We write $x \in A$, if x is an element of A. If x is not an element of set A we write $x \notin A$. If every element of B is also an element of A then we write $B \subset A$ or $A \supset B$ which we read as B is a subset of A or A contains B.

In addition, the set B is a proper ~~set~~ subset of ~~B~~ the set A. If every element of A is an element of B. We define it by $B \subset A$, meaning B is a proper subset of A. For example, consider two sets;

$A = \{x; x \text{ is a natural } \#\}$

$B = \{y; y \text{ is an even } \#\}$

In this case then all the elements in B are also contained in A ie $B \subset A$.

→ Two sets A and B are equal if and only if they contain same elements. That $A = B$ if both $A \subset B$ and $B \subset A$ hold for example,

$A = \{a, b, c, d\}$ and

$B = \{b, d, a, c\}$

$\therefore A = B$

→ Two sets which can be put into one to one correspondence with one another are said to be equivalent, for instance the sets $\{a, b, c, d\}$ and $\{\#, \&, 8, \%\}$ are equivalent since a one to one correspondence can be established.

The set containing no element is called an empty set or Null set and it is denoted by either empty or blackets

$\{\ \}$ or $\emptyset$

The set containing the totality of elements for any particular discussion or situations is called the universal set and it is denoted by the symbol U or E.

The complement set of A denoted by A' or A^c is the set of elements which do not belong to A that is, for a given universal set U and A subset A of U, the complement of A (denoted) A' is the element U which are not elements of A, for example, let the universal set

$U = \{a, b, c, d, e, f, g, h\}$

and $A = \{a, d, c\}$ then

$A' = \{b, e, f, g, h\}$

BASE OPERATION OF SETS

- The union of two sets A and B is the set containing all the elements which are in A or B or in Both. That is if A and B are two sets, then the union of A and B written $A \cup B$, is analystically defined as

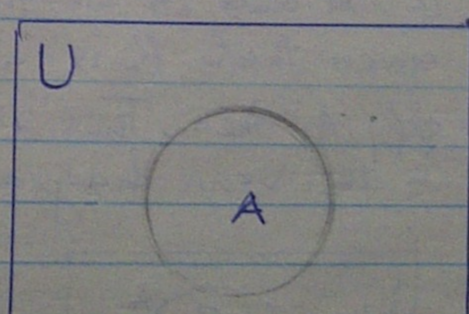
$A \cup B = \{x : x \in A \text{ or } x \in B\}$

- The intersection of two sets A and B is the set which contain all the element which are ~~given~~ In A and B. That is written as $A \cap B$, and defined analystically as

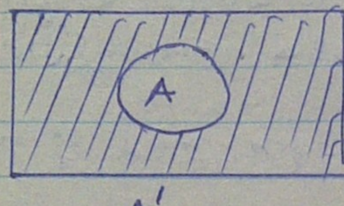
$A \cap B = \{x : x \in A \text{ \& } x \in B\}$

- If two sets A and B are such that $A \cap B = \emptyset$, the sets are said to be Disjoint sets, that is, Disjoint set contain no elements.

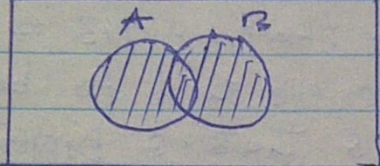
- A Venn Diagram is a pictorial representation of a set if for example U is the universal set containing the set A is a subset then, this relation can be represented in a venn diagram as shown below



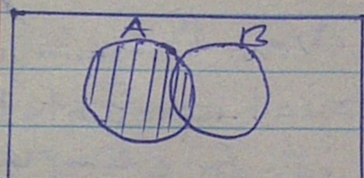
In other words venn diagrams are used to help visualise sets and the relations between sets for example.

a) 

A'

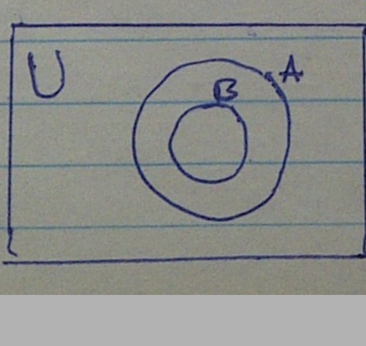
b) 

$A \cup B$

c) 

$A \cap B'$

Another relation that can be represented by Venn diagram is that of "Is a subset of" If A and B are two sets given that B is a subset of A then, This is shown in the venn diagram below



Note if, $B \subset A$, then $A' \subset B'$

PRACTICE QUESTIONS

1. list the elements of

a) $\{x; x \text{ is an integer}\} \cap \{1, \sqrt{2}, 5, 14, 7\}$

b) $\{x; x \text{ is a male}\} \cap \{y; y \text{ is a female}\}$

c) $\{x; x$ is a positive even # less than $20\} \cap \{1,2,8,9,13,14,17,18\}$

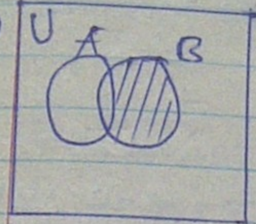
2. If $A \subset B$ then Simplify the following of possible

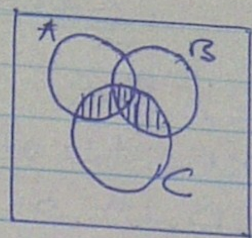
a) $A \cap B$ b) $A' \cup B'$

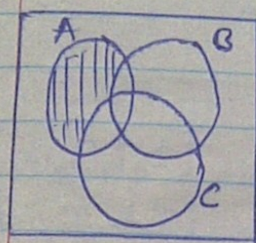
c) $(A \cap B)'$ d) $(A \cup B)'$

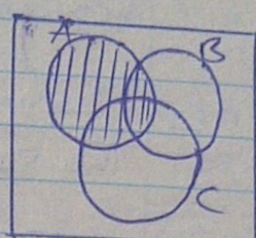
3. Set $A = \{F, U, N, C, H\}$, $B = \{N, A, P\}$ and $D = \{D, A, N, C, E\}$, Draw a Venn diagram to illustrate these sets.

4. In each of these Venn diagram below, What set designation describes the portion that is shaded

a) 

b) 

c) 

d) 

Sets under the observation of union intersection, Complement satify the following laws

① for any set A

$A \cup B = A \quad A \cap B = A$

② Commutatwety for any sets A and B We write

a) $A \cup B = B \cup A$

b) $A \cap B = B \cap A$

③ Associativity for any sets A, B, and D

a) $(A \cup B) \cup D = A \cup (B \cup D)$

b) $(A \cap B) \cap D = A \cap (B \cap D)$

④ Distributive property for any set A, B and D

a) $A \cap (B \cup D) = (A \cap B) \cup (A \cap D)$

b) $A \cup (B \cap D) \cap (A \cup D)$

⑤ DF Morgan's law

For any sets A & B are we have

a) $(A \cup B)' = A' \cap B'$

b) $(A \cap B)' = A' \cup B'$

Example

let $U = \{1,2,3,4,5,6,7,8,9,10\}$
$A = \{1,2,3,4,5\}$, $B = \{2,3,5,7\}$
and $D = \{3,4,5,6,7,8\}$
verify the following

i) $(A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

iii) $A \cup (B \cap D) = (A \cup B) \cap (A \cup D)$

iv) $A \cap (B \cup D) = A \cap B \cup (A \cap D)$

We can also prove these properties analytically if we so wish

Example: prove that if A, B, D are any sets then $A \cap (B \cup D) = (A \cap B) \cup (A \cap D)$

proof: We must show that

$A \cap (B \cup D) \subset (A \cap B) \cup (A \cap D)$

and also that $(A \cap B) \cup (A \cap D) \subset A \cap (B \cup D)$

to show the first

inclussion, let $x \in A \cap (B \cup D)$ then $x \in A$ and $x \in B \cup D$; $x \in A$ and $x \in B$ or $x \in D$.
If $x \in B$ then $x \in A \cap B$ or if $x \in D$ then $x \in A \cap D$ and either of cases of the two cases we have
$x \in (A \cap B) \cup (A \cap D)$.
Thus $A \cap (B \cup D) \subset (A \cap B) \cup (A \cap D)$

for the reverse inclusion we note that since $B \subset B \cup D$ then $A \cap B \subset A \cap (B \cup D)$. Similarly $D \subset B \cup D$ then $A \cap D \subset A \cap (B \cup D)$.
therefore, $(A \cap B) \cup (A \cap B) \subset A \cap (B \cup D) \cup A \cap (B \cup D) \Rightarrow (A \cap B) \cup (A \cap D) \subset A \cap (B \cup D)$
$A \cap (B \cup D) = (A \cap B) \cap (A \cap D)$.

EQUIVALENT SETS

Def:
Two sets are equivalent if there is a one-to-one correspondence between them

Example
The sets $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$ are equivalent
$n(A) = 4 = n(B)$

Questions

1. List all the subsets of
a) $M = \{1\}$, b) $N = \{1, 2\}$
c) $S = \{1, 2, 3\}$

2. Give a description notation for the set
a) $A = \{1, 8, 27, 64\}$
b) $B = \{1, 3, 5, 7, 9, 11\}$
c) $C = \{4, 6, 8\}$

Solutions

1. a) $\{\ \}, \{1\}$
b) $\{\ \}, \{1\}, \{2\}, \{1, 2\}$
c) $\{\ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

2. a) Let $S = \{1, 2, 3, \ldots 65\}$
$A = \{x \in S : x = n^3, n = 1, 2, 3, 4\}$

b) Let $M = \{1, 2, 3, \ldots n\}$

$B = \{y \in M : y = 2n - 1, n = 1,2,3,4,5,6\}$

c) Let $P = \{1, 2, 3, 4, 5, \ldots 8\}$

$C = \{a \in P : a = 2n + 2, n = 1,2,3\}$

THE UNIVERSAL SET

— this is the set which contains all elements under discussion. the universal set usually denoted by U or ε, contains every set under consideration.

Example

Given $A = \{1, 2, 3, 4\}$
$B = \{2, 5\}$ and $C = \{2, 4, 6\}$

possible universal set is:

$U = \{0, 1, 2, 3, \ldots 10\}$

THE COMPLEMENT OF A SET

The word, Complement means 'to complete the other'

Defn: For a given universal set U and a subset A of U, the complement of A (denoted A' or A^c) is the set of elements of U which are not in A

Examples

[a]. If $U = \{0, 1, 2, 3, \ldots 7\}$ and $A = \{0, 2, 4, 6\}$ find A'

[b] If $B = \{0, 1, 2, 3\}$ and $B' = \{4, 5, 6\}$ find U

Solutions

a) $A' = \{1, 3, 5, 7\}$

b) $U = B \cup B'$
$= \{0, 1, 2, 3, 4, 5, 6\}$

THE SET DIFFERENCE

- If A and B are sets, then $A - B$ or $(A \cap B')$ is the set of all elements which belong to set A but do not belong to B

Example

Let $A=\{2,3,4,5,\ldots 8\}$ and $B=\{5,6,7,8,9,10,d,f,g\}$

$A-B=\{2,3,4\}$

$B-A=\{9,10,d,f,g\}$

Q. If $M=\{0,1,2,3,\ldots 6\}$ and $N=\{x : x \text{ is a natural number}, 2<x<10\}$, find a) $M-N$ b) $N-M$

Solution:

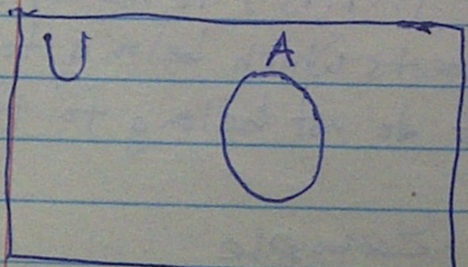
a) $M-N=\{0,1,2\}$

b) $N-M=\{7,8,9\}$

VENN DIAGRAMS

A venn diagram is a pictorial representation of a set

Let a set $A \subset U$, then

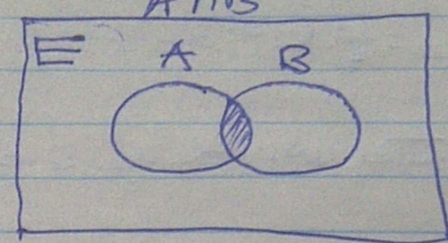


OPERATIONS ON SETS

1) INTERSECTION OF SETS

$A\cap B=\{x \mid x\in A \text{ and } x\in B\}$

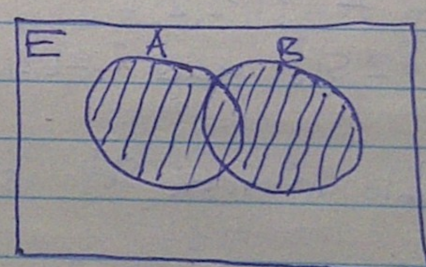
$A\cap B$



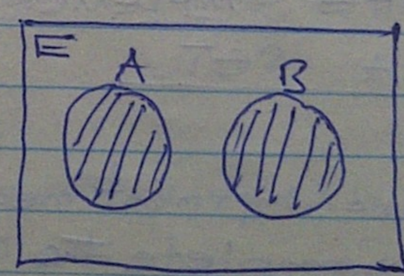
2) UNION OF SETS

$A\cup B=\{x \mid x\in A \text{ or } x\in B \text{ or both}\}$

- The union of two sets A and B, denoted $A\cup B$ is the set of all the elements which belong to set A or to set B or belongs to both sets A and B



$A\cup B$



$(A\cup B)$ Disjoint sets

Example

Let $A = \{1,2,3,---8\}$,
$B = \{3,6,9,12,15\}$ and
$C = \{2,5,7\}$, find

a) $A \cap B$ b) $B \cup C$ c) $B \cap C$

Solution

a) $A \cap B = \{3,6\}$

b) $B \cup C = \{2,3,5,6,7,9,12,15\}$

c) $B \cap C = \{\ \}$ or $\emptyset$

Example

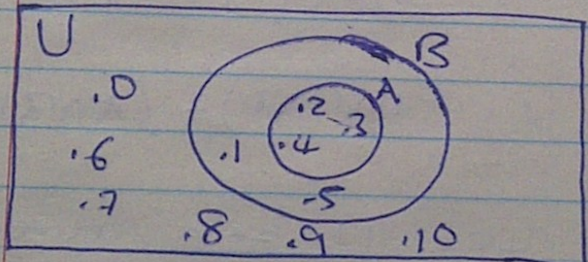
Given the sets $U = \{0,1,2---10\}$

$A = \{2,3,4\}$ and $B = \{1,2,3,4,5\}$

a) Find $A \cap B$ (i) ~~A ∪ B~~

ii) $A \cup B$ iii) A' iv) B'

b) What is the relationship between the set A' & B'



a) i) $A \cap B = \{2,3,4\} = A$

ii) $A \cup B = \{1,2,3,4,5\} = B$

iii) $A' = \{0,1,5,6,7,8,9,10\}$

iv) $B' = \{0,6,7,8,9,10\}$

SOME PROPERTIES OF INTERSECTION AND UNION OF SETS

1) COMMUTATIVITY

(talks of two sets)

For any sets A and B we have

① $A \cap B = B \cap A$

② $A \cup B = B \cup A$

2) ASSOCIATIVITY

(talks of more than two)

to associate is to group. It doesn't matter which two figures you start with cause the answer will still be the same. It only operates on one operation at a time.

For any sets A, B and C

i) $A \cap (B \cap C) = (A \cap B) \cap C$

ii) $A \cup (B \cup C) = (A \cup B) \cup C$

3) THE DISTRIBUTIVE PROPERTIES

- For any a sets A, B & C

a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4) DE MORGAN'S LAW

For any sets A and B

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

EXAMPLES

Let $U = \{1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$

$B = \{2,3,5,7,\}$ and

$C = \{3,4,5,6,7,8\}$

Verify that

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

SOLUTIONS

(i) $A \cup B = \{1,2,3,4,5,7\}$

$\therefore (A \cup B)' = \{6,8,9,10\}$

$A' = \{6,7,8,9,10\}$

$B' = \{1,4,6,8,9,10\}$

$A' \cap B' = \{6,8,9,10\}$

$\therefore (A \cup B)' = A' \cap B'$

(ii) $A \cap B = \{2,3,5\}$

$(A \cap B)' = \{1,4,6,7,8,9,10\}$

$A' = \{6,7,8,9,10\}$

$B' = \{1,4,6,8,9,10\}$

$A' \cup B' = \{1,4,6,7,8,9,10\}$

$\therefore (A \cap B)' = A' \cup B'$

(iii) $B \cup C = \{2,3,4,5,6,7,8\}$

$A = \{1,2,3,4,5\}$

$A \cap (B \cup C) = \{2,3,4,8\}$

$A \cap B = \{2,3,5\}$

$A \cap C = \{3,4,5\}$

$(A \cap B) \cup (A \cap C) = \{2,3,4,5\}$

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

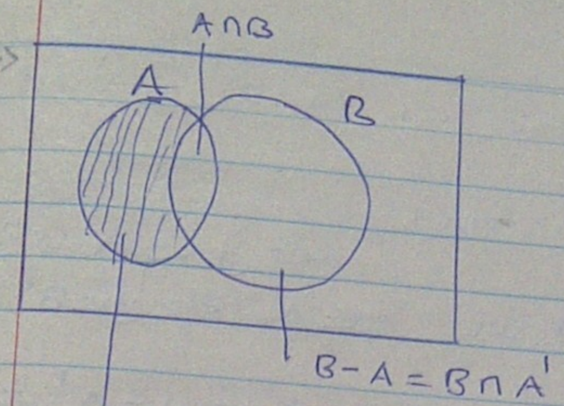
Let U be a universal set, $\emptyset$ be an empty set and set A be a set $A \subset U$

(1) $A \cap U = A$

(2) $A \cup U = U$

(3) $A \cap \emptyset = \emptyset$

(4) $A \cup \emptyset = A$



* $A - B = A \cap B'$
* $(A \cap B) \subset A$
* $(A \cap B') \subset A$
* $(A \cap B) \subset B$
* $(B \cap A') \subset B$

Examples

① Simplify $(A \cap B) \cup (A - B)$

SOLUTION

$(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B')$

$[(A \cap B) \cup A] \cap [(A \cap B) \cup B']$

(using the distributive law)

$= A \cap [(A \cup B') \cap (B \cup B')]$

$= A \cap [(A \cup B') \cap U]$

$= A \cap (A \cup B')$ Distribute

$= (A \cap A) \cup (A \cap B')$

$= A \cup (A \cap B')$

Since $A \cap B' \subset A$

$A \cup (A \cap B') = \underline{\underline{A}}$

SOLUTION B

$A \cap (B \cup B') = A \cap U$

$= \underline{\underline{A}}$

Example

Given the set A, B and C, prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Sol.

proof

We need to show that

$\cancel{A \cap (B \cup C) \subset A \cap (A}$

$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

and

$(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$

Suppose $x \in A \cap (B \cup C)$, Then $x \in A$ & $x \in B \cup C$

Thus $x \in A$ & $x \in B$ or $x \in C$. This results into two cases:

$x \in A$ & $x \in B$ or $x \in A$ & $x \in C$

$\therefore x \in (A \cap B) \cup (A \cap C)$

Hence, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ --- (i)

Let $y \in (A \cap B) \cup (A \cap C)$

Then $y \in (A \cap B)$ or $y \in (A \cap C)$.
thus $y \in A$ & $y \in B$ or
$y \in C$. Therefore, $y \in A \cap (B \cup C)$.

Hence, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$
--- ②

By statements (1) & (2) we have shown that:
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example

for two sets A and B, prove that $(A \cap B)' = A' \cup B'$

Solution

We need to show that
$(A \cap B)' \subset A' \cup B'$ and
$A' \cup B' \subset (A \cap B)'$

let $x \in (A \cap B)'$. Then $x \notin (A \cap B)$. Thus $x \notin A$ or $x \notin B$

Case 1. If $x \notin A$, then $x \in A'$ and therefore $x \in A' \cup B'$

Case 2. If $x \notin B$, then $x \in B'$ & therefore $x \in A' \cup B'$

Combining the two Cases we see that
$x \in A' \cup B'$

$\therefore (A \cap B)' \subset A' \cup B'$ --- (1)

Let $y \in A' \cup B'$. Then $y \in A'$ or $y \in B'$

Case 1. If $y \in A'$, then $y \notin A$ and therefore $y \notin A \cap B$

Case 2. If $y \in B'$, then $y \notin B$ and therefore $y \notin A \cap B$

The two cases both mean $y \notin A \cap B$ and thus $y \in (A \cap B)'$

Hence, $A' \cup B' \subset (A \cap B)'$ --- (2)

therefore we conclude that

$(A \cap B)' = A' \cup B'$

SETS OF NUMBERS

Sets of numbers we are familiar with ~~or~~ include

1. $N = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$ - Natural numbers (counting numbers)
2. $W = \{0, 1, 2, 3, 4, 5, 6, 7 \ldots\}$ - Whole numbers
3. $Z = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$ - Intergers

- Integers Comes from word integral (which means whole)
- Integers exclude fractions

(4) RATIONAL NUMBERS (Q)

The word rational comes from the word 'ratio', therefore the set of rational numbers denoted by Q consists of all numbers of the form $\frac{a}{b}$ (a ratio of intergers) Where a and b are integers and $b \neq 0$

DECIMAL REPRESENTATION OF RATIONAL NUMBERS

the decimal representation of any rational number is either a terminating decimal or a repeating decimal (of some pattern)

E.g $\frac{1}{8} = 0.125 \longrightarrow$ terminating

$\frac{1}{3} = 0.3333 \ldots = 0.\overline{3}$

$\frac{19}{11} = 1.727272 \ldots = 1.\overline{72}$

Examples

Express the following numbers in the form $\frac{a}{b}$ where a & b are integers

a) $0.\overline{3}$ b) $1.\overline{3}$ c) $1.\overline{72}$ d) $4.\overline{83}$ e) $0.\overline{354}$

Solution

a) $0.\overline{3}$

let $x = 0.\overline{3}$ --- (i)

$10x = 3.\overline{3}$ --- (ii)

Subtract eqn (i) from (ii)

$$-\begin{pmatrix} 10x = 3.\overline{3} \\ x = 0.\overline{3} \end{pmatrix}$$

$$\frac{9x}{9} = \frac{3}{9}$$

$$x = \frac{1}{3}$$

b) $1.\overline{3}$

let $x = 1.\overline{3}$ --- (i)

$10x = 13.\overline{3}$ --- (ii)

$$-\begin{pmatrix} 10x = 13.\overline{3} \\ x = 1.\overline{3} \end{pmatrix}$$

$$\frac{9x}{9} = \frac{12}{9}$$

$$x = 4/3$$

c) $1.\overline{72}$

let $x = 1.\overline{72}$

$100x = 172.\overline{72}$

$100x = 172.\overline{72}$

$x = 1.\overline{72}$

$$\frac{99x}{99} = \frac{171}{99}$$

$$x = 19/11$$

d) $4.8\overline{3}$

let $x = 4.8\overline{3}$

$10x = 48.\overline{3}$ -- (i)

$100x = 483.\overline{3}$ -- (ii)

$$-\begin{pmatrix} 100x = 483.\overline{3} \\ 10x = 48.\overline{3} \end{pmatrix}$$

$$\frac{90x}{90} = \frac{435}{90}$$

$$x = \frac{29}{6}$$

e) $0.\overline{354}$

let $x = 0.\overline{354}$ --- (i)

$1000x = 354.\overline{354}$ --- (ii)

$$-\begin{pmatrix} 1000x = 354.\overline{354} \\ x = 0.\overline{354} \end{pmatrix}$$

$$\frac{999x}{999} = \frac{354}{999}$$

$$x = \frac{118}{333}$$

IRRATIONAL NUMBERS

An irrational number is a number which cannot be expressed in the form $\frac{a}{b}$

for some integer a and b with $b \neq 0$

DECIMAL REPRESENTATION OF IRRATIONAL NUMBERS

The decimal representation of an irrational number is non-terminating and non-repeating (no parttern → crazy)

Example

prove that a) $\sqrt{2}$ b) $\sqrt{3}$ is irrational.

Solution

a) Suppose $\sqrt{2}$ is a rational. then $\sqrt{2} = \frac{a}{b}$ where $a>0$ and $b>0$ are integers with no common factors, (apart from 1)

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides we get;

$$2 = a^2/b^2$$

$$a^2 = 2b^2 \text{ --- (i)}$$

we see that a^2 is an even number. therefore, since it is a perfect square its square root is also even

$\therefore$ a = 2K where K is an integer

$$a = 2K \text{ --- (ii)}$$

Substitute eqn (ii) in (i)

$$(2K)^2 = 2b^2$$

$$4K^2 = 2b^2$$

$$b^2 = 2K^2 \text{ ---- (iii)}$$

Eqns (iii) shows that b^2 is an even square. Its square root b must also be even

Hence, a & b have a common factor 2 contradicting our earlier assumption. By contradiction $\sqrt{2}$ is ~~not~~ irrational.

Example

If $\sqrt{3}$ is an irrational number Show that $2+\sqrt{3}$ is not rational

Solution

Suppose $2+\sqrt{3}$ is rational

then $2+\sqrt{3} = \frac{a}{b}$ where a and b are integers & $b \neq 0$

$$2+\sqrt{3} = \frac{a}{b}$$

$$\Rightarrow \sqrt{3} = \frac{a}{b} - 2$$

$$\Rightarrow \sqrt{3} = \frac{a-2b}{b}$$

the last statement suggests that $\sqrt{3}$ is a rational number since $a-2b$ is an integer. This is a contradiction

$\therefore$ $2+\sqrt{3}$ is not rational

REAL NUMBERS

We know that

$N \subset W \subset Z \subset Q$

N.B A number can never be both a rational number an irrational number

Real numbers = {rational #s} $\cup$ {Irrational #s}

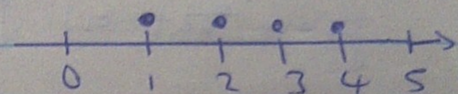
Real numbers
- Rational #s
- Irrational #s

difference btn N & $\mathbb{R}$ #s

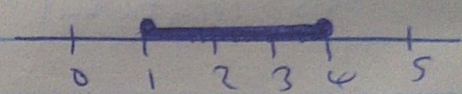
$N = \{1, 2, 3, 4\}$ (i)

$A = \{x \in \mathbb{R} \mid 1 \leq x \leq 4\}$ (ii)

(i)



(ii)



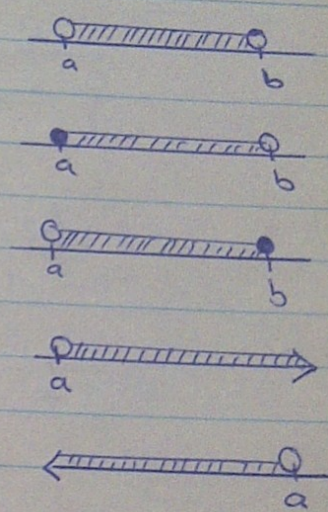
Real numbers are denoted by $\mathbb{R}$

INTERVALS

- This is a special subset of real numbers.
- The intervals can be represented on the real number line.

N.B We use open brackets ( ) when the numbers at the boundary are not included and a closed bracket [ ] when the boundary numbers are included

Interval notation	Set builder notation	Number line
1) (a, b)	$\{x \in \mathbb{R} \mid a < x < b\}$	
2) $[a, b)$	$\{x \in \mathbb{R} \mid a \leq x < b\}$	
3) $(a, b]$	$\{x \in \mathbb{R} \mid a < x \leq b\}$	
4) (a, ∞)	$\{x \in \mathbb{R} \mid x > a\}$	
5) $(-\infty, a)$	$\{x \in \mathbb{R} \mid x < a\}$	



EXAMPLES

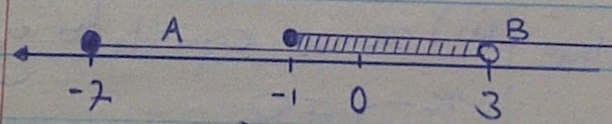
1 let $A = \{x \in \mathbb{R} \mid -7 \leq x < 3\}$ and $B = \{x \in \mathbb{R} \mid x \geq -1\}$,

find a) $A \cap B$ b) A' c) B'

SOLUTION

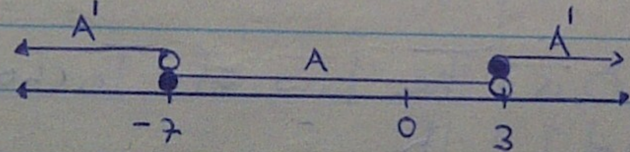
$A = [-7, 3)$ $B = [-1, \infty)$

a)



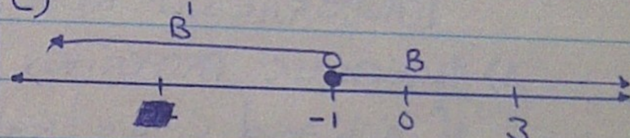
$A \cap B = [-1, 3)$

b)



$A' = (-\infty, -7) \cup [3, \infty)$

c)



$B' = (-\infty, -1)$

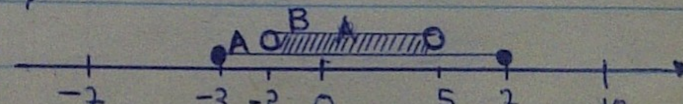
2 If ~~the~~

let $U = (-7, 10]$ be the universal set, $A = [-3, 7]$, $B = (-2, 5)$ and $C = (-6, 10]$

find a) $A \cap B$ b) $A' \cap B'$ c) $A \cup B'$

Solution

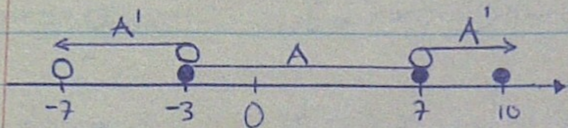
a)



$A \cap B = (-2, 5)$

b) $A' \cap B' = (A \cup B)' = A'$

$B \subset A \Leftrightarrow A \cup B = A$



$A' = (-7, -3) \cup (7, 10] = A' \cap B'$

PROPERTIES OF REAL NUMBERS

A) ALGEBRAIC PROPERTIES OF $\mathbb{R}$

If a, b and c are real numbers, then

i) $a+b = b+a$, $ab = ba$ – Commutative laws of addition and multiplication

ii) $a + (b+c) = (a+b)+c$,
$a(bc) = (ab)c$ – Associative laws of addition and multiplication

iii) $a+0 = 0+a = a$

$1.a = a.1 = a$ → 0 is additive identity and 1 is a multiplicative identity element.

iv) For each $a \in \mathbb{R}$, there corresponds $-a \in \mathbb{R}$ such that

$a + (-a) = -a + a = 0$

Note that $-a$ is known as as the additive inverse of a

v) Each non-zero $a \in \mathbb{R}$ has a corresponding element $a^{-1} \in \mathbb{R}$ such that:

$a^{-1}.a = a.a^{-1} = 1$

vi) $a(b+c) = ab + ac$ } ~~Distr~~ Distribution law
or $a(b-c) = ab - ac$

B) OLDER RELATIONS

Let $x, y \in \mathbb{R}$, then we say that

i) x is greater than y written $x > y$ if $\underbrace{x-y}_{\text{positive}} > 0$

ii) x is less than y written $x < y$ if $x - y < 0$

iii) x is equal to y if $x - y = 0$

The law of Trichotomy for $\mathbb{R}$

If $x, y \in \mathbb{R}$, then either

$x > y$ or $x = y$ $x < y$

COMPLEX NUMBERS

Let us agree that;

$\sqrt{-1} = i$

Def. = $\sqrt{-1} = i$, then $i^2 = -1$
Define

Simplying the powers of i

If $i^2 = -1$, what i^4?

$i^4 = i^2 \times i^2 = -1(-1) = 1$

Example 1

Simplify

a) i^5 b) i^7 c) i^8 d) i^{53} e) i^{80}

Solution

a) $i^5 = i^4 \times i$
$= 1 \times i = i$

b) $i^7 = i^4 \times i^3 = 1 \times i^2 \times i = 1 \times (-1) \times i$
$= -1$

c) $i^8 = (i^4)^2 = 1^2 = 1$

d) $i^{53} = i^{52} \times i$
$= (i^4)^{13} \times i = 1^{13} \times i = 1 \times i = i$

e) $i^{80} = (i^4)^{20} = 1^{20} = 1$

f) $i^{95} = i^{92} \times i^3$
$= 1 \times i^2 \times i = 1 \times (-1) \times i$
$= -i$

IMAGINARY NUMBERS

A number of the form bi where $i = \sqrt{-1}$, & where $b \in \mathbb{R}$ is known as an imaginary number.

e.g $\sqrt{-25} = \sqrt{-1 \times 25} = \sqrt{-1} \times \sqrt{25} = 5i$

Defn

A number of the form $a + bi$ where $a, b \in \mathbb{R}$ is called a complex number

e.g $3 + 4i$.

A complex number is usually denoted by z (z).

N.B $a + bi$

1) If $a = 0$, then we have imaginary numbers bi

2) If $b = 0$, hence real numbers are complex numbers with $b = 0$

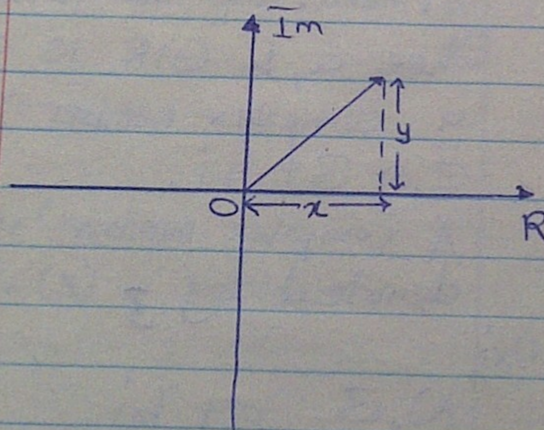
Defn.
let $z = x + iy$, $i = \sqrt{-1}$, then we can write

$Re(z) = x$ (real part of z)
$Im(z) = y$ (Imaginary part of z)

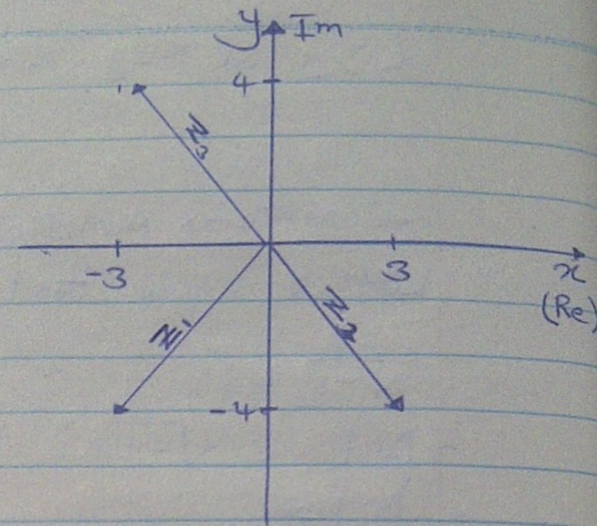
Example
If $z = -5 + \frac{3}{4}i$, then

$Re(z) = -5$
$Im(z) = 3/4$

ARGAND DIAGRAM
A Complex number $z = x + iy$ Can be represented on an argand diagram as:



Example
Illustrate $z_1 = -3 - 4i$ on the argand diagram
ii) $z_2 = 3 - 4i$ iii) $z_3 = -3 + 4i$



EQUALITY OF COMPLEX NUMBERS

Defn. Two complex numbers are equal if and only if both their real parts and imaginary parts are equal

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 = z_2$ iff $x_1 = x_2$ & $y_1 = y_2$

Example
Find x and y such that
$x + iy = 7 - 4i$

Solution
$x = 7$, $y = -4$

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

Let $Z_1 = x_1 + iy_1$ &

$Z_2 = x_2 + iy_2$, then

$Z_1 + Z_2 = (x_1 + x_2) + i(y_1 + y_2)$

and $Z_1 - Z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$= x_1 + iy_1 - x_2 - iy_2$

$= x_1 - x_2 + iy_1 - iy_2$

$= (x_1 - x_2) + i(y_1 - y_2)$

EXAMPLE

If $Z_1 = 3 - 4i$ and $Z_2 = -4 - 2i$, Simplify

a) $Z_1 + Z_2$ b) $Z_2 - Z_1$

Solution

a) $Z_1 + Z_2 = (3 + (-4)) + i(-4 + (-2))$

$= 3 - 4 + i(-4 - 2)$

$= -1 - 6i$

b) $Z_2 - Z_1 = (-4 - 3) + i(-2 - (-4))$

$= -7 + 2i$

Q. If $(3x - iy) + (2 + 13i) = -7 + 3i$, find x & y

Solution

$(3x + 2) + i(-y + 13) = -7 + 3i$

$3x + 2 = -7$

$3x + 2 - 2 = -7 - 2$

$\frac{3x}{3} = \frac{-9}{3}$

$x = -3$

$-y + 13 = 3$

$y = 13 - 3$

$y = 10$

PRODUCT OF COMPLEX NUMBERS

let $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$ then

$Z_1Z_2 = (x_1 + iy)(x_2 + iy_2)$

$= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$

$= x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)$

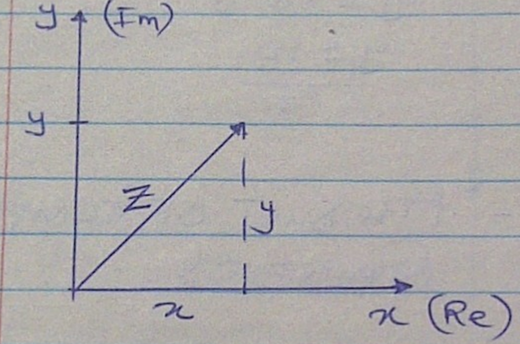
Example

If $Z_1 = -3 + 4i$ and $Z_2 = 1 + 5i$ find Z_1Z_2

Solution

$$Z_1 Z_2 = (-3+4i)(1+5i)$$
$$= -3(1+5i)+4i(1+5i)$$
$$= -3-15i+4i-20$$
$$= -3-20-15i+4i$$
$$= \underline{\underline{-23-11i}}$$

MODULUS OF A COMPLEX NUMBER



$$Z = x+iy$$
$$|Z|^2 = x^2+y^2$$
$$|Z| = \sqrt{x^2+y^2}$$

N.B $|Z|^2 = x^2+y^2$

EXAMPLE

Find the modulus of

a) $Z_1 = 5+12i$ b) $Z_2 = -8+6i$

c) $Z_3 = 15i$

Solution

a) $|Z_1| = \sqrt{5^2+12^2}$
$= \sqrt{169}$
$= \underline{\underline{13}}$

b) $|Z_2| = \sqrt{(-8)^2+6^2}$
$= \sqrt{100}$
$= \underline{\underline{10}}$

c) $|Z_3| = \sqrt{0^2+15^2}$
$= \sqrt{15^2}$
$= \underline{\underline{15}}$

THE CONJUGATE OF A COMPLEX NUMBER

The conjugate of a complex number $Z = x+iy$ denoted by $\bar{Z}$ is given by

$$\bar{Z} = x-iy.$$

EXAMPLE

$Z_1 = 4+3i$ then $\bar{Z}_1 = 4-3i$ and $Z_2 = -4-2i$ then $\bar{Z}_2 = -4+2i$

Take note; If $Z = x+iy$

and $\bar{Z} = x - iy$, then

$$Z\bar{Z} = (x+iy)(x-iy)$$
$$= x(x-iy) + iy(x-iy)$$
$$= x^2 - ixy + ixy - i^2y^2$$
$$= x^2 + y^2 \longrightarrow \text{This is a real \#}$$

DIVISION OF TWO COMPLEX NUMBERS

let $Z_1 = x_1 + iy$ and $Z_2 = x_2 + iy_2$ then

$$\frac{Z_1}{Z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$
$$= \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}$$
$$= \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$
$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

EXAMPLE

Simplify $\frac{3+2i}{4+3i}$

$$= \frac{(3+2i)(4-3i)}{(4+3i)(4-3i)}$$
$$= \frac{12 - 9i + 8i + 6}{4^2 + 3^2} = \frac{18 - i}{16+9}$$
$$= \frac{18}{25} - \frac{i}{25}$$

PROPERTIES OF CONJUGATION

If $Z = x + iy$

$\bar{Z} = x - iy$

i) $\overline{Z_1 Z_2} = \bar{Z_1} \cdot \bar{Z_2}$

ii) $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\bar{Z_1}}{\bar{Z_2}}$

Examples

Express each of the following in the form $a + bi$ where a and b rational numbers

a) $\frac{3+i}{4-3i}$ b) $\frac{2}{1-i}$

Solution

a) $$\frac{3+i}{4-3i} = \frac{(3+i)(4+3i)}{(4-3i)(4+3i)}$$
$$= \frac{12 - 3 + i(9+4)}{4^2 + (-3)^2}$$
$$= \frac{9 + 13i}{25}$$
$$= \frac{9}{25} + \frac{13}{25}i$$

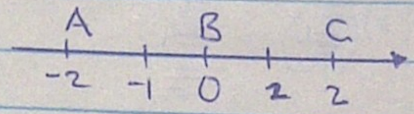
b) $$\frac{2}{1-i} = \frac{2(1+i)}{(1-i)(1+i)}$$

$$= \frac{2+2i}{1^2+(-i)^2}$$

$$= \frac{2+2i}{2} = 1+i$$

THE ABSOLUTE VALUE OF A REAL NUMBER

Distance on the number line



Distance $BA = 2$

$BC = 2$

$BB = 0$

The absolute value of a real number x is denoted by $|x|$

Example

Evaluate: (i) $|-3| = 3$

(ii) $|0| = 0$

(iii) $|4| = 4$

Defn: If x is a real number, then the absolute value of x is

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$= 4$

4 if $x < 0$

$+$ if $x > 0$

EXAMPLE

Solve the equation

a) $|2x+5| = 7$

b) $|5x-2x| = 3$

Solution

a) $|2x+5| = 7$

$2x+5 = 7$ if $2x+5 > 0$

$\frac{2x}{2} = \frac{7-5}{2}$

$x = 1$

$-(2x+5) = 7$ if $2x+5 < 0$

$-2x-5 = 7$

$\frac{-2x}{-2} = \frac{12}{-2}$

$x = -6$

b) $|5x-2x| = 3$

$5x-2x = 3$ if $5x-2x > 0$

$\frac{3x}{3} = \frac{3}{3}$

$x = 1$

$-(5x-2x) = 3$ if $5x-2x < 0$

$-5x+2x = 3$

$\frac{-3x}{-3} = \frac{3}{-3}$

$x = -1$

SURDS

Numbers of the form $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc are called surds.

N.B i) These are irrational #s

ii) $\sqrt{2} = 2^{1/2}$

$$\sqrt{2}\times\sqrt{2} = 2^{1/2}\times 2^{1/2}$$
$$= 2^{1/2+1/2} = 2^1 = 2$$

Examples

1. Express each of the following in the simplest form

a) $\sqrt{12}$

$$=\sqrt{12} = \sqrt{4\times 3} = \sqrt{4}\sqrt{3}$$
$$= \underline{\underline{2\sqrt{3}}}$$

b) $\sqrt{63}$

$$=\sqrt{9\times 7} = \sqrt{9}\sqrt{7} = \underline{\underline{3\sqrt{7}}}$$

c) $\sqrt{80}$

$$=\sqrt{4\times 20} = \sqrt{4\times 4\times 5} = \underline{\underline{4\sqrt{5}}}$$

d) $\sqrt{50}$

$$=\sqrt{50} = \sqrt{25\times 2} = \underline{\underline{5\sqrt{2}}}$$

2. If $A = 1+\sqrt{2}$ and $B = 3-2\sqrt{2}$ find

a) $A+B$ b) $B-A$

Solution (Addition & Subtraction of surds)

a) $A+B = (1+\sqrt{2}) + (3-2\sqrt{2})$
$$= (1+3) + \sqrt{2}(1-2)$$
$$= \underline{\underline{4-\sqrt{2}}}$$

b) $B-A = (3-2\sqrt{2}) - (1+\sqrt{2})$
$$= (3-1) - \sqrt{2}(2+1)$$
$$= \underline{\underline{2-3\sqrt{2}}}$$

MULTIPLICATION OF SURDS

Example

Simplify (1) $(1+2\sqrt{5})(2+\sqrt{5})$

$$= 1(2+\sqrt{5}) + 2\sqrt{5}(2+\sqrt{5})$$
$$= 2+\sqrt{5}+4\sqrt{5}+10$$
$$= 2+10+\sqrt{5}(1+4)$$
$$= \underline{\underline{12+5\sqrt{5}}}$$

(2) $(7-\sqrt{2})(7+\sqrt{2})$

$$= 7(7+\sqrt{2}) - \sqrt{2}(7+\sqrt{2})$$
$$= 49+7\sqrt{2}-7\sqrt{2}-2$$
$$= 49-2$$
$$= \underline{\underline{47}}$$

Number	Conjugate
$4+3i$	$4-3i$
$0+4i$	$0-4i = -4i$
$4+0i$	$4-0i = 4$

RATIONALIZING THE DENOMINATOR

When a fraction contains a surd in the denominator we can simplify it by removing the surd from the denominator.

E.g

a) $\frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{\sqrt{7}\sqrt{7}} = \frac{3\sqrt{7}}{7}$

b) $\frac{6}{\sqrt{45}} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}}$

$= \frac{2\sqrt{5}}{\sqrt{5}\sqrt{5}}$

$= \frac{2\sqrt{5}}{5}$

Example

1. Rationalize the denominator of $\frac{1-5\sqrt{2}}{3-\sqrt{2}}$

Solution

$$\frac{1-5\sqrt{2}}{3-\sqrt{2}} = \frac{(1-5\sqrt{2})(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}$$

$$= \frac{3+\sqrt{2}-15\sqrt{2}-10}{3^2-(\sqrt{2})^2}$$

$$= \frac{-7-14\sqrt{2}}{9-2}$$

$$= \frac{7(-1-2\sqrt{2})}{7}$$

$$= \frac{-1-2\sqrt{2}}{}$$

2. Rationalize the denominator of $\frac{2+\sqrt{3}}{3+\sqrt{3}}$

Solution

$$\frac{(2+\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}$$

$$= \frac{6-2\sqrt{3}+3\sqrt{3}-3}{3^2-(\sqrt{3})^2}$$

$$= \frac{3+\sqrt{3}}{9-3} = \frac{3+\sqrt{3}}{6}$$

$$= \cancel{\frac{3(1+\sqrt{3})}{}}$$

Surd	Conjugate
$3-\sqrt{2}$	$3+\sqrt{2}$
$\sqrt{3}+2$	$2-\sqrt{3}$
$\sqrt{2}+\sqrt{3}$	$\sqrt{2}-\sqrt{3}$

$$(a+b)(a-b) = a^2-b^2$$

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) =$$

$$(\sqrt{a})^2-(\sqrt{b})^2$$

$$= a-b$$

Example

Products of radical expressions may be simplified by applying

$\sqrt{a}\sqrt{b} = \sqrt{a.b}$

Q1. Write the products in simplest radical form

a) $\sqrt{3xy}\sqrt{6y}$

$= \sqrt{18xy^2}$

$= \sqrt{9 \times 2 \times x \times y^2}$

$= 3y\sqrt{2x}$

b) $\sqrt[3]{4}\left(\sqrt[3]{6} + \sqrt[3]{10}\right)$

$= \sqrt[3]{4 \times 6} + \sqrt[3]{4 \times 10}$

$= \sqrt[3]{24} + \sqrt[3]{40}$

$= \sqrt[3]{8 \times 3} + \sqrt[3]{8 \times 5}$

$= 2\sqrt[3]{3} + 2\sqrt[3]{5}$

Q2 (a) Simplify $\frac{\sqrt{35x^3y^2}}{\sqrt{5x}}$

(b) $\frac{\sqrt[3]{160x^3y^2}}{\sqrt[3]{\frac{20}{y}}}$

Solution

(a) $\sqrt{\frac{35x^3y^2}{5x}}$

$= \sqrt{7x^2y^2}$

$= \sqrt{7}\,xy$

b) $\frac{\sqrt[3]{160x^3y^2}}{\sqrt[3]{20y^{-1}}}$

$= \sqrt[3]{\frac{160x^3y^2}{20y^{-1}}}$

$= \sqrt[3]{8x^3y^3} = 2xy$

EXPONENTS (INDICES)

Defn

If $a \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$a^n = \underbrace{a \ldots a \ldots\ldots a}_{n \text{ factors}}$

1. THEOREMS

i) $a^m \cdot a^n = a^{m+n}$

ii) $a^m \div a^n = a^{m-n}$

iii) $(a^m)^n = a^{mn}$

iv) $(a^p b^q)^n = a^{pn} b^{qn}$

v) $\left(\frac{a^p}{b^q}\right)^m = \frac{a^{pm}}{b^{qm}} = (b \neq 0)$

② INTEGERS AS EXPONENTS

$\frac{1}{a} \longrightarrow$ the reciprocal of a!

Defn If $a \in \mathbb{R}$, $a \neq 0$, then $a^0 = 1$

Defn If n is a positive integer and a is non-zero real number, then

$$a^{-n} = \frac{1}{a^n}$$

EXAMPLE

Write each expression without negative exponents

a) x^{-3} b) $\frac{1}{y^{-2}}$ c) $\frac{x^{-2}}{y^{-2}}$

Solution

a) $x^{-3} = \frac{1}{x^3}$

b) $\frac{1}{y^{-2}} = \frac{1}{\frac{1}{y^2}} = \frac{y^2}{1} = y^2$

c) $\frac{x^{-2}}{y^{-2}} = \frac{\frac{1}{x^2}}{\frac{1}{y^2}} = \frac{1}{x^2} \div \frac{1}{y^2}$

$= \frac{1}{x^3} \times \frac{y^2}{1}$

$= \left(\frac{y}{x}\right)^2$

Q. Simply

① $\left(\frac{2}{y}\right)^3$ ② $(-2x^2)^3$

③ $(-3x^2)^2$ ④ $\left(\frac{x^4}{x}\right)^{-2}$

⑤ $\left(\frac{2}{x}\right)^3\left(\frac{2}{x}\right)^{-4}$ ⑥ $(x^{-2}y^3)^{-1}$

⑦ $(x^2y^2)(x^{-2}y^3)$

⑧ $\frac{(5x)^2}{(3x^2)^{-3}}$ ⑨ $x^{n-1}x^{2-n}$

Solution

① $\left(\frac{2}{y}\right)^3 = \frac{2^3}{y^3} = \frac{8}{y^3} = 8y^{-3}$

② $(-2x^2)^3 = -2^3 \times x^{2\times 3}$

$= -8x^6$

③ $(-3x^2)^2 = -3^2 \times x^4$

$= 9x^4$

y^2

$^{-1}$

(4) $\left(\frac{x^4}{x}\right)^{-2} = \frac{x^{-8}}{x^{-2}}$

$= \frac{\frac{1}{x^8}}{\frac{1}{x^2}} = 1/x^8 \div 1/x^2$

$= 1/x^8 \times \frac{x^2}{1}$

$= \frac{x^2}{x^8}$

$= x^{2-8} = x^{-6}$

(5) $\left(2/x\right)^3 \left(2/x\right)^{-4} = \left(2/x\right)^{3+(-4)}$

$= \left(2/x\right)^{-1} = \frac{1}{(2/x)}$

$= \frac{x}{2}$

(6) $(x^{-2}y^3)^{-1}$

$= x^{-2(-1)} y^{3(-1)} = x^2 y^{-3}$

(7) $(x^2y^2)(x^2y^3) = (x^{2+2} y^{2+3})$

$= (x^4 y^5)$

(8) $\frac{(5x)^2}{(3x^2)^{-3}} = \frac{(5^1 \times x^1)^2}{(3^1 \times x^2)^{-3}}$

$= \frac{5^2 x^2}{3^{-3} x^{-6}} = \frac{25}{x^{-}}$

$= \frac{25x^2}{\frac{1}{3^3 x^6}} = 25x$

$= 25x$

$= \frac{25x^2}{\frac{1}{27x^6}} = 25$

$= 25x^2 \div \frac{1}{27x^6}$

$= 25x^2 \times \frac{27x^6}{1}$

$= 25x^2 27x^6$

$= 675x^8$

(9) $x^{n-1} x^{2-n}$

$= x^{(n-1)+(2-n)}$

$= x^{n-1+2-n}$

$= x^1$

$= x$

Examples

Evaluate

i) $8^{2/3}$ ii) $\left(\frac{8}{27}\right)^{-1/3}$

iii) $(-1)^{5/3}$ iv) $(27\times 8)^{-5/3}$

v) $125^{-2/3}$ (vi) $81^{-1/2}\times 3^{2}$

Examples

Simplify

a) $\left(x^{3/2}\right)^{2/3}$ b) $\left(y^{1/2}\,y^{3/4}\right)^{2/5}$

c) $\left(\frac{x^{3/2}}{x^{-1/2}}\right)^{1/2}$

Solutions

a) $\left(x^{3/2}\right)^{2/3} = x^{3/2\times 2/3}$

$= x^{1} = x$

b) $\left(y^{1/2}\times y^{3/4}\right)^{2/5} = \left(y^{2/4}\times y^{3/4}\right)^{2/5}$

$= \left(y^{5/4}\right)^{2/5}$

$= y^{1/2}$

$= \sqrt{y}$

c) $\left(\frac{x^{3/2}}{x^{-1/2}}\right)^{1/2} = \left(x^{3/2-(-1/2)}\right)^{1/2}$

$= \left(x^{2}\right)^{1/2}$

$= x^{1} = x$

Example

Simplify $\sqrt[3]{64\,\frac{x^{2}y^{7/2}}{x^{-1}\sqrt{y}}}$

Solution

$$\sqrt[3]{64\,\frac{x^{2}y^{7/2}}{x^{-1}y^{1/2}}} = \sqrt[3]{64x^{3}y^{3}}$$

$$= 4xy$$

Example

Expand and Simplify

i) $\sqrt{x}\left[(\sqrt{x})^{3} + \sqrt{x}\right]$

ii) $\left(x^{1/3} + x^{-1/3}\right)^{2}$

Solutions

(i) $x^{1/2}\left((x^{1/2})^3 + x^{1/2}\right)$

$= x^{1/2}\left(x^{3/2} + x^{1/2}\right)$

$= x^{1/2} \times x^{3/2} + x^{1/2} \times x^{1/2}$

$= x^2 + x$

(ii) $\left(x^{1/3} + x^{-1/3}\right)\left(x^{1/3} + x^{-1/3}\right)$

$x^{1/3}\left(x^{1/3} + x^{-1/3}\right) + x^{-1/3}\left(x^{1/3} + x^{-1/3}\right)$

$= x^{2/3} + x^0 + x^0 + x^{-2/3}$

$= \left(x^{1/3} + x^{-1/3}\right) = x^{2/3} + 2 + \frac{1}{x^{2/3}}$

EXPONENTS WITH NEGATIVE BASES

i) $(-2)^3 = -2 \times (-2) \times (-2) = -8$

ii) $-2^3 = -(2 \times 2 \times 2) = -8$

iii) $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2)$
$= 16$

iv) $-2^4 = -(2 \times 2 \times 2 \times 2)$
$= -16$

$\left(\sqrt[3]{\frac{8}{27}}\right)^{-1}$

BINARY OPERATIONS

Bi means two (2)

Defn: A binary operation, denoted 'o', on a non-empty set G is a rule which associates to each pair of elements, $a, b \in G$ a unique element. $a \circ b \in G$

Examples

1) ADDITION (+) is a binary operation on the set of natural numbers since $m, n \in N$ then $m + n \in N$. However, subtraction is not a binary operation on N since $m - n$ may not be a natural number even if $m, n \in N$
e.g. $6, 5 \in N$ but $5 - 6 \notin N$

2) Both addition and subtraction are binary operations on $\mathbb{Z}$ the set of integers

N.B Whenever 'o' is a binary operation on a set G we say G is closed or the closure property is satisfied in G with respect to the operation 'o'

* The binary operation x is closed on the set G = {0, 1}

x	0	1
0	0	0
1	0	1

* But the operation '+' is not closed on the set G = {0, 1}

+	0	1
0	0	0
1	0	2

Not closed because 2 is not a member of set G ie $2 \notin G$
Therefore, addition + is not a binary operation on the set G = {0, 1}

THE OPERATION 'o' on a set G is said to be

a) Commutative if for every pair $a, b \in G$ we have $a \circ b = b \circ a$

b) Associative if for all $a, b, c \in G$ we have

$$a \circ (b \circ c) = (a \circ b) \circ c$$

Examples

1. Let the operation 'o' be a binary operation on the IR such that

$$a \circ b = \frac{a+b}{2}$$

a) Is it Commutative?
b) Is it associative?

Solution

a) $a \circ b = \frac{a+b}{2}$

$a \circ b = \frac{b+a}{2}$

$a+b = b+a$ since $a, b \in IR$, this is one of the properties of real #s

Therefore, $\frac{a+b}{2} = \frac{b+a}{2}$

$\Longrightarrow a \circ b = b \circ a$

it is commutative

b) $a \circ (b \circ c) = (a \circ b) \circ c$

L.H.S

$a \circ (b \circ c) = a \circ \left(\frac{b+c}{2}\right)$

$$= \frac{a + \frac{b+c}{2}}{2}$$

$$= \frac{2a+b+c}{2} \div \frac{2}{1}$$

$$= \frac{\frac{2a+b+c}{4}}{\quad}$$

R.H.S

$$(a \circ b) \circ c = \left(\frac{a+b}{2}\right) \circ c$$

$$= \frac{\frac{a+b}{2} + c}{2}$$

$$= \frac{a+b+2c}{2} \div \frac{2}{1}$$

$$= \frac{\frac{a+b+2c}{4}}{\quad}$$

therefore, the operation is not associative on $\mathbb{R}$

Examples

Let the operation $a \circ b = a - 2b$ be defined on $\mathbb{R}$

a) Is '$\circ$' Commutative

b) Is '$\circ$' a binary operation

c) Is '$\circ$' associative

Solution

a) $a \circ b = a - 2b$

$b \circ a = b - 2a$

Since $a - 2b \neq b - 2a \Rightarrow$ $a \circ b \neq b \circ a$, Hence the operation '$\circ$' is not commutative

b) $a - 2b \in \mathbb{R}$, ie the operation '$\circ$' is Closed Hence, It is a binary operation

c)

Example

Let the operation '$\circ$' on $\mathbb{R}$ be defined by $a \circ b = 2^{a+b}$

i) is the operation a binary operation on $\mathbb{R}$?

ii) Is It Commutative

iii) Is It associative

iv) Evaluate $3 \circ 2$

Solutions

i) It is a binary operation since 2^{a+b} is $\in \mathbb{R}$ for any $a, b \in \mathbb{R}$

ii) $a \circ b = 2^{a+b}$

and $b \circ a = 2^{b+a}$

This is a property of $\mathbb{R}$ since $a+b = b+a$

i.e. $a \circ b = b \circ a$

it is commutative

iii)

iv) $3 \circ 2 = 2^{3+2}$

$= 2^5$

$= 32$

FUNCTIONS

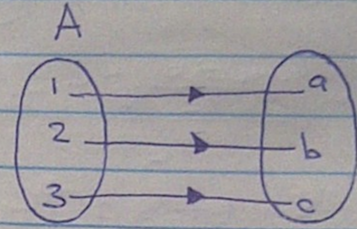
① RELATION

Defn

Relation is a rule that associates elements of two sets

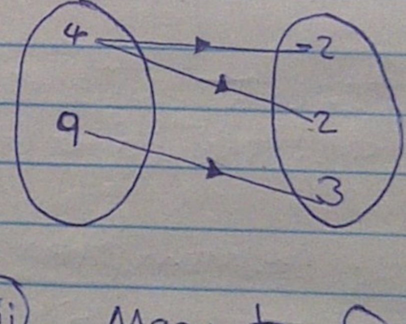
FOUR TYPES OF RELATIONS

(i) One-to-One



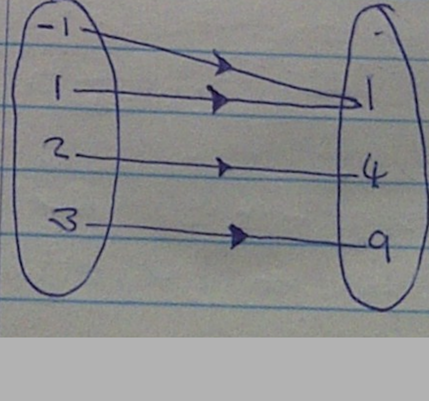
(ii) One-to-Many

A "square of" B

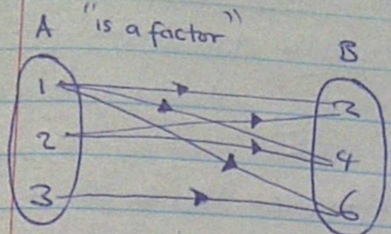


(iii) Many-to-One

A "square root of" B

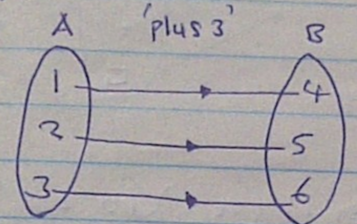


iv) MANY – TO – MANY



Defn (Product set)

Let A and B be two sets. Let $a \in A$ and $b \in B$. Then (a, b) is called an ordered pair



$(1,4), (2,5), (3,6)$

CARTESIAN PRODUCT

$A = \{1, 2, 3\}$ and $B = \{2, 4\}$

$A \times B =$
$\{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)\}$

$B \times A = \{(2,1), (2,2), (2,3), (4,1), (4,2), (4,3)\}$

Therfore, $A \times B \neq B \times A$

The set of distinct ordered pairs whose first coordinate is an element of A and whose second coordinate is an element of B is called Cartesian product of A and B denoted by

$A \times B = \{(a, b) ; a \in A, b \in B\}$

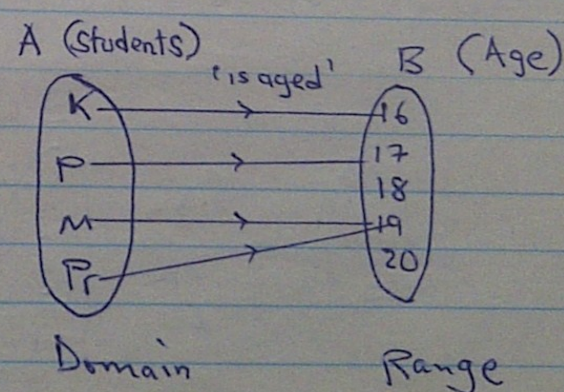
what is $B \times A =$
$\{(b, a) : a \in A, b \in B\}$

Defn:

A function is a relation such that for each first component (object) of the domain there is one and only one second component (image) of the range

N.B No object in the domain should have more than 1 image in the range

e.g



Take note that:

1) Each element of set A (domain) has an image in set B (range)
2) Some elements in set B (range) may not be connected to any object in set A (domain)
3) Each object in the domain set A cannot be mapped to more than one image in set B

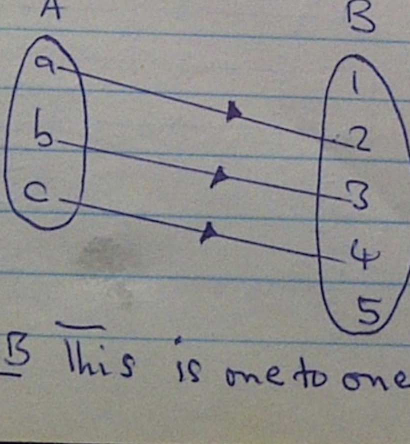
TESTING FOR FUNCTIONS

Example 1

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4, 5\}$

$A \times B = \{(a,1), (a,2), (a,3), (a,4), (a,5), (b,1), (b,2), (b,3), (b,4), (b,5), (c,1), (c,2), (c,3), (c,4), (c,5)\}$ not a function (cartesian product)

a) $\{(a,2), (b,3), (c,4)\}$ function

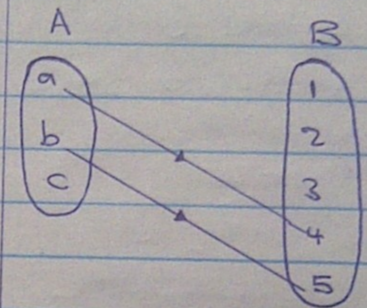


N.B This is one to one relation

1) Each object is mapped exactly to one image
2) Every object has an image

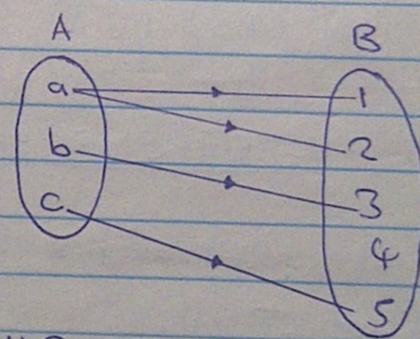
Hence, this is a function

b) $\{(a,4), (b,5)\}$



C has no image. Therefore, this is not a function.

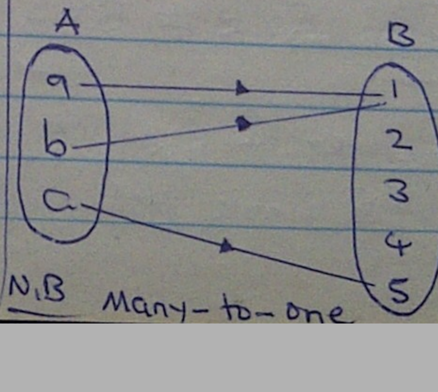
c) $\{(a,1), (a,2), (b,3), (c,5)\}$



The object a is mapped to two images. Therefore, this is not a function.

N.B one-to-many

d) $\{(a,1), (b,1), (c,5)\}$



N.B Many-to-one

1) Each object has exactly one image
2) Each object has an image
This is a function

Example 2

Test for functions represented by Equations

Which of the equations represent(s) y as a function of x?

a) $x^2 + y = 1$ b) $-x + y^2 = 1$

Solution

a) $x^2 + y = 1$

$y = 1 - x^2$

if $(x = 1, 0)$

$1 \longrightarrow 0$

This is a function $x \to 1 - x^2$ (a single image)

b) $-x + y^2 = 1$

$y^2 = 1 + x$

$y = \pm\sqrt{1+x}$

mapped to $\sqrt{1+x} = y$ (image)

x
object

$-\sqrt{1+x} = y$ (image)

Not a function, because each object is mapped to 2 images

FUNCTION NOTATION

$y = 1 - x^2$

Input (objects)	Output (images)
x	$y = f(x)$

Equation

$f(x) = 1 - x^2$

or $f: x \longrightarrow 1 - x^2$

Example

If $f(x) = 3 - 2x$, evaluate

a) $f(-1)$ b) $f(0)$ c) $f(2)$

Solution

a) $f(-1) = 3 - 2(-1)$

$= 5$

b) $f(0) = 3 - 2(0)$

$= 3$

c) $f(2) = 3 - 2(2)$

$= 3 - 4$

$= -1$

Q. If $f(x) = -x^2 + 4x + 1$, find

a) $f(2)$ b) $f(t)$ c) $f(x+2)$.

Solution

a) $f(2) = -2^2 + 4(2) + 1$

$= -(2\times 2) + 8 + 1$

$= -4 + 9$

$= 5$

b) $f(t) = -t^2 + 4t + 1$

c) $f(x+2) = -(x+2)^2 + 4(x+2) + 1$

$= \cancel{-(x^2+4) + 4x + 8 + 1}$

$= \cancel{-x^2 - 4 + 4x + 9}$

$= -(x^2 + 4x + 4) + 4x + 8 + 1$

$= -x^2 - 4x - 4 + 4x + 9$

$= -x^2 + 5$

Example

Given the function

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Evaluate

a) $f(-1)$ b) $f(0)$ c) $f(1)$

Solution

$x = -1 < 0$

a) $f(-1)$

$f(x) = x^2 + 1$

$f(-1) = (-1)^2 + 1 = 2$

b) $x = 0$

$f(x) = x - 1$

$f(0) = 0 - 1$

$= -1$

c) $f(x)$

$f(1) = 1 - 1$

$= 0$

Example

For $f(x) = |x| - 4$, evaluate

a) $f(2)$ b) $f(-2)$ c) $f($[illegible]

Solution

a) $f(2) = |2| - 4$

$= 2 - 4$

$= -2$

b) $f(-2) = |-2| - 4$

$= 2 - 4$

$= -2$

c) $f(-7) = |-7| - 4$
$= 7 - 4$
$= \underline{\underline{3}}$

THE DOMAIN AND THE RANGE OF A FUNCTION

Find the domain of each function below

i) $f: \{(-3,0), (-1,6), (0,2), (2,2)(4,-1)\}$

Domain $= \{-3, -1, 0, 2, 4\}$

ii) $f(x) = x^3 - 2x^2 + 4$

Domain $= \mathbb{R}$ (all real numbers)

iii) $f(x) = \dfrac{1}{x+5}$

$(x+5=0,\ x=-5)$

Domain = all real numbers except $x = -5$

iv) a) $f(x) = \sqrt{x}$

Domain $= x \geq 0$ where x is a real number

b) $f(x) = \sqrt{4-x^2}$

Domain $= 4 - x^2 \geq 0$
$-x^2 \geq -4$
$x^2 \leq 4$

$-2 \leq x \leq 2$

where $x \in \mathbb{R}$

GRAPHS OF COMMON FUNCTIONS

The graph of a function f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f.

THE ZERO OF A FUNCTION

N.B The value of a function at a point (x_0, y_0) is the Y-coordinate y_0. Where (x_0, y_0) is on the function

If the graph of a function has x-intercept $(a,0)$, then a is a zero of the function. In other words, the zeros of a function are the values of x for which $f(x) = 0$

Eg (a) $f(x) = x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$x = -2$ or $x = 2$

Therefore, -2 and 2 are the zeros of the function $f(x) = x^2 - 4$

b) $f(x) = 2x - 4$

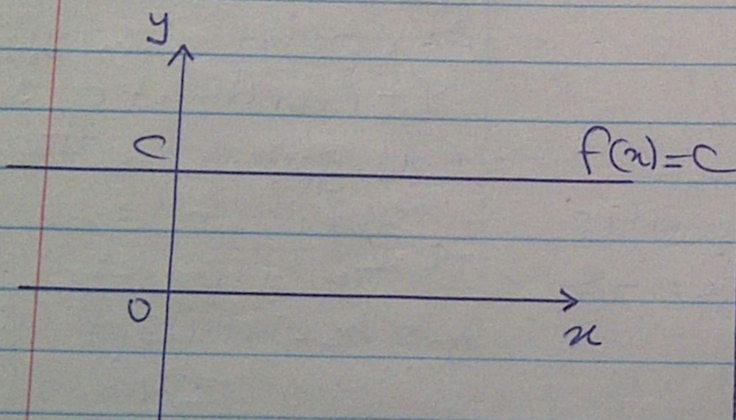
$\Rightarrow 2x - 4 = 0$

$\frac{2x}{2} = \frac{4}{2}$

$x = 2$

2 is the zero of a function $f(x) = 2x + 4$

1. $f(x) = c$ where c is a constant



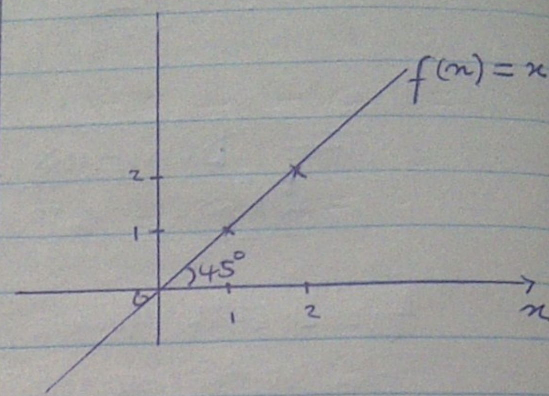
Domain = all real numbers

Range = $\{c\}$

Given that $f(x) = 4$, find $f(2)$

$f(2) = 4 \longrightarrow$ Constant function

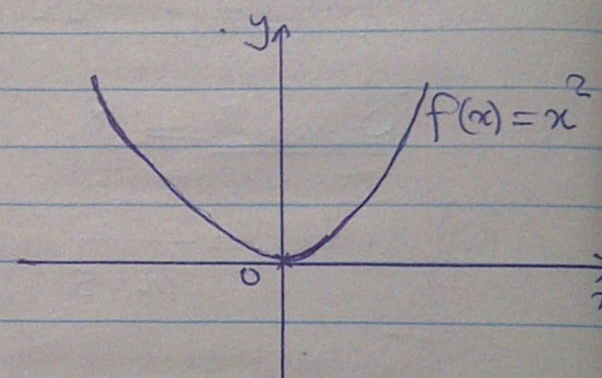
2) $f(x) = x$ or $y = x$



Domain = all real #s

Range = all real #s

3) $f(x) = x^2$



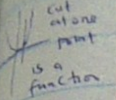
Domain = all real #s

Range = $y \geq 0$ where $y \in \mathbb{R}$

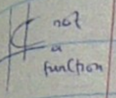
VERTICAL LINE TEST

FUNCTIONS

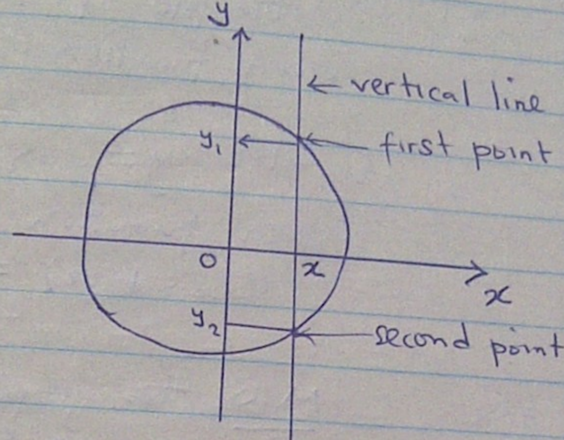
A set of points in a Coordinate plane is the



graph of y as a function of x if and only if no vertical line intersects the graph at more than one point



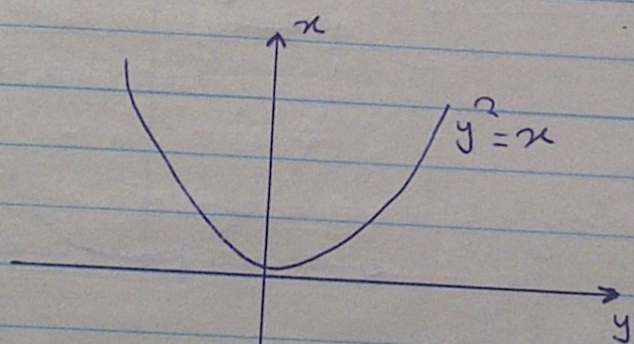
E.g



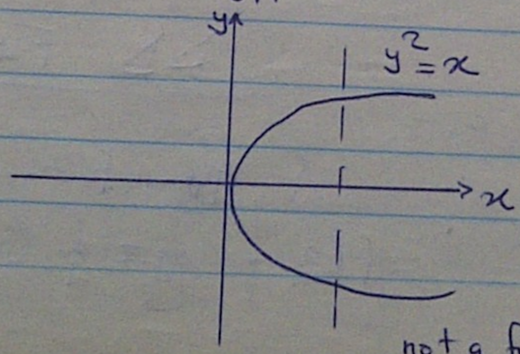
Not a function

4) $y=\sqrt{x} \iff y^2=x$

a)



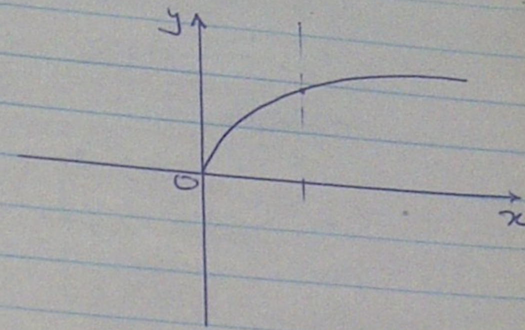
OR



not a function

To make it a function consider the positive part only & the -ve only

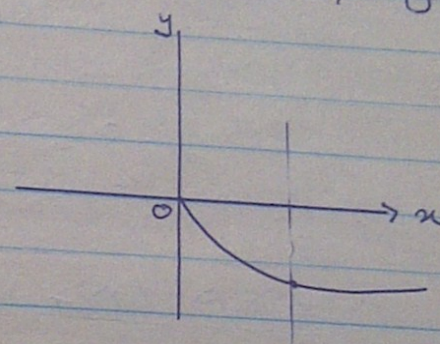
b) $y=+\sqrt{x}=\sqrt{x}$



where $y \geq 0$ and $y \in \mathbb{R}$

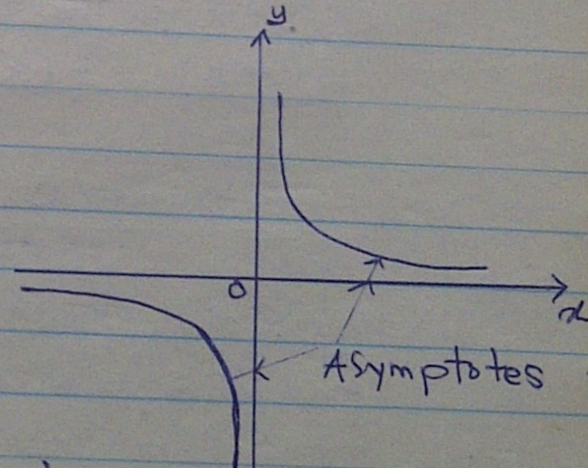
OR $y=-\sqrt{x}$

where $y \leq 0$, & $y \in \mathbb{R}$



5) $f(x)=\dfrac{1}{x}$ (Rational function)

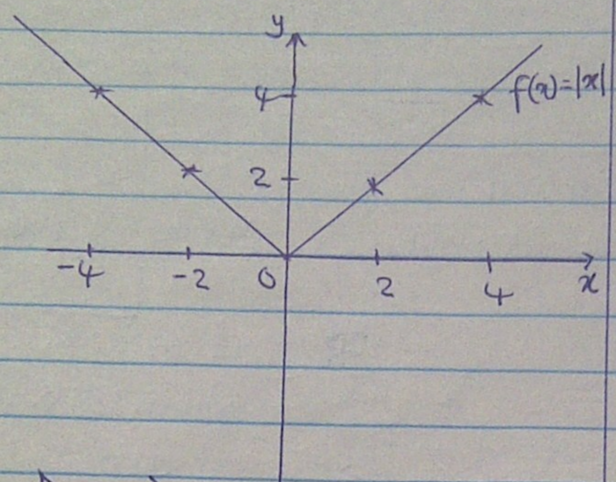
Note that $x \neq 0$ & $y \neq 0$



Domain = all $\mathbb{R}$ except $x=0$ (asymptotes
Range = all $\mathbb{R}$ except $y=0$

b). $f(x) = |x|$

x	-4	-2	0	2	4
$f(x)$	4	2	0	2	4



Domain = all $\mathbb{R}$ #s

Range = $y \geq 0$ and $y \in \mathbb{R}$

EVEN AND ODD FUNCTIONS

a) EVEN FUNCTIONS

A function given by $y = f(x)$ is ~~odd~~ even if, for each x in the domain of f,

$f(-x) = f(x)$

Example

$f(x) = x^2$ is an even function since

$f(-x) = (-x)^2 = x^2 = \boxed{f(x)}$

ie; $f(-2) = 4 = f(2)$

ii) $f(-1) = 1 = f(1)$

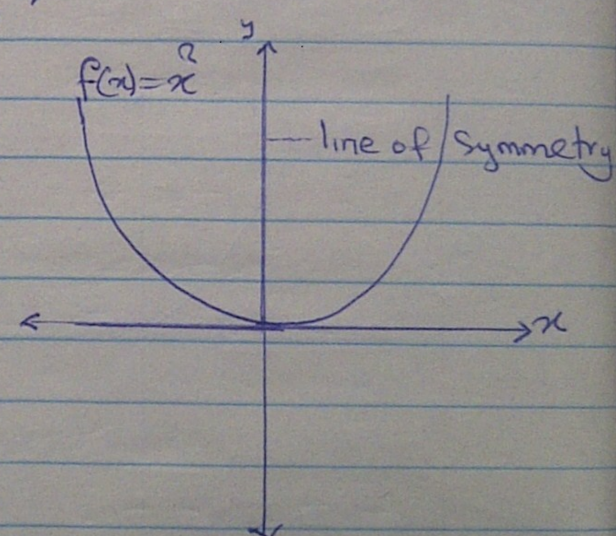
$f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ therefore $f(x) = x^2$ is an even function

also $y = \cos x$ is an even function

$\cos(-x) = \cos x$

N.B the graph of an even function is always symmetrical about the y-axis



ODD FUNCTIONS

A function given by $y = f(x)$ is odd

If, for each x in the domain of f,

$$f(-x) = -f(x)$$

Example

Show that $f(x) = x^3$ is odd

solutions

$f(x) = x^3$

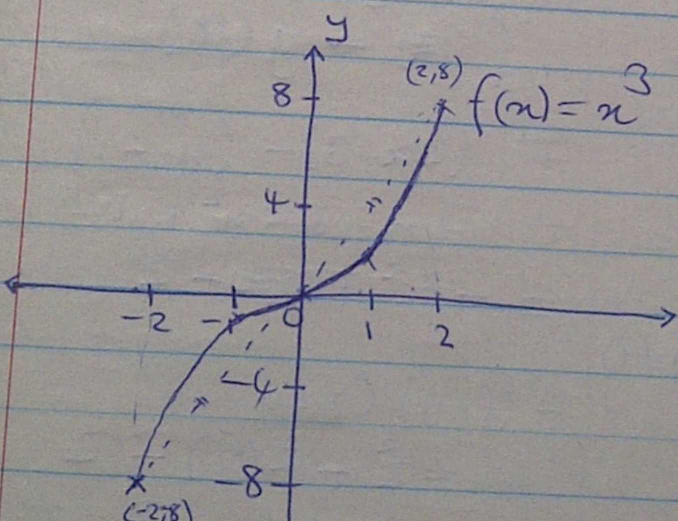
$f(-x) = (-x)^3 = -(x^3)$

$= -f(x)$

$\therefore f(x) = x^3$ is odd.

Graphically

x	-2	-1	0	1	2
$f(x)=x^3$	-8	-1	0	1	8



The symmetry of this graph is rotational

$\therefore$ the symmetry is about the point.

N.B - for even $f(-x) = f(x)$
- for odd $f(-x) = -f(x)$

The graph of an odd function is not symmetry about the y-axis Instead it is symmetricall about the point $(0,0)$

Some functions are neither even nor odd

e.g $f(x) = 2x+5$

$f(-x) = 2(-x)+5$

$= -2x+5$

$= -(2x-5)$

This is not $f(x)$ meaning that it is not even or odd -

Thus $f(x) = 2x+5$ is neither even nor odd

COMBINATIONS OF FUNCTIONS

Functions can be combined in two ways; When functions overlap each other they can be added, subtracted or multiplied.

1) ARITHMETIC COMBINATIONS OF FUNCTIONS

SUM, DIFFERENCE, PRODUCT AND QUOTIENTS OF FUNCTIONS

Let f and g be two functions (in x) with overlapping domains, then for all x common to both domains.

1) SUM: rule notation $(f+g)(x) = f(x) + g(x)$

2) DIFFERENCE: $(f-g)(x) = f(x) - g(x)$

3) PRODUCT: $(fg)(x) = f(x)g(x)$

4) QUOTIENT: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $\cancel{g(x) \neq 0}$ $g(x) \neq 0$

#(i) Examples

Given that $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ have overlapping domains, find

a) $(f+g)(x)$ b) $(f-g)(2)$

c) $\left(\frac{f}{g}\right)(x)$ d) $(fg)(x)$

Solution

a) $(f+g)(x)$

$= f(x) + g(x)$

$= (2x-3) + (x^2-1)$

$= 2x - 3 + x^2 - 1$

$= \underline{\underline{x^2 + 2x - 4}}$

b) $(f-g)(2)$

$= f(x) - g(x)$

$= (2x-3) - (x^2-1)$

$= 2x - 3 - x^2 + 1$

$= -x^2 + 2x - 2$

$f(f-g)(2)$

$= -2^2 + 2(2) - 2$

$= -4 + 4 - 2$

$= \underline{\underline{-2}}$

or

$f(2) - g(2)$

$(2(2)-3) - (2^2-1) = \underline{\underline{-2}}$

c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$= \frac{2x-3}{x^2-1}$

$= \frac{2x-3}{(x+1)(x-1)}$

Domain: all real #s except $x=-1$ or $x=1$

or $\{x \in \mathbb{R} ; x \neq -1$ and $x \neq 1\}$

d) $(fg)(x)$

$= f(x)g(x)$

$= (2x-3)(x^2-1)$

$= 2x^3 - 2x - 3x^2 + 3$

$= \cancel{2x^3 - 3x^2 - 2x + 3}$

$= 2x(x^2-1) - 3(x^2-1)$

$= \underline{\underline{(x^2-1)(2x-3)}}$

2) <u>COMPOSITION OF FUNCTIONS</u>

Another way of combining two functions is to form the composition of one with the other

E.g $f(x) = x^2$ and $g(x) = x+1$

$f[g(x)] = f(x+1) = (x+1)^2$

This is also denoted as $f \circ g$

Take note that ① the order in which letters which represent functions is very important

② One function is in the stomach of the other.

$f[g(x)] = f \circ g = (x+1)^2$

$x \xrightarrow{g} x+1 \xrightarrow{f} (x+1)^2$

inside

$\therefore f \circ g = (x+1)^2$

find $g \circ f$

$g \circ f = g[f(x)]$

$x \xrightarrow{f} x^2 \xrightarrow{g} x^2+1$

inside

$\therefore g \circ f = g[f(x)] = x^2 + 1$

ONE TO ONE (OR INJECTIVE FUNCTION)

Defn

A function $f: X \to Y$ is one to one (or injective) if whenever $f(a) = f(b)$ then $a = b$

Example

Let $f(x) = \frac{6x+4}{2x-3}$

Show that f is one to one.

Solution

Let $f(a) = f(b)$

$\Leftrightarrow \frac{6a+4}{2a-3} = \frac{6b+4}{2b-3}$

$(6a+4)(2b-3) = (2a-3)(6b+4)$

$12ab - 18a + 8b - 12 =$
$12ab + 8a - 18b - 12$

$8b - 18a = 8a - 18b$

$8b + 18b = 8a + 18a$

$26b = 26a$

$b = a$

or $a = b$

we conclude that f is one to one

② If $f(x) = x^2$. Is f one to one

Solution

Suppose $f(a) = f(b)$

$a^2 = b^2$

$a^2 - b^2 = 0$

$(a+b)(a-b) = 0$

$a = -b$ or

$a = b$

Hence $f(x) = x^2$ is not one to one

THE INVERSE OF A FUNCTION

Any one to one function has an inverse

E.g

Suppose we have a function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$

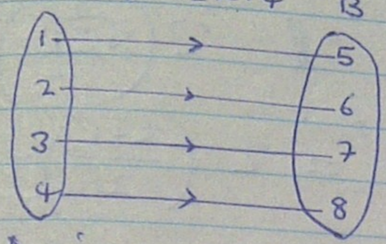
$A \times B$

$f(x) = x + 4 : \{(1,5), (2,6), (3,7)(4,8)\}$

set builder notation

$= \{(x,y) : y = x + 4, x = 1,2,3,4\}$

A $\quad f(x) = x + 4 \quad$ B



Domain $\qquad$ Range

one – to – one

$B \times A = \{(5,1), (6,2), (7,3), (8,4)\}$

$f^{-}(x) = x - 4 : \{(5,1), (6,2), (7,3), (8,4)\}$

<u>N.B</u> Domain of inverse $f^{-}(x)$ is $C = \{5, 6, 7, 8\}$ and the range $= \{1, 2, 3, 4\}$.

<u>Actual formula</u>

$f(x) = x + 4$

$y = x + 4$

$x = y + 4$ (Swapped x & y)

make y the Subject (after swapping x & y)

$x = y + 4$

$y = x - 4$

$$f^{-}(x) = x - 4$$

<u>Example 2</u>

Find the inverse of

$f(x) = 2x - 3$

<u>Solution</u>

$y = 2x - 3$

$x = 2y - 3$ (swap x & y)

$x + 3 = 2y$

$\therefore y = \dfrac{x+3}{2}$

$$f^{-1}(x) = \frac{x+3}{2}$$

$x \xrightarrow{\times 2} \boxed{2x} \xrightarrow{-3} f(x)$

$f^{-}(x) \xleftarrow{\div 2} \boxed{x+3} \xleftarrow{+3} x$

$f^{-}(x) = \dfrac{x+3}{2}$

The function $f(x) = x$ is known as the identity function

<u>Composition between a function $f(x)$ and the Inverse $f^{-}(x)$</u>

$f(x) = 2x - 3, \quad f^{-}(x) = \dfrac{x+3}{2}$

$f_0 f^{-}(x) = f\left[f^{-}(x)\right] = 2\left(\dfrac{x+3}{2}\right) - 3$

$= x + 3 - 3$

$= \underline{\underline{x}}$

$f^{-1} \circ f(x) = f^{-1}[f(x)]$

$$x \xrightarrow[\substack{\uparrow \\ \text{inside}}]{f} 2x-3 \xrightarrow{f^{-1}} \frac{2x-3+3}{2}$$

$$= \frac{2x-3+3}{2} = \frac{2x}{2} = \underset{\substack{\downarrow \\ \text{identity} \\ \text{function}}}{x}$$

Defn

Let f and g be two functions such that $f[g(x)] = x$ for every x in the domain of g and $g[f(x)] = x$ for every x in the domain of f. Then under this conditions, the function g is the inverse of f. The function g is thus denoted f^{-}.

N.B The domain of f must be equal to the range of f^{-1} and the range of f must be equal to the to the domain of f^{-1}

Q. Show that the functions are inverses of each other

$f(x) = 2x^3 - 1$ & $g(x) = \sqrt[3]{\frac{x+1}{2}}$

solution

$$f \circ g = f[g(x)]$$

$$= f\left(\sqrt[3]{\frac{x+1}{2}}\right)$$

$$= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$

$$= 2\left(\frac{x+1}{2}\right) - 1$$

$$= x+1-1 = x$$

$$g \circ f = g[f(x)]$$

$$= g(2x^3 - 1)$$

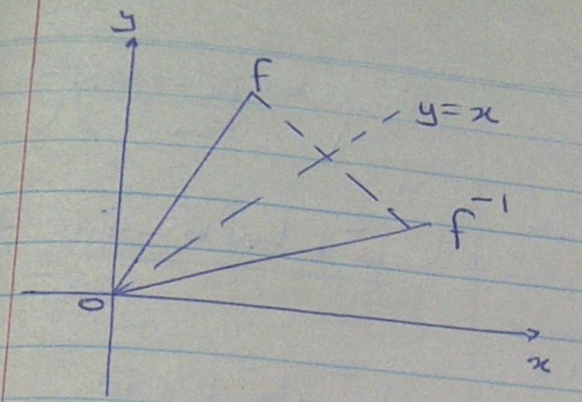
$$= \sqrt[3]{\frac{2x^3 - 1 + 1}{2}}$$

$$= \sqrt[3]{\frac{2x^3}{2}}$$

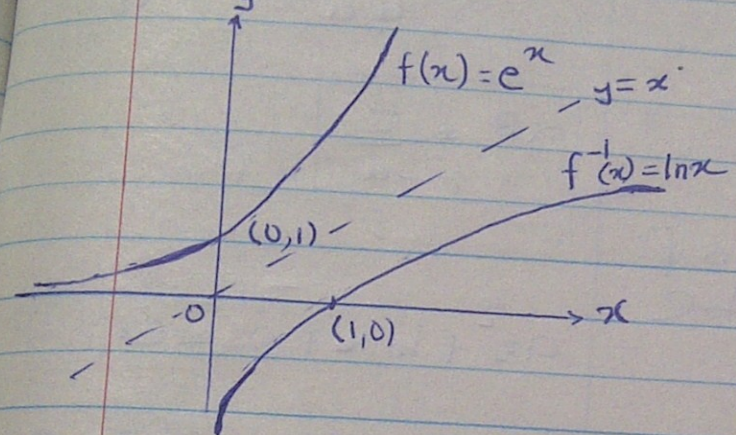
$$= \sqrt[3]{x^3} = x$$

Hence, f and g are inverse of each other

THE GRAPH OF THE INVERSE FUNCTION



$f(x) = e^x$, $f^{-1}(x) = \ln x$.



THE QUADRATIC FUNCTIONS

A quadratic function is of the form

$$f(x) = ax^2 + bx + c \text{ where}$$

$a, b, c \in \mathbb{R}$ and $x \neq 0$

1) ROOTS OF THE QUADRATIC EQUATION

Some useful Identities

① $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\iff \alpha^2 + \beta^2 = \boxed{(\alpha+\beta)^2 - 2\alpha\beta}$

② $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$\iff \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2$

$= \boxed{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}$

③ $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$

$= \alpha^2 + \beta^2 - 2\alpha\beta$

$= \boxed{(\alpha+\beta)^2 - 4\alpha\beta}$

$(x+p)^2 = x^2 + 2px + p^2$

$x^2 + 2px + p^2$

$(x+p)^2 + 2px$

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square roots on both sides:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let the two roots of $ax^2 + bx + c = 0$ be α and β then

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a})$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = \frac{-b}{a}$$

$$\boxed{\alpha + \beta = \frac{-b}{a}}$$

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$\frac{b^2 + b\sqrt{b^2 - 4ac} - b\sqrt{b^2 - 4ac} - (b^2 - 4ac)}{4a^2}$$

$$\frac{b^2 - b^2 + 4ac}{4a^2}$$

$$\alpha\beta = \frac{4ac}{4a^2} = \frac{c}{a}$$

$$\boxed{\alpha\beta = \frac{c}{a}}$$

$$ax^2 + bx + c = 0$$

$$\alpha + \beta = \frac{-b}{a}, \quad \alpha\beta = \frac{c}{a}$$

Example

Let α and β be the roots of the equation $7x^2 + 2x - 5 = 0$. Find the value of

a) $\frac{1}{\alpha} + \frac{1}{\beta}$ b) $\alpha^2 + \beta^2$

c) $\alpha - \beta$

Solution

$7x^2 + 2x - 5 = 0$

$a = 7, \; b = 2, \; c = -5$

a) $\alpha + \beta = \frac{-b}{a}$

$= \frac{-2}{7}$

b) $\alpha\beta = \frac{c}{a} = \frac{-5}{7}$

4 a) $\frac{1}{\alpha} + \frac{1}{\beta}$

$\frac{\beta + \alpha}{\alpha\beta}$

$= \frac{-2/7}{-5/7} = 2/7 \div 5/7$

$= 2/7 \times 7/5$

$= 2/5$

b) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\left(-2/7\right)^2 - 2\left(-5/7\right) = \frac{4}{49} + \frac{70}{49}$

$= \frac{74}{49}$

c) $\alpha - \beta$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\alpha + \beta = \frac{-b}{a} = -2/7$

$\alpha\beta = \frac{c}{a} = -5/7$

$= \left(-2/7\right)^2 - 4\left(-5/7\right)$

$= \frac{4}{49} + \frac{140}{49}$

$= \sqrt{\frac{144}{49}} = \frac{\pm 12}{7}$

Example

Let α and β be the roots of the eqn $5x^2 + 6x + 1 = 0$

Find, of the value of

a) $\frac{1}{\alpha} + \frac{1}{\beta}$ b) $\alpha^2 + \beta^2$

c) $\alpha^3+\beta^3$ d) $(\alpha-\beta)$

Solution

$5x^2+6x+1=0$

$a=5,\ b=6,\ c=1$

$\alpha+\beta=\frac{-b}{a}=\frac{-6}{5}$

$\alpha\beta=\frac{c}{a}=\frac{1}{5}$

a) $\frac{\alpha+\beta}{\alpha\beta}$

$=\frac{-6}{5}\div\frac{1}{5}$

$=\frac{-6}{5}\times 5=\underline{\underline{-6}}$

b) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$=\left(\frac{-6}{5}\right)^2-2\left(\frac{1}{5}\right)$

$=\frac{36}{25}-\frac{2}{5}$

$=\frac{36}{25}-\frac{10}{25}$

$=\underline{\underline{\frac{26}{25}}}$

c) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$=\left(\frac{-6}{5}\right)^3-3\left(\frac{1}{5}\right)\left(\frac{-6}{5}\right)$

$=-\frac{216}{125}+\frac{18}{25}$

$=\frac{-216+90}{125}$

$=\underline{\underline{\frac{-126}{125}}}$

d) $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

$=\left(\frac{-6}{5}\right)^2-4\left(\frac{1}{5}\right)$

$=\frac{36}{25}-\frac{4}{5}$

$=\frac{36-20}{25}$

$\sqrt{(\alpha-\beta)^2}=\sqrt{\frac{16}{25}}$

$=\underline{\underline{\pm\frac{4}{5}}}$

$ax^2 + bx + c = 0$

α and β

$(x-\alpha)(x-\beta) = 0$

$x^2 - \alpha x - \beta x + \alpha\beta = 0$

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$x^2 -$ (sum of roots) x + (product of roots) $= 0$

Example

Let α and β be the root of $5x^2 + 6x + 1 = 0$. Find the equation of a quadratic with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Solution

$5x^2 + 6x + 1 = 0$

$a = 5, b = 6, c = 1$

$\alpha + \beta = -\frac{b}{a} = -\frac{6}{5}$

$\alpha\beta = \frac{c}{a} = \frac{1}{5}$

Another quadratic with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Sum of new roots

$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$

$= \frac{-6}{5} \div \frac{1}{5}$

$= -6$

product of new roots

$\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta}$

$= \frac{1}{1/5} = 5$

$x^2 -$ (sum of roots) x + (product of roots) $= 0$

$x^2 - (-6)x + 5 = 0$

$x^2 + 6x + 5 = 0$

Example

Let α and β be the roots of the eqn $2x^2 - x - 10 = 0$. Find the equation of a quadratic with roots α^2 and β^2

Solution

$2x^2 - x - 10 = 0$

$a = 2,\ b = -1,\ c = -10$

$\alpha + \beta = \frac{-b}{a} = \frac{1}{2}$

$\alpha\beta = \frac{c}{a} = \frac{-10}{2} = -5$

Sum of new roots

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{1}{2}\right)^2 - 2(-5)$

$= \frac{1}{4} + \frac{10}{1}$

$= \frac{1+40}{4} = \frac{41}{4}$

Product of new roots

$\alpha^2 \times \beta^2 = \alpha^2\beta^2 = (\alpha\beta)^2$

$= (-5)^2$

$= 25$

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - \frac{41}{4}x + 25 = 0$

$4x^2 - 41x + 100 = 0$

THE DISCRIMINANT OF A QUADRATIC EQUATION

$-2i \times (-2i) = 4i^2 = -4$

$2i \times 2i = 4i^2 = -4$

Quadratic	roots	Type of root	$b^2 - 4ac$
$5x^2+6x+1$	$-1, -\frac{1}{5}$	Two real distinct roots	$16 > 0$
x^2-4x+4	$2, 2$	one real repeated roots	0
$x^2+4=0$	$-2i, 2i$	Two complex conjugate roots	$-16 < 0$

THE DISCRIMINANT OF A QUADRATIC EQUATION

In the equation
quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let $D = b^2 - 4ac$, then

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

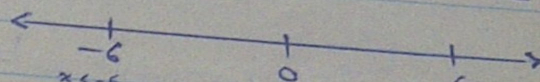
The nature of the roots of a quadratic equ is determined by the term under the radical: $D = b^2 - 4ac$ where $a, b, c, \in \mathbb{R}$. The term $D = b^2 - 4ac$ is called the discriminant

Situation	Nature of roots
$b^2 - 4ac > 0$	Two real distinct roots
$b^2 - 4ac = 0$	Two equal real roots
$b^2 - 4ac < 0$	Two Complex Conjugate

$f(x) = \sqrt{x^2 - 36}$, find the domain of $f(x)$

$$x^2 - 36 \geq 0$$
$$(x+6)(x-6) \geq 0$$

Critical numbers -6 and 6



factors	$x<-6$	$-6<x<6$	$x>6$
$x+6$	$-$	$+$	$+$
$x-6$	$-$	$-$	$+$
product	$+$	$-$	$+$

The domain $\{x \leq -6\} \cup \{x \geq 6\}$

THE DISCRIMINANT

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = b^2 - 4ac$$

if $b^2 - 4ac = 0$

$$= \frac{-b \pm \sqrt{0}}{2a}$$

$$= \frac{-b \pm 0}{2a} = \frac{-b}{2a}$$

Example 1

Determine the nature of the roots of the equation

$2x^2+2x+5=0$

Solution

$ax^2+bx+c=0$

$a=2,\ b=2,\ c=5$

$$\text{Discriminant} = b^2-4ac$$
$$= 2^2-4(2)(5)$$
$$= 4-40$$
$$= -36$$
$$= -36<0$$

We have a pair of Complex Conjugate roots

Example 2

For what values of K will the quadratic eqn $Kx^2+(K+1)x+K=0$ have real & equal roots?

Solution

$ax^2+bx+c=0$

$a=K,\ b=K+1,\ c=K$

$$D = b^2-4ac=0$$
$$=(K+1)^2-4(K)(K)=0$$

$$(K+1)^2-4K^2=0$$
$$K^2+2K+1-4K^2=0$$
$$-3K^2+2K+1=0$$

$$3K^2-2K-1=0$$
$$3K^2-3K+K-1=0$$
$$3K(K-1)+1(K-1)=0$$
$$(3K+1)(K-1)=0$$
$$K=-\tfrac{1}{3} \text{ or } K=1$$

Any quadratic function can be expressed in the form

$$f(x)=a(x+p)+q$$

Where $p=\frac{b}{2a}$, $q=\frac{4ac-b^2}{4a}$

$$ax^2+bx+c=0$$

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0$$

$$a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$$

$$a\left(x^2+\left(\frac{b}{2a}\right)^2\right)-\frac{b^2}{4a^2}+\frac{c}{a}\Big)=0$$

$$a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right)=0$$

$$a\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right)=0$$

$$a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}=0$$

Let $p=\frac{b}{2a}$, $q=\frac{4ac-b^2}{4a}$

$$a(x+p)^2+q=0$$

Summary

Any quadratic equation $ax^2+bx+c=0$ can be expressed in the form

$$a(x+p)^2+q=0$$

where $p=\frac{b}{2a}$, $q=\frac{4ac-b^2}{4a}$

Example

Express each the following quadratic functions below in the form $f(x)=a(x+p)^2+q$

a) $f(x)=2x^2+3x+1$

b) $f(x)=1-6x-x^2$

Solutions

a) $f(x)=2x^2+3x+1$

$f(x)=ax^2+bx+c$.

$a=2$, $b=3$, $c=1$

$$P=\frac{b}{2a}=\frac{3}{2(2)}=\frac{3}{4}$$

$$q=\frac{4ac-b^2}{4a}=\frac{4(2)(1)-(3)^2}{4(2)}$$

$$=\frac{8-9}{8}$$

$$=-1/8$$

$f(x)=$

$a(x+p)^2+q$

$=2(x+3/4)^2+(-1/8)$

$$f(x)=2(x+3/4)^2-1/8$$

b) $f(x)=ax^2+bx+c$

$a=-1$, $b=-6$, $c=1$

$$p=\frac{b}{2a}=\frac{-6}{2(-1)}=\frac{-6}{-2}=3$$

$$q=\frac{4ac-b^2}{4a}=\frac{4(-1)(1)-(-6)^2}{4(-1)}$$

$$=\frac{-4-(36)}{-4}$$

$$\frac{-40}{-4} = 10$$

$$f(x) = a(x+p)^2 + q$$

$$f(x) = -1(x+3)^2 + 10$$

$$f(x) = -(x+3)^2 + 10$$

GRAPHS OF QUADRATIC FUNCTIONS

Sketch the graph of

1) $y=2x^2$ and $y=x^2$ on the same axes

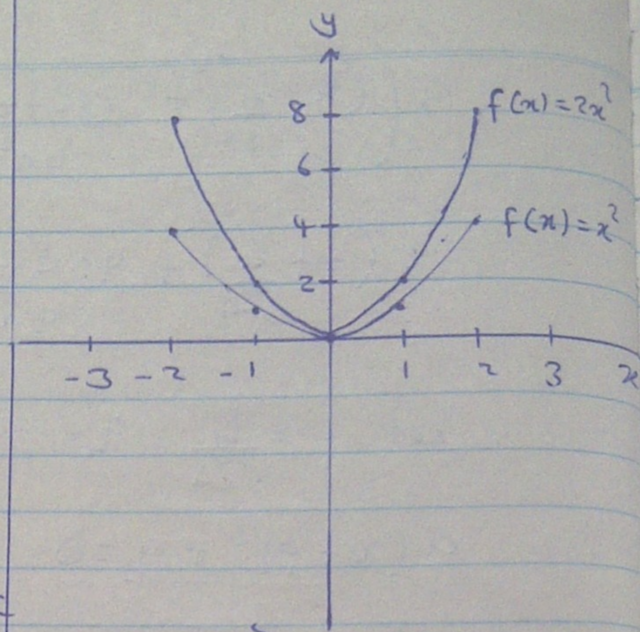
Solution

$f(x) = x^2$

x	-2	-1	0	1	2
$y=f(x)$	4	1	0	1	4

$f(x) = 2x^2$

x	-2	-1	0	1	2
$f(x)$	8	2	0	2	8



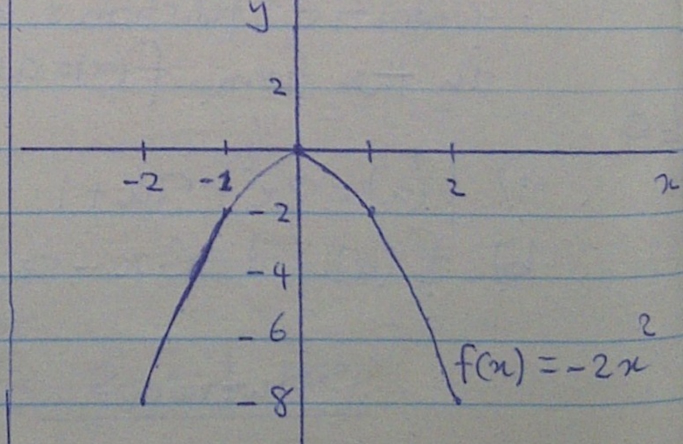
Conclusion

Given the function $f(x) = ax^2 + bx + c$ we note that if $a > 0$, then the graph of $f(x) = ax^2 + bx + c$ opens upwards.

If $f(x) = -2x^2$

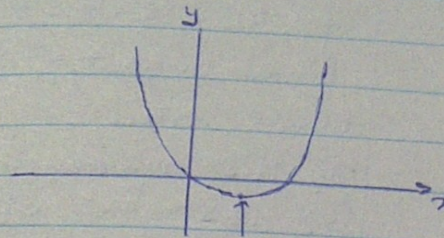
$$f(-2) = -2(-2)^2$$
$$= -8$$

x	-2	-1	0	1	2
$-2x^2$	-8	-2	0	-2	-8



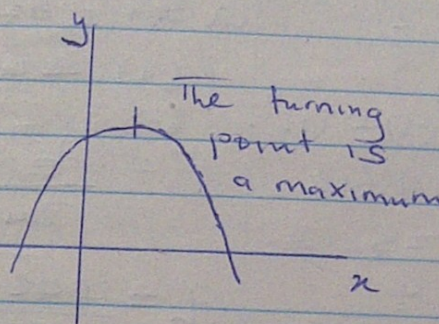
a) $f(x) = ax^2 + bx + c$

① If $a > 0$ opens upwards



The turning point is a minimum

② If $a < 0$, the graph opens downwards



b) The Meaning of C in the function

$f(x) = ax^2 + bx + c$

$x = 0$ along the y-axis

$f(0) = a(0)^2 + b(0) + c$

$= 0 + 0 + c$

$f(0) = c$

the graph of $f(x) = ax^2 + bx + c$ always cuts the y-axis at $(0, c)$

a quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the form

a $f(x) = a(x+p)^2 + q$

$p = \frac{b}{2a}$, $q = \frac{4ac - b^2}{4a}$

$f(x) = x^2 + x - 6$

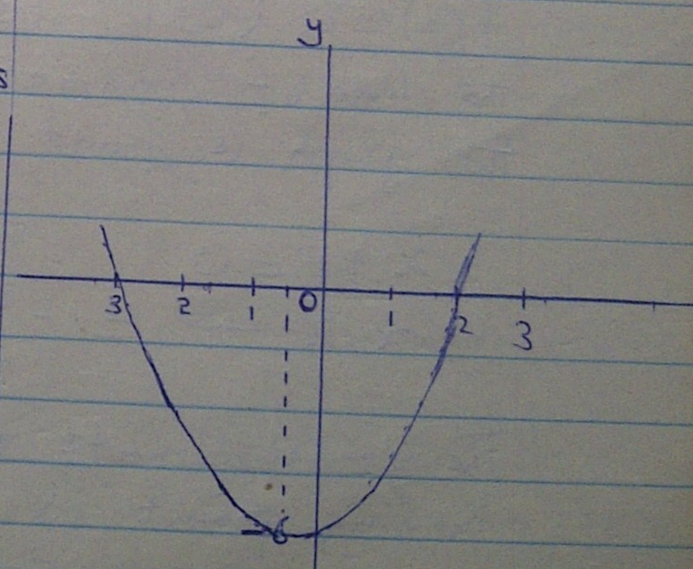
$f(x) = (x+3)(x-2)$

- $a = 1 > 0$ the graph opens upward
- It has a minimum turning point
- It cuts the y-axis at $(0, -6)$

Zeros of $f(x)$

$f(x) = (x+3)(x-2) = 0$

$x = -3$ and $x = 2$



line of symmetry $= \frac{-3+2}{2} = -\frac{1}{2} = -0.5$

$f(x) = x^2 + x - 6 \qquad x = -\frac{1}{2}$

$f(-\frac{1}{2}) = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 6$

$= \frac{-25}{4}$

the Significance of p & q

$f(x) = x^2 + x - 6$

$a = 1, \; b = 1 \quad c = -6$

$p = \frac{b}{2a} = \frac{1}{2\times 1} = \frac{1}{2}$

$q = \frac{4ac - b^2}{4a} = \frac{4(1)(-6) - (1)^2}{4(1)}$

$= \frac{-25}{4}$

If the $f(x) = ax^2 + bx + c$ is expressed in the form

$f(x) = a(x+p)^2 + q$, then the line of symmetry of the graph is found by

$x + p = 0$

$x = -p = -\frac{b}{2a} =$

The y-Coordinate of the turning point is

$q = \frac{4ac - b^2}{4a}$

Example

Express the function $f(x) = x^2 - 8x + 12$ in the form $a(x+p)^2 + q = f(x)$

Hence find

a) Its minimum value

b) Its line of Symmetry

c) Sketch it

Solution

$f(x) = x^2 - 8x + 12$

$a = 1, \; b = -8, \; c = 12$

$f(x) = a(x+p)^2 + q$

$p = \frac{b}{2a} = \frac{-8}{2\times 1} = -4$

$q = \frac{4ac - b^2}{4a} = \frac{4(1)(12) - (-8)^2}{4(1)}$

$= -4$

$f(x) = a(x+p)^2 + q$

$= 1(x-4)^2 + (-4)$

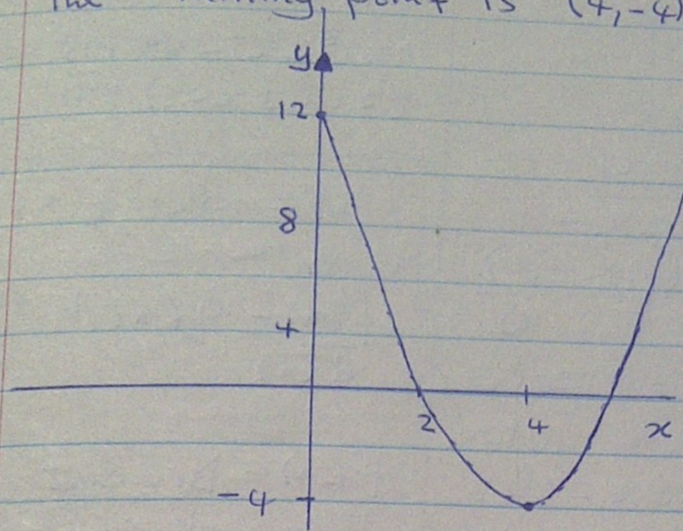
$= (x-4)^2 - 4$

a) Minimum value $= q = -4$

b) $x - 4 = 0$

$x = 4$

c) The y-intercept is $(0,12)$
The turning point is $(4,-4)$



THE REMAINDER AND FACTOR THEOREMS

USEFUL IDENTITIES

① $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

② $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Example

Factorise a) $x^3 - 8 = x^3 - 2^3$
$= (x-2)(x^2+2x+4)$

b) $x^3 + 1$

Defn of a polynomial

An expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

where a_k, $k = 0,1,2,\dots n$ are real numbers and $n \in \mathbb{N}$ and $a_n \neq 0$ is called a polynomial of degree n.

E.g. $2x^3 - 6x^2 + \frac{1}{2}x + 6$ of degree 3 (the highest power

- $6x^2 + 4x - 7$ of degree 2
- $7x^7 - 6$ of degree 7
- $6x^8 + 7x^5 - 6 + \frac{2}{x}$

is not a polynomial because the term $\frac{2}{x} = 2x^{-1}$ but $-1 \notin \mathbb{N}$

LONG DIVISION

EXAMPLE 1

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$

$$
\begin{array}{r}
6x^2 - 7x + 2 \\
x-2 \overline{\big) 6x^3 - 19x^2 + 16x - 4} \\
-(6x^3 - 12x^2) \\
\hline
-7x^2 + 16x \\
-(-7x^2 + 14x) \\
\hline
2x - 4 \\
-(2x - 4) \\
\hline
0
\end{array}
$$

<u>N.B</u> The remainder is zero, meaning $x-2$ is a factor of $6x^3 - 19x^2 + 16x - 4$.

Q. Divide $2x^3 - x^2 - 8x + 15$ by $x-2$.

<u>Solution</u>

$$
\begin{array}{r}
2x^2 + 3x - 2 \\
x-2 \overline{\big) 2x^3 - x^2 - 8x + 15} \\
-(2x^3 - 4x^2) \\
\hline
3x^2 - 8x \\
-(3x^2 - 6x) \\
\hline
-2x + 15 \\
-(-2x + 4) \\
\hline
11
\end{array}
$$

The remainder is 11, meaning $x-2$ is not a factor of $\cancel{6x^3 - 19x^2 + 16x}$ $2x^3 - x^2 - 8x + 15$

THE DIVISION ALGORITHM

Note that

$$2x^3 - x^2 - 8x + 15 = (x-2)(2x^2 + 3x - 2) + 11$$

$$\boxed{\begin{array}{l} f(x) = d(x)q(x) + r(x) \\ \text{or} \\ \dfrac{f(x)}{d(x)} = q(x) + \dfrac{r(x)}{d(x)} \\ \\ f(x) - \text{Dividend} \\ d(x) - \text{divisor} \\ q(x) - \text{quotient} \\ r(x) - \text{remainder} \end{array}}$$

<u>N.B</u> (1) The rational expression $\dfrac{f(x)}{dx}$ is improper because the degree of $d(x)$ is less than or equal to the degree of $f(x)$

But $\dfrac{r(x)}{d(x)}$ is proper because the degree of $r(x)$ is always lower than the degree of $d(x)$

<u>Example</u>

Divide $x^3 - 1$ by $x - 1$

Solution

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x-1 & x^3 - 0x^2 + 0x - 1 \\ & -(x^3 - x^2) \\ \hline & \quad x^2 + 0x \\ & \quad -(x^2 - x) \\ \hline & \qquad x - 1 \\ & \qquad -(x-1) \\ \hline & \qquad\quad 0 \end{array}$$

$$x^3 - 1 = (x-1)(x^2 + x + 1) + 0$$

If we divide a polynomial $f(x)$ by a linear factor $x - a$ for some $a \in \mathbb{R}$, then

$$f(x) = (x-a)\,q(x) + r(x)$$

Let $x - a = 0$

$x = a$

$$f(a) = (a-a)\,q(x) + r(x)$$

$$= 0 \cdot q(x) + r(x)$$

$$= r(x)$$

$$\boxed{r(x) = f(a)}$$

$$f(a) = (a-a)\,q(x) + r(x)$$

$$f(a) = r(x)$$

THE REMAINDER THEOREM

If a polynomial $f(x)$ is divided by $x - a$, the remainder is $r(x) = f(a)$

Example

Find the remainder when $f(x) = 2x^3 + 2x + 31$ is divided by $x + 2$

Solution

$x + 2 = 0$

$x = -2$

$$R(x) = f(-2) = 2(-2)^3 + 2(-3) + 31$$

$$= -16 - 4 + 31$$

$$= 11$$

Using long division

$$\begin{array}{r|l} & 2x^2 - 4x + 10 \\ \hline x+2 & 2x^3 + 0x^2 + 2x + 31 \\ & -(2x^3 + 4x^2) \\ \hline & \quad -4x^2 + 2x \\ & \quad -(-4x^2 - 8x) \\ \hline & \qquad 10x + 31 \\ & \qquad -(10x + 20) \\ \hline & \qquad\quad 11 \end{array}$$

Example

If the polynomial

$$f(x) = x^3 - 7x^2 + bx - 2$$

is divided by the factor $x-2$, the remainder is -10. Find the value of b

Solution

By the remainder theorem the remainder is $r(x) = f(a)$ for the linear factor $x-a$

Let $x-2=0 \Rightarrow x=2$

$r(x) = f(2) = -10$

$f(x) = (2)^3 - 7(2)^2 + 2b - 2 = -10$

$\Rightarrow 8 - 28 + 2b - 2 = -10$

$\Rightarrow 2b - 22 = -10$

$\Rightarrow 2b = 12$

$b = 6$

Example

Find the remainder when the polynomial $f(x) = x^4 - x^3 + x^2 - 3x + 2$ is divided by $x-1$

Solution

$f(1) = 0$, this means $x-1$ is a factor of $f(x)$

$x-1=0$
$x=1$

THE FACTOR THEOREM

The polynomial $f(x)$ has a factor $(x-a)$ if and only if $f(a)=0$

Example

Is $x-2$ a factor of $f(x) = 6x^3 - 19x^2 + 16x - 4$?

Solution

$x-2=0$

$x = 2$

$r(x) = f(2) = 6(2)^3 - 19(2)^2 + 16(2) - 4$

$= 48 - 76 + 32 - 4$

$= 0$

Hence, $x-2$ is a factor of $f(x) = 6x^3 - 19x^2 + 16x - 4$

Example

When $x^5 + 4x^2 + ax + b$ is divided by $x^2 - 1$ the remainder is $2x+3$. Find a and b.

Solution

Let $x^2 - 1 = 0$

$(x+1)(x-1) = 0$

$x = -1$ or $x = 1$

When $x = -1$

$r(x) = f(-1) = (-1)^5 + 4(-1)^2 + a(-1) + b = 2(-1) + 3$

$\Rightarrow -1 + 4 - a + b = 1$

$\Rightarrow 3 - a + b = 1$

$-a + b = -2 \;\ldots\; i$

When $x = 1$

$r(x) = f(1) = (1)^5 + 4(1)^2 + a(1) + b = 2(1) + 3$

$\Rightarrow 1 + 4 + a + b = 2 + 3$

$\Rightarrow 5 + a + b = 5$

$\Rightarrow a + b = 0 \;\ldots\; ii$

$a + b = 0$

$a = -b \;\ldots\; iii$

$-(-b) + b = -2$

$2b = -2$

$b = -1$

$a = -(-1)$

$= 1$

SYNTHETIC DIVISION

An alternative method for dividing a polynomial by a linear factor

N.B. This method should be used when the divisor is a linear factor.

Example

Divide $x^3 - 2x^2 + 5x - 3$ by $x - 2$

Solution

$$
\begin{array}{r|l}
 & x^2 + 0 + 5 \\
\hline
x - 2 & x^3 - 2x^2 + 5x - 3 \\
 & -(x^3 - 2x^2) \\
\hline
 & 0 + 5x \\
 & (0 + 0) \\
\hline
 & 5x - 3 \\
 & -(5x - 10) \\
\hline
 & 7
\end{array}
$$

let $x - 2 = 0 \Rightarrow x = 2$

$$
\begin{array}{c|cccc}
2 & 1 & -2 & 5 & -3 \\
 & & 2 & 0 & 10 \\
\hline
 & 1 & 0 & 5 & 7
\end{array}
$$

-3 ← degree of 3

7 – the remainder

degree of 2

$x^3 - 2x^2 + 5x - 3 = (x - 2)(x^2 + 5) + 7$

Example

Show that $(x-2)$ and $(x+3)$ are factors of $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$, Hence factorise $f(x)$ completely

Solution

let $x-2 = 0 \Rightarrow x = 2$

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array}$$

degree of 4

degree of 3

remainder is zero

$\therefore x-2$ is a factor of $f(x)$

Let $x+3 = 0 \Rightarrow x = -3$

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

remainder is zero

$\therefore x+3$ is a factor of $f(x)$

$$f(x) = (x-2)(2x^3 + 11x^2 + 18x + 9)$$

Factorise completely

$$f(x) = (x-2)(x+3)(2x^2 + 5x + 3)$$

$$= (x-2)(x+3)(x+1)(2x+3)$$

THE RATIONAL ZERO TEST

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ has integer coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

Where p and q are relatively prime (have no common factors other than 1)

p = a factor of the Consultant term a_0

q = a factor of the leading coefficient a_n

Example

Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$ and factors $f(x)$ completely

Solution

p is a factor of $3 = \pm 1, \pm 3$

q is a factor of $2 = \pm 1, \pm 2$

$$\frac{p}{q} = \pm 1, \pm 3, \pm \tfrac{1}{2}, \pm \tfrac{3}{2}$$

(possible zeros)

$$\frac{p}{q} = \frac{-1, +1, -3, +3}{-1, +1, -2, +2}$$

$$= \pm 1, \pm 3, \pm \tfrac{1}{2}, \pm \tfrac{3}{2}$$

Example

Use the rational zero test to completely factorise

1) $2x^3 + 3x^2 - 8x + 3$
2) $3x^3 - 4x^2 - 3x + 4$
3) $x^4 - 5x^2 + 4$
4) $3x^3 - 10x^2 + 9x - 2$
5) $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120$

Solution

there is a difference between a zero and a factor

e.g $f(x) = x^2 + x - 6$

its factors

$f(x) = (x+3)(x-2)$

The zeros

These are the x-values for $f(x) = 0$ ie

$f(x) = (x+3)(x-2) = 0$

$x = -3$, $x = 2$ (zeros)

$$\text{Rational zeros} = \frac{\text{factors of } a_0}{\text{factors of } a_n}$$

a_0 = Constant term

a_n = Coefficient of the leading term

Q1. $2x^3 + 3x^2 - 8x + 3$

$$\text{Rational zero} = \frac{\text{factor of } 3}{\text{factor of } 2}$$

$$= \frac{\pm 1, \pm 3}{\pm 1, \pm 2}$$

$$= \pm 1, \pm 3, \pm \tfrac{1}{2}, \pm \tfrac{3}{2}$$

(possible zeros)

1 (zero)

1	2	3	−8	3 → Degree of 3
		2	5	−3 — Degree of 2
	2	5	−3	0

0 → remainder

Zero means 1 is a zero of $f(x)$

$x = 1 \Rightarrow x - 1$ is a factor

$2x^2+5x-3$

$2x^2+6x-x-3$

$2x(x+3)-1(x+3)$

$(x+3)(2x-1)$

$f(x)=2x^3+3x^2-8x+3 = (x+3)(2x-1)(x-1)$

Q3. $x^4-5x^2+4 = x^4+0x^3-5x^2+0x+4$

Rational Zeros $= \dfrac{\text{factors of } 4}{\text{factors of } 1}$

$= \dfrac{\pm 1, \pm 2, \pm 4}{\pm 1}$

$= \pm 1, \pm 2, \pm 4$ (possible zeros)

1 is a zero

1	1	0	-5	0	4	Degree of 4
		1	1	-4	-4	Degree of 3
	1	1	-4	-4	0	

remainder is zero

1 is zero

$x = 1$

$x-1$ is a factor.

x^3+x^2-4x-4

$(-x)$ x^3+x^2-4x-4

$(-x)^3+(-x)^2-4(-x)-4$

$-x^3+x^2+4x-4$

-1	1	1	-4	-4	Degree 3
		-1	0	4	Degree 2
	1	0	-4	0	

-1 is a zero

$x = -1$

$x+1$ is a factor

$x^2-4 = x^2-2^2$

$= (x+2)(x-2)$

$x^4-5x^2+4 = (x-1)(x+1)(x+2)(x-2)$

SOLVING QUADRATIC INEQUALITIES

a) LINEAR INEQUALITIES

If $x \in \mathbb{R}$, solve:

i) $3x-2 \leq -1$ ii) $2x+3 \geq 5x-2$

iii) $\frac{2}{3}x+5 > 3$

Solution

i) $3x - 2 \leq -1$

$3x \leq -1 + 2$

$3x \leq 1$

$\frac{3x}{3} \leq \frac{1}{3}$

$x \leq \frac{1}{3}$

Solution set $S.S = \{x \in \mathbb{R} \mid x \leq \frac{1}{3}\}$

or

$(-\infty, \frac{1}{3}] = S.S$

ii) $2x + 3 \geq 5x - 2$

$2x - 5x \geq -2 - 3$

$-3x \geq -5$

$\frac{-3x}{-3} \leq \frac{-5}{-3}$

$x \leq \frac{5}{3}$

Solution set $= \{x \in \mathbb{R} \mid x \leq \frac{5}{3}\}$

or

$(-\infty, \frac{5}{3}]$

iii) $\frac{2}{3}x + 5 > 3$

$\frac{2}{3}x > 3 - 5$

$\frac{2}{3}x > -2$

$\frac{3}{2} \times \frac{2}{3}x > -2 \times \frac{3}{2}$

$x > -3$

Solution set $= \{x \in \mathbb{R} \mid x > -3\}$

b) QUADRATIC INEQUALITIES

Solve $x^2 + x - 6 < 0$

$f(x) = x^2 + x - 6$

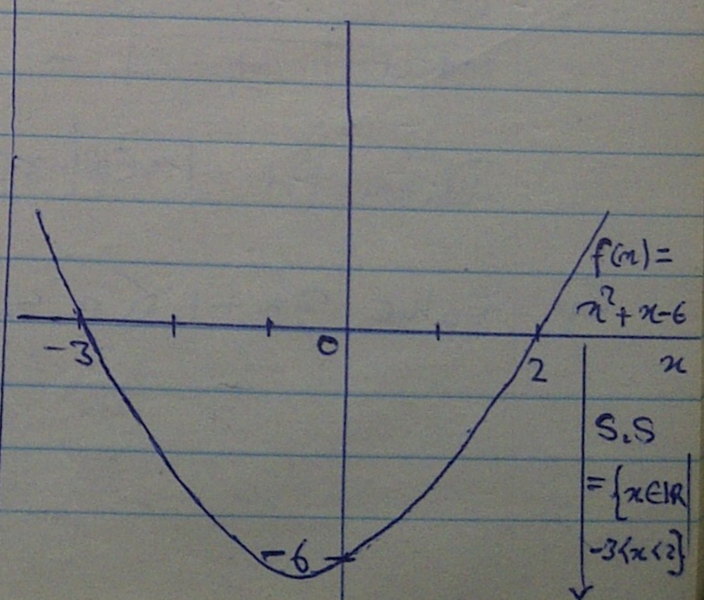
$a = 1 > 0$ (open upwards)

$f(x) = x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

Y-Intercept $(0, -6)$



If the e inequ. was $x^2+x-6>0$

$S.S = \{x \in \mathbb{R} \mid x<-3 \text{ or } x>2\}$

The recommended method

Example

Resolve $x^2+x-6<0$

We proceed as follows:

i) Factorise the quadratic function on L.H.S

ii) Equate the factors obtained in (i) to find Critical values

$$x^2+x-6<0$$
$$(x+3)(x-2)<0$$
$$(x+3)(x-2)=0$$
$$x=-3 \text{ and } x=2$$

Before | Between | after

-3 2

factors	$x<-3$ (-4)	$-3<x<2$	$x>2$ (3)
$(x+3)$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$+$
product	$+$	$-$	$+$

Solution Set $=\{x\in\mathbb{R} \mid -3<x<2\}$

Q. Solve $2x-1 \leq x^2-4$

c) **RATIONAL INEQUALITIES**

Example

Solve the inequality $\frac{x}{x+2} \geq 3$

(Avoid cross multiplying)

$$\frac{x}{x+2} \geq 3$$

$$\frac{x}{x+2} - \frac{3}{1} \geq 0$$

$$\frac{x-3(x+2)}{x+2} \geq 0$$

$$\frac{x-3x-6}{x+2} \geq 0$$

$$\frac{-2x-6}{x+2} \geq 0$$

Critical values

$$-2x-6=0$$
$$x=-3$$

$$x+2=0$$
$$x=-2$$

Before | Between | after

-3 -2

factors	$x<-3$ (-5)	$-3<x<-2$ (-2.5)	$x>-2$ (10)
$-2x-6$	$+$	$-$	$-$
$x+2$	$-$	$-$	$+$
~~Product~~ Quotient	$-$	$+$	$-$

$S.S = \{x \in \mathbb{R} \mid -3 \leq x \leq -2\}$

Absolute Value In Inequalities

① $|x| < a$, then,

$\Rightarrow -a < x < a$

② $|x| > a$ then,

$\Rightarrow x < -a$ and $x > a$

Solve (i) $|2x-1| < 3$

(ii) $|4x+2| \geq 4$

Solution

i) $|2x-1| < 3$ $\qquad -a < x < a$

$-3 < 2x-1 < 3$

$-3+1 < 2x-1+1 < 3+1$

$-2 < 2x < 4$

$\frac{-2}{+2} < \frac{2x}{2} < \frac{4}{2}$

$-1 < x < 2$

$S.S = \{x \in \mathbb{R} \mid -1 < x < 2\}$

ii) $|4x+2| \geq 4$ $\qquad |x| > a$

$x < -a$ & $x > a$

$4x+2 \leq -4$ or $4x+2 \geq 4$

$4x \leq -6$ $\qquad 4x \geq 4-2$

$x \leq -6/4$ $\qquad 4x \geq 2$

$x \leq -\frac{3}{2}$ $\qquad x \geq 2/4$

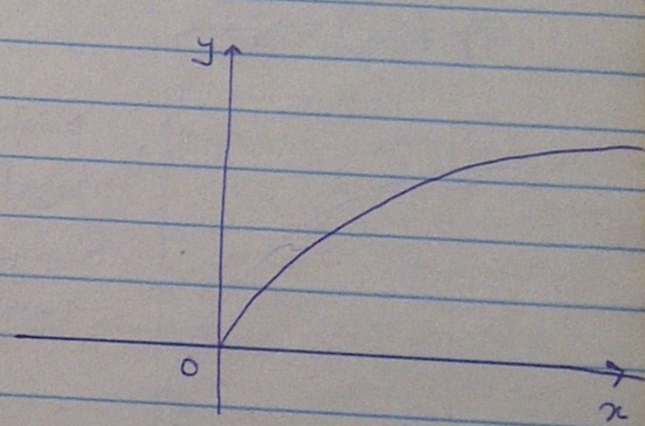
$x \geq 1/2$

$S.S = \{x \in \mathbb{R} \mid x \leq -3/2 \text{ or } x \geq 1/2\}$

SKETCHING GRAPHS

① RADICAL FUNCTIONS

a) $f(x) = \sqrt{x}$



$f(x) = \sqrt{x}$

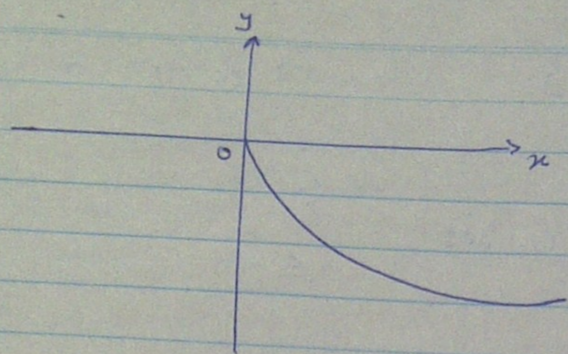
Domain $= \{x \in \mathbb{R} \mid x \geq 0\}$

Range (y-values) $= \{y \in \mathbb{R} \mid y \geq 0\}$

Domain – non negative

Range – non negative

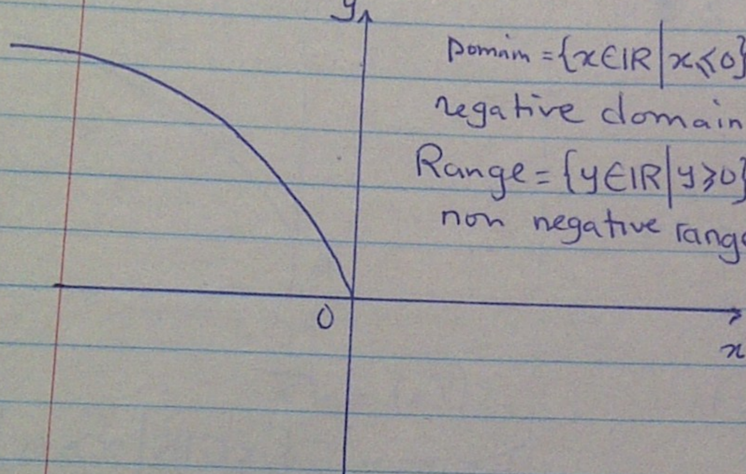
b) $f(x) = -\sqrt{x}$



Domain = $\{x \in \mathbb{R} \mid x \geq 0\}$
non negative domain

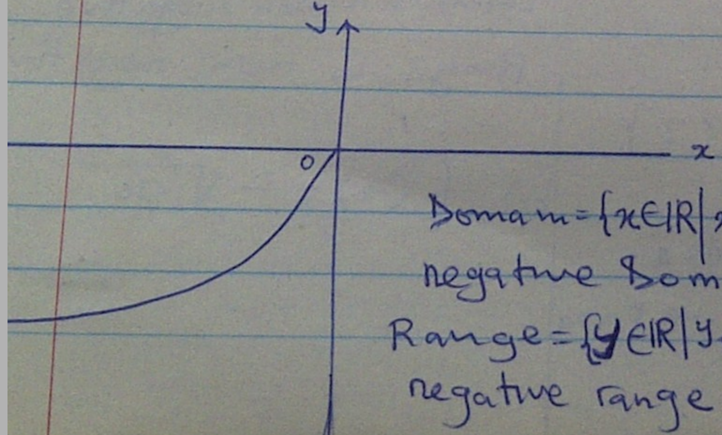
Range = $\{y \in \mathbb{R} \mid y \leq 0\}$
negative range

c) $f(x) = \sqrt{-x}$



Domain = $\{x \in \mathbb{R} \mid x \leq 0\}$
negative domain
Range = $\{y \in \mathbb{R} \mid y \geq 0\}$
non negative range

d) $f(x) = -\sqrt{-x}$



Domain = $\{x \in \mathbb{R} \mid x \leq 0\}$
negative Domain
Range = $\{y \in \mathbb{R} \mid y \leq 0\}$
negative range

TRANSFORMATIONS OF THE RADICAL FUNCTION

① Transformation of form $y = \sqrt{x} + a$ or $y = f(x) + a$

Example 1

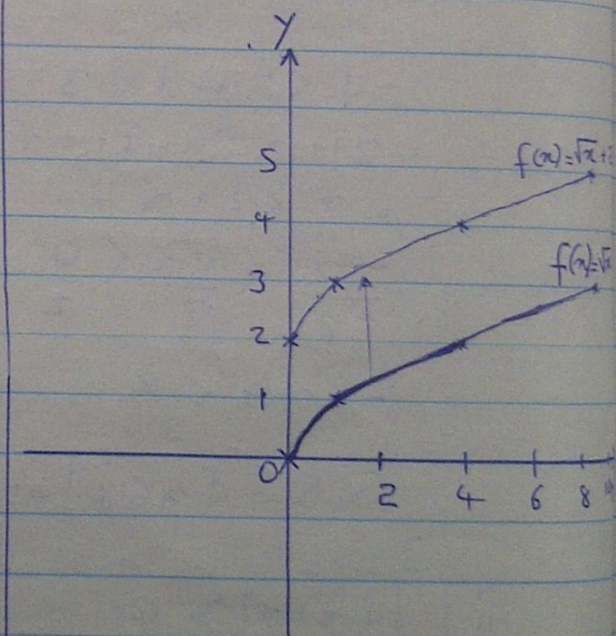
Sketch the graphs of $f(x) = \sqrt{x}$ and $f(x) = \sqrt{x} + 2$ on the same axes.

$f(x) = \sqrt{x}$

x	0	1	4	9
$f(x)$	0	1	2	3

$f(x) = \sqrt{x} + 2$

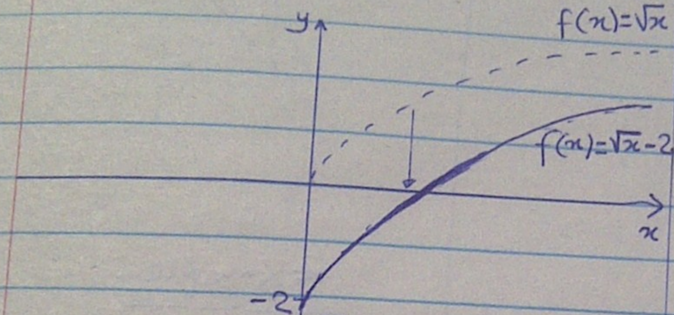
x	0	1	4	9
$f(x)$	2	3	4	5



We see that the graph of

$f(x)=\sqrt{x}+2$ is a shift of the graph $f(x)=\sqrt{x}$ 2 steps in the y-direction or upwards.

Q. Sketch the graph of $f(x)=\sqrt{x}-2$



The graph of $f(x)=\sqrt{x}-2$ is simply the shift of $f(x)=\sqrt{x}$ 2 steps downwards

(2) TRANSFORMATION OF THE FORM $y=f(x+a)$ or
$f(x)=\sqrt{x+a}$

Example

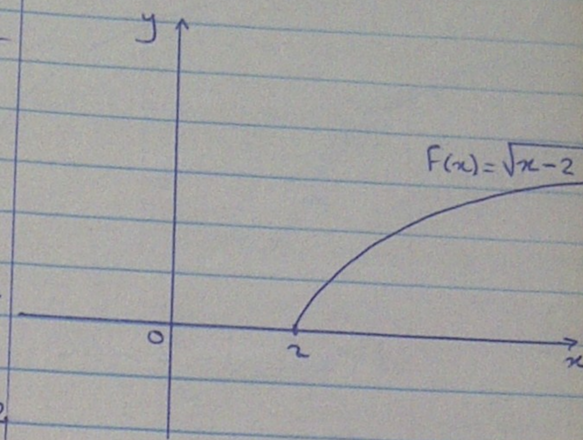
Sketch the graph of $f(x)=\sqrt{x-2}$

Domain of $f(x)=\sqrt{x-2}$

$$x-2\geqslant 0$$

and then solve for x

$$x\geqslant 2$$

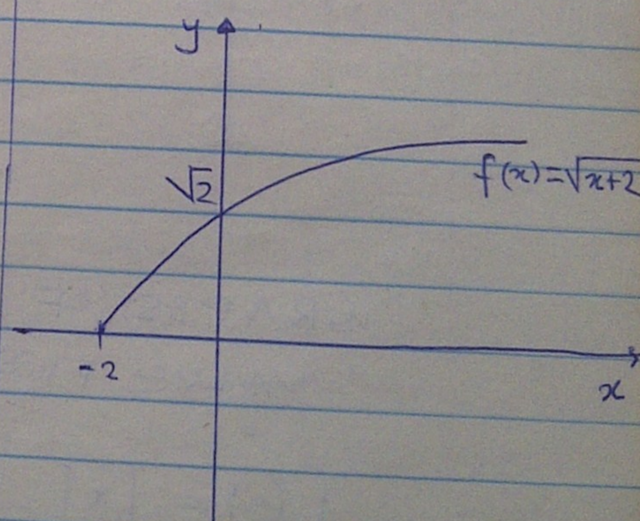


Domain $=\{x\in\mathbb{R} \mid x\geqslant 2\}$

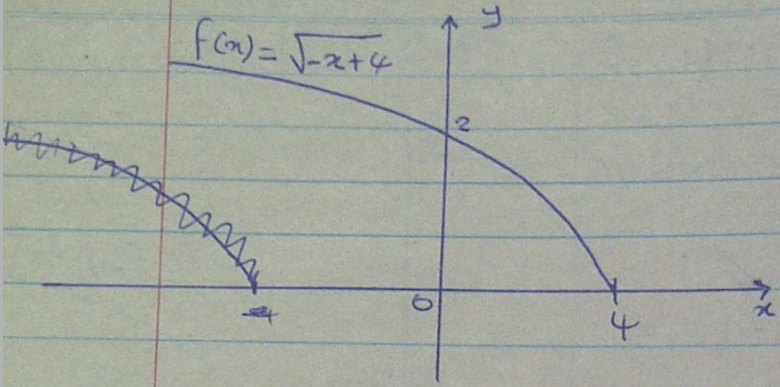
Sketch the graph of

$f(x)=\sqrt{x+2}$

$$x+2\geqslant 0$$
$$x\geqslant -2$$



Sketch the graph of
$f(x)=\sqrt{4-x}$
$=\sqrt{-x+4}$

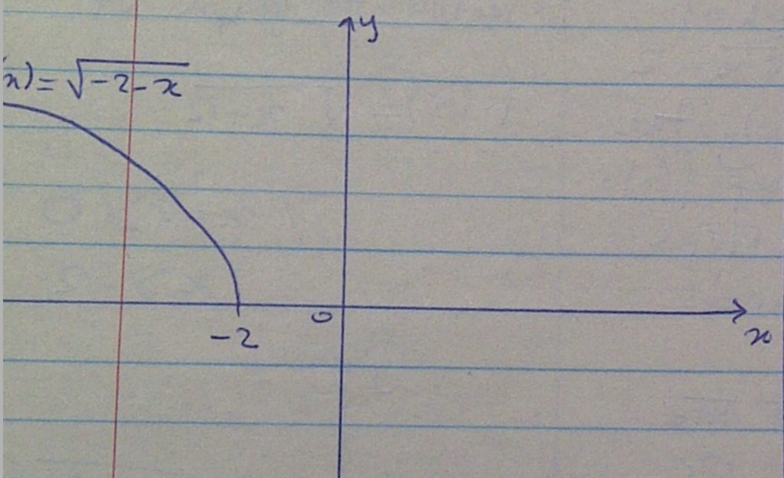


$$f(x) = \sqrt{-2-x}$$

$$-2 - x \geq 0$$

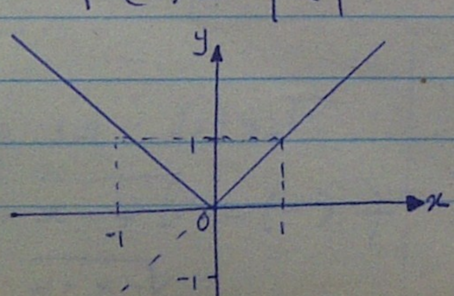
$$-x \geq 2$$

$$x \leq -2$$

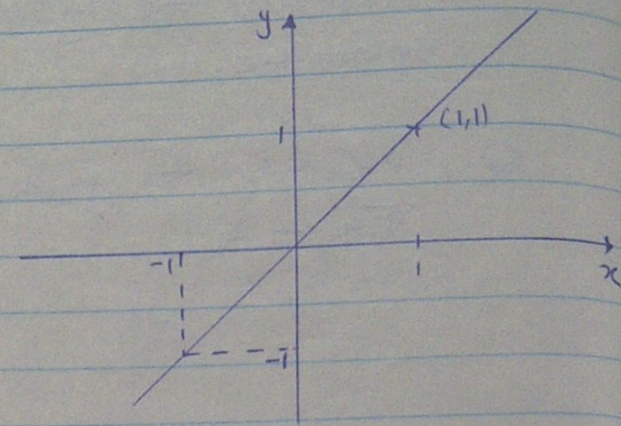


GRAPHS OF ABSOLUTE VALUE FUNCTION

$$f(x) = |x|$$



$f(x) = x$ or $y = x$

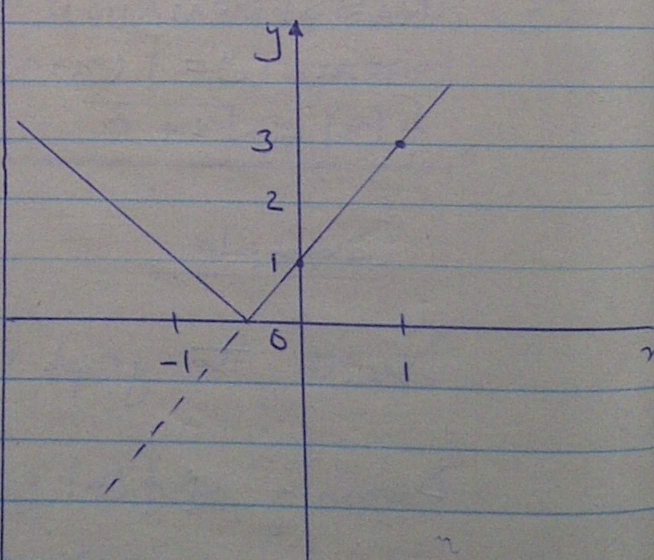


Example

Sketch the graph of $f(x) = |2x+1|$

$$y = 2x + 1$$

x	0	1	$-\frac{1}{2}$
y	1	3	0

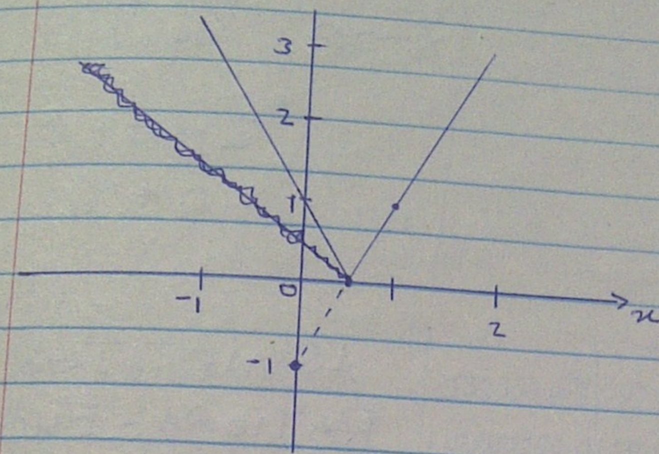


Sketch the graph of $f(x)=|2x-1|$

$y=|2x-1|$

$y=2x-1$

x	0	1	$\frac{1}{2}$
y	-1	1	0



Example

Sketch the graph of $y=|x^2+x-6|$

$y=x^2+x-6$

$a=1,\ b=1,\ c=-6$

$p=\frac{b}{2a}=\frac{1}{2}$

$q=\frac{4ac-b^2}{4a}=\frac{-25}{4}$

$a=1>0$

$y=x^2+x-6=(x-2)(x+3)=0$

zeros $\to x=2,\ x=-3$

y-intercept $(0,-6)$

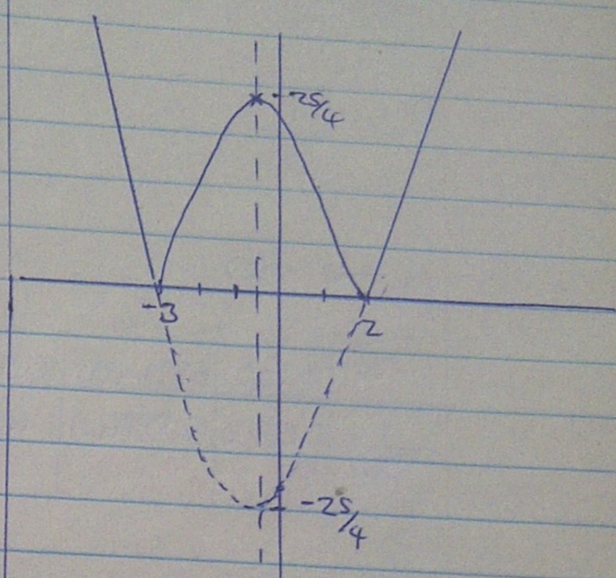
line of symmetry

$=-p=-\frac{b}{2a}=\frac{-1}{2}$

Turning point $\left(-\frac{1}{2}, \frac{-25}{4}\right)$

$(-p, q)$

$y=|x^2+x-6|$



GRAPHS OF RATIONAL FUNCTIONS

* A rational function is a ratio of two polynomial functions

* Domain of rational functions - generally they have restrictions on domain. The zeros of the ~~domain~~ denominator are excluded from their domain

The Idea of a limit of a function

$\cancel{\text{limit } \frac{1}{X^P}}$ $\lim\limits_{X \to \infty} \frac{1}{X^P}$ where $p = 1, 2, 3, 4$

If $p = 1$

$$\lim_{X \to \infty} \frac{1}{X} =$$

x	1000 000	10 000 000	100 000 000	1000 000 000
$\frac{1}{x}$	0.000001	0.0000001	0.00000001	0.000000001

$$\boxed{\lim_{X \to \infty} \frac{1}{X^P} = 0 \quad \text{where } p = 1, 2, 3, \ldots}$$

$$\lim_{x \to \infty} 2 = 2$$

$$\lim_{X \to \infty} C = C$$

where C is a constant

Evaluate $\lim\limits_{X \to \infty} \dfrac{2x^2 - 3x + 6}{4x + 6x^2}$

$$\lim_{X \to \infty} \frac{(2x^2 - 3x + 6)\frac{1}{x^2}}{(4x + 6x^2)\frac{1}{x^2}}$$

$$\lim_{X \to \infty} \frac{2 - 3/x + 6/x^2}{4/x + 6}$$

$$\frac{2 - 0 + 0}{0 + 6} = 2/6 = \frac{1}{3}$$

ASYMPTOTES OF A RATIONAL FUNCTION

1) VERTICAL ASYMPTOTES

A vertical asymptote has equation of the form $x = C$ (where C is a zero of the denominator of the rational function)

Example

Find the vertical asymptot of the function

$$f(x) = \frac{x}{x^2 + x - 6}$$

Solution

let

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

$x = -3$ and $x = 2$

2) HORIZONTAL ASYMPTOTES

Horizontal asymptotes of the rational function $y = f(x)$ are found by taking limits to infinity

ie $$Y = \lim_{x \to \infty} f(x)$$

Where $f(x)$ is the rational function under consideration

N.B

Not all rational functions have horizontal asymptotes

Example

Find the horizontal asymptotes of the rational function:

i) $f(x) = \frac{x+2}{x^2-2}$

degree of 1

(This is a proper rational function)

of degree 2.

$$Y = \lim_{x \to \infty} \frac{x+2}{x^2-2}$$

$$= \frac{\frac{1}{x} + \frac{2}{x^2}}{1 - \frac{2}{x^2}}$$

$$= \frac{0+0}{1-0} = \frac{0}{\underline{\underline{1}}}$$

ii) $f(x) = \frac{2x^2+2x+7}{3x^2-5x+2}$

$$Y = \lim_{x \to \infty} \frac{2x^2+2x+7}{3x^2-5x+2}$$

$$= \frac{2 + 2/x + 7/x^2}{3 - 5/x + 2/x^2}$$

$$= \frac{2+0+0}{3+0+0}$$

$$y = \underline{\underline{2/3}}$$

iii) $f(x) = \frac{x^3-1}{x^2-4}$

$$Y = \lim_{x \to \infty} \frac{x^3-1}{x^2-4}$$

$$= \frac{1 - 1/x^3}{\frac{1}{x} - 4/x^3}$$

$$= \frac{1-0}{0-0} = \frac{1}{0}$$

Undefined

Hence $f(x) = \frac{x^3-1}{x^2-4}$ has no horizontal asymptotes

OBLIQUE ASYMPTOTES

This exist only on improper rational functions which satisfy the condition;
The degree of the polynomial at the top (numerator) is greater than that of the denominator by a degree of 1.

e.g $f(x) = \frac{x^3 - 1}{x^2 - 4}$ has oblique asymptotes.

Example

Find the oblique asymptotes of:

$$f(x) = \frac{x^2 + x - 2}{x - 2}$$

Solution

$$\begin{array}{r|l} & x + 3 \\ \hline x - 2 & x^2 + x - 2 \\ & -(x^2 - 2x) \\ \hline & 3x - 2 \\ & -(3x - 6) \\ \hline & 4 \end{array}$$

$$\frac{x^2 + x - 2}{x - 2} = x + 3 + \frac{4}{x - 2}$$

$$\boxed{\begin{array}{l} f(x) = d(x)q(x) + r(x) \\ \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \end{array}}$$

$x + 3 + \frac{4}{x - 2}$ — $x \to \infty$

$y = x + 3$

SUMMARY

The graph of the rational function defined by the equation;

$$y = \frac{a_n x^n + a_{n-1} X^{n-1} + \dots + a_0}{b_m X^m + b_{m-1} X^{m-1} + \dots + b_0}$$

where a_0 and b_0 are non zero real numbers and $m \geq 0$ and $n \geq 0$ (non negative integers) has the following asymptotes

1) A vertical asymptote whose equation is $x = c$ for each c that is a zero of the denominator.

2) A horizontal asymptote whose equation is $y = 0$ (x-axis) if $n < m$

3) A horizontal asymptote whose equation is $y = \frac{a_n}{b_m}$ if $n = m$

4)

5)

(i)

4) No horizontal asymptote if $n > m$. (undefined)

5) An oblique asymptote if $n - m = 1$

$$\frac{\ldots + a_0}{\ldots + b_0}$$

Example

Sketch the graph of

(i) $f(x) = \frac{1}{x}$

(ii) $f(x) = \frac{x}{x-3}$

Showing all the relevant asymptotes Clearly

Solution

(i) $f(x) = \frac{1}{x}$

Vertical asymptote $x = 0$

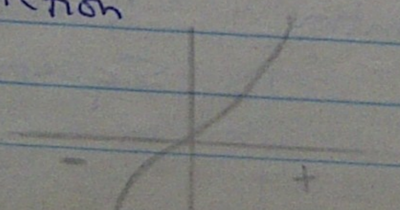
horizontal asymptote $y = 0$

$x = 0$ (y-axis)

$y = 0$ (x-axis)

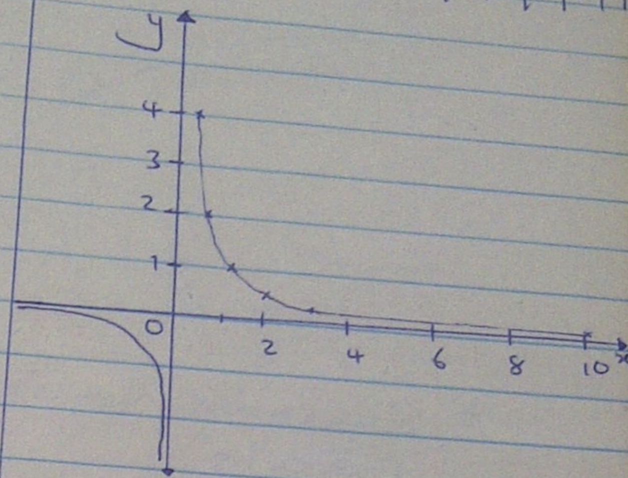
$$f(-x) = \frac{1}{-x} = -\frac{1}{x} = -f(x)$$

odd function



$f(x) = \frac{1}{x}$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	10
$\frac{1}{x}$	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{10}$



$f(x) = \frac{x}{x-3}$

Vertical asymptote $x - 3 = 0$

$x = 3$

horizontal asymptote $y = \frac{a_n}{b_m}$

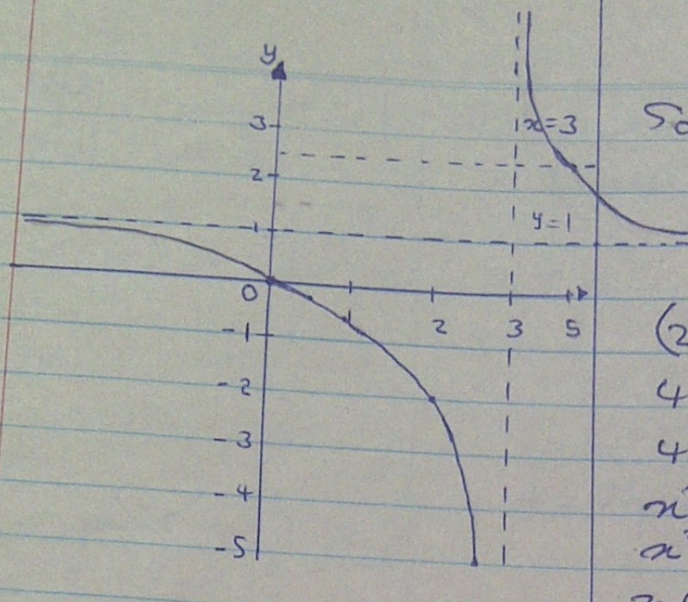
$y = \frac{1}{1} = 1$

When $x < 3$

x	0	$\frac{1}{2}$	1	2	2.5
$\frac{x}{x-3}$	0	$-\frac{1}{5}$	$-\frac{1}{2}$	-2	-5

When $x > 3$

x			
$\frac{x}{x-3}$			



* Equations Containing radicals

$x-3=1 \Rightarrow x=4$

Square both sides

$(x-3)^2 = 1$

$x^2-6x+9=1$

$x^2-2x-4x+8=0$

$x(x-2)-4(x-2)=0$

$(x-2)(x-4)=0$

$x=2$ or $x=4$

Checking

$2-3 \neq 1$, 2 is not a solution,

$S.S = \{4\}$

Hence squaring both sides of an equation is not always the best solution

Example

Solve $2\sqrt{x+4} - x = 1$

$2\sqrt{x+4} = 1+x$

Square both sides

$(2\sqrt{x+4})^2 = (1+x)^2$

$4(x+4) = 1+2x+x^2$

$4x+16 = 1+2x+x^2$

$x^2-2x-15=0$

$x^2-5x+3x-15=0$

$x(x-5)+3(x-5)=0$

$x=5$, $x=-3$

(tentative solution)

Checking

$2\sqrt{5+4} - 5 = 1$

$2\sqrt{9} - 5 = 1$

$2(3)-5=1$

$6-5=1$

Hence 5 is a true solution

$2\sqrt{-3+4} - (-3) = 1$?

$2(1)+3 = 1$?

$5 \neq 1$

Hence -3 is a false solution

$\therefore$ Solution set $= \{5\}$

Example 2

Solve $\sqrt{x+1} - \sqrt{x+6} = 1$

$\sqrt{x+1} = 1+\sqrt{x+6}$

Squaring both sides

$$(\sqrt{x+1})^2 = (1+\sqrt{x+6})^2$$
$$x+1 = 1+2\sqrt{x+6}+x+6$$
$$-6 = 2\sqrt{x+6}$$
Squaring both sides
$$(-6)^2 = (2\sqrt{x+6})^2$$
$$36 = 4(x+6)$$
$$36 = 4x+24$$
$$36-24 = 4x$$
$$4x = 12$$
$x = 3$ (tentative)

Checking by replacing Tentative solution in original equation

$$\sqrt{3+1} - \sqrt{3+6} = 1$$
$$\sqrt{4} - \sqrt{9} = 1$$
$$2-3 \neq 1$$

Solution set = { }

(1) Solve;
$$\sqrt{x+6} + x - 6 = 0$$
$$\sqrt{x+6} = -x+6$$
Squaring both sides
$$(\sqrt{x+6})^2 = (-x+6)^2$$
$$x+6 = x^2-12x+36$$
$$x^2-13x+30 = 0$$
$$x^2-10x-3x+30 = 0$$
$$x(x-10)-3(x-10) = 0$$
$$(x-10)(x-3) = 0$$
$x = 10$ or $x = 3$
(tentative solution)

Checking
$$\sqrt{3+6}+3-6 = 0$$
$$\sqrt{9}+3-6 = 0$$
$$3-3 = 0$$
$$0 = 0$$
Hence 3 is a true solution

(2) $\sqrt{3x+16} - \sqrt{2x+9} = -1$

(3) $\sqrt{x+1} = 4$

(4) $\sqrt{x}\sqrt{x+1} = \sqrt{x} - \sqrt{6}$

(5) $\sqrt{3x+1} = \sqrt{x} - 1$

(6) $\sqrt{x+4} + \sqrt{x-4}$

PARTIAL FRACTIONS

i) EQUATION VS IDENTITIES

* $3x + 5 = 7 \longrightarrow$ equation

* $x^2 - y^2 \equiv (x+y)(x-y) \longrightarrow$ identity

$\equiv$ ↓ equivalent

N.B
The major difference between an equation and an identity is that the latter is true for any value substitution of the variable.

- An identity can take on any value substitution.

Q. Factorise:
i) $x^2 - 2x - 8$ ii) $x^2 + 5x + 8$

Solution

i) $x^2 + 2x - 4x - 8$
$x(x+2) - 4(x+2)$
$(x+2)(x-4)$

ii) $x^2 + 5x + 8$
Can not be factorised over $\mathbb{Q}$, (Rational #s).
This is irreducible over $\mathbb{Q}$

PARTIAL FRACTIONS

Q Simplify $\frac{8}{x-4} - \frac{4}{x+2}$

Solution

$$\frac{8}{x-4} - \frac{4}{x+2}$$

$$\frac{8(x+2) - 4(x-4)}{(x-4)(x+2)}$$

$$= \frac{8x + 16 - 4x + 16}{(x-4)(x+2)}$$

$$= \frac{4x + 32}{(x-4)(x+2)}$$

The reverse Process

$$\frac{4x+32}{(x-4)(x+2)} = \underbrace{\frac{8}{x-4} - \frac{4}{x+2}}_{\text{Partial fraction}}$$

Proper fraction (algebraic)

*Remember that an algebraic fraction is 'proper' if the degree of the numerator is lower than that of the denominator.

* The first role on breaking up an algebraic fraction into partial fractions

is make sure the fraction under Consideration is 'proper' if not proper force it to become proper by long division

Example

Express $\frac{4x+32}{x^2-2x-8}$ in partial fractions

Step 1

we see that the fraction above is 'proper'.

Step 2

* Factorice the denominator if it is not in a factored form

$x^2-2x-8 = (x+2)(x-4)$

$$\frac{4x+32}{(x+2)(x-4)}$$

$$\equiv \frac{A}{x+2} + \frac{B}{x-4}$$

N.B A and B are Constant polynomials

$$4x+32 \equiv A(x-4)+B(x+2)$$

$$\frac{\cancel{(x+2)}\cancel{(x-4)}\,4x+32}{\cancel{(x+2)}\cancel{(x-4)}} = \frac{A\,\cancel{(x+2)}(x-4)}{x+2} + \frac{B}{x-4}$$

Never Show this step $\times(x+2)(x-4)$

$$4x+32 \equiv A(x-4)+B(x+2)$$

if $x=4$

$$4(4)+32 \equiv B(4+2)$$

$$\frac{48}{6} = \frac{6B}{6}$$

$$B = 8$$

if $x=-2$

$$4(-2)+32 \equiv A(-2-4)$$

$$\frac{24}{-6} = \frac{-6A}{-6}$$

$$A = -4$$

$$\therefore \frac{4x+32}{(x+2)(x-4)} \equiv \frac{A}{x+2} + \frac{B}{x-4}$$

$$= \frac{-4}{x+2} + \frac{8}{x-4}$$

$$= \frac{8}{x-4} - \frac{4}{x+2}$$

Example 2

Express in ~~fracl~~ partial fractions:

$$\frac{8x-28}{x^2-6x+8}$$

Solution

$$\frac{8x-28}{x^2-6x+8} \equiv \frac{8x-28}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}$$

$$\Rightarrow 8x-28 = A(x-4) + B(x-2)$$

if $x = 4$

$$8(4)-28 = B(4-2)$$
$$4 = 2B$$
$$B = 2$$

if $x = 2$

$$8(2)-28 = A(2-4)$$
$$-12 = -2A$$
$$A = 6$$

$$\therefore \frac{8x-28}{(x-2)(x-4)} \equiv \frac{A}{x-2} + \frac{B}{x-4}$$

$$= \frac{6}{x-2} + \frac{2}{x-4}$$

Example 3

Express $\dfrac{x^2+3x-10}{x^2-2x-3}$ in partial fractions

Solution

We know that this fraction is not 'proper'

$$\begin{array}{r|l} & \quad 1 \\ x^2-2x-3 & x^2+3x-10 \\ & -(x^2-2x-3) \\ \hline & \quad 5x-7 \end{array}$$

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

$$\frac{x^2+3x-10}{x^2-2x-3} = 1 + \frac{5x-7}{x^2-2x-3} \quad (1)$$

$$\frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x-3)(x+1)} \Rightarrow \quad (2)$$

$$\frac{A}{x-3} + \frac{B}{x+1}$$

$$\Rightarrow 5x-7 = A(x+1) + B(x-3)$$

If $x = -1$

$$5(-1)-7 = B(-1-3)$$
$$-12 = -4B$$
$$B = 3$$

if $x = 3$

$$5(3)-7 = A(3+1)$$
$$8 = 4A$$
$$A = 2$$

$$\therefore \frac{x^2+3x-10}{x^2-2x-3} = 1 + \frac{2}{x-3} + \frac{3}{x+1}$$

Exercise

Express in partial fractions;

Ans $+\frac{3}{x+2}+\frac{5}{x+3}$

① $\dfrac{2x^2+18x+31}{x^2+5x+6}$

Ans $1+\frac{1}{x+4}-\frac{5}{x+6}$

② $\dfrac{2x^3+3x^2-54x+50}{x^2+2x-24}$

PARTIAL FRACTIONS

Q. Express $\dfrac{15x^2-x+2}{(x-5)(3x^2+4x-2)}$ in partial fraction

Solution

* $3x^2+4x-2$ cannot be factorised
* But x^2+2x-8 can be factorised

$D = b^2-4ac =$ perfect square which is positive

$3x^2+4x-2$

$a=3$, $b=4$, $c=-2$

$$D = 4^2-4(3)(-2) = 16+24$$
$$= 40$$

Not a perfect square

Hence, Cannot be factorised

$3x^2+4x-2$ is irreducible over $\mathbb{Q}$

x^2+2x-8

$a=1$, $b=2$, $c=-8$

$$D = b^2-4ac$$
$$= 2^2-4(1)(-8)$$
$$= 4+32$$
$$= 36 \text{ perfect square}$$

An irreducible quadratic factor in the denominator of the original rational expression of the form (ax^2+bx+c) gives rise to the partial fraction it is of the form

$$\frac{Ax+B}{ax^2+bx+C} \quad \text{where } a \neq 0$$

IRREDUCIBLE QUADRATIC FACTOR IN THE DENOMINATOR

Example

Express $\dfrac{x^2-3}{(x-1)(x^2+1)}$ in partial fraction

Solution

$$\frac{x^2-3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2-3 = A(x^2+1) + (Bx+C)(x-1)$$

if $x=1$

$$1^2-3 = A(1^2+1)$$

$$-2 = 2A$$

$$A = -1$$

$$x^2-3 = A(x^2+1) + (Bx+C)(x-1)$$

$$x^2-3 = A(x^2+1) + Bx^2 - Bx + Cx - C$$

$$x^2-3 = (A+B)x^2 + (C-B)x + (A-C)$$

$*_1$ $A+B=1$

$*_2$ $C-B=0$

$*_3$ $A-C=-3$

$A=-1$

$A+B=1$

$-1+B=1$

$B=2$

$C=B=2$

$$\frac{x^2-3}{(x-1)(x^2+1)} = \frac{-1}{x-1} + \frac{2x+2}{x^2+1}$$

THE REPEATED FACTOR IN THE DENOMINATOR

$*_1$ $\dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} = \dfrac{x+1+1}{(x+1)^2}$

$$= \frac{x+2}{(x+1)^2}$$

$*_2$ $\dfrac{1}{x+1} + \dfrac{1}{(x+1)^3}$

$$= \frac{x^2+2x+1+1}{(x+1)^3} = \frac{x^2+2x+2}{(x+1)^3}$$

$*_3$ $\dfrac{1}{x+1} + \dfrac{\varphi}{(x+1)^2} + \dfrac{1}{(x+1)^3}$

$$= \frac{x^2 + 2x + 1 + 4x + 4 + 1}{(x+1)^3}$$

$$= \frac{x^2 + 6x + 6}{(x+1)^3}$$

*₂ In partial fraction is

$$\frac{x^2 + 2x + 2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$* \quad \frac{x^2 - 3}{(x+1)^4} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4}$$

Example

Express $\frac{x-1}{(x+1)(x-2)^2}$ in partial fractions

N.B $(x-2)^2$ is a repeated factor.

Solution

$$\frac{x-1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\Rightarrow x - 1 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

If $x = 2$

$$2 - 1 = 3C$$
$$3C = 1$$
$$C = \frac{1}{3}$$

If $x = -1$

$$-1 - 1 = A(-1-2)^2$$
$$9A = -2$$
$$A = -\frac{2}{9}$$

x^2

$$0x^2 = (A+B)x^2$$

$$A + B = 0$$
$$-\frac{2}{9} + B = 0$$
$$B = \frac{2}{9}$$

$$\frac{x-1}{(x+1)(x-2)^2} = -\frac{2}{9(x+1)} + \frac{2}{9(x-2)} + \frac{1}{3(x-2)^2}$$

THE BINOMIAL

EXPANSION

THE BINOMIAL EXPANSION

Bi means two

$\underbrace{(a+b)}_{\text{Two terms}}^2$

RAISING BINOMIALS TO POWERS

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2+2ab+b^2$

n (Power)	number of terms
0	1
1	2
2	3
3	4
4	5

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

Several patterns appear in the above expressions

1) Each expression has one more term than the power (index) of the binomial

2) The degree of each term in each expression is equal to the exponent (index) of the binomial that is being expanded.

3) The first term in each expression is ~~a~~ raised to the power of the binomial

PASCAL'S TRIANGLE

0 ⟶ 1

1 ⟶ 1 1

2 ⟶ 1 2 1

3 ⟶ 1 3 3 1

4 ⟶ 1 4 6 4 1

5 ⟶ 1 5 10 10 5 1

Example

Expand $(x+y)^5$

Solution

Use Column 5

$$(x+y)^5 = 1(x)^5 + 5(x)^4y^1 + 10(x)^3y^2 + 10(x)^2y^3 + 5x^1y^4 + 1(y)^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

FACTORIAL NOTATION

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

$3! = 3 \times 2 \times 1$

> If $n \in N$, then
>
> $$n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$$

Example

Find (i) $5!$ (ii) $(n-2)!$

(iii) $-9!$

Solution

i) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

ii) $(n-2)! = (n-2)(n-3)(n-4)\ldots(3)(2)(1)$

$v_f = 0$
$x = ?$

$x = \frac{-v_0^2}{2a} = \frac{-2^2}{2\times(-1,154)}$

iii) $-9! = -9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = -362880$

Example

Simplify

a) $\frac{n!}{(n-1)!}$ b) $\frac{n!}{(n-3)!}$

Solution

a) $\frac{n!}{(n-1)!}$

$$= \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$$

b) $\frac{n!}{(n-3)!}$

$$= \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}}$$

$$= n(n-1)(n-2)$$

c) $\frac{(n+1)!}{(n-4)!}$

Zero factorial

$0! = 1$

N.B $0! = 1 = 1!$

COMBINATIONS

The Symbol ${}^nC_r$ or $\binom{n}{r}$ means "n Combinations or selections of objects taking r at a time"

Defn

$${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

N.B $n \geq r$.

Example

Evaluate i) ${}^6C_2$

Solution

$${}^6C_2 = \frac{6!}{(6-2)!\,2!}$$

$$= \frac{6!}{4!\,2!} = \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!}\,2!}$$

$$= \frac{6 \times 5}{2} = \underline{\underline{15}}$$

ii) $\binom{6}{3} = \frac{6!}{3!\,3!} = \frac{6 \times 5 \times 4 \times 3!}{3!\,3!}$

$$= \underline{\underline{20}}$$

THE BINOMIAL THEOREM

In the expression of $(x+y)^n$

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}y^n$$

Example

Find the binomial expansion of $(x-y)^4$

Solution

$$(x-y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3(-y)^1 + \binom{4}{2}x^2(-y)^2 + \binom{4}{3}x^1(-y)^3 + \binom{4}{4}(-y)^4$$

$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Q. Expand $\left(x + \frac{1}{x}\right)^4$

Solution

$$\left(x+\frac{1}{x}\right)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3\left(\frac{1}{x}\right)^1 + \binom{4}{2}x^2\left(\frac{1}{x}\right)^2 + \binom{4}{3}x^1\left(\frac{1}{x}\right)^3 + \binom{4}{4}\left(\frac{1}{x}\right)^4$$

$$= x^4 + 4x^2 + \underline{\underline{6}} + 4\frac{1}{x^2} + \frac{1}{x^4}$$

The term independent of x

$\cancel{(a+b)^3 = a^3 + 3a^2b +}$

$$(a+b)^3 = \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3$$

less 1^{st} position less 2^{nd} position 3^{rd} position

$$\boxed{\begin{array}{l}\text{The } (r+1)^{th} \text{ position of the binomial expansion } (a+b)^n \text{ is} \\ T_{r+1} = \binom{n}{r}a^{n-r}b^r\end{array}}$$

Example

Find the sixth term in the binomial expression of $(x-y)^7$

Solution

$6 = r+1 \Rightarrow r = 6-1 = 5$

$T_{r+1} = \binom{n}{r} a^{n-r} b^{r}$

$T_6 = \binom{7}{5} x^{7-5} (-y)^5$

$T_6 = \frac{7!}{(7-5)!\,5!} x^2 (-y^5)$

$= -21x^2y^5$

Q. In the binomial expansion of $\left(x + \frac{1}{x}\right)^{10}$, find

a) The 4th term b) The Coefficient of x^4

c) The constant term

Solution

$(r+1)^{th}$ term $(a+b)^r$

$T_{r+1} = \binom{n}{r} a^{n-r} b^{r}$

$n = 10,\ r = ?$

$a = x,\ b = \frac{1}{x}$

$T_{r+1} = \binom{10}{r} x^{10-r} \left(\frac{1}{x}\right)^r = \binom{10}{r} x^{10-r-r}$

$$= \binom{10}{r} x^{10-2r}$$

$$x^4 = x^{10-2r}$$

$$4 = 10 - 2r$$
$$2r = 10 - 4$$
$$r = 5 - 2 = 3$$

a) $T_{r+1} = \binom{10}{r} x^{10-2r}$

$$T_{3+1} = T_4 = \binom{10}{4} x^{10-2(3)}$$

$$T_4 = \frac{10!}{(10-3)!\,3!} = 120x^4$$

b) 120

c) $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$n = 10$, $a = x$, $b = \frac{1}{x}$

$$= \binom{10}{r} x^{10-r} \left(\frac{1}{x}\right)^r$$

$$= \binom{10}{r} x^{10-r} \times \frac{1}{x^r}$$

$$= \binom{10}{r} x^{10-r-r} \Rightarrow T_{r+1} = \binom{10}{r} x^{10-2r}$$

$10 - 2r = 0$

$2r = 10$

$r = 5$

$$T_{r+1} = \binom{10}{r} x^{10-2r}$$

$$T_{5+1} = T_6 = \binom{10}{5} x^{10-2(5)}$$

$$= \frac{10!}{5!5!} = \underline{252}$$

SEQUENCE AND SERIES

Defn

a sequence is a function whose domain is the set of natural numbers

e.g $F(n) = 3n+2$, where $n \in N$ is a sequence

$n = \{1, 2, 3, 4, 5, 6 \ldots\}$

$F(1) = 3(1) + 2 = 5$

$f(2) = 3(2) + 2 = 8$

$f(3) = 3(3) + 2 = 11$

$f(4) = 3(4) + 2 = 14$

$f(5) = 3(5) + 2 = 17$

$f(6) = 3(6) + 2 = 20$

It is common to call the list as well as the function, a sequence. Each number in the list is called a term of the sequence

1, 1, 2, 3, 5, 8, 13, 21, ---

The fibonacci sequence

ARITHMETIC SEQUENCE

E.G 1, 3, 5, 7, 9, 11, ---

In an arithmetic sequence the first term and the common difference are denoted by a and d respectively

a, $a+d$, $a+2d$, $a+3d$, $a+4d$, --- →

1st position

2nd position

3rd position

4th position

5th position

$a + (n-1)d$ $\quad n = n^{th}$ term

Therefore, the n^{th} term in an arithmetic sequence is $T_n = a + (n-1)d$

Example 1

An arithmetic sequence has a first term

of 5 and a common difference of 4.

a) Write down the first 5 terms of the sequence

b) Write the 25th term of the sequence

Solution

a) $a = 5, \quad d = 4$

$T_n = a + (n-1)d$

$T_n = 5 + (n-1)4$

$n = 1, 2, 3, 4, 5, ..$

$f(n) = 5 + 4(n-1)$

$f(1) = 5 + 4(1-1) = 5$

$f(2) = 5 + 4(2-1) = 9$

$f(3) = 5 + 4(3-1) = 13$

$f(4) = 5 + 4(4-1) = 17$

$f(5) = 5 + 4(5-1) = 21$

The first 5 terms are 5, 9, 13, 17, 21

b) $n = 25$

$T_{25} = 5 + 4(25-1)$

$= 5 + 100 - 4$

$= \underline{\underline{101}}$

Example 3

The first term of an arithmetic sequence is 12, and the 50th term is 3099, write the first 6 terms of the sequence

Solution

$f(n) = a + (n-1)d$

50^{th} term $= 50$

$f(50) = a + (50-1)d$

But, $f(50) = 3099$

$3099 = 12 + 49d$

$\frac{49d}{49} = \frac{3099 - 12}{49}$

$d = \underline{\underline{63}}$

$f(n) = 12 + 63(n-1)$

$n = 1, 2, 3, 4, 5, 6,$

$f(1) = 12 + 63(1-1)$
$= 12$

THE DIFFERENCE BETWEEN A SEQUENCE AND A SERIES

1. 4, 7, 10, 13, 16, 19, 22, ... is an example of a sequence
But the sum of the above terms
$4 + 7 + 10 + 13 + 16 + 19 + 22 ...$
is a Series.
Hence, a Sequence is a list of terms according to some formula or rule. On the other hand a series is the sum of terms in a sequence.

Examples on A.P

Find the 25th term of an arithmetic sequence whose third and sixth terms are 10 and 19 respectively.

Solution

$T_n = a + (n-1)d$

Third term

$n = 3 \quad T_3 = 10$

$T_3 = a + (3-1)d$

$10 = a + 2d \quad ---(i)$

Sixth term

$n = 6, T_6 = 19$

$T_6 = a + (6-1)d$

$19 = a + 5d \quad ---(ii)$

$$-\begin{pmatrix} a + 2d = 10 \\ a + 5d = 19 \end{pmatrix}$$

$-3d = -9$

$d = 3$

$a + 2d = 10$

$a = 10 - 2d$

$a = 10 - 2(3)$

$\underline{\underline{a = 4}}$

$T_n = a + (n-1)d$

$T_n = 4 + 3(n-1)$

25th term

$n = 25$

$T_{25} = 4 + 3(25-1)$

$= 4 + 3 \times 24$

$= 4 + 72$

$= \underline{\underline{76}}$

ARITHMETIC MEANS

If numbers are inserted between numbers a and b to form an arithmetic sequence, then the ~~inverse~~ inserted numbers are called arithmetic means. If a single number is inserted between a and b to form an arithmetic sequence, that Number is called the Arithmetic mean

Examples

Insert two arithmetic means between 6 & 27

Solution

$6, 6+d, 6+2d, 27$

1st — 4th position

$a = 6, d = ?$

Last term

$n = 4 \quad T_4 = 27$

$T_n = a + (n-1)d$

$T_4 = 6 + (4-1)d = 27$

$6 + 3d = 27$

$3d = 21$

$\underline{\underline{d = 7}}$

1st arithmetic mean $=$

$6 + d = 6 + 7 = 13$

2^{nd} arithmetic mean $= 6+2d = 6+2(7) = 20$

N.B
The last term of arithmetic sequence may be denoted by L.

THE SUM OF THE FIRST n TERMS OF AN ARITHMETIC SEQUENCE

Terms in an arithmetic sequence are:

$a,\ a+d,\ a+2d,\ a+3d,$

$----\ a+(n-2)d,\ a+(n-1)d.$

N.B $L = a+(n-1)d$

$S_n = (a) + (a+d) + (a+2d) + \ldots + [a+(n-3)d] + [a+(n-2)d] + [a+(n-1)d]$

$S_n = [a+(n-1)d] + [a+(n-2)d] + [a+(n-3)d] + \ldots (a+2d) + (a+d) + (a)$

$2S_n = 2a+(n-1)d + 2a+(n-1)d + 2a+(n-1)d + \ldots + 2a+(n-1)d + 2a+(n-1)d$

$2S_n = n(2a+(n-1)d$

$$\boxed{S_n = \frac{n}{2}(2a+(n-1)d}$$

partial sum ~~of~~ up to the n^{th} term

$$S_n = \frac{n}{2}\left[a + \underbrace{a+(n-1)d}_{L}\right]$$

$$\boxed{S_n = \frac{n}{2}(a+L)}$$

Where a and L are the first and last terms respectively.

Example

Find the sum of the first 40 terms of the arithmetic sequence:
4, 10, 16, 22, 28, ----

The n^{th} term in terms of S_n

$$\boxed{T_n = S_n - S_{n-1}}$$

Solution

$S_n = \frac{n}{2}(2a + (n-1)d)$

$n = 40, \ a = 4, \ d = 10 - 4 = 6$

$S_{40} = \frac{40}{2}[2 \times 4 + (40-1)6]$

$= 20[8 + 39 \times 6]$

$= 20(8 + 234)$

$= 20 \times 242$

$= 4840$

SUMMATION NOTATION

We can use short hand notation for indicating the sum of a finite number of consecutive terms of the sequence. This notation, called summation notation, involves the Greek letter Σ (sigma)

Example

Given the series: $3+6+9+12+15+18$ Express in summation notation

Solution

$T_n = 3n$

$$3+6+9+12+15+18 = \sum_{K=1}^{6} 3K$$

Example

Evaluate $\sum_{K=1}^{5} 2K$

Solution

$K = 1, 2, 3, 4, 5$

$$\sum_{K=1}^{5} 2K = 2(1) + 2(2) + 2(3) + 2(4) + 2(5)$$

$$= 2 + 4 + 6 + 8 + 10$$

$$= \underline{\underline{30}}$$

Q. Evaluate $\sum_{K=2}^{5} K^2$

Solution

$K = 2, 3, 4, 5,$

$$\sum_{K=2}^{5} K^2 = (2)^2 + (3)^2 + (4)^2 + (5)^2$$

$$= 4 + 9 + 16 + 25$$

$$= \underline{\underline{54}}$$

Example

Evaluate $\sum_{K=2}^{50} (2K+1)$

Solution

$$\sum_{K=2}^{50} (2K+1) = 5 + 7 + 9 + 11 + \dots 101$$

$n = 49$

$$S_n = \frac{n}{2}(a + l)$$

$$S_{49} = \frac{49}{2}(5 + 101)$$

$$= \underline{\underline{2597}}$$

Exercise

1. Evaluate

i) $\sum_{K=1}^{4} (2K^2 - 1)$

ii) $\sum_{K=3}^{200} (3K + 1)$

2. Express in summation notation:

a) $1^2 + 2^2 + 3^2 + 4^2 + \dots + 8^2$

b) $4 + 8 + 12 + 16 + 20 + 24$

c) $10 + 12 + 14 + 16 + 18$

Q. Evaluate (i) $\sum_{i=1}^{6} i^2$ (ii) $\sum_{i=1}^{100} i^2$

Solutions

i) $\sum_{i=1}^{6} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 1 + 4 + 9 + 16 + 25 + 36$

$= \underline{\underline{91}}$

ii) $\sum_{i=1}^{100} i^2 = 1^2 + 2^2 + 3^2 + \dots 100^2$

~~n = 100~~ $\cancel{S_n = \frac{n}{2}(a + l)}$

$\cancel{S_{100} = \frac{100}{2}(1 + 100) = 5050}$

GEOMETRIC SEQUENCES

Complete the following table

n	$5(2^n)$
1	$5(2^1) = 10$
2	$5(2^2) = 20$
3	$5(2^3) = 40$
4	$5(2^4) = 80$
5	$5(2^5) = 160$

The numbers

10, 20, 40; 80, 160 _ _ are in a geometric series

N.B. The next term of the above

sequence is obtained by multiplying the preceeding term by 2, Hence 2, is known as the common ratio.

THE n^{th} TERM OF A GEOMETRIC SEQUENCE

Let r be the common ratio of a geometric sequence and let a be the ~~ratio~~ first term of the geometric sequence, then;

$$a, \; ar, \; ar^2, \; ar^3, \; ----- \; ar^{n-1}$$

$$1^{st} \quad 2^{nd} \quad 3^{rd} \quad 4^{th}, \; ----- \; n^{th}$$

$$\boxed{T_n = ar^{n-1}}$$

Example 1

A geometric sequence has a first term of 2 and a common ratio of 3.

a) Write down the first four terms of the sequence.

b) find the ninth term

Solution

a) $T_n = ar^{n-1}$, $a = 2$ and $r = 3$

$T_n = 2(3)^{n-1}$

$n = 1, 2, 3, 4$

$T_1 = 2(3)^{1-1} = 2 \times 3^0 = 2 \times 1 = \underline{\underline{2}}$

$T_2 = 2(3)^{2-1} \Rightarrow 2 \times 3 = \underline{\underline{6}}$

$T_3 = 2(3)^{3-1} \Rightarrow 2 \times 3^2 = 2 \times 9$
$= \underline{\underline{18}}$

$T_4 = 2(3)^{4-1} = 2 \times 3^3$
$= 2 \times 27$
$= \underline{\underline{54}}$

b) Find the ninth term

$T_n = ar^{n-1}$
$T_9 = ar^{9-1}$
$= 2(3)^{9-1}$
$= 2 \times 3^8$
$= \underline{\underline{13122}}$

<u>Example 2</u>

The first three terms of a geometric sequence are 16, 8 and 4, Find the seventh term.

<u>Solution</u>

$a_1 = 16 \quad r = \frac{T_n}{T_{n-1}} = \frac{4}{8} = \frac{1}{2} \quad n = 7$

$T_n = ar^{n-1} \Rightarrow T_7 = 16\left(\frac{1}{2}\right)^{n-1}$

$$T_7 = 16\left(\tfrac{1}{2}\right)^{7-1}$$
$$= 16 \times \frac{1}{2^6}$$
$$= 16 \times \frac{1}{64} = \frac{1}{4}$$

GEOMETRIC MEAN

If numbers are inserted between two numbers a and b to form a geometric sequence, the inserted numbers are called GEOMETRIC MEANS between a and b to form a geometric sequence, that number is called a GEOMETRIC MEAN between a and b.

Example 3

Insert two geometric means between 1 and 8.

Solution

$1, \underline{r}, \underline{r^2}, 8$

$T_n = ar^{n-1}$, $a = 1$, $n = 3$ (position of r^2), $T_4 = 8$

$T_4 = ar^{4-1}$

$T_4 = 1 \times r^3$

$r^3 = 8$

$r = \sqrt[3]{8} = 2$

$T_2 = r = 2$

$T_3 = r^2 = 2^2 = 4$

The two geometric means are 2 and 4

THE SUM OF THE FIRST n TERMS OF A GEOMETRIC SEQUENCE

$a, ar, ar^2, ar^3, \ldots, ar^{n-4}, ar^{n-3}, ar^{n-2}, ar^{n-1}$

① $S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \ldots + \cancel{ar^{n-4}} + \cancel{ar^{n-3}} + \cancel{ar^{n-2}} + \cancel{ar^{n-1}}$

② $rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{ar^4} \ldots \cancel{ar^{n-3}} + \cancel{ar^{n-2}} + \cancel{ar^{n-1}} + ar^n$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a - ar^n$$

$$S_n = \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r}$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

This is the sum of the first n th terms of a geometric sequence

Example 4

Find the sum of the first six terms of the geometric sequence

$$250, 50, 10, \ldots$$

Solution

$a = 250$, $r = \frac{T_n}{T_{n-1}} = \frac{10}{50} = \frac{1}{5}$, $n = 6$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_6 = \frac{250\left[1-\left(\frac{1}{5}\right)^6\right]}{\left(1-\frac{1}{5}\right)}$$

$$= \frac{250\left(1-\frac{1}{5^6}\right)}{4/5}$$

$$= \frac{250\left(1-\frac{1}{15625}\right)}{4/5}$$

$$= 250\left(\frac{15624}{15625}\right) \div \frac{4}{5}$$

$$= 250 \times \frac{15624}{15625} \times \frac{5}{4} = \cancel{499.68}$$

$$= 312.48$$

Example 5

Evaluate $\sum_{i=1}^{100} 6\left(\frac{1}{2}\right)^i$

Solution

$$\sum_{i=1}^{100} 6\left(\frac{1}{2}\right)^i = 6\left(\frac{1}{2}\right)^1 + 6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right)^3 + \dots + 6\left(\frac{1}{2}\right)^{100}$$

$$= 3 + \frac{3}{2} + \frac{3}{4} + \dots + 6\left(\frac{1}{2}\right)^{100}$$

$a = 3, \quad r = 1/2, \quad n = 100$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{100} = \frac{3\left[(1-\cancel{100}\left(\frac{1}{2}\right)^{100}\right]}{1-1/2}$$

$$= 3\left[(1-\left(\tfrac{1}{2}\right)^{100}\right] \div 1/2$$

$$= 3\left[1-\left(\tfrac{1}{2}\right)^{100}\right] \times 2$$

$$= 6\left[1-\left(\tfrac{1}{2}\right)^{100}\right] \approx 6$$

THE SUM OF INFINITY OF A GEOMETRIC SERIES

WHEN $0<|r|<1$

$$S = \frac{a(1-r^n)}{r-1}$$

If $0<|r|<1$, then $n \rightarrow \infty$, $r^n \Rightarrow 0$

Hence, $\boxed{S_\infty = \frac{a}{1-r}}$ Sum to infinity of a geometric series when $0<|r|<1$

e.g $0 < \frac{1}{5} < 1$ ie $1/5$ is between 0 and 1

$*_1$ $\left(\frac{-1}{2}\right)^{1000} = \frac{1}{2^{1000}} \longrightarrow 0$

$*_2$ $\left(-1/2\right)^{1001} = -\frac{1}{2^{1001}} = -0.00000 \ldots 1$

Eample 7

If we have a geometric sequence,
64, 32, 16, 8
Find the sum of infinity

Solution

$r = \frac{T_n}{T_{n-1}} = \frac{8}{16} = 1/2$, we note that $0 < r < 1$

ie $0 < 1/2 < 1$

$$S_\infty = \frac{a}{1-r} \qquad a = 64, \; r = 1/2$$

$$= \frac{64}{1-1/2} = \frac{64}{1/2} = 64 \times 2$$

$$= \underline{\underline{128}}$$

Example 8

Convert $0.\overline{3}$ to a rational number

Solution

$$0.\overline{3} = 0.333333\ldots$$

$$= 0.3 + 0.03 + 0.003 + 0.0003 + \ldots$$

$(0.3, 0.03, 0.003, 0.0003, \ldots)$ Geometric sequence

$$r = \frac{T_n}{T_{n-1}} = \frac{0.03}{0.3} = 0.1$$

$$0 < 0.1 < 1$$

$$a = 0.3, \quad r = 0.1 = \frac{1}{10}$$

$$S_\infty = \frac{a}{1-r} = \frac{0.3}{1-0.1}$$

$$= \frac{0.3}{0.9}$$

$$= \frac{1}{3}$$

Q1. $\sum_{i=1}^{100} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + 100^2$

$[1 + 4 + 9 + 16 + \ldots + 10000]$

$S_n = \frac{n}{2}(a+L) \longrightarrow$ Cannot be used on squared terms or sequences

MATHEMATICS FOR FIRST YEAR STUDENTS

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BOOK ONE