MAT 1110:

Chapter 3

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Functions

1.1. Relations

Definition 1.1.1 *Let A and B be two sets. Let $a \in A$ and $b \in B$. Then the pair (a, b) is called an ordered pair. The elements a and b are called coordinates. Two ordered pairs (a, b) and (c, d) are equal, that is, $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.*

The set of all distinct ordered pairs whose first coordinate is in A and the second coordinate is an element of B is called the Cartesian product of A and B. The Cartesian product of A and B is denoted by $A \times B$. Thus,

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Example 1.1.0.1 *Let $A = \{2, 3, 4\}$ and $B = \{6, 7\}$. Then,*

$$A \times B = \{(2, 6), (2, 7), (3, 6), (3, 7), (4, 6), (4, 7)\}.$$

In general,

$$A \times B \neq B \times A.$$

Definition 1.1.2 *A relation R from a set X to a set Y is a set of ordered pairs (x, y) such that for each $x \in X$, there corresponds atleast one $y \in Y$.*

We write xRy to mean that x is related y. We shall call X the *input* set and Y the *output* set.

Generally speaking, a relation is simply a rule which connects two elements from different sets.

1.1.1 Classification of Relations

Let R be a relation from X to Y.

Relations for which each element of the set X is mapped to a unique element of the set Y are said to be *one-to-one.*

Example 1.1.1.1 *Let $X = \{3, 5, 7\}$ and $Y = \{6, 10, 14\}$, and let the relation R from X to Y be defined by : " is a factor of".Display this on an arrow diagram. Clearly, this is a one-to-one.*

A relation can map more than one element of the set X to the same element of the set Y. Such a type of relation is said to be *many-to-one.*

Example 1.1.1.2 *Let* $X = \{4, 5, 7\}$ *and* $Y = \{2, 3, 6\}$, *and let the relation* R *from* X *to* Y *be defined by : " is greater than".Display this on an arrow diagram. Clearly, this is a many-to-one.*

A relation can map one element of the set X to more than one element of the set Y. Such a type of relation is said to be *one-to-many.*

Example 1.1.1.3 *Let* R *be a relation from* X *to* Y *defined by* $' <'$ *given that* $X = \{1, 2, 3, 4\}$ *and* $Y = \{1, 3, 5\}$. *Display this on an arrow diagram. Clearly, this is a one-to-many.*

The inverse of the relation $' <'$, defined in the previous example is called $' >'$: and is given by

$$R^{-1} = \text{ inverse of } R = \{(3, 1), (3, 2), (5, 1), (5, 2), (5, 3), (5, 4)\}$$

1.1.2 Domain, Range and Co-domain

Let R be a relation from X to Y.

1. The set of all values for which the relation is defined (set of input values) is called the domain (D).
2. The range (R) is the set of all values that it can produce. It is the set of output values.

3. The co-domain of a relation is a set of values that includes the range as described above, but may also include additional values beyond those in the range.

Example 1.1.2.1 *Let R be a relation from X to Y defined by $' <'$ given that $X = \{1, 2, 3, 4\}$ and $Y = \{1, 3, 5\}$. Display this on an arrow diagram and hence find the domain, range and the co-domain.*

1.2. Functions

Definition 1.2.1 *A function from a set X to a set Y is a set of ordered pairs (x, y) such that for each $x \in X$, there corresponds a unique $y \in Y$.*

Clearly, a function is a relation but the converse is not true. The domain, range and the co-domain for a function are defined as for a relation. Throughout this course, all functions will be defined on the subset of $\mathbb{R}$.

Since a function assigns every element $x \in X$ to exactly one element in set Y, we will denote the output elements (images) by $y = f(x)$, where x is the input and is called the *independent variable* and y is called the *dependent variable*. $f(x)$ is read as "f of x" or the value of f at x.

Example 1.2.0.2 *The function*

$$f(x) = \frac{x}{2} + 7$$

is the rule that takes a number, divides it by 2*, and then adds* 7 *to the quotient. For example, if* $x = 4$, $f(4) = 9$.

1.2.1 Classification of functions

Let f be a function from X to Y.

Functions for which each element of the set X is mapped to a unique element of the set Y are said to be *one-to-one.*

Example 1.2.1.1 *Let* $X = \{3, 5, 7\}$ *and* $Y = \{6, 10, 14\}$, *and let the relation R from* X *to* Y *be defined by : " is a factor of". Display this on an arrow diagram. Since every input element has a unique image, and so it is a function. Clearly this is one-to-one.*

A function can map more than one element of the set X to the same element of the set Y. Such a type of function is said to be *many-to-one.*

Example 1.2.1.2 *Let* $X = \{4, 5, 7\}$ *and* $Y = \{3, 6\}$, *and let the relation R from* X *to* Y *be defined by : " is greater than".Display this on an arrow diagram. Clearly, this is a many-to-one.*

Definition 1.2.2 *A function f is one-to-one if and only if $f(a) = f(b)$ for any a and b in the domain means $a = b$.*

Example 1.2.1.3 *Let $f(x) = 3x + 1$. Show that f is one-to-one.*

Example 1.2.1.4 *Which of the following are functions:*

1. $y = f(x) = -2x + 7$
2. $y^2 = x$
3. $y = 4$
4. $y = x^2$.

Example 1.2.1.5

1. *Given that $f(x) = x^2 + 5x - 6$, find $f(3)$ and $f(-3)$.*
2. *Given that $f(x) = \frac{4x^2-9x+17}{x+7}$, find $f(5)$ and $f(-4)$.*

Example 1.2.1.6 *For each of the following functions, find the domain.*

1. $y = 4x^2 + 7x - 19$
2. $y = \sqrt{t - 5}$
3. $y = \frac{7}{x(x-4)}$
4. $y = \frac{6x}{(x-5)(x-9)}$

Example 1.2.1.7 *Find the range of the following functions given that* $-2 \leq x \leq 2$.

1. $y = 2x$
2. $y = x^2$
3. $y = \frac{1}{1-x}$

Listed below are examples of different functions.

1. Linear: $f(x) = ax + c,\ \ a, c \in \mathbb{R}$.
2. Quadratic: $f(x) = ax^2 + bx + c,\ \ a, b, c \in \mathbb{R}, a \neq 0$.
3. Polynomial: $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$.
4. Rational: $f(x) = \frac{x^2-9}{x+4},\ \ x \neq -4$.
5. Exponential: $f(x) = a^x,\ \ a \neq 0$.
6. Logarithmic function $f(x) = \log_a x$

1.2.2 Inverse of a function

An inverse function is a function that " reverses" another function. This means that if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, and vice versa. That is $f(x) = y$ and iff $g(y) = x$.

Example 1.2.2.1 *Find the inverse of each of the following functions, and each case state the range of f and the domain of f^{-1}.*

1. $f(x) = 3x + 1$

2. $f(x) = \frac{x-1}{x+2}, \ \ x \neq -2.$

3. $f(x) = \frac{1}{x-1}, \ \ x \neq 1.$

1.2.3 Composite functions

Definition 1.2.3 *Given two functions f and g, the composite function, denoted by $f \ og$ (read f composed with g" or "f of g") is defined by*

$$(f \ \ og) = f(g(x)).$$

In the definition above, we refer to g as the inside function and f as the outside function.

When determining the domain for the composite function, the domain for the inside function and the domain for the resultant composite function must be accounted for.

Example 1.2.3.1 *Let f and g be two functions defined by $f(x) = 2x - 1$ and $g(x) = 3x$. Find*

1. $(gof)(x)$

2. $(fog)(x)$

3. $(fof)(x) = f^2(x)$

4. $(gog)(x) = g^2(x)$

Example 1.2.3.2 *Find* $(g\ of)(x)$ *if* $f(x) = \frac{5}{x+4}$ *and* $g(x) = \frac{3x}{2x-1}$. *State the domain of* $g\ of$.

1.2.4 Odd and even functions

Definition 1.2.4 *An even function is one which is unchanged when the sign of its argument changes,* $f(-x) = f(x)$.

Examples of such function include $f(x) = x^2+2$, $f(x) = 3x^4+x^2$. Thus polynomials with even powers are even functions.

Definition 1.2.5 *An odd function is a function such that* $f(-x) = -f(x)$.

Examples of odd function include $f(x) = 3x$, $f(x) = 2x^3-x$, that is polynomials with odd powers of x.

Example 1.2.4.1 *State whether each of the following functions is even, odd or neither.*

1. $f(x) = \frac{x^2+4}{x^3-x}$

2. $f(x) = x^4 - 3x^2 + 7$

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Polynomial Functions

2.1. Linear function

Definition 2.1.1 *A function f is said to be linear if it is of the form $y = ax + b$, where a and b are real numbers.*

2.1.1 Linear equations and solutions

Definition 2.1.2 *An equation is a statement that two algebraic expressions are equal.*

They include

1. $3x - 5 = 7$
2. $\frac{x}{3} + \frac{3x}{4} = 2$.
3. $\sqrt{2x} = 4$.

To solve an equation in x means to find all values of x for which the equation is TRUE. Such values are called **solutions**.

2.1.2 Linear equations in one variable

Definition 2.1.3 *A linear equation in one variable is an equation that can be written in the standard form*

$$ax + b = 0,$$

where a and b are real numbers with $a \neq 0$.

A linear equation in x has exactly one solution. For example, from $ax + b = 0$, we get that $ax = -b$, so that $x = -\frac{b}{a}$.

Example 2.1.2.1 *Solve each of the following equations*

1. $3x - 6 = 0$
2. $5 + 5x = 15$
3. $10 - x = 11.$

2.2. Equations involving fractional expressions

To solve an equation involving fractional expressions you can multiply every term in the equation by the least common denominator (LCD) of the terms.

Example 2.2.0.2 *Solve each of the following fractional equations*

1. $\frac{x}{3} + \frac{3x}{4} = 2.$

2. $\frac{4x}{5} - \frac{x}{2} = 9.$

An equation with a single fraction on both sides can be cleared of denominators by **cross-multiplying,** which is equivalent to multiplying each side of the equation by the least common denominator and then simplify.

Example 2.2.0.3 *Cross-multiply to solve each of the following fractional equations*

1. $\frac{3y-2}{2y+1} = \frac{6y-9}{4y+3}.$

2. $\frac{3x-6}{x+10} = \frac{3}{4}$

2.2.1 Graphs of linear functions

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. The graph of an equation in the two variables x and y is the set of all points (x, y) whose coordinates satisfy the equation.

To graph a linear function, substitute various values of x into the equation and solve for y to produce some of the ordered pairs that satisfy the equation.

Example 2.2.1.1 *Draw the graph of each of the following by plotting the points.*

1. $y = 2x - 1$

2. $y = -x + 3$

3. $y = 4$

4. $x = 3$.

2.3. Quadratic functions

Definition 2.3.1 *A function f, in x, is said to be quadratic if it is of the form $f(x) = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.*

2.3.1 Quadratic equations

Definition 2.3.2 *A quadratic equation in the variable x is defined as the equation of the form $ax^2 + bx + c = 0$, where a, b and c are constants with $a \neq 0$.*

Such equations can be solved using the following methods:

1. Factorization
2. Quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3. Completing the square

Example 2.3.1.1 *1. Use factorization method to solve $3x^2 - 5x + 2 = 0$.*

2. By using the quadratic formula, solve $2x^2 + 7x + 4 = 0$.

To solve a quadratic equation ($ax^2 + bx + c = 0$) by completing the square method, the following steps will be required.

1. If a does not equal 1, divide each side by a so that the coefficient of x^2 is 1.
2. Rewrite the equation with the constant term on the right side.
3. Complete the square by adding the square of one-half of the new coefficient of x to both sides.
4. Write the left side as a square and simplify the right side.
5. Finally, simplify both sides so that you can take square root of both sides.

The following example will help us understand the above steps.

Example 2.3.1.2 *Using completing the square method, solve*

1. $3x^2 - 5x + 2 = 0$

2. $2x^2 + 7x + 4 = 0.$

By using the quadratic formula, we observe that, we obtain different solutions depending on the value of the radicand $b^2 - 4ac$. b^2-4ac is called the **discriminant** of the quadratic equation. The discriminant is used to determine the nature of solutions/roots of a quadratic equation as follows:

1. If $b^2 - 4ac > 0$, then the quadratic equation has two distinct real roots.

2. If $b^2 - 4ac < 0$, then the quadratic equation has no real roots (has complex roots).

3. If $b^2 - 4ac = 0$, then the quadratic equation has one real root.

Example 2.3.1.3 *Determine the nature of roots for the following equations:*

1. $4x^2 - 7x - 1 = 0$

2. $4x^2 + 12x + 9 = 0.$

3. $5x^2 + 2x + 1 = 0.$

4. $9x^2 - 16 = 0.$

5. $2x^2 + 2x - 2 = 4x.$

2.4. Graph of a quadratic function

The following steps will be needed to be able to sketch the graphs of quadratic functions.

1. Decide on the shape.

   (a) When $a > 0$, the curve will be a $\cup$ shape.

   (b) When $a < 0$, the curve will be a $\cap$ shape.

2. Work out the points where the curve crosses the $x-$ and $y-$axes.

   (a) Put $y = 0$ to find the point(s) where it crosses the $x-$axis.

   (b) Put $x = 0$ to find the point where it crosses the $y-$axis.

Example 2.4.0.4 *Sketch the graph of each of the following:*

1. $f(x) = x^2 - 5x + 4$.
2. $f(x) = -2x^2 - 7x + 4$.

By the process of completing the square, all quadratic functions, $f(x) = ax^2 + bx + c,\ a \neq 0$, may be transposed into what will be called the turning point form:

$$y = a(x - h) + k, \quad \text{where } (h, k) \text{ is the turning poing.}$$

We determine the x- and y- intercepts as before. i.e. by equation $y = 0$ and $x = 0$ respectively.

Example 2.4.0.5 *Sketch the graph of each of the following:*

1. $f(x) = x^2 - 5x + 4$.
2. $f(x) = -2x^2 - 7x + 4$.
3. *Let* $p(x) = -\frac{x^2}{10} + 50x - 750$ *be the profit function in dollars that a company earns as a function of* x, *number of products of a given type that are sold, and is valid for values*

x greater than or equal to 0 and less than or equal to 500. Find the maximum loss and maximum profit.

2.5. Polynomial functions and rational functions

In this section, we will study polynomial functions and rational functions.

Definition 2.5.1 *A polynomial function is a function defined by*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

For instance, $f(x) = x^3 + 4x^2 - x + 1$ is a polynomial function, and is called a third-degree, or cubic polynomial function because the largest power is 3.

Definition 2.5.2 *A rational function is a function defined by the quotient of two polynomials.*

For example,

$$f(x) = \frac{7x}{2x^2 + 5}$$

is a rational function.

If p is a polynomial function, then the values of x for which $p(x)$ is equal to 0, is called the zeros of p. For instance, -1 is a zero of $p(x) = 2x^3 - x + 1$ because $p(-1) = 0$.

Much of this section concerns finding the zeros of polynomial functions. Sometimes the zero of a polynomial function are determined by dividing one polynomial by the another.

2.5.1 Division of polynomials

Consider the polynomial $x^3 - 2x^2 + 4x - 3$. Divide this by $x - 3$. Using long division as in arithmetic, we get the following result

and the steps are as follows.

1. Divide x^3 by x to give x^2.
2. Multiply $x - 3$ by x^2.
3. Subtract to get x^2 and bring down the next term, $4x$.
4. Divide x^2 by x to give x.
5. Multiply $x - 3$ by x.
6. Subtract to get $7x$ and bring down the next term, -3.
7. Divide $7x$ by x to give 7.
8. Multiply $x - 3$ by 7.

9. Subtract; this gives the remainder 18.

Thus

$$\frac{x^3 - 2x^2 + 4x - 3}{x - 3} = (x^2 + x + 7) + \frac{18}{x - 3}.$$

Hence

$$f(x) = (x^2 + x + 7)(x - 3) + 18.$$

This means that when we divide $f(x)$ by $(x - a)$ the quotient is Q, and the remainder is R, then

$$f(x) = Q \times (x - a) + R,$$

where $R = f(a)$. In general, if the polynomial $f(x)$ is divided by the linear expression $px + q$, then

$$f(x) = Q \times (px + q) + R,$$

where $R = f(\frac{-q}{p})$. This is called the **remainder theorem** for a polynomial $f(x)$.

Note that the theorem only applies to polynomials and linear divisors.

Example 2.5.1.1 *What are the remainders when* $x^3 - x^2 + 3x - 2$ *is divided by*

1. $x - 1$

2. $x + 2$

3. $2x - 1$?

Example 2.5.1.2 *1. The polynomial* $x^3 + ax^2 - 3x + 4$ *is divided by* $x - 2$ *and the remainder is* 14. *What is the value of* a?

2. *Let* $f(x) = x^3 + ax^2 + bx - 3$. *When* $f(x)$ *is divided by* $x - 1$ *and* $x + 1$, *the remainders are* 1 *and* -9 *respectively. Find the values of* a *and* b.

2.5.2 The factor theorem

If $(x - a)$ is a factor of $f(x)$, then there will be no remainder when $f(x)$ is divided by $(x - a)$. So $f(a) = 0$. Similarly, if $px + q$ is a factor of $f(x)$, then $f\left(\frac{-q}{p}\right) = 0$. This is called the factor theorem for a polynomial $f(x)$.

Example 2.5.2.1 *Determine whether* $x + 1$ *is a factor of:*

1. $f(x) = x^2 + 2x + 1$.
2. $f(x) = x^3 - 6x^2 - x + 6$.
3. $f(x) = x^3 - 2x^2 - x + 2$.

Example 2.5.2.2 *Given that* $p(x) = x^3 + kx^2 + x + 6$, *find the value of* k *if* $x+1$ *is a factor of* p. *Hence, find the other factors.*

Example 2.5.2.3 *Factorize* $x^3 - 6x^2 - x + 6$.

Solution

As $f(x)$ is of degree 3, it will have atmost three linear factors of the general form $px + a$, $qx + b$ and $rx + c$, so that

$$\begin{aligned} x^3 - 6x^2 - x + 6 &= (px + a)(qx + b)(rx + c) \\ &= pqrx^3 + \cdots + abc \end{aligned}$$

Since the coefficient of x^3 is 1 and the last term is $+6$, we observe that $pqr = 1$ and $abc = +6$. So the possible factors come from the factors of $+6$. The first factor has to be found by trial which in this case is $x - 1$.

We could continue like this to get the other factors but the best method is to use long division, considered earlier and show that $x^3 - 6x^2 - x + 6 = (x - 1)(x^2 - 5x - 6) = (x - 1)(x + 1)(x - 6)$. or use synthetic division stated below.

Given a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

we can divide it by a linear factor $x - a$, where $'a'$ is a real number, using the following steps.

1. Write the a on the left side of a vertical bar and the coefficients in decreasing degree.

$$a | a_n \qquad a_{n-1} \qquad \cdots \qquad a_1 \qquad a_0$$

2. Pass the coefficient of the highest degree below the horizontal line as below

$$\begin{array}{c|ccccc} a & a_n & a_{n-1} & \cdots & a_1 & a_0 \\ & a_n & & & & \end{array}$$

3. Multiply the passed coefficient by a and add to the next coefficient

$$\begin{array}{c|ccccc} a & a_n & a_{n-1} & \cdots & a_1 & a_0 \\ & a_n & aa_n + a_{n-1} & & & \end{array}$$

4. Continue step 3 until all coefficients are covered.
5. The numbers below $a_n, a_{n-1}, \dots a_1$ give the coefficients of the quotient whose degree is 1 less than that of the polynomial. The number below a_0 is the remainder.

Example 2.5.2.4 *Solve each of the following:*

1. $x^3 - 6x^2 + 8x = 0$
2. $x^3 - 2x^2 - x + 2 = 0$

Example 2.5.2.5 *Sketch each of the following*

1. $f(x) = x^3 - 6x^2 + 8x$
2. $f(x) = x^3 - 2x^2 - x + 2$

2.5.3 Rational functions; Partial fractions

Definition 2.5.3 *A rational function* $R(x)$ *is the quotient of two polynomials* $N(x),\ D(x)$ *such that* $R(x) = \frac{N(x)}{D(x)}$ *provided that* $D(x) \neq 0$.

$R(x)$ is called proper if the degree of $N(x)$ is less than the degree of $D(x)$. If the degree of $N(x)$ is greater than the degree of $D(x)$, then $R(x)$ is called improper.

Partial fractions

Express $\frac{1}{x+2} + \frac{2}{x-3}$ as a single fraction. This gives

$$\frac{1}{x+2} + \frac{2}{x-3} = \frac{3x+1}{(x+2)(x-3)} = \frac{3x+1}{x^2-x-6}.$$

$\frac{1}{x+2}$ and $\frac{2}{x-3}$ called partial fractions of $\frac{3x+1}{x^2-x-6}$, and the ability to represent a complicated algebraic fraction in terms of its partial fractions is the purpose of this section.

Rules of partial fractions

1. The numerator must be of lower degree than the denominator. That is, it must be a proper rational function. If it is not, then we first divide out.

2. Factorize the denominator into its prime factors. These determine the shapes of the partial fractions.

The following list summarizes the forms of all the possible partial fractions.

Denominator containing	Expression	Form of partial fraction
Linear factors	$\dfrac{f(x)}{(x+a)(x-b)(x+c)}$	$\dfrac{A}{x+a}+\dfrac{B}{x-b}+\dfrac{C}{x+c}$
Repeated linear factors	$\dfrac{f(x)}{(x+a)^3}$	$\dfrac{A}{x+a}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+a)^3}$
Quadratic factors	$\dfrac{f(x)}{(ax^2+bx+c)(x+d)}$	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{C}{x+d}$

Example 2.5.3.1 *Split into partial fractions:*

1. $\frac{x}{(x+2)(x+3)}$.
2. $\frac{11-3x}{x^2+2x-3}$.
3. $\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$.

Example 2.5.3.2 *Split into partial fractions:*

1. $\frac{2x}{(x-2)^2(x+2)}$.
2. $\frac{x^2+2}{(x+2)^2(x+3)}$.

Example 2.5.3.3 *Split into partial fractions:*

1. $\frac{1}{x(x^2+5)}$.
2. $\frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}$.

Example 2.5.3.4 *Split into partial fractions:*

1. $\frac{x^4+1}{x^3+9x}$.
2. $\frac{6x^3+x^2+5x-1}{x^3+x}$.

2.6. Inequalities

2.6.1 introduction

Simple inequalities are used to order real numbers. The inequality symbols $<, \leq, >$ and $\geq$ are used to compare two real numbers and to denote subsets of real numbers. For example, the simple inequality $x \geq 3$ denotes all real numbers x that are greater than or equal to 3.

In this section, we will expand our work with inequalities to include more involved statements such as

$$5x - 7 > 3x + 9 \text{ and } -3 \leq 6x - 1 < 3.$$

As is the case with equations, we solve an inequality in the variable x by finding all values of x for which the inequality is true. Such values are solutions and are said to satisfy the inequality. The set of all real numbers that are solutions of an inequality. For example, the solution set of $x + 3 > 4$ is

$$S = \{x : x >, x \in \mathbb{R}\}.$$

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2.6.2 Solving linear inequalities

The simplest type of inequality to solve is a linear inequality in x. For example $2x + 3 > 4$ is a linear inequality in x.

As we solve the following examples, remember that when you multiply and divide by a negative number, you must reverse the inequality symbol.

Example 2.6.2.1 *Solve each of the following linear inequalities.*

1. $5x - 7 > 3x + 9$
2. $3x < 2x + 1$
3. $1 - \frac{3x}{2} \geq x - 4.$
4. $-3 \leq 6x - 1 < 3.$

2.6.3 Quadratic inequalities

Definition 2.6.1 *A quadratic inequality is any inequality that can be put in one of the forms*

1. $ax^2 + bx + c < 0$
2. $ax^2 + bx + c > 0$
3. $ax^2 + bx + c \leq 0$

4. $ax^2 + bx + c \geq 0$

where a, b and c real numbers and $a \neq 0$.

Procedure for solving Quadratic inequalities

- Express the given inequality in the standard form.
- Solve the equation $ax^2 + bx + c = 0$. The real solutions are the boundary points.
- Locate these boundary points on a number line, thereby dividing the number line into test intervals.
- Choose one representative number within each test interval. If substituting that value into the original inequality produces a true statement, then all the real numbers in that test interval belong to the solution set. Otherwise, no real numbers in the test interval belong to the solution set.
- Write the solution set; the intervals that produce a true statement.

Example 2.6.3.1 *Solve and graph the solution on a number line.*

1. $2x^2 - 3x \geq 2$.
2. $5x^2 - 9x > 5$.

Example 2.6.3.2 *Sketch the graph of* $-x^2 + 2x \geq -5$. *Find the values of* x *for which* $-x^2 + 2x + 5 \geq 0$.

2.6.4 Polynomial inequalities

Procedure for solving polynomial (cubic) inequalities

- Express the given inequality in the standard form.
- Solve the equation $ax^3 + bx^2 + cx + d = 0$. The real solutions are the boundary points.
- Locate these boundary points on a number line, thereby dividing the number line into test intervals.
- Choose one representative number within each test interval. If substituting that value into the original inequality produces a true statement, then all the real numbers in that test interval belong to the solution set. Otherwise, no real numbers in the test interval belong to the solution set.
- Write the solution set; the intervals that produce a true statement.

Example 2.6.4.1 *Solve and graph the solution on a number line.*

1. $x^3 + x^2 \geq 8x + 12$.

2. $x^3 + 8 \leq 5x^2 - 2x$.

Example 2.6.4.2 *Sketch each of the following, and hence state the set(s) of value of x for which $f(x) \geq 0$.*

1. $f(x) = x^3 - 6x^2 + 8x$

2. $f(x) = x^3 - 2x^2 - x + 2$

2.6.5 Rational inequalities

Definition 2.6.2 *Asymptote*

An asymptote is a line that a graph gets closer and closer to, but never touches or crosses it.

To solve a rational inequality:

- Get all the terms on the left hand side.
- Combine all the terms into one single rational expression.
- Factor the ration expression so you know the exact location of the zeroes and vertical asymptotes, and thus where the y-values change sign.
- State the solution as the interval(s) that have the desired y-values as demanded by the inequality.

Example 2.6.5.1 *Solve the inequalities:*

1. $\frac{1}{x+5} < \frac{x-2}{x-7}$.
2. $\frac{x+1}{x+3} \leq 2$.
3. $\frac{x+5}{x^2-7x+12} \leq 0$.

Example 2.6.5.2 *In each of the following find/state the domain. Hence sketch the graph.*

1. $f(x) = \frac{1}{x}$.
2. $f(x) = \frac{2}{x-3}$.
3. $f(x) = \frac{x+1}{x-1}$.