10$, we have,

$$\begin{aligned} y =& \frac{200}{1+199e^{-0.2t}} \\ =& \frac{200}{1+199e^{-0.2(10)}} \\ =& \frac{200}{1+199e^{-2}} \\ =& \frac{200}{1+199(0.135)} && \text{since } e^{-2} \approx 0.1353353 \\ =& \frac{200}{27.932} \\ =& 7.160 && \text{Hence, 7 individuals will be infected after 10 days} \end{aligned}$$

iii) Taking $y = \frac{1}{2}(200) = 100$ and solving for t, we have,

$$\begin{aligned} 100 =& \frac{200}{1+199e^{-0.2t}} \\ 100\left(1+199e^{-0.2t}\right) =& 200 \\ 1+199e^{-0.2t} =& 2 \\ e^{-0.2t} =& \frac{1}{199} \\ \ln e^{-0.2t} =& \ln(\frac{1}{199}) && \text{after taking logs on both sides} \\ -0.2t =& -\ln 199 \\ t =& \frac{\ln 199}{0.2} \\ t =& 26.5 \text{ days} \end{aligned}$$

The above example shows the basic use of the exponential function. For more on the application of the exponential and logarithmic functions, see the exercise below.

Exercise 10

1. Solve the following equations:

   a)$2^x = 32$ b)$(\frac{1}{2})^x = \frac{1}{16}$ c)$27^{4x} = 9^{x+1}$ d)$(\frac{1}{8})^{-2y} = 2^{y+3}$ e)$2^{2x+1} = 3(2^x) - 1$

2. Sketch the graphs of the following functions:

   a) $y = -4^x + 2$ b) $y = 3^{-x} - 1$ c) $y = (\frac{1}{4})^x$ d) $y = -2 + e^{-x}$ e) $y = 1 - e^x$

3. Write each of the following in logarithmic form:

   (a) $2^3 = 8$ b) $(\frac{2}{3})^{-3} = \frac{27}{8}$ c) $4^{-2} = \frac{1}{16}$ d) $(0.4)^3 = 0.064$ e) $5^1 = 5$

4. Write each of the following in exponential form:

   (a)$\log_2 64 = 6$ b) $\log_2 (\frac{1}{16}) = -4$ c)$\log_{10} 0.00001 = -5$ d) $\log_5 0.5 = -1$

5. Solve each of the following:

   a)$\log_3 81 = x$ b) $\log_x 81 = 2$ c)$\log_8 \frac{x}{2} = \frac{\log_8 x}{\log_8 2}$ d) $\log_4 y = \frac{3}{2}$ e) $\log_x 3 = \frac{1}{2}$

   g)$3^x = 5$ h) $2^x = 9$ i)$5^{2x+1} = 3^{2-x}$ j) $2^{2x-1} = 3^{2-x}$ k) $6^{x+1} = 8$

6. Simplify the following logarithms or write as a single quantity.

   (a)$2\log_b x + 4\log_b y - 3\log_b z$ (b) $2\log_b x + \frac{1}{2}\log_b(x-1) - 4\log_b(2x+5)$ (c)$-\ln x - 3\ln 2$

   (d)$2\ln 8 + 5\ln 2$ (e) $\frac{1}{2}\ln x - 5\ln x + 4\ln y$ (f)$\ln 64 - \ln 128 + \ln 32$

7. Find the possible values of x for which:

   a) $\log 5 + \log x = 0$ b) $\log_3 x - 2\log_x 3 = 1$ c) $\ln(x-4) + \ln(x-1) = 1$

   d) $\log(x-1) + \log(x+1) = 2\log(x+2)$ e) $\log_3(2-3x) = \log_9(6x^2 - 19x + 2)$

8. Write $\ln\left(\frac{125 \times \sqrt[4]{16}}{\sqrt[4]{81^3}}\right)$ in terms of $\ln 2$, $\ln 3$ and $\ln 5$

9. Show that $\log_4 3 = \log_2 \sqrt{3}$. Hence solve the following simultaneous equations

   $$3^y = 9^x \quad , \qquad 2\log_2 y = \log_4 3 + \log_2 x$$

10. Express $\log_9 xy$ in terms of $\log_3 x$ and $\log_3 y$. Hence solve for x and y in the simultaneous equations

    $$\log_9 xy = \frac{5}{2} \quad , \qquad \log_3 x \log_3 y = -6$$

11. In a certain bacterial culture, the equation $p(t) = 1000e^{0.4t}$ expresses the number of bacteria present as a function of time t, where t is expressed in hours.

    (a) How many bacteria are present initially?

    (b) How many bacteria are present after 2 hours?

    (c) After how many hours will the number of bacteria be double the initial number?

12. The rate at which a body cools is given by $\theta = 250e^{-0.05t}$ where the excess of the temperature of a body above its surrounding at time t minutes is θ°C:

    (a) Sketch the graph showing the natural decay curve for the first hour of cooling

    (b) Determine temperature after 25 minutes and the time when temperature is 195°C.

13. Suppose that a certain substance has a half life of 20 years. If there are presently 2500 milligrams of the substance, then the equation $Q(t) = 2500(2)^{-\frac{t}{20}}$ yields the amount remaining after t years

    (a) How much remains after 40 years?

    (b) How many years will it take for only half the initial amount to remain?

11
Binomial Expansions

11.1. Introduction

A binomial expression is one which contains two terms connected by a plus or minus sign, such as $(a+b)$, $(x-y)$, $(2x+y)^2$, etc. If the binomial $(x+y)$ is squared, the result is the expansion of $(x+y)^2$. Similarly, if its cubed, the result is the expansion of $(x+y)^3$ and so on.

We will be interested in the binomial expansions of the form $(a+x)^n$, where $n \in \mathbb{R}$, and usually rational. We start with the case where n is a positive integer. Consider the expansion of $(a+x)^n$ for integer values of n from 0 to 6:

$$(a+x)^0 = 1$$
$$(a+x)^1 = a + x$$
$$(a+x)^2 = a^2 + 2ax + x^2$$
$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$
$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$$
$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$$
$$(a+x)^6 = a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$$

Note 11.1.1. From this expansion, we are able to notice the following:

- The powers of a decrease from n to 0 moving from left to right
- The powers of x are increasing from 0 to n moving from left to right
- The coefficients of each term of the expansions are symmetrical about the middle coefficient when n is even and symmetrical about the two middle coefficients when n is odd.
- For each term of the expansion, the powers of a and x add up to n
- The expansion of $(a+x)^n$ has $n+1$ terms

Isolating the coefficients of the above expansion gives the pattern below called the **Pascal's Triangle**, named after the French Mathematician Pascal (1623-1662). Working through the pattern in the triangle can give us the coefficients of the expansion $(a+x)^n$, and hence, the expansion can be determined. The figure below shows part of the **Pascal's Triangle**. Each number in the triangle is obtained by adding together the two numbers directly above.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

The second row is 1 2 1. This represents the coefficients of $(a+x)^2 = a^2 + 2ax + x^2$.

Similarly, the third row is 1 3 3 1 which has coefficients of $(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$

The last row, 1 6 15 20 15 6 1 has coefficients of $(a+x)^6$

Studying the pattern of the Pascal's Triangle, enables us to obtain coefficients of $(a+x)^n$ for $n \in \mathbb{N}$. For example, we can see that the coefficients of $(a+x)^7$ will make the next row in our diagram above. Verify that this is given by

1 7 21 35 35 21 7 1

Hence, the expansion of $(a+x)^7$ is given by

$$
\begin{aligned}
(a+x)^7 =& \mathbf{1}a^7 + \mathbf{7}a^6x + \mathbf{21}a^5x^2 + \mathbf{35}a^4x^3 + \mathbf{35}a^3x^4 + \mathbf{21}a^2x^5 + \mathbf{7}ax^6 + \mathbf{1}x^7 \\
=& a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + 1x^7
\end{aligned}
$$

Note that the coefficients of a decreases from 7 to 0, while the coefficients of x increases from 0 to 7. Also, the sum of the powers is always equal to 7.

Example 11.1.1. Expand $(a+x)^8$

Sol: Exercise

Example 11.1.2. Expand the binomial $(3x-1)^4$

Sol: Using the Pascal's Triangle for $n=4$, the coefficients are: 1 4 6 4 1. Hence

$$
\begin{aligned}
(3x-1)^4 =& \mathbf{1}(3x)^4(-1)^0 + \mathbf{4}(3x)^3(-1) + \mathbf{6}(3x)^2(-1)^2 + \mathbf{4}(3x)(-1)^3 + \mathbf{1}(3x)^0(-1)^4 \\
=& 81x^4 - 108x^3 + 54x^2 - 12x + 1
\end{aligned}
$$

Example 11.1.3. Find in ascending powers of x the expansion of $(2-\frac{x}{2})^6$

Sol: For $n = 6$, the coefficients are: 1 6 15 20 15 6 1. Therefore,

$$\begin{aligned}(2-\frac{x}{2})^6 =& \mathbf{1}(2)^6 + \mathbf{6}(2)^5(-\frac{x}{2}) + \mathbf{15}(2)^4(-\frac{x}{2})^2 + \mathbf{20}(2)^3(-\frac{x}{2})^3 + \mathbf{15}(2)^2(-\frac{x}{2})^4 + \mathbf{6}(2)(-\frac{x}{2})^5 + \mathbf{1}(-\frac{x}{2})^6 \\ =& 64 - 6(2^4)x + 15(2^2)x^2 - 20x^3 + 15(\frac{1}{2^2})x^4 - 6(\frac{1}{2^4})x^5 + \frac{1}{2^6}x^6 \\ =& 64 - 96x + 60x^2 - 20x^3 + \frac{15}{4}x^4 - \frac{3}{8}x^5 + \frac{1}{64}x^6 \\ =& 64 - 96x + 60x^2 - 20x^3 + \frac{15x^4}{4} - \frac{3x^5}{8} + \frac{x^6}{64}\end{aligned}$$

Example 11.1.4. Find, in ascending powers of x, the first three terms in the expansion of the binomial $(1-\frac{3x}{2})^5$

Sol: For $n = 5$, the Coefficients from Pascal's Triangle are: 1 5 10 10 5 1 Hence,the first three terms end up to the coefficient of 10

$(1-\frac{3x}{2})^5 = 1(1)^5 + 5(1)^4(-\frac{3x}{2}) + 10(1)^3(-\frac{3x}{2})^2 + \cdots = 1 - \frac{15x}{2} + \frac{45x^2}{2} + \cdots$

Hence, the first three terms are $1 - \frac{15x}{2} + \frac{45x^2}{2}$

Example 11.1.5. Use the binomial expansion of $(1-\frac{x}{2})^4$ to find the exact value of $(0.995)^4$

Sol: $n = 4$. From Pascal's Triangle, the required coefficients are: 1 4 6 4 1. Hence,

$$\begin{aligned}(1-\frac{x}{2})^4 =& \mathbf{1}(1)^4(-\frac{x}{2}) + \mathbf{4}(1)^3(-\frac{x}{2}) + \mathbf{6}(1)^2(-\frac{x}{2})^2 + \mathbf{4}(1)(-\frac{x}{2})^3 + \mathbf{1}(-\frac{x}{2})^4 \\ =& 1 + 4(-\frac{x}{2}) + 6(-\frac{x}{2})^2 + 4(-\frac{x}{2})^3 + (-\frac{x}{2})^4 \\ =& 1 - 2x + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{x^4}{16}\end{aligned}$$

To find the exact value of $(0.995)^4$, equate it to $(1-\frac{x}{2})^4$ and determine the required value of x

Thus, $0.995 = 1 - \frac{x}{2} \implies x = 0.01$. Using this value of x in the expansion gives the results. Hence,

$$\begin{aligned}(0.995)^4 =& \left(1 - \frac{0.01}{2}\right)^4 \\ =& 1 - 2(0.01) + \frac{3(0.01)^2}{2} - \frac{(0.01)^3}{2} + \frac{(0.01)^4}{16} \\ =& 1 + \frac{3(0.01)^2}{2} + \frac{(0.01)^4}{16} - 2(0.01) - \frac{(0.01)^3}{2} \\ =& 1 + 0.00015 + 0.000000000625 - 0.02 - 0.0000005 \\ =& 1.000150000625 - 0.0200005 \\ =& 0.980149500625\end{aligned}$$

Hence, $(0.995)^4 = 0.980149500625$

11.2. The Binomial Theorem

The use of Pascal's Triangle in determining the coefficients of the Binomial expansion can be very tedious if the value of n is large. The approach works well for n values from 0 up to 8. For larger values, it does not work quite well as the computation becomes burdensome. Hence, a more general, yet non-burdensome method is needed. The binomial series or binomial theorem is a more general formula for raising a binomial expression to any power without lengthy multiplication. Below is the formula for the Binomial series expansion of $(a+bx)^n$.

$$(a+bx)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}(bx)^r$$

OR

$$(a+bx)^n = a^n + \binom{n}{1} a^{n-1}(bx) + \binom{n}{2} a^{n-2}(bx)^2 + \cdots + \binom{n}{r} a^{n-r}(bx)^r + \cdots + (bx)^n$$

where the expression $\binom{n}{r} = \frac{n!}{(n-r)!r!}$, denotes the Binomial coefficients.

When $a = 1$ in the binomial $(a+bx)^n$, the the formula for expansion simplifies to

$$(1+bx)^n = \sum_{r=0}^{n} \binom{n}{r} (bx)^r$$

OR

$$(1+bx)^n = 1 + \binom{n}{1} bx + \binom{n}{2} (bx)^2 + \cdots + \binom{n}{r} (bx)^r + \cdots + (bx)^n$$

Binomial Coefficients

Before we apply the Binomial Theorem to the Binomial Expansions, let us study the coefficients of the Binomial Series, $\binom{n}{r}$.

Definition 11.2.1. *The factorial of a positive integer, n, denoted $n!$ is defined by,*

$$n! = n(n-1)(n-2)(n-3)\cdots 2.1$$

Example 11.2.1. i) $5! = 5(4)(3)(2)(1) = 120$ ii) $2! = 2(1) = 2$ iii) $1! = 1$ iv) $0! = 1$

The factorial of a positive integer n is just the product of all integers from n down to 1

Therefore, the Binomial coefficient is now defined as

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Alternatively,

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Example 11.2.2. Evaluate the following

i) $\binom{8}{3}$ ii) $\binom{8}{5}$ iii) $\binom{8}{0}$ iv) $\binom{8}{8}$ v) $\binom{8}{1}$

Sol: Using $\binom{n}{r} = \frac{n!}{(n-r)!r!}$, we get

i) $\binom{8}{3} = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{(5\times4\times3\times2\times1)(3\times2\times1)} = \frac{8\times7\times6}{3\times2\times1} = \frac{8\times7}{1} = 56$

ii) $\binom{8}{5} = \frac{8!}{(8-5)!5!} = \frac{8!}{3!5!} = \frac{8\times7\times6}{3!} = 8 \times 7 = 56$

iii) $\binom{8}{0} = \frac{8!}{(8!)!0!} = \frac{8!}{8!\times0!} = \frac{8!}{8!\times1} = 1$

iv) $\binom{8}{8} = \frac{8!}{(8-8)!8!} = \frac{8!}{0!8!} = \frac{8!}{8!} = 1$

v) $\binom{8}{1} = \frac{8!}{(8-1)!1!} = \frac{8!}{7!1!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1} = 8$

Using the Binomial Theorem

By definition, we have

$$(a+bx)^n = a^n + \binom{n}{1}a^{n-1}(bx) + \binom{n}{2}a^{n-2}(bx)^2 + \cdots + \binom{n}{r}a^{n-r}(bx)^r + \cdots + (bx)^n$$

Alternatively, this can be written as

$$(a+bx)^n = a^n + na^{n-1}(bx) + \frac{n(n-1)}{2!}a^{n-2}(bx)^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}a^{n-r}(bx)^r + \cdots + (bx)^n$$

Note that the first four terms of the expansion of $(a+bx)^n$ are noticed as

$$(a+bx)^n = a^n + na^{n-1}(bx) + \frac{n(n-1)}{2!}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}(bx)^3 + \cdots$$

The coefficient of the r^{th} term in the expansion is given by $\binom{n}{r-1}$

OR in other words,

The coefficient of the $(r+1)^{th}$ term in the expansion is given by $\binom{n}{r}$

Hence,

$$\text{the } \mathbf{(r+1)^{th}} \text{ term of the expansion } \mathbf{(a+bx)^n} \text{ is } \binom{\mathbf{n}}{\mathbf{r}}\mathbf{(a)^{n-r}(bx)^r}$$

Example 11.2.3. Expand $(2+x)^6$

Sol: We can either use Pascal's Triangle or The Binomial Theorem. We use the later.

$(2+x)^6 = 2^6 + 6(2)^5x + \frac{6(5)}{2!}2^4x^2 + \frac{6(5)(4)}{3!}2^3x^3 + \frac{6(5)(4)(3)}{4!}2^2x^4 + \frac{6(5)(4)(3)(2)}{5!}2x^5 + \frac{6(5)(4)(3)(2)(1)}{6!}x^6$

$= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

Example 11.2.4. Use the Binomial Theorem to expand $(1-2x)^5$

Sol: using $(a+bx)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}(bx)^r$, we have

$$\begin{aligned}(1-2x)^5 =& \sum_{r=0}^{n} \binom{5}{r}(1)^{n-r}(-2x)^r \\ =& \sum_{r=0}^{n} \binom{5}{r}(-2x)^r \\ =& \binom{5}{0}(-2x)^0 + \binom{5}{1}(-2x)^1 + \binom{5}{2}(-2x)^2 + \binom{5}{3}(-2x)^3 + \binom{5}{4}1(-2x)^4 + \binom{5}{5}(-2x)^5 \\ =& 1 - 2x + 40x^2 - 80x^3 + 80x^4 - 32x^5\end{aligned}$$

Example 11.2.5. Find the first four terms in the expansion of $(x-2)^{12}$

Sol: Using the Binomial Theorem,

$$\begin{aligned}(x-2)^{12} =& \sum_{r=0}^{n} \binom{12}{r}(x)^{n-r}(-2)^r \\ =& x^{12} + 12x^{11}(-2) + \frac{12 \times 11}{2!}x^{10}(-2)^2 + \frac{12 \times 11 \times 10}{3!}x^9(-2)^3 + \cdots \\ =& x^{12} - 24x^{11} + 264x^{10} - 1760x^9 + \cdots\end{aligned}$$

Example 11.2.6. Find the 5^{th} term in the expansion of $\left(2x - \frac{1}{2}\right)^{12}$

Sol: Since we need the 5^{th} term, we take $r = 5 - 1 = 4$

The 5^{th} term $= \binom{12}{4}(2x)^{12-4}(-\frac{1}{2})^4 = \frac{10 \times 9 \times 8 \times 7}{4!}(2x)^6(\frac{1}{2})^4 = 840x^6$

Example 11.2.7. Find the term that is independent of x in the expansion of $(2x - \frac{1}{x})^{10}$

Sol: The $(r+1)^{th}$ term $= \binom{10}{r}(2x)^{10-r}(-\frac{1}{x})^r$. Hence

$$\begin{aligned}(r+1)^{th} \text{ term } =& \binom{10}{r}(2x)^{10-r}(-\frac{1}{x})^r \\ =& \binom{10}{r}(2)^{10-r}x^{10-r}(-1)^r(\frac{1}{x})^r \\ =& \binom{10}{r}(-1)^r(2)^{10-r}x^{10-r}x^{-r} \\ =& \binom{10}{r}(-1)^r(2)^{10-r}x^{10-2r}\end{aligned}$$

If any term is to be independent of x, then $x^{10-2r} = 1 \Longrightarrow 10 - 2x = 0 \Longrightarrow x = 5$

Hence, the 6^{th} term is independent of x

So far, our discussion of the Binomial Theorem has made use of only positive integer values of n. What if n is not a positive integer? The next subsection discusses the general application of the Binomial Theorem, provided n is rational.

Binomial Theorem for Any Rational Power

The Binomial Expansion formula,

$$(a+bx)^n = a^n + \binom{n}{1}a^{n-1}(bx) + \binom{n}{2}a^{n-2}(bx)^2 + \cdots + \binom{n}{r}a^{n-r}(bx)^r + \cdots + (bx)^n$$

applies to all rational powers provided that $|x| < \frac{a}{b}$ or $-\frac{a}{b} < x < \frac{a}{b}$. Further, for the series to converge, an additional point is needed that $|x| < a$.

Example 11.2.8. Use the Binomial expansion to find the first four terms in the polynomial approximation for $\frac{1}{(2-3x)^2}$

Sol: We rewrite $\frac{1}{(2-3x)^2}$ as $(2-3x)^{-2}$. Hence, using the Binomial Theorem, we have

$$\begin{aligned}\frac{1}{(2-3x)^2} =&(2-3x)^{-2}\\ =&(2)^{-2} + (-2)(2)^{-2-1}(-3x) + \frac{(-2)(-3)}{2!}(2)^{-2-2}(-3x)^2 + \frac{(-2)(-3)(-4)}{3!}(2)^{-5}(-3x)^3 + \cdots\\ =&\frac{1}{4} + \frac{6}{8}x + \frac{27}{16}x^2 + \frac{(-2)(-3)(-4)}{3!}(2)^{-5}(-3x)^3 + \cdots\\ =&\frac{1}{4} + \frac{3}{4}x + \frac{27}{16}x^2 + \frac{27}{8}x^3 + \cdots\end{aligned}$$

This is valid for $|3x| < 2 \Longrightarrow |x| < \frac{2}{3}$

Example 11.2.9. Expand $\frac{1}{(1+2x)^3}$ in ascending powers of x as far as the term in x^3, using the binomial series. State the limits of x for which the expansion is valid.

Sol: Rewrite $\frac{1}{(1+2x)^3}$ as $(1+2x)^{-3}$ and then use the Binomial Theorem.

$$\begin{aligned}\frac{1}{(1+2x)^3} =&(1+2x)^{-3}\\ =&1 + (-3)(2x) + \frac{(-3)(-3-1)}{2!}(2x)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}(2x)^3 + \cdots\\ =&1 - 6x + 24x^2 - 80x^3 + \cdots\end{aligned}$$

This expansion is valid for $|x| < \frac{1}{2}$, ie, it is valid provided $-\frac{1}{2} < x < \frac{1}{2}$

Example 11.2.10. Expand $\sqrt{1-x}$ in ascending powers of x as far as the term in x^3, using the binomial series. State the limits of x for which the expansion is valid.

Sol: Rewrite $\sqrt{1-x}$ as $(1-x)^{\frac{1}{2}}$ and then use the Binomial Theorem with $n = \frac{1}{2}$.

$$\begin{aligned}\sqrt{1-x} =&(1-x)^{\frac{1}{2}}\\ =&1 + (\frac{1}{2})(-x) + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!}(-x)^2 + \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(-x)^3 + \cdots\\ =&1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \cdots\end{aligned}$$

This expansion is valid for $|x| < 1$, ie, it is valid provided $-1 < x < 1$

Example 11.2.11. Use the Binomial expansion to find a series expansion for the rational expression $\frac{3x+5}{x^2+2x-3}$ up to the term in x^3.

Sol: Using partial fraction decomposition, we can show that $\frac{3x+5}{x^2+2x-3} = \frac{1}{x+3} + \frac{2}{x-1}$.

To obtain the required series expansion, we obtain the expansion for $\frac{1}{x+3}$ and the expansion for $\frac{2}{x-1}$. Then add the two expansions.

$$\begin{aligned}
\frac{1}{x+3} =&(3+x)^{-1}\\
=&(3)^{-1}(1+\frac{x}{3})^{-1}\\
=&\frac{1}{3}[1+(-1)(\frac{x}{3})+\frac{(-1)(-2)}{2!}(\frac{x}{3})^2+\frac{(-1)(-2)(-3)}{3!}(\frac{x}{3})^3+\cdots]\\
=&\frac{1}{3}[1-\frac{x}{3}+\frac{x^2}{9}-\frac{x^3}{27}+\cdots]\\
=&\frac{1}{3}-\frac{x}{9}+\frac{x^2}{27}-\frac{x^3}{81}+\cdots
\end{aligned}$$

This expansion is valid for $|x| < 3$.

$$\begin{aligned}
\frac{2}{x-1} =&2(x-1)^{-1}\\
=&-2(1-x)^{-1}\\
=&-2[1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\frac{(-1)(-2)(-3)}{3!}(-x)^3+\cdots]\\
=&-2[1+x+x^2+x^3+\cdots]\\
=&-2-2x-2x^2-2x^3-\cdots
\end{aligned}$$

This expansion is valid for $|x| < 1$.

Hence, the expansion is:

$$\begin{aligned}
\frac{3x+5}{x^2+2x-3} =&(\frac{1}{3}-\frac{x}{9}+\frac{x^2}{27}-\frac{x^3}{81}+\cdots)+(-2-2x-2x^2-2x^3-\cdots)\\
=&\frac{1}{3}-2-\frac{x}{9}-2x+\frac{x^2}{27}-2x^2-\frac{x^3}{81}-2x^3+\cdots\\
=&-\frac{5}{3}-\frac{19x}{9}-\frac{53x^2}{27}-\frac{163x^3}{81}-\cdots
\end{aligned}$$

This expansion is valid for $|x| < 1$

Example 11.2.12. Use the Binomial expansion to find a series expansion for the rational expression $\frac{1-x-x^2}{(1-2x)(1-x)^2}$ up to the term in x^3.

Sol: Exercise

12
Introduction to Calculus

12.1. Introduction

One of the most fundamental topics studied in mathematics is Calculus. Calculus is simply a branch of mathematics that studies changing quantities, such as velocity, acceleration, rate of inflation, rate of spread of disease, etc. Calculus is widely used by many different professionals to model real life phenomenon. Economists, Engineers and scientists in general apply calculus to solve the real word problems. Since calculus is the science of change, the rate at which a chemical reaction is taking place and the mode of that reaction can be determined using the concepts of calculus. Take the rate at which the population of a particular bacteria in a given culture is growing. All these and many other problems are simplified through the use of calculus to solve them.

12.2. Limits of Functions

The notion of a limit is fundamental to the study of calculus. Thus, it is important to acquire a good working knowledge of limits before proceeding to the other topics in calculus. Suppose that a function $f(x)$, is defined for all points in the domain of $f(x)$, except possibly at a particular point say c. We are interested in knowing the behaviour of this function as the domain values get closer and closer to this point c. If as the domain values approach c, the function $f(x)$ approaches a real number say L, then we say that the limit of $f(x)$ as x approaches c is L. We write this limit as

$$\lim_{x \to c} f(x) = L$$

We can determine the limit of a function using three basic methods;

1. **Numerical Approach:** This involves constructing a table of values of x getting closer and closer to c, with the corresponding $y = f(x)$ values.

2. **Graphical Approach:** Draw the graph of the function, then observe the limit of the y values as x values approach c.

3. **Analytic Approach:** Here, we use the concepts of algebra and calculus to determine the limits

We will focus on the use of graphical method and analytical approach as the methods of determining the limits. Let us now give a formal definition of a limit.

Definition 12.2.1. *Let $f(x)$ be a function defined on an open interval containing a point c, but not necessarily at c. If L is a real number such that as the x values approach c from either the left hand side or right hand side of c, $f(x)$ gets arbitrarily close to L, then L is called the limit of $f(x)$ as x approaches c. This is written as*

$$\lim_{x \to c} f(x) = L$$

Note that x can actually approach c for two directions, the L.H.S and the R.H.S. Hence we can talk of two kinds of limits:

- If as x approaches c from the L.H.S, $f(x)$ is approaching L_1, we write
$$\lim_{x \to c^-} f(x) = L_1$$
to denote the *left limit* of $f(x)$ as x approaches c. The minus sign on c^- indicates that the x values are being taken from the negative side of c, but are not necessarily negative.
- If as x approaches c from the R.H.S, $f(x)$ is approaching L_2, we write
$$\lim_{x \to c^+} f(x) = L_2$$
to denote the *right limit* of $f(x)$ as x approaches c. The plus sign on c^+ indicates that the x values are being taken from the positive side of c, but are not necessarily positive.

Theorem 12.2.2. *Let $f(x)$ be a function defined on an open interval containing a point c, but not necessarily at c. The limit of $f(x)$ as x approaches c exists if and only if the left limit and the right limit of $f(x)$ at c exist and are equal. ie*

$$\lim_{x \to c} f(x) = L \quad iff \quad \lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L$$

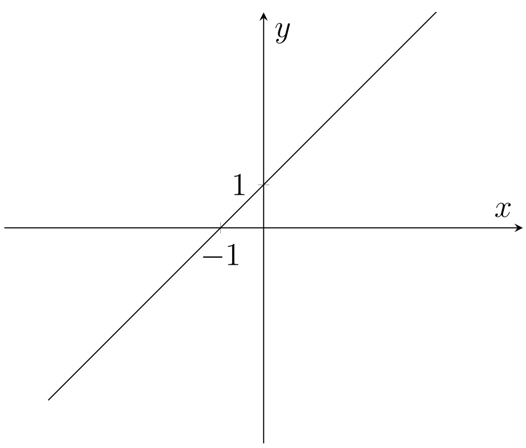
Example 12.2.1. Given the function $f(x) = x + 1$,

i) sketch the graph $y = x + 1$

ii) Hence, determine $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$

iii) Does the limit exist?

Sol: i) The sketch of this function is a straight line $y = x + 1$, shown below.



ii) From the graph, we can see that

$$\lim_{x\to 0^-} f(x) = 1 \text{ and } \lim_{x\to 0^+} f(x) = 1$$

iii) Using the concept from the theorem above, since

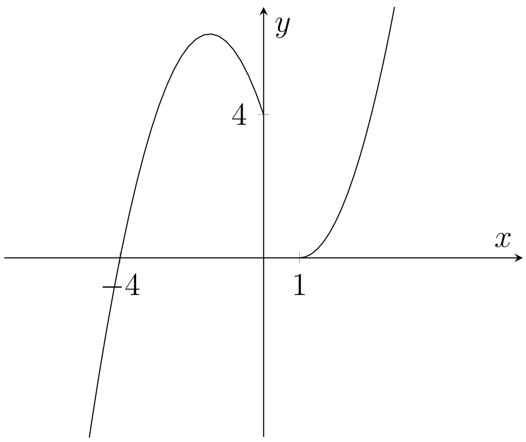
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1$$

Therefore, we conclude that the limit exist and that, $\lim_{x\to 0} f(x) = 1$

Example 12.2.2. Sketch the graph of the function $f(x) = x^3 - 4x^2 - x + 4$. Hence, determine;

i) $\lim_{x\to -\frac{1}{2}} f(x)$ *ii)* $\lim_{x\to 0} f(x)$ *iii)* $\lim_{x\to 1} f(x)$ *iv)* $\lim_{x\to 2} f(x)$

Sol: This is a polynomial of degree 3. The sketch is shown below.



i) From the graph,

$$\lim_{x\to -\frac{1}{2}^-} f(x) = \lim_{x\to -\frac{1}{2}^-} x^3 - 4x^2 - x + 4 = \frac{27}{8} \quad \text{and} \quad \lim_{x\to -\frac{1}{2}^+} f(x = \lim_{x\to -\frac{1}{2}^+} x^3 - 4x^2 - x + 4) = \frac{27}{8}$$

Hence, the limit exists at $c = -\frac{1}{2}$ and we say $\lim_{x\to -\frac{1}{2}} f(x) = \frac{27}{8}$

ii) similarly, from the graph,

$$\lim_{x\to o^-} f(x) = 4 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = 4$$

Hence, the limit exists at $c = 0$ and we write $\lim_{x\to 0} f(x) = 4$

iii) From the graph,

$$\lim_{x\to 1^-} f(x) = 0 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = 0$$

Hence, the limit exists at $c = 1$ and we write $\lim_{x\to 1} f(x) = 0$

iv) From the graph,

$$\lim_{x\to 2^-} f(x) = -6 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = -6$$

Hence, the limit exists at $c = 2$ and we write $\lim_{x\to 2} f(x) = -6$

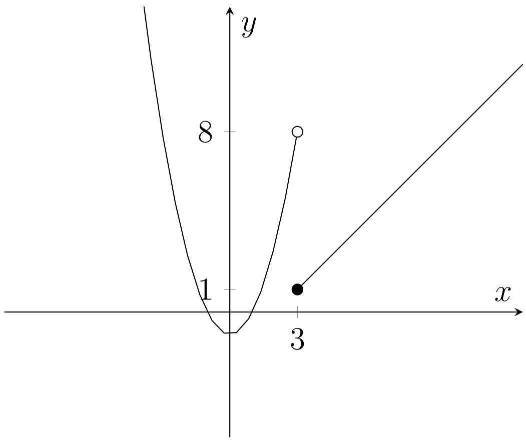
Example 12.2.3. Sketch the graph of the function $f(x)$ where

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x < 3; \\ x - 2 & \text{if } x \geq 3. \end{cases}$$

Hence, determine

i) $\lim_{x\to 2} f(x)$ *ii)* $\lim_{x\to 3} f(x)$ *iii)* $\lim_{x\to 1} f(x)$ *iv)* $\lim_{x\to 5} f(x)$

Sol:



i) From the graph,

$$\lim_{x\to 2^-} f(x) = 3 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = 3$$

Since the left limit equals the right limit, limit exists at $x = 2$. Therefore,

$$\lim_{x\to 2} f(x) = 3$$

ii) From the graph,

$$\lim_{x\to 3^-} f(x) = 8 \quad \text{while} \quad \lim_{x\to 3^+} f(x) = 1$$

Since the left limit is not equal to the right limit, limit does NOT exists at the point where $c = 3$. In mathematical terms, since

$$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$$

the limit at $c = 3$ does not exists

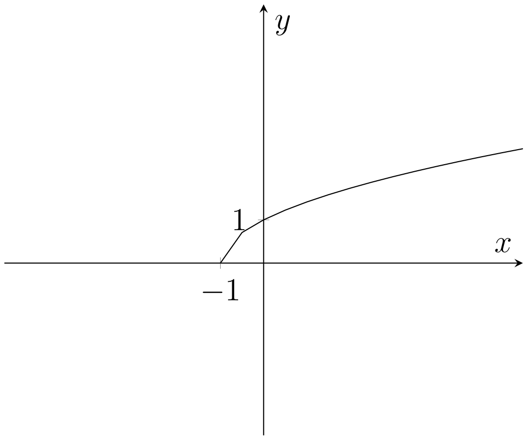
iii) Exercise

iv) Exercise

Example 12.2.4. Sketch the graph of the function $f(x) = \sqrt{x+1}$. Hence, determine

$$i)\ \lim_{x\to 3} f(x) \qquad ii)\ \lim_{x\to -\frac{1}{4}} f(x) \qquad iii)\ \lim_{x\to -1} f(x) \qquad iv)\ \lim_{x\to 0} f(x)$$

Sol: This is a radical function whose domain is $[-1, \infty)$



i) From the graph,

$$\lim_{x\to 3^-} f(x) = 2 \quad \text{and} \quad \lim_{x\to 3^+} f(x) = 2$$

Hence, the limit exists at $c = 3$ and we say

$$\lim_{x\to 3} f(x) = 2$$

ii) From the graph,

$$\lim_{x\to -\frac{1}{4}^-} f(x) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \lim_{x\to -\frac{1}{4}^+} f(x) = \frac{\sqrt{3}}{2}$$

Hence, the limit exists at $c = -\frac{1}{4}$ and we say

$$\lim_{x\to -\frac{1}{4}} f(x) = \frac{\sqrt{3}}{2}$$

iii) From the graph,

$\lim_{x\to -1^-} f(x)$ does not exists because $f(x) = \sqrt{x+1}$ is not defined on the L.H.S of -1

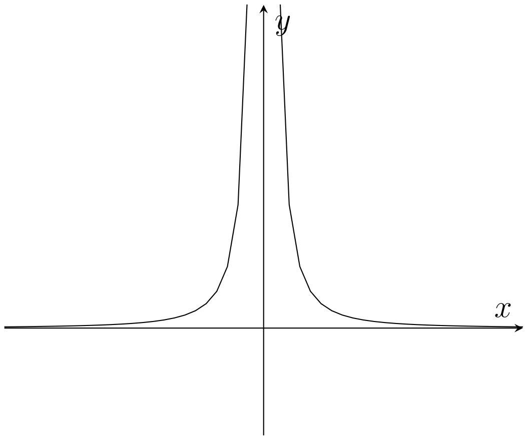
However, $\lim_{x\to -1^+} f(x) = 0$. Thus, the right limit exists, but the left limit does not exist.

Therefore, we conclude that $\lim_{x\to -1} f(x)$ does not exist for the function $f(x) = \sqrt{x+1}$

Example 12.2.5. Sketch the graph of $f(x) = \frac{1}{x^2}$. Hence, determine the existence of the limit

$$\lim_{x\to 0} \frac{1}{x^2}$$

Sol: This is a rational function. $D_f = \{x|\ x \neq 0, \quad x \in \mathbb{R}\}$



We can see that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x^2} = \infty$$

Because $f(x)$ is not approaching a real number L as x approaches 0, we conclude that the limit does not exists. ie

$$\lim_{x\to 0} \frac{1}{x^2} = \infty \qquad \text{Hence, the limit does not exist.}$$

12.2.1 Properties of Limits

So far, we have determined our limits through graph sketching. Recall that the limit of a function $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$. In such cases the, the limit can be evaluated by **direct substitution.** Hence, we can evaluate limits analytically as

$$\lim_{x\to c} f(x) = f(c)$$

This applies to functions that are continuous at the point $x = c$. The following theorem gives some basic properties of Limits:

Theorem 12.2.3. *Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.*

$$\lim_{x\to c} f(x) = L \quad \textit{and} \quad \lim_{x\to c} g(x) = K$$

- *limit of a constant function* $\lim_{x\to c} b = b$
- *Limit of the identity function $f(x) = x$* $\lim_{x\to c} x = c$
- *Limit of a term of a polynomial* $\lim_{x\to c} x^n = c^n$
- *Limit involving a radical function* $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

- *Multiplication by a scalar quantity* $\lim_{x\to c}[bf(x)] = bL$
- *Limit raised to a power* $\lim_{x\to c}[f(x)]^n = L^n$
- *Product of two limits* $\lim_{x\to c}[f(x)g(x)] = LK$
- *Sum or difference of two limits* $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$
- *Quotient of two limits* $\lim_{x\to c}[\frac{f(x)}{g(x)}] = \frac{L}{K}$, *provided* $K \neq 0$

Example 12.2.6. Evaluate the following limits

i) $\lim\limits_{x\to 2} -5$ *ii)* $\lim\limits_{x\to -1}(x+7)$ *iii)* $\lim\limits_{x\to 3}(5x^2-2x+9)$ *iv)* $\lim\limits_{x\to 0}\sqrt{x^2-3x+2}$

Sol: Using analytic method, we have

i) $\lim_{x\to 2} -5 = -5$

ii) $\lim_{x\to -1}(x+7) = (-1)+7 = 6$

iii) $\lim_{x\to 3}(5x^2-2x+9) = 5(3)^2 - 2(3) + 9 = 48$

iv) $\lim_{x\to 0}\sqrt{x^2-3x+2} = \sqrt{(0)^2-3(0)+2} = \sqrt{2}$

From the above example, we see how easy it is to obtain limits by simply applying our theorem above. We just substitute directly, provided the function remains defined for that direct substitution.

Example 12.2.7. Evaluate the following limits

i) $\lim\limits_{x\to 1}\frac{x^2+x+2}{x+1}$ *ii)* $\lim\limits_{x\to 1}\frac{x^3-1}{x-1}$ *iii)* $\lim\limits_{x\to -3}\frac{x^2+x-6}{x+3}$ *iv)* $\lim\limits_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

Sol: We use analytic methods.

i) Here, a direct substitution holds as the function is still defined for $x = 1$

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+x+2}{x+1} =& \frac{(1)^2+(1)+2}{(1)+1}\\ =& \frac{4}{2}\\ =& 2\end{aligned}$$

ii) Here, a direct substitution does NOT hold as the function is undefined for $x = 1$. Hence, we first factorise and then cancel out the common. Then we can substitute in the resulting expression.

$$\begin{aligned}\lim_{x\to 1}\frac{x^3-1}{x-1} =& \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}\\ =& \lim_{x\to 1}(x^2+x+1)\\ =& (1)^2+(1)+1\\ =& 1+1+1\\ =& 3\end{aligned}$$

iii) Again, direct substitution does NOT hold as the function $f(x) = \frac{x^2+x-6}{x+3}$ is undefined for $x = -3$. Hence, we factorise and then cancel out some common factors.

$$\begin{aligned}\lim_{x\to-3} \frac{x^2+x-6}{x+3} &= \lim_{x\to-3} \frac{(x+3)(x-2)}{x+3} \\ &= \lim_{x\to-3}(x-2) \\ &=(-3)-2 \\ &=-5\end{aligned}$$

iv) Again, direct substitution does NOT hold as the function $f(x) = \frac{\sqrt{x+1}-1}{x}$ is undefined for $x = 0$. Hence, we need to rationalize the numerator.

$$\begin{aligned}\lim_{x\to0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to0} \frac{\sqrt{x+1}-1}{x} \times \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} \\ &= \lim_{x\to0} \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} \\ &= \lim_{x\to0} \frac{(x+1)-1}{x\sqrt{x+1}+1} \\ &= \lim_{x\to0} \frac{x}{x\sqrt{x+1}+1} \\ &= \lim_{x\to0} \frac{1}{\sqrt{x+1}+1} \\ &= \frac{1}{\sqrt{(0)+1}+1} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2}\end{aligned}$$

Limits of Trigonometric Functions

Let c be a real number in the domain of a given trigonometric function. Then:

$i)$ $\lim_{x\to c} \sin x = \sin c$ $ii)$ $\lim_{x\to c} \cos x = \cos c$ $iii)$ $\lim_{x\to c} \tan x = \tan c$

$iv)$ $\lim_{x\to c} \sec x = \sec c$ $ii)$ $\lim_{x\to c} \csc x = \csc c$ $iii)$ $\lim_{x\to c} \cot x = \cot c$

Example 12.2.8. evaluate the following limits.

$i)$ $\lim_{x\to0} \sin x$ $ii)$ $\lim_{x\to\pi}(x\cos x)$ $iii)$ $\lim_{x\to0} \tan^2 x$

Sol: The results of the theorem above applies to this case. As before, we can have a direct substitution.

i) Hence, $\lim_{x\to 0} \sin x = \sin 0 = 0$

ii) We separate the two functions.

$$\begin{aligned}\lim_{x\to\pi}(x\cos x) =& \left(\lim_{x\to\pi} x\right)\left(\lim_{x\to\pi}\cos x\right)\\ =&(\pi)(\cos\pi)\\ =&(\pi)(-1)\\ =&-\pi\end{aligned}$$

iii) We uses product of a limit.

$$\begin{aligned}\lim_{x\to\pi}\tan^2 x =& \lim_{x\to\pi}(\tan x)^2\\ =& \left(\lim_{x\to\pi}\tan x\right)\left(\lim_{x\to\pi}\tan x\right)\\ =&(\tan\pi)(\tan\pi)\\ =&(0)(0)\\ =&0\end{aligned}$$

Two Special Trigonometric Limits

The following two limits are special cases. We will not prove them, but we must know them as we shall use them often. For a proof, see *The Squeeze Theorem:*

(i) $\lim_{x\to 0}\frac{\sin x}{x} = 1$ (ii) $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

Example 12.2.9. Evaluate the following limits

i) $\lim_{x\to 0}\frac{\tan x}{x}$ *ii)* $\lim_{x\to 0}\frac{\sin 4x}{x}$ *iii)* $\lim_{\theta\to 0}\frac{3(1-\cos\theta)}{\theta}$

Sol: We see that direct substitution will not apply here.

i) Direct substitution gives a zero in the denominator which is indeterminate. Therefore,

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\left(\frac{1}{x}\right)\left(\frac{\tan x}{1}\right)\\
&= \lim_{x\to 0}\left(\frac{1}{x}\right)\left(\frac{\sin x}{\cos x}\right)\\
&= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right)\\
&= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos x}\right)\\
&=(1)\left(\frac{1}{\cos 0}\right)\\
&=(1)(1)\\
&=1
\end{aligned}$$

ii) Direct substitution gives a zero in the denominator which is indeterminate. Thus,

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 4x}{x} &=1\times\lim_{x\to 0}\frac{\sin 4x}{x}\\
&=\frac{4}{4}\times\lim_{x\to 0}\frac{\sin 4x}{x}\\
&=\frac{4}{4}\left(\lim_{x\to 0}\frac{\sin 4x}{x}\right)\\
&=4\left(\lim_{x\to 0}\frac{\sin 4x}{4x}\right)\\
&=(4)(1)\\
&=4
\end{aligned}$$

iii) Using the special Trig Limit, $\lim_{\theta\to 0}\frac{3(1-\cos\theta)}{\theta} = 3\left(\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\right) = 3(0) = 0$

Limits at Infinity:

If $f(x)$ is a function and c is a real number in the domain of f such that

$$\lim_{x\to c} = \pm\infty$$

then $x = a$ is a vertical asymptote to the curve $y = f(x)$.

On the other hand, if $f(x)$ is a function and L is a real number of such that

$$\lim_{x\to\infty} = L \qquad \text{or} \qquad \lim_{x\to-\infty} = L$$

then $y = L$ is a horizontal asymptote to the curve $y = f(x)$.

Note: The following results hold for the limits at infinity

$-i)\ \lim_{x\to\infty}\frac{1}{x^2}=0$ $\quad ii)\ \lim_{x\to-\infty}\frac{1}{x^2}=0$ $\quad iii)\ \lim_{x\to0^-}\frac{1}{x^2}=\infty$ $\quad iv)\ \lim_{x\to0^+}\frac{1}{x^2}=\infty$

$-i)\ \lim_{x\to\infty}\frac{1}{x}=0$ $\quad ii)\ \lim_{x\to-\infty}\frac{1}{x}=0$ $\quad iii)\ \lim_{x\to0^-}\frac{1}{x}=-\infty$ $\quad iv)\ \lim_{x\to0^+}\frac{1}{x}=\infty$

Example 12.2.10. Evaluate the following limits.

$i)\ \lim_{x\to\infty}\frac{x+1}{x^2+1}$ $\quad ii)\ \lim_{x\to-\infty}\frac{x^3+x^2-2x-11}{x^3-7}$ $\quad iii)\ \lim_{x\to\infty}\frac{x^3+2}{x^2+1}$

Sol:

$i)$

$$\begin{aligned}\lim_{x\to\infty}\frac{x+1}{x^2+1}&=\lim_{x\to\infty}\frac{x\left(1+\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x^2}\right)}\\&=\lim_{x\to\infty}\left(\frac{1}{x}\right)\left(\frac{1+\frac{1}{x}}{1+\frac{1}{x^2}}\right)\\&=\left(\lim_{x\to\infty}\frac{1}{x}\right)\left(\lim_{x\to\infty}\frac{1+\frac{1}{x}}{1+\frac{1}{x^2}}\right)\\&=(0)\left(\frac{1+0}{1+0}\right)\\&=0\end{aligned}$$

$ii)$

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^3+x^2-2x-11}{x^3-7}&=\lim_{x\to-\infty}\frac{x^3\left(1+\frac{1}{x}-\frac{2}{x^2}-\frac{11}{x^3}\right)}{x^3\left(1-\frac{7}{x^3}\right)}\\&=\lim_{x\to-\infty}\left(\frac{x^3}{x^3}\right)\left(\frac{1+\frac{1}{x}-\frac{2}{x^2}-\frac{11}{x^3}}{1-\frac{7}{x^3}}\right)\\&=\lim_{x\to-\infty}\left(\frac{1+\frac{1}{x}-\frac{2}{x^2}-\frac{11}{x^3}}{1-\frac{7}{x^3}}\right)\\&=\left(\frac{1+0-0-0}{1-0}\right)\\&=\frac{1}{1}\\&=1\end{aligned}$$

iii)

$$\begin{aligned}\lim_{x\to\infty}\frac{x^3+2}{x^2+1} &= \lim_{x\to\infty}\frac{x^3\left(1+\frac{2}{x^3}\right)}{x^2\left(1+\frac{1}{x^2}\right)}\\ &= \lim_{x\to\infty}\left(\frac{x^3}{x^2}\right)\left(\frac{1+\frac{2}{x^3}}{1+\frac{1}{x^2}}\right)\\ &= \lim_{x\to\infty}(x)\left(\frac{1+\frac{2}{x^3}}{1+\frac{1}{x^2}}\right)\\ &= \left(\lim_{x\to\infty} x\right)\left(\lim_{x\to\infty}\frac{1+\frac{2}{x^3}}{1+\frac{1}{x^2}}\right)\\ &=(\infty)\left(\frac{1+0}{1+0}\right)\\ &=\infty\end{aligned}$$

12.3. Continuity of Functions

In Mathematics, the term continuity has much the same meaning as it has in everyday usage. To say that a function is continuous, at $x = c$ means that there is no interruption in the graph of $y = f(x)$ at c. This means that its graph is unbroken at $x = c$ and there are no holes, jumps or gaps. However, below is the definition of continuity that is acceptable in mathematics.

Definition 12.3.1. *A function f is **continuous at a point** c if the following three conditions are satisfied.*

1. *$f(c)$ is defined. ie, the function $f(x)$ is defined when $x = c$*
2. *$\lim_{x\to c} f(x)$ exists. ie, the limit of $f(x)$ exists at $x = c$*
3. *$\lim_{x\to c} f(x) = f(c)$*

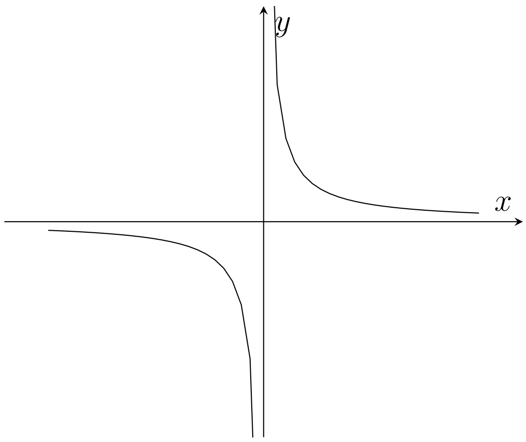
Definition 12.3.2. *A function $f(x)$ is **continuous at on an open interval** (a, b) if it is continuous at each point in the interval. If a function is continuous on the entire real line, $\mathbb{R} = (-\infty, \infty)$, then it is **everywhere continuous***

Example 12.3.1. Discuss the continuity of the following functions

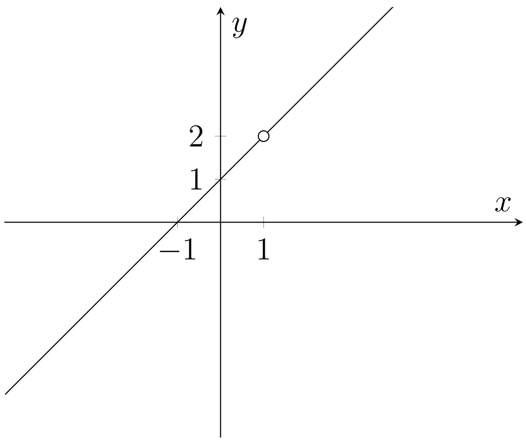
a) $f(x) = \dfrac{1}{x}$ *b)* $g(x) = \dfrac{x^2-1}{x-1}$ *c)* $h(x) = \cos x$ *d)* $f(x) = \begin{cases} x^2-1 & \text{if } x < 3; \\ x-2 & \text{if } x \geq 3. \end{cases}$

Sol:

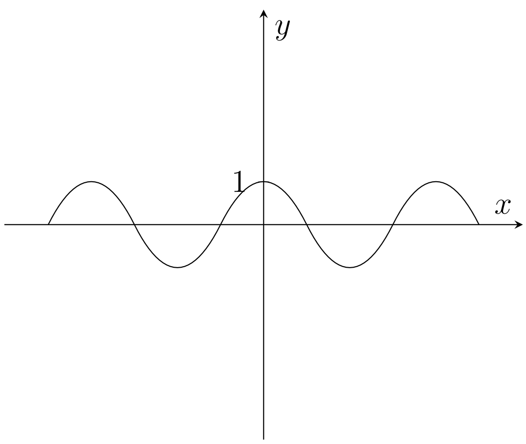
a) For $f(x) = \frac{1}{x}$, the domain, $D_f = \{x|\ x \neq 0, \quad x \in \mathbb{R}\}$. See the graph below. We see that the function is continuous for all points in its domain. At $x = 0$, the function is discontinuous. It has non-removable discontinuity. In other words, there is no way we can redefine $f(0)$ to make the function continuous at $x = 0$



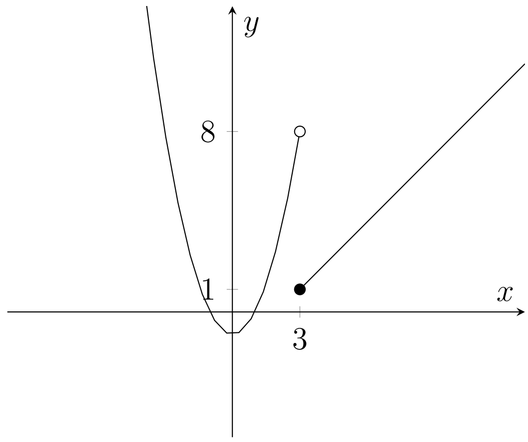
b) For $g(x) = \frac{x^2-1}{x-1}$, $D_g = \{x|\ x \neq 1, \quad x \in \mathbb{R}\}$. For all points in this domain, the function is continuous. At $x = 1$, the function has a removable discontinuity, ie we can **redefine** $f(1)$ to make the function continuous at $x = 1$. See the graph below. Notice that if we define $f(1) = 2$, the function becomes continuous at $x = 1$.



c) For $h(x) = \cos x$, the domain is $D_h = (-\infty, \infty)$ and we can see that it is continuous for all points in D_h.



d) For this function, $D_f = (-\infty, \infty)$ and we see that it is continuous for all real numbers except for $x = 3$. It has a non-removable discontinuity at $x = 3$. See the sketch below

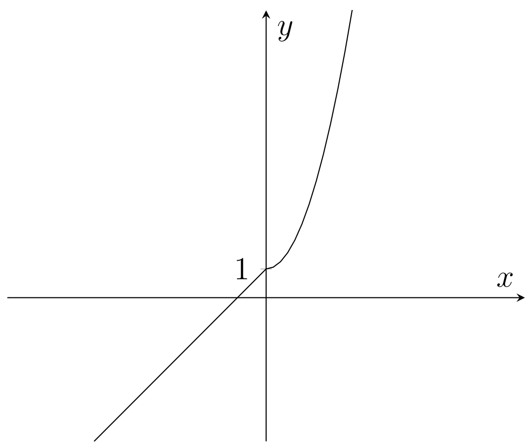


Example 12.3.2. Discuss the continuity of the following functions

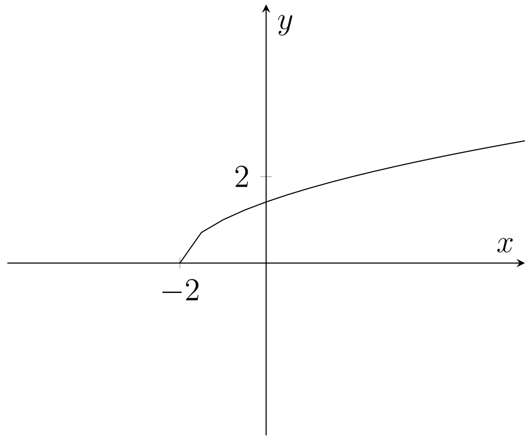
a) $f(x) = \begin{cases} x+1 & \text{if } x \leq 0; \\ x^2+1 & \text{if } x > 0. \end{cases}$ *b)* $g(x) = \sqrt{2+x}$ *c)* $h(x) = \sqrt{1-x}$ *d)* $k(x) = \dfrac{1}{x+2}$

Sol:

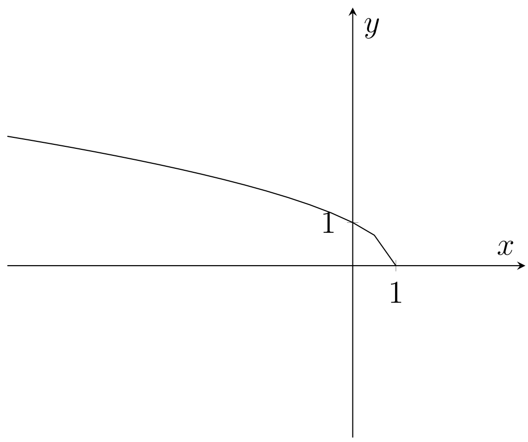
a) For this function, $D_f = (-\infty, \infty)$ and we see that it is continuous for all real numbers. See the graph below.



b) This function has domain $D_g = [-2, \infty)$. For all points in the open interval $(-2, \infty)$, the function is continuous. Since the right limit at $x = -2$ exists and it is $f(-2)$, we conclude that the function is also continuous on the closed interval $[-2, \infty)$. see graph below



c) This function has domain $D_h = (-\infty, 1]$. For all points in the open interval $(-\infty, 1)$, the function is continuous. Since the left limit at $x = 1$ exists and it is $f(1) = 0$, we conclude that the function is also continuous on the closed interval $(-\infty, 1]$. see graph below



d) Exercise

Example 12.3.3. Discuss the continuity of the function $f(x) = \sqrt{1 - x^2}$

Sol: Exercise

Note 12.3.1. We take note of the following on continuity.

- All polynomial functions are continuous on $\mathbb{R}$
- The radical functions and rational functions are continuous on their respective domains
- Trigonometric functions $\sin x$ and $\cos x$ are continuous on $\mathbb{R}$
- The tangent function is not continuous for points $x = \frac{n\pi}{2}$ where n is an odd integer

12.4. Differentiation

We have looked at the basic prerequisites for one to study calculus, the limit and continuity of a function. The notion of a limit and continuity are very vital for one to understand calculus. Now, we have now arrived at a crucial point in our study of calculus. Recall the notion of gradient of a straight line. The concept of a limit can be used to extend the notion of a gradient to curve. The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus −**differentiation.**

Definition 12.4.1. *The derivative of a continuous function f at a point x is denoted by $f'(x)$ and defined as*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative $f'(x)$ of a function $f(x)$, is itself a function. The process of finding the derivative of a function is called **differentiation.** A function f is **differentiable** at x if the derivative exists at x and **differentiable on an open interval (a,b)** if it is differentiable at every point in the interval.

In addition to the symbol $f'(x)$, other notations are used to denote the derivative of a function:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}[f(x)] = D_x[y]$$

Any of these notations are acceptable and can be used. We will mostly adopt the use of

$$f'(x), \qquad y' \qquad \text{and} \qquad \frac{dy}{dx}$$

12.4.1 Differentiation From First Principle

This is the standard procedure used to determine the derivative of any function. The process makes us of the definition of a derivative

Example 12.4.1. Find the derivative of the function $f(x) = x^3 + 2x$ from first principle

Sol:

$$\begin{aligned}
f'(x) =& \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
=& \lim_{h\to 0} \frac{[(x+h)^3 + 2(x+h)] - (x^3 + 2x)}{h} \\
=& \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x}{h} \\
=& \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 2h}{h} \\
=& \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 + 2)}{h} \\
=& \lim_{h\to 0} (3x^2 + 3xh + h^2 + 2) \\
=& 3x^2 + 3x(0) + (0)^2 + 2 \\
=& 3x^2 + 2
\end{aligned}$$

The approach above can be used to determine the derivative of any polynomial function. All differentiable functions that we will consider in our study of calculus can have their derivatives determined from first principle. This approach, however, is not always straight forward for some functions as determining the limit can be difficult at times. Hence, we will analyse other methods later.

Example 12.4.2. Differentiate the function $f(x) = \sqrt{x+1}$ from first principle.

Sol:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(x+h)+1}-\sqrt{x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(x+h)+1}-\sqrt{x+1}}{h} \times \frac{\sqrt{(x+h)+1}+\sqrt{x+1}}{\sqrt{(x+h)+1}+\sqrt{x+1}} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{(x+h)+1}-\sqrt{x+1}\right)\left(\sqrt{(x+h)+1}+\sqrt{x+1}\right)}{h\left(\sqrt{(x+h)+1}+\sqrt{x+1}\right)} \\
&= \lim_{h\to 0} \frac{[(x+h)+1]-(x+1)}{h\left(\sqrt{(x+h)+1}+\sqrt{x+1}\right)} \\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{(x+h)+1}+\sqrt{x+1}\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{(x+h)+1}+\sqrt{x+1}} \\
&= \frac{1}{\sqrt{(x+0)+1}+\sqrt{x+1}} \\
&= \frac{1}{2\sqrt{x+1}}
\end{aligned}
$$

Example 12.4.3. Differentiate the function $g(x) = \frac{1}{x+1}$ from first principle

Sol:

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)+1}-\frac{1}{x+1}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{h(x+h+1)(x+1)} \\
&= \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)} \\
&= \frac{-1}{(x+0+1)(x+1)} \\
&= -\frac{1}{(x+1)^2}
\end{aligned}
$$

The two examples above show the technique for differentiating from first principle for radical and rational functions respectively.

12.4.2 Basic Differentiation Rules

In this section we look at differentiation rules that allow us to find derivatives without the **direct** use of the limit definition.

1. **The Constant Rule:** The derivative of a constant is 0. That is, if $c \in \mathbb{R}$, then

$$\frac{d}{dx}[c] = 0$$

Example 12.4.4. Differentiate the following functions

i) $y = 7$ ii) $f(x) = 0$ iii) $s(t) = -3$ iv) $y = 3\pi^2$

Sol: All functions here are constants

i) $\frac{dy}{dx} = 0$ ii) $f'(x) = 0$ iii) $s'(t) = 0$ iv) $y' = 0$

2. **The Power Rule:** If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$f'(x) = nx^{n-1}$$

Example 12.4.5. Differentiate the following functions

i) $y = x^3$ ii) $f(x) = x$ iii) $s(t) = \sqrt[3]{x^2}$ iv) $g(x) = \frac{1}{x^5}$

Sol: We use the power rule

i) $\frac{dy}{dx} = 3x^2$ ii) $f'(x) = 1$ iii) $s'(t) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$ iv) $g'(x) = -5x^{-4} = -\frac{5}{x^4}$

3. **The constant Multiple Rule:** If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Example 12.4.6. Differentiate $y = \frac{2}{x}$ with respect to x.

Sol: Using the constant multiple rule

$$\frac{dy}{dx} = \frac{d}{dx}[\frac{2}{x}] = 2\frac{d}{dx}[\frac{1}{x}] = 2\frac{d}{dx}[x^{-1}] = 2(-x^{-2}) = -\frac{2}{x^2}$$

4. **The Sum and Difference Rule:** Let $u(x)$ and $v(x)$ be two differentiable functions of x. If $f(x) = u(x) + v(x)$, then

$$f'(x) = u'(x) + v'(x) \qquad \text{Sum Rule}$$

If $f(x) = u(x) - v(x)$, then

$$f'(x) = u'(x) - v'(x) \qquad \text{Difference Rule}$$

Example 12.4.7. Differentiate $y = x^3 - 4x + 5$

Sol: Using the sum and difference rule, we differentiate term by term

$$\frac{dy}{dx} = 3x^2 - 4 + 0 = 3x^2 - 4$$

5. **The Product Rule:** Let $u(x)$ and $v(x)$ be two differentiable functions of x. If the function $f(x) = u(x)v(x)$ is the product of $u(x)$ and $v(x)$, then

$$f'(x) = u(x)v'(x) + v(x)u'(x)$$

Example 12.4.8. Differentiate the function $f(x) = (2x - x^4)(x^3 - x - 12)$

Sol: Use the product rule: Let $u(x) = 2x - x^4$ and $v(x) = x^3 - x - 12$ Then, $u'(x) = 2 - 4x^3$ and $v'(x) = 3x^2 - 1$. Thus

$$\begin{aligned} f'(x) =& u(x)v'(x) + v(x)u'(x) \\ =& (2x - x^4)(3x^2 - 1) + (x^3 - x - 12)(2 - 4x^3) \\ =& 6x^3 - 2x - 3x^6 + x^4 + 2x^3 - 2x - 24 - 4x^6 + 4x^4 + 48x^3 \\ =& -7x^6 + 5x^4 + 56x^3 - 4x - 24 \end{aligned}$$

6. **The Quotient Rule:** Let $u(x)$ and $v(x)$ be two differentiable functions of x. If the function $f(x) = \frac{u(x)}{v(x)}$ is the quotient of $u(x)$ and $v(x)$, then

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

Example 12.4.9. Find the derivative of the function $f(x) = \frac{x^2 - \sqrt{x+1}}{\sqrt{x}}$

Sol: Let $u(x) = x^2 - \sqrt{x+1}$ and $v(x) = \sqrt{x}$. Then $u'(x) = 2x - \frac{1}{2\sqrt{x+1}}$ and $v' = \frac{1}{2\sqrt{x}}$. Using the Quotient Rule, we have

$$\begin{aligned} f'(x) =& \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} \\ =& \frac{(\sqrt{x})\left(2x - \frac{1}{2\sqrt{x+1}}\right) - \left(x^2 - \sqrt{x+1}\right)\left(\frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x}\right)^2} \\ =& \frac{1}{2}\left[3\sqrt{x} + \frac{1}{x\sqrt{x^2+x}}\right] \end{aligned}$$

Example 12.4.10. Differentiate $y = \frac{3x-1}{x^2+5x}$

Sol: Let $u(x) = 3x - 1$ and $v(x) = x^2 + 5x$. Then $u'(x) = 3$ and $v' = 2x + 5$. Using the quotient rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}\\ &= \frac{(x^2+5x)(3) - (3x-1)(2x+5)}{[x^2+5x]^2}\\ &= \frac{3x^2 + 15x - 6x^2 - 13x + 5}{(x^2+5x)^2}\\ &= \frac{-3x^2 + 2x + 5}{(x^2+5x)^2}\end{aligned}$$

Derivatives of Trigonometric Functions

We start with the derivative of the sin and cos functions. The derivatives for the other Trigonometric functions will be derived from these two using the discussed properties of derivatives.

Example 12.4.11. Differentiate the following functions from first principle

i) $f(x) = \sin x$ *ii)* $g(x) = \cos x$

Sol:

i) By definition;

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h\to 0} \frac{\sin(x+h) - (\sin x)}{h}\\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}\\ &= \lim_{h\to 0} \frac{\cos x \sin h - (\sin x)(1 - \cos h)}{h}\\ &= \lim_{h\to 0} \left[(\cos x)\left(\frac{\sin h}{h}\right) - (\sin x)\left(\frac{1-\cos h}{h}\right)\right]\\ &= (\cos x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) - (\sin x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)\\ &= (\cos x)(1) - (\sin x)(0)\\ &= \cos x\end{aligned}$$

$$\text{Therefore, if } f(x) = \sin x, \text{ then } f'(x) = \cos x$$

ii) **Exercise:** Show using first principle that if $g(x) = \cos x$, then $g'(x) = -\sin x$.

Example 12.4.12. Differentiate the following functions

i) $y = 2\sin x$ *ii)* $y = \dfrac{\cos x}{5}$ *iii)* $y = x^2 + 3\cos x$

Sol:

i) $\frac{dy}{dx} = 2\cos x$ ii) $y = \frac{\cos x}{5} = \frac{1}{5}\cos x \Longrightarrow \frac{dy}{dx} = -\frac{1}{5}\sin x$ iii) $\frac{dy}{dx} = 2x - 3\sin x$

Example 12.4.13. Find the derivative for each of the following

i) $y = x^3 \sin x$ *ii)* $y = \dfrac{\cos x}{x^2}$ *iii)* $y = \sin x \cos x$ *iv)* $y = \dfrac{\sqrt{x-2}}{\sin x}$

Sol:

i) Let $u(x) = x^3$ and $v(x) = \sin x$. Then $u'(x) = 3x^2$ and $v'(x) = \cos x$. Using product rule,

$$\begin{aligned}\frac{dy}{dx} &= uv' + vu' \\ &= (x^3)(\cos x) + (\sin x)(3x^2) \\ &= x^3 \cos x + 3x^2 \sin x \\ &= x^2 (x \cos x + 3 \sin x)\end{aligned}$$

ii) Let $u(x) = \cos x$ and $v(x) = x^2$. Then $u'(x) = -\sin x$ and $v'(x) = 2x$. By quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{(x^2)(-\sin x) - (\cos x)(2x)}{[x^2]^2} \\ &= \frac{-x^2 \sin x - 2x \cos x}{x^4} \\ &= \frac{-x(x \sin x + 2\cos x)}{x^4}\end{aligned}$$

iii) Let $u(x) = \sin x$ and $v(x) = \cos x$. Then $u'(x) = \cos x$ and $v'(x) = -\sin x$. By product rule

$$\begin{aligned}\frac{dy}{dx} &= uv' + vu' \\ &= (\sin x)(-\sin x) + (\cos x)(\cos x) \\ &= -\sin^2 x + \cos^2 x \\ &= \cos^2 x - \sin^2 x \\ &= \cos 2x\end{aligned}$$

iv) Let $u(x) = \sqrt{x-2}$ and $v(x) = \sin x$. Then $u'(x) = \frac{1}{2\sqrt{x-2}}$ and $v'(x) = \cos x$. By quotient rule,

$$\begin{aligned}\frac{dy}{dx} =& \frac{vu' - uv'}{v^2}\\ =& \frac{(\sin x)(\frac{1}{2\sqrt{x-2}}) - (\sqrt{x-2})(\cos x)}{[\sin x]^2}\\ =& \frac{\frac{\sin x}{2\sqrt{x-2}} - (\sqrt{x-2})(\cos x)}{[\sin x]^2}\\ =& \frac{\sin x - 2(x-2)\cos x}{2(\sin^2 x)\sqrt{x-2}}\end{aligned}$$

We can now use the quotient rule to determine the derivatives of the other trigonometric functions.

Example 12.4.14. Differentiate $f(x) = \tan x$

Sol: Recall that $\tan x = \frac{\sin x}{\cos x}$. Hence $f(x) = \tan x$ can be written as $f(x) = \frac{\sin x}{\cos x}$ and then use the quotient rule.

Let $u(x) = \sin x$ and $v(x) = \cos x$. Then $u'(x) = \cos x$ and $v'(x) = -\sin x$. By quotient rule,

$$\begin{aligned}\frac{dy}{dx} =& \frac{vu' - uv'}{v^2}\\ =& \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{[\cos x]^2}\\ =& \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ =& \frac{1}{\cos^2 x}\\ =& \left(\frac{1}{\cos x}\right)^2\\ =& (\sec x)^2\\ =& \sec^2 x\end{aligned}$$

$$\text{Therefore, if } f(x) = \tan x, \text{ then } f'(x) = \sec^2 x$$

Exercise: Show using quotient rule that if $f(x) = \cot x$, then $f'(x) = -\csc x$.

Example 12.4.15. Differentiate $f(x) = \sec x$

Sol: Recall that $\sec x = \frac{1}{\cos x}$. Hence $f(x) = \sec x$ can be written as $f(x) = \frac{1}{\cos x}$ and then use the quotient rule.

Let $u(x) = 1$ and $v(x) = \cos x$. Then $u'(x) = 0$ and $v'(x) = -\sin x$. By quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{(\cos x)(0) - (1)(-\sin x)}{[\cos x]^2} \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{1}{\cos x}\frac{\sin x}{\cos x} \\ &= \sec x \tan x\end{aligned}$$

$$\text{Therefore, if } f(x) = \sec x, \text{ then } f'(x) = \sec x \tan x$$

Exercise: Show using quotient rule that if $f(x) = \csc x$, then $f'(x) = -\csc x \cot x$.

Example 12.4.16. Differentiate the following functions

i) $y = 3x^4 - 2\tan x$ ii) $x^2 \sec x$

Sol:

i) $\frac{dy}{dx} = 12x^3 - \sec^2 x$ ii) $\frac{dy}{dx} = (x^2)(\sec x \tan x) + (\sec x)(2x) = x^2 \sec x \tan x + 2x \sec x$

The Chain Rule

Let us now discuss one of the most powerful rules of differentiation, the **chain Rule**. It enables us to differentiate composite functions. The theorem is stated below.

Theorem 12.4.2. *Let $y = f(u)$ be a differentiable function of u. Further, if $u = g(x)$ is itself a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 12.4.17. Differentiate the function $y = (x^2 + x - 2)^5$

Sol: Let $u = x^2 + x - 2$. Then $y = u^5$, $\frac{dy}{du} = 5u^4$ and $\frac{du}{dx} = 2x + 1$. By Chain Rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (5u^4)(2x + 1) \\ &= 5u^4(2x + 1) \\ &= 5(x^2 + x - 2)^4(2x + 1) \\ &= 5(2x + 1)(x^2 + x - 2)^4\end{aligned}$$

Example 12.4.18. Given the function $y = \sqrt[3]{(x^2-1)^2}$, find $\frac{dy}{dx}$

Sol: Note that $\sqrt[3]{(x^2-1)^2} = (x^2-1)^{\frac{2}{3}}$. Hence, $y = (x^2-1)^{\frac{2}{3}}$. Using the chain rule, we let $u = x^2 - 1$, then $y = u^{\frac{2}{3}}$ so that $\frac{dy}{du} = \frac{2}{3}u^{-\frac{1}{3}}$ and $\frac{du}{dx} = 2x$.

$$\begin{aligned}\frac{dy}{dx} =& \frac{dy}{du}\frac{du}{dx}\\ =& \left(\frac{2}{3}u^{-\frac{1}{3}}\right)(2x)\\ =& \left(\frac{2}{3}\left(x^2-1\right)^{-\frac{1}{3}}\right)(2x)\\ =& \frac{4x}{3\sqrt[3]{x^2-1}}\end{aligned}$$

Example 12.4.19. Differentiate the following

i) $y = \cos^2 x$ ii) $y = \sin^3 4x$ iii) $y = \tan^2(x^2 - 3x + 1)$ iv) $y = \csc^3(2x^5 - 1)$

Sol: We use the chain rule

i) Recall that $y = \cos^2 x = (\cos x)^2$. Thus,

$$\begin{aligned}\frac{dy}{dx} =& (2\cos x)(-\sin x)\\ =& -2\cos x \sin x\\ =& -\sin 2x\end{aligned}$$

ii) Recall that $y = \sin^3 4x = (\sin 4x)^3$. Thus,

$$\begin{aligned}\frac{dy}{dx} =& (3\sin 4x)^2(\cos 4x)(4)\\ =& 12\sin^2(4x)\cos(4)\\ =& 12\sin^2 4x\cos 4x\end{aligned}$$

iii) Recall that $y = \tan^2(x^2 - 3x + 1) = [\tan(x^2 - 3x + 1)]^2$. Thus,

$$\begin{aligned}\frac{dy}{dx} =& 2[\tan(x^2-3x+1)]^2 \times \sec(x^2-3x+1) \times (2x-3)\\ =& 2(2x-3)\tan^2(x^2-3x+1)\sec(x^2-3x+1)\end{aligned}$$

iv) Recall that $y = \csc^3(2x^5 - 1) = [\csc(2x^5 - 1)]^3$. Thus,

$$\begin{aligned}\frac{dy}{dx} =& 3[\csc(2x^5-1)]^2(-\cot(2x^5-1)\csc(2x^5-1))(10x^4)\\ =& -30x^4\csc^3(2x^5-1)\cot(2x^5-1)\end{aligned}$$

Implicit Differentiation

Most functions we have looked at now have been expressed explicitly, ie, they have been written in such a way that y can easily be written as the subject of the formula. For example, $y = 3x^2 = 2x + 7$ is an explicit function as y is an explicit function of x. Some functions can not have y written explicitly as a function of x. For example $y^3 + y^2 - 5xy - x^2 = -4$ is an **implicit** function since y can not easily be made the subject of the formula. To differentiate such functions, we use implicit methods. The following guidelines can be used:

- differentiate both sides with respect to x
- collect all terms involving $\frac{dy}{dx}$ on the left side of the equation, and all other terms to the right side.
- factor $\frac{dy}{dx}$ out of the left side of the equation.
- solve for $\frac{dy}{dx}$

Example 12.4.20. Find $\frac{dy}{dx}$ given that $y^3 - y + xy - 3x + 2x^2 = -7$

Sol: This is an implicit function since we can not make y the subject of the formula.

$$\begin{aligned} y^3 - y + xy - 3x + 2x^2 =& -7 \\ \frac{d}{dx}[y^3 - y + xy - 3x + 2x^2] =& \frac{d}{dx}[-7] \\ 3y^2\frac{dy}{dx} - \frac{dy}{dx} + x\frac{dy}{dx} + y - 3 + 4x =& 0 \\ (3y^2 - 1 + x)\frac{dy}{dx} + y + 4x - 3 =& 0 \\ (3y^2 - 1 + x)\frac{dy}{dx} =& -(y + 4x - 3) \\ \frac{dy}{dx} = -\frac{(y + 4x - 3)}{(3y^2 - 1 + x)} \end{aligned}$$

Example 12.4.21. Find $\frac{dy}{dx}$ if $\sin\sqrt{y} = x$

Sol: Making y the subject is not an easy task. We differentiate implicitly.

$$\begin{aligned} \sin\sqrt{y} =& x \\ \left(\frac{1}{2\sqrt{y}}\cos\sqrt{y}\right)\frac{dy}{dx} =& 1 \\ \frac{dy}{dx} =& \frac{2\sqrt{y}}{\cos\sqrt{y}} \end{aligned}$$

Higher Order Derivatives

Let $y = f(x)$ denote a function. The derivative $\frac{dy}{dx} = f'(x)$ is called the first order derivative. Recall that $f'(x)$ is itself a function as well. Hence, it can be differentiated to obtain $\frac{d^2y}{dx^2} = f''(x)$, which is called the second order derivative. Similarly, we can differentiate $f''(x)$ to obtain the third order derivative $\frac{d^3y}{dx^3} = f'''(x)$.

In general then, if $y = f(x)$ is a differentiable function, then we can obtain the n^{th} order derivative by differentiating $f(x)$ n times. Hence, the n^{th} order derivative has the notation given below:

$$\frac{d^ny}{dx^n} = y^n = f^n(x) = \frac{d^n}{dx^n}[f(x)] = D_x^n[y]$$

Example 12.4.22. Given the function $f(x) = x^4 - 3x^2 + 12$, find $f^{(4)}(x)$, the fourth order derivative.

Sol: $f'(x) = 4x^3 - 6x$. Differentiating further,we have $f''(x) = 12x^2 - 6$, $f'''(x) = 24x$ so that $f^{(4)}(x) = 24$.

Example 12.4.23. Find $\frac{d^3y}{dx^3}$ given that $y = \sin(x^2)$.

Sol:

$\frac{dy}{dx} = 2x\cos(x^2)$

$\frac{d^2y}{dx^2} = 2[x(-2x\sin(x^2)) + \cos(x^2)] = 2\cos(x^2) - 4x^2\sin(x^2)$

$\frac{d^3y}{dx^3} = 2[-4x\sin(x^2) - 2x^2\cos(x^2)2x - \sin(x^2)2x] = 2[-6x\sin(x^2) - 4x^3\cos(x^2)]$

Example 12.4.24. Find $\frac{d^2y}{dx^2}$ given that $x^2 + y^2 = 25$

Sol: differentiate implicitly

$2x + 2y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$. Therefore, using the quotient rule,

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\left[\frac{(y)(1) - (x)(\frac{dy}{dx})}{y^2}\right] \\ &= -\left[\frac{(y)(1) - (x)(-\frac{x}{y})}{y^2}\right] \\ &= -\frac{x^2 + y^2}{y^3} \\ &- \frac{25}{y^3}\end{aligned}$$

Derivatives of Exponential Functions

Let $f(x) = e^x$. We can find the derivative of the exponential function from first principle as shown below.

$$\begin{aligned} f'(x) =& \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ =& \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\ =& \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\ =& \lim_{h\to 0} \frac{e^x \left(e^h - 1\right)}{h} \\ =& \lim_{h\to 0} (e^x) \left(\frac{e^h - 1}{h}\right) \\ =& (e^x) \left(\lim_{h\to 0} \frac{e^h - 1}{h}\right) \\ =& (e^x)(1) \\ =& e^x \end{aligned}$$

Therefore, if $f(x) = e^x$, then the derivative is given as $f'(x) = e^x$

Example 12.4.25. Differentiate the following

i) $y = e^{3x}$ *ii)* $y = 5e^{-x^2}$ *iii)* $y = x^2 - e^{\sqrt{x}-1}$ *iv)* $y = xe^{\sin x}$ *v)* $y = \dfrac{e^{-x} + 1}{\sec x}$

Sol:

i) Let $u = 3x$. Then $y = e^u$ so that $\frac{du}{dx} = 3$ and $\frac{dy}{du} = e^u$. By the chain Rule,

$$\begin{aligned} \frac{dy}{dx} =& \frac{dy}{du}\frac{du}{dx} \\ =& (3)(e^u) \\ =& 3e^{3x} \end{aligned}$$

ii) Let $u = -x^2$. Then $y = 5e^u$ so that $\frac{du}{dx} = -2x$ and $\frac{dy}{du} = 5e^u$. By the chain Rule,

$$\begin{aligned} \frac{dy}{dx} =& \frac{dy}{du}\frac{du}{dx} \\ =& (-2x)(5e^u) \\ =& -2xe^{-x^2} \end{aligned}$$

iii) Similarly, applying the chain rule and the sum/difference rule, we have

$$\begin{aligned} \frac{dy}{dx} =& 2x - \frac{1}{2\sqrt{x}} e^{\sqrt{x}-1} \\ =& \frac{4x\sqrt{x} - e^{\sqrt{x}-1}}{2\sqrt{x}} \end{aligned}$$

iv) We use the product rule together with the chain rule.

$$\begin{aligned}\frac{dy}{dx} =&(x)\left[(\cos x)e^{\sin x}\right] + \left(e^{\sin x}\right)(1)\\ =&x\cos x e^{\sin x} + e^{\sin x}\\ =&(x\cos x + 1)e^{\sin x}\end{aligned}$$

v) Exercise [*Hint:* Use the quotient Rule and the Chain Rule]

Derivatives of Logarithmic Functions

We start with the natural logarithmic function, $f(x) = \ln x$. We will differentiate this function implicitly. Let $y = \ln x$. Converting this to exponential form, we have $x = e^y$. Hence, using implicit differentiation, we have

$$\begin{aligned} x =& e^y\\ 1 =& e^y\frac{dy}{dx}\\ \frac{dy}{dx} =& \frac{1}{e^y}\\ \frac{dy}{dx} =& \frac{1}{x} && \text{since } e^y = x\end{aligned}$$

$$\text{Therefore, if } f(x) = \ln x, \text{ then the derivative is given as } f'(x) = \frac{1}{x}$$

Example 12.4.26. Find the derivative for each of the following functions

$i)\ y = x^3\ln x \qquad ii)\ y = \log_{10}(3x^2 - 4) \qquad iii)\ y = \dfrac{\ln\sqrt{x}}{x^2} \qquad iv)\ y = x^x \qquad v)\ y = 2^{3x}$

Sol:

i) Let $u = x^3$, $v = \ln x$. Then $u' = 3x^2$ and $v' = \frac{1}{x}$. By the product rule, we have

$$\begin{aligned}\frac{dy}{dx} =&(u)(v') + (v)(u')\\ =&(x^3)\left(\frac{1}{x}\right) + (\ln x)(3x^2)\\ =&x^2 + 3x^2\ln x\end{aligned}$$

ii) We change the base to the natural logarithm:

$$\begin{aligned} y =& \log_{10}(3x^2 - 4)\\ =&\frac{\ln(3x^2-4)}{\ln 10}\\ =&\frac{1}{\ln 10}\ln(3x^2 - 4)\end{aligned}$$

We can differentiate the function $y = \ln(3x^2 - 4)$. Let $u = 3x^2 - 4$. Then $y = u$ so that $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 6x$. By the Chain Rule, we have

$$\begin{aligned}\frac{dy}{dx} =& \frac{dy}{du}\frac{du}{dx} \\ =& \left(\frac{1}{u}\right)(6x) \\ =& \left(\frac{1}{3x^2-4}\right)(6x) \\ =& \frac{6x}{3x^2-4}\end{aligned}$$

$$\text{Hence, for } y = \frac{1}{\ln 10}\ln(3x^2-4), \quad \frac{dy}{dx} = \left(\frac{1}{\ln 10}\right)\left(\frac{6x}{3x^2-4}\right) = \frac{6x}{(\ln 10)(3x^2-4)}$$

iii) Let $u = \ln\sqrt{x}$ and $v = x^2$. Then, by chain rule, $u' = \frac{1}{2x}$ and $v' = 2x$. By quotient rule,

$$\begin{aligned}\frac{dy}{dx} =& \frac{vu' - uv'}{v^2} \\ =& \frac{(x^2)\left(\frac{1}{2x}\right) - (\ln\sqrt{x})(2x)}{x^4} \\ =& \frac{\frac{x}{2} - 2x\ln\sqrt{x}}{x^4} \\ =& \frac{1 - 2\ln x}{2x^3}\end{aligned}$$

iv) We take the natural log on both sides, then differentiate implicitly. $y = x^x \implies \ln y = \ln x^x \implies \ln y = x\ln x$. Differentiating this implicitly, we have

$$\begin{aligned}\ln y =& x\ln x \\ \frac{1}{y}\frac{dy}{dx} =& (x)(\frac{1}{x}) + (\ln x)(1) \\ \frac{1}{y}\frac{dy}{dx} =& 1 + \ln x \\ \frac{dy}{dx} =& y(1 + \ln x) \\ \frac{dy}{dx} =& (x^x)(1 + \ln x) \\ \frac{dy}{dx} =& x^x(1 + \ln x)\end{aligned}$$

v) Exercise

12.4.3 Applications of Differentiation

Differential calculus can be applied to numerous problems in applied fields. We start with its application to the Tangents and Normal to curve.

Tangents and Normal to a curve:

If $y = f(x)$ is an equation of a curve, then the derivative $\frac{dy}{dx} = f'(x)$ is the gradient function as it gives the gradient of the tangent to the curve at any given point where the derivative exists.

Example 12.4.27. Find the gradient of the tangent to the curve $y = \frac{1}{x}$ at the point where $x = -1$. Hence, find the equation of this tangent and the corresponding normal.

Sol: $\frac{dy}{dx} = -\frac{1}{x^2}$. Hence, the gradient at $x = -1$ is given as $\frac{dy}{dx}|_{x=-1} = -\frac{1}{(-1)^2} = -1$. Hence, the equation of the tangent is

$$\begin{aligned} y - y_1 =& m(x - x_1) \\ y - (-1) =& -1(x - (-1)) \\ y + 1 =& -x - 1 \\ y =& -x - 2 \qquad \text{is the equation of the tangent line at } x = -1 \end{aligned}$$

Normal: Recall that the gradient of the normal is $-\frac{1}{m} = -\frac{1}{(-1)} = 1$

$$\begin{aligned} y - y_1 =& -\frac{1}{m}(x - x_1) \\ y - (-1) =& 1(x - (-1)) \\ y + 1 =& x + 1 \\ y =& x \qquad \text{is the equation of the normal line at } x = -1 \end{aligned}$$

Example 12.4.28. Find equation of the tangent to the curve $x^2 + y^2 = 10$ at the point $(1, 3)$.

Sol: Differentiate implicitly to get $2x+2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$. Gradient at $(1, 3)$: $\frac{dy}{dx}|_{(1,3)} = -\frac{1}{3}$

$$\begin{aligned} y - y_1 =& m(x - x_1) \\ y - 3 =& -\frac{1}{3}(x - 1) \\ 3y - x =& 10 \end{aligned}$$

Example 12.4.29. Find the equation of the tangent line to the curve $y^3 - 3xy^2 + \cos xy = 2$ at the point $(0, 1)$

Sol: Differentiate implicitly to get: $3y^2\frac{dy}{dx} - 3\left(x2y\frac{dy}{dx} + y^2\right) + \left(y + x\frac{dy}{dx}\right) - \sin xy = 0$

$$\text{Thus, } \frac{dy}{dx} = \frac{3y^2 + y\sin(xy)}{3y^2 - 6xy - x\sin(xy)} \qquad \text{so that, } \frac{dy}{dx}|_{(0,1)} = \frac{3(1)^2 + (1)\sin(0)}{3(1)^2 - 6(0) - (0)\sin(0)} = 1$$

Hence the required tangent is given by $y - 1 = 1(x - 0) \implies y = x + 1$ is the tangent line of interest.

Increasing and Decreasing Functions:

Let $y = f(x)$ be a curve. On any interval where $\frac{dy}{dx} > 0$, the function $y = f(x)$ is increasing on this interval. If $\frac{dy}{dx} < 0$, then the function is decreasing. If $\frac{dy}{dx} = 0$, then the function has a critical point or stationary point, where the function is neither increasing nor decreasing.

Example 12.4.30. Determine the range of values of x for which the function

$y = x^3 - 3x^2 - 9x + 4$ is   i) increasing   ii) decreasing   iii) stationary

Sol: The derivative: $\frac{dy}{dx} = 3x^2 - 6x - 9$

i) We need $\frac{dy}{dx} > 0$. So we solve this inequality.

$$\begin{aligned} 3x^2 - 6x - 9 >& 0 \\ x^2 - 2x - 3 >& 0 \\ (x+1)(x-3) >& 0 \end{aligned}$$

This function is increasing in the interval $(-\infty, -1) \cup (3, \infty)$ since $\frac{dy}{dx} > 0$ in this interval

ii) We need $\frac{dy}{dx} < 0$. ie, we need $(x+1)(x-3) < 0$. Verify that the function is decreasing in the interval $(-1, 3)$.

iii) The stationary points are points where $\frac{dy}{dx} = 0$. Hence, the stationary points are obtained by solving the equation $\frac{dy}{dx} = 0$, ie $(x+1)(x-3) = 0 \Longrightarrow x = -1$ and $x = 3$

Example 12.4.31. Determine the range of values of x for which the function

$y = x$ is   i) increasing   ii) decreasing   iii) stationary

Stationary Points: Maximum, Minimum and Point of Inflexion

Let $f(x)$ be a function. The curve $y = f(x)$ has a stationary point where $\frac{dy}{dx} = 0$. There are three types of stationary points

- **Maximum point:** The derivative here moves from positive through zero to negative values. Thus, at maximum, we have

$$\frac{dy}{dx} = 0 \qquad \text{and} \qquad \frac{d^2y}{dx^2} < 0$$

- **Minimum point:** The derivative here moves from negative through zero to positive values. Thus, at minimum, we have

$$\frac{dy}{dx} = 0 \qquad \text{and} \qquad \frac{d^2y}{dx^2} > 0$$

- **Point of Inflexion:** The derivative $\frac{dy}{dx} = 0$ but has the same value on both sides of the zero value

To determine the nature of the stationary points, the following steps may be used:

- Determine the stationary points by solving $\frac{dy}{dx} = 0$. This gives the values of x, which we can use to determine the corresponding y values.
- Determine the values of $\frac{d^2y}{dx^2}$. If the result is:
  (a) positive, the point is a minimum one,
  (b) negative, the point is a maximum one,
  (c) zero, the point is indecisively unknown.
  OR
- Determine the sign of the gradient of the curve just before and just after the stationary points. If the sign change for the gradient of the curve is:
  (a) positive to negative, the point is a maximum one
  (b) negative to positive, the point is a minimum one
  (c) positive to positive or negative to negative, the point is a point of inflexion

Example 12.4.32. Find the nature of the stationary points on the curve $y = 4x^3 - 3x^2 - 6x + 2$

Sol: The derivative, $\frac{dy}{dx} = 12x^2 - 6x - 6$ and $\frac{d^2y}{dx^2} = 24x - 6$. To get the critical values, set $\frac{dy}{dx} = 0$

$$\begin{aligned} \frac{dy}{dx} &=0 \\ 12x^2 - 6x - 6 &=0 \\ 2x^2 - x - 1 &=0 \\ (2x+1)(x-1) &=0 \end{aligned}$$

$$\text{Stationary points occur at } x = -\tfrac{1}{2} \text{ and at } x = 1$$

Since $\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{1}{2}} = 24(-\frac{1}{2}) - 6 = -18 < 0$, $x = -\frac{1}{2}$ gives a maximum point.

Since $\left.\frac{d^2y}{dx^2}\right|_{x=1} = 24(1) - 6 = 18 > 0$, $x = 1$ gives a minimum point.

Hence the turning points or stationary points of the curve $y = 4x^3 - 3x^2 - 6x + 2$ are:

when $x = -\frac{1}{2}$, then $y = 4(-\frac{1}{2})^3 - 3(-\frac{1}{2})^2 - 6(-\frac{1}{2}) + 2 = \frac{15}{4}$

when $x = 1$, then $y = 4(1)^3 - 3(1)^2 - 6(1) + 2 = -3$

The points are: $(-\frac{1}{2}, \frac{15}{4})$ which is a maximum point and $(1, -3)$ which is a minimum point

For more on maximum and other applications of calculus, see the references.

12.5. Integral Calculus

See the References or consult Mr. Mulenga F.

Exercise 15

1. Evaluate the following limits:

   $(a)\lim\limits_{x\to 2}(3x^2+x-5)$ $(b)\lim\limits_{x\to\pi}\tan x$ $(c)\lim\limits_{x\to 1}\sqrt{x+2}$ $(d)\lim\limits_{x\to 0}\dfrac{\sqrt{x+9}-3}{x}$ $(e)\lim\limits_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}$

   $(f)\lim\limits_{x\to -2}\dfrac{x^3+8}{x+2}$ $(g)\lim\limits_{x\to -3}\dfrac{x^2+5x+6}{x+3}$ $(h)\lim\limits_{x\to -2}\dfrac{2x^2+5x+2}{x^2+9x+14}$ $(i)\lim\limits_{x\to 1}\dfrac{x^3+x^2-8x+6}{x-1}$

2. Determine the following limits:

   $(a)\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$ $(b)\lim\limits_{x\to 0}e^x$ $(c)\lim\limits_{x\to 0}|x^2-x-2|$ $(d)\lim\limits_{x\to 0}\dfrac{1}{\cos x}$ $(e)\lim\limits_{x\to\infty}\dfrac{x^4-1}{x^3+x^2}$.

   $(f)\lim\limits_{x\to 0}\dfrac{x}{3x^2-2x}$ $(g)\lim\limits_{x\to 0}\dfrac{x^2}{2-x^2}$ $(h)\lim\limits_{x\to\infty}\dfrac{3x^2+7x^3}{x^2+5x^4}$ $(i)\lim\limits_{x\to\infty}\dfrac{3x^3+2x}{4-2x^2-7x^3}$ $(j)\lim\limits_{x\to\infty}\dfrac{x+1}{x^2+1}$

3. Differentiate the following functions from first principle:
   (a)$f(x)=x^2-4$ (b) $f(x)=\sqrt{x}-4x^2$ (c)$f(x)=\frac{1}{x}-\frac{1}{x^2}$ (d) $f(x)=\frac{1}{\sqrt{x}}+\sqrt{2-x}$
   e) $f(x)=\sqrt{x+2}$ (f) $f(x)=\frac{1}{\sqrt{1-x}}$ (g) $f(x)=\frac{1}{x-4}$ (h) $f(x)=\sin 5x$ (i) $f(x)=\cos 2x$

4. Differentiate the following with respect to x:

   $a)\ y=2x^3-7x-12$ $b)\ y=x^2\sqrt{x+2}$ $c)\ y=\dfrac{x^2-4x+1}{x^4-2}$ $d)\ y=(4x^3-x^2)^6$ $e)\ y=\dfrac{(x^2+1)^3}{x^{\frac{3}{2}}}$

5. Differentiate the following with respect to x:

   $a)\ y=\sec 2x$ $b)\ y=\cos^3(5x)$ $c)\ y=\tan\sqrt{x}$ $d)\ y=\tan^5(3x^2-1)$ $e)\ y=\dfrac{\sin(3x^2-1)}{\cos(3-2x^3)}$

6. For each of the following functions, find the derivative f':

   $(a)f(x)=e^{3x^2}$ $(b)f(x)=e^{2-5x}$ $(c)f(x)=e^{\sin x+\cos 2x}$ $(d)f(x)=e^x\sin^2(\cos x)$ $(e)f(x)=x\ln x$

7. For each of the following, find $\frac{dy}{dx}$:

   $a)\ y=\sin^2(2x)\cos^3(5x)$ $b)\ y=\cos(e^x)$ $c)\ y=2^{x^2}5^{x-1}$ $d)\ y=\ln(3x-2)$ $e)\ y=\ln(x^2\sin x)$

   $f)\ y=\log_3(x^2+e^x)$ $g)\ y=x^3\ln x^2$ $h)\ y=(x^3+2x^2-x-1)e^{x^2}$ $i)\ y=e^{x\ln x}$ $j)\ y=6^{5x}$

8. Using the rules of logarithms, simplify the following expressions. Hence, find $f'(x)$:

   (a) $f(x) = \ln \frac{(x+1)^{16}(2x^2+x)^8}{\sqrt{x^2+4}}$ (b) $f(x) = \ln \frac{(e^{2x}+6)^7\sqrt{x+4}}{(e^{-x}+e^x)^5}$

9. For each of the following functions, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

   $(a) y+xy+y = 2$ $(b) y^3+xy^2+x^2-1 = 0$ $(c) \sin x \cos y = 2$ $(d) xe^y-x-1 = 0$ $(e) x^y+ye^x-2x = 5$

10. For each of the given functions below, find $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$:

    $(a) f(x) = \frac{1}{x^2}$ $(b) f(x) = \sqrt{1-2x^2}$ $(c) f(x) = 2\ln\sqrt{x}$ $(d) f(x) = \ln(3x-2)$ $(e) f(x) = \sin 4x$

11. An open rectangular box is made from a square sheet of cardboard by removing a square from each corner and joining the cut edges. If the cardboard is of edge $0.5m$, find the maximum volume of the box.

12. If the selling price x is related to the profit y by the equation $y = 5000x - 125x^2$, determine the value of x for which the profit is maximum. Find that profit.

13. Find the critical values and determine the relative minimum and relative maximum for:

    (a) $f(x) = x^3 + 3x^2 - 12x + 7$ (b) $f(x) = (x-1)(x+1)(x+2)$

14. The concentration C, of hydrogen ions in a solution is given by $C = H + \frac{10^{-5}}{H}$. Find the value of H for which the concentration is a minimum.

15. A farmer wants to make a rectangular enclosure using a wall as one side and a $120M$ of fencing for the other three sides. Let x denote the width of the enclosure measured in meters.

    i) Find the area in terms of x and state the domain of the area function.

    ii) Find the value of x that gives maximum area.

16. Integrate the following functions with respect to x:

    $a)\ x^5$ $b)\ x^{\frac{2}{3}}$ $c)\ \frac{1}{x^8}$ $d)\ \sqrt{x}+3x^2-5$ $e)\ (2x-1)^2$ $f)\ 3\sqrt{2-5x}$ $g)\ \frac{2}{(4x+5)^3}$ $h)\ \frac{1}{\sqrt{x+2}}$

17. Evaluate the following integrals

    $a)\int \frac{x^4+x^2-2x+\sqrt{x}}{x^2}\,dx$ $b)\int \frac{x^2-x+2}{\sqrt{x}}\,dx$ $c)\int (2x-1)^5\,dx$ $d)\int \frac{1}{(3-2x)^4}\,dx$

18. Evaluate the following integrals:

    $a)\int x(x^2+3)^3\,dx$ $b)\int 5e^x\,dx$ $c)\int (3x^2+e^{4x})\,dx$ $d)\int x^2\ln x\,dx$ $e)\int \frac{x^2}{1+x^3}\,dx$

    $f)\int \frac{2x}{\sqrt{x^2+1}}\,dx$ $g)\int \frac{x^2}{\sqrt{1-2x^3}}\,dx$ $h)\int \frac{2x+3}{(x^2+3x+4)^3}\,dx$ $i)\int \frac{2x}{e^{x^2c}}\,dx$ $j)\int \sec^2 x\,dx$

19. Evaluate the following integrals:

$$a) \int \sin 2x \, dx \quad b) \int \sec x \tan x \, dx \quad c) \int 2 \sin x \, dx \quad d) \int \csc^2 x \, dx \quad e) \int \ln x \, dx \quad f) \int x^3 \ln x \, dx$$

$$g) \int x e^{-x^2} \, dx \quad h) \int e^x \cos x \, dx \quad i) \int \cos^3 x \, dx \quad j) \int x^2 \sin x \, dx \quad k) \int (1-x) e^x \, dx$$

20. By decomposing the following into partial fractions, find the following integrals:

$$a) \int \frac{x}{x+1} \, dx \quad b) \int \frac{1}{x} \, dx \quad c) \int \frac{x^2-2}{x^2-1} \, dx \quad d) \int \frac{x^2+x+5}{x(x+1)^2} \, dx \quad e) \int \frac{x^2+2x+4}{(2x-1)(x^2-1)} \, dx$$

$$g) \int \frac{12x}{(2-x)(3-x)(4-x)} \, dx \quad h) \int \frac{x^3+x^2+2}{(x^2+2)^2} \, dx \quad i) \int \frac{x^2+2}{x^2-1} \, dx \quad j) \int \frac{11x-10}{(x-2)(x+1)} \, dx$$

21. Evaluate the following definite integrals:

$$(a) \int_{-1}^{3} (x^2+2x-1) \, dx \quad (b) \int_{1}^{2} x e^{x^2} \, dx \quad (c) \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 2t \, dt \quad (d) \int_{0}^{\frac{\pi}{6}} \sin^2 x \, dx \quad (e) \int_{-3}^{1} x^2(x^3-1)^6 \, dx$$

22. Find the area of the region bounded by the graphs of $f(x) = -\frac{1}{4}x^2 + 1$ and $g(x) = \frac{1}{2}x^2 - 3$ for x in the interval $[-1, 2]$.

23. Find the area of the region bounded by the curve $y = x^2 - 9$ and the $x - axis$

24. In an idealized experiment, a colony of bacteria is introduced to a limited food supply. If the rate of change in the number N of live bacteria with respect to time t is given by

$$N'(t) = 6000t^2 - 75t^4,$$

Find the size of the population of the bacteria at time t if initially 1000 bacteria were introduced to the food supply.

25. An object on the ground is projected vertically with initial velocity of $32m/s$. If the acceleration $a(t) = -10.6m/s^2$, find:

    (a) the velocity function, $v(t)$

    (b) the distance at time t

    (c) the height the object will attain

    (d) the height of the object in 5 seconds

13

Sample Exam Questions

July-2016

1. a) Define a function. [2]

   b) Let f be a function defined as $f(x) = \frac{x-1}{x+1}$.

   (i) Find the domain of $f(x)$ [2]

   (ii) Find the range of $f(x)$. [3]

   (iii) Sketch the graph of $f(x)$, clearly indicating the intercepts and asymptotes. [4]

   c) Solve the following equations

   (i) $|3x - 6| = 12$ [2]

   (ii) $|2x + 1| = |4x - 3|$ [4]

   (iii) $\sqrt{2x - 1} - \sqrt{x + 3} = 1$ [4]

   d) Show that the function $f(x) = 2x^2 + 4$ is not a one to one function. [4]

2. a) (i) When is a function $f(x)$ said to be even? [2]

   (ii) Determine whether $f(x) = 2x^3 + 4x$ is odd, even or neither. [3]

   b) Let $g(x) = \frac{2}{x+1}$ and $f(x) = 3x - 2$ be two functions.

   (i) Find $fog(x)$ and state its domain. [3]

   (ii) Find $(fog)^{-1}(x)$ [3]

   c) Find the solution sets for each of the following inequalities:

   (i) $\sqrt{x - 2} \leq 1$ [4]

   (ii) $|6x - 11| < -5$ [2]

   d) Sketch the graph and state the range of $f(x) = \begin{cases} 1 - x & \text{if } x \leq 0; \\ x + 1 & \text{if } x > 0. \end{cases}$ [4]

   e) Decompose the following fraction into its partial fractions

   $\frac{x}{(x+1)(x^2+2x+2)}$ [4]

3. a) Let g be a function defined as $g(x) = \sqrt{x^2 - 3x + 2}$.

   (i) Find the domain of $g(x)$ [3]

   (ii) Sketch the graph of $g(x)$. Hence, state its range. [3]

b) Find $\frac{dy}{dx}$ for each of the following

$$(i)\;\; y = x^3 \ln 3x \qquad (ii)\;\; y = \frac{(x^2+1)^4}{\sin x} \qquad (iii)\;\; y^3 + xy^2 + x^2 = 1$$

[3,3,2]

c) The concentration C, of hydrogen ions in a solution is given by $C = H + \frac{10^{-5}}{H}$. Find the value of H for which the concentration is a minimum. [3]

d) solve the following equations

(i) $2^{2x+1} = 3(2^x) - 1$ [2]

(ii) $\log_2(x^2 - x + 2) = 1 + 2\log_2 x$ [3]

(iii) $\log_3 x - 2\log_x 3 = 1$ [3]

4. a) Find the exact value, leaving your answer in surd form where necessary.

(i) $\sin 150°$ [2]

(ii) $\tan(-\frac{5\pi}{3})$ [2]

b) Prove the following identities

(i) $\frac{1}{1-\sin x} + \frac{1}{1+\sin x} = \sec^2 x$ [2]

(ii) $(\csc x - \cot x)^2 = \frac{1-\cos x}{1+\cos x}$ [3]

c) Solve each of the following for $0 \leq \theta \leq 2\pi$

(i) $\cos\theta = \sin 2\theta$ [4]

(ii) $2\sin^2\theta - \cos\theta - 1 = 0$ [4]

d) (i)Express $f(x) = \sqrt{3}\cos x - \sin x$ in the form $f(x) = r\cos(x+\alpha)$ [4]

(ii) Sketch the graph of $f(x) = 3\sin(x - \frac{\pi}{2})$. State the amplitude and period. [4]

5. a) Evaluate the following limits.

$$(i)\;\; \lim_{x\to -3} \frac{x^2+5x+6}{x+3} \qquad (ii)\;\; \lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} \qquad (iii)\;\; \lim_{x\to\infty} \frac{x^4-1}{x^3+x^2}$$

[2,3,2]

b) Differentiate the function $f(x) = \frac{1}{\sqrt{x+1}}$ from first principle. [4]

c) Determine the following integrals

(i) $\int 3x^2 - 2x + \frac{1}{x} + 5\,dx$ [2]

(ii) $\int \frac{x^2}{\sqrt{1-2x^3}}\,dx$ [3]

(iii) $\int \frac{3}{(x+1)(x+2)}\,dx$ [4]

d) Given the function$f(x) = \ln\frac{(x+1)^{16}(2x^2+x)^8}{\sqrt{x^2+4}}$,

(i) Use the rules of logarithms to simplify $f(x)$. [2]

(iii) Hence or otherwise, find $f'(x)$ [3]

END!

July-2015

1. a) Given that X and Y are subsets of the universal set E, simplify the following as far as possible:

   (i) $X - (Y - X)$

   (ii) $X' \cup (X' \cap Y)'$

   b) Let $\mathbb{R} = (-\infty, \infty)$ be the universal set. Further, let $A = (-8, 6]$, $B = [4, \infty)$ and $C = [0, 1)$, be subsets of the universal set, $\mathbb{R}$. Find:

   (i) $A \cup B$

   (ii) $(A \cap B)'$.

   (iii) C'

   c) Express the following rational numbers in the form $\frac{a}{b}$ where a and b are integers, with $b \neq 0$.

   (i) $0.\overline{1}$

   (ii) $12.\overline{13}$.

2. a) Find the derivatives of the following functions:

   (i) $f(x) = 12x^4 + \sqrt{x} - \frac{2}{x^3}$

   (ii) $g(x) = \frac{x^5 + x^{\frac{2}{3}} - x^3 + x}{x^3}$

   b) The line $y = x + 1$ meets the curve $y = x^2 - x - 2$ at the points A and B. Find the:

   (i) coordinates of A and B

   (ii) equations of the tangent lines at A and B

   (iii) equations of the normal lines at A and B

   c) Given the equation of the curve $y = x(x-2)^2$,

   (i) find $\frac{dy}{dx}$

   (ii) hence, find the coordinates of the point on the curve at which the gradient is zero.

3. a) Given the polynomial function $p(x) = x^3 + 4x^2 + x - 6$;

   (i) factorize completely $p(x) = x^3 + 4x^2 + x - 6$

   (ii) Sketch the graph of the polynomial $p(x)$

   (iii) Hence or otherwise, find the values of x for which $x^3 + 4x^2 + x - 6 \geq 0$

   b) Solve the following equations

   (i) $3(2x - 5) + 2x \leq 32 + x4$

   (ii) $2x^2 - 11x + 5 \geq 0$

   c) Factorize completely $x^4 - 1$

4. a) Given the polynomial $f(x) = x^3 + x^2 - 5x - 2$;

   (i) Show that $x - 2$ is a factor of $f(x)$.

   (ii) solve the equation $x^3 + x^2 = 5x + 2$

b) The polynomial $h(x) = 3x^3 + 2x^2 - px + q$ is divisible by $x - 1$, but leaves a remainder of 10 when divided by $x + 1$. Find the values of p and q.

c) Solve the following pair of simultaneous equations:

$$y - x = -2$$
$$2x^2 - 10x = y - 3$$

5. a) Simplify the following as far as possible:

(i) $(\frac{27}{64})^{\frac{2}{3}}$

(ii) $9x^{\frac{11}{5}} \div 3x^{\frac{1}{5}} \times 2x$

b) (i) Given that $x^2 + 4x - 2 = (x + a)^2 + b$ where a and b are constants, find the values of a and b

(ii) Solve for x given that $2\sqrt{x} = \sqrt{40}$

(iii) Rationalize the denominator and simplify

$$\frac{8}{3 + \sqrt{5}} + \sqrt{45}$$

c) Given the quadratic function $f(x) = 2x^2 - x - 5$,

(i) Complete the square of $f(x) = 2x^2 - x - 5$

(ii) Determine the type of roots of the quadratic $f(x) = 2x^2 - x - 5$

(iii) Sketch the graph of the given quadratic.

6. a) The line l_1 has equation $2y = x - 3$ and the line l_2 has equation $5y + 2x - 18 = 0$. Another line, l_3 is perpendicular to l_1 and passes through the point $(0, 3)$. Find:

(i) the gradient of l_1

(ii) the coordinates of the point of intersection of l_1 and l_2.

(iii) the equation of l_3

b) The points A and B have coordinates $(2k, 1)$ and $(9, k - 1)$ respectively where k is a constant. Given that the gradient is $\frac{1}{3}$,

(i) Show that $k = 3$

(ii) Find the equation of the line through A and B.

c) Express $\sqrt{12} + \sqrt{147} - \sqrt{27}$ in the form $r\sqrt{3}$ where r is a constant

END!

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