MAT 1110:

Chapter 4

2019/2020

Contents

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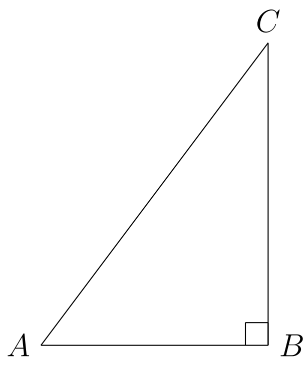
Trigonometry

1.1. Radian measure and the circle

1.1.1 Trigonometric Ratios

The trigonometric ratios of sine , cosine and the tangent give a relationship between the sides of a right angled triangle and their corresponding angles. Consider the right angled triangle ABC below:

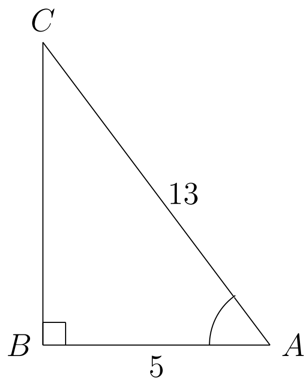


By Pythagoras theorem we have

$$AC^2 = AB^2 + BC^2. \tag{1.1}$$

This means that if in the triangle above we are given two of the sides, we can use equation (1.1) to find the value of the third side.

Example 1.1.1.1 *Consider the right angled triangle ABC below:*



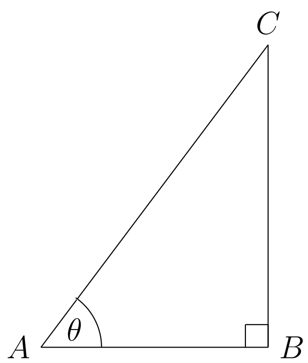
If $AB = 5cm$ and $AC = 13cm$, find the length of BC.

We use equation (1.1) to find the length of BC as follows:

$$\begin{aligned}
AC^2 &= AB^2 + BC^2 \\
13^2 &= 5^2 + BC^2 \\
169 &= 25 + BC^2 \\
169 - 25 &= BC^2 \\
144 &= BC^2 \\
BC &= \sqrt{144} \\
BC &= 12cm.
\end{aligned}$$

Corresponding to the sides of a right angled triangle are the trigonometric ratios, sine, cosine and tangent of the angles of the triangle which we now define.

Consider again the right angled triangle ABC below:



$$\begin{aligned} \sin\theta &= \frac{BC}{AC} \\ \cos\theta &= \frac{AB}{AC} \\ \tan\theta &= \frac{BC}{AB} \end{aligned}$$

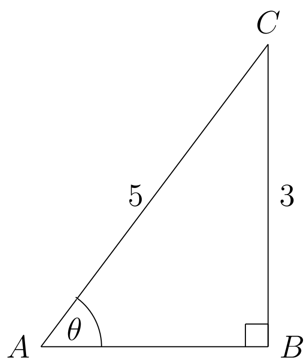
Where sin is short for sine, cos is short for cosine and tan is short for tangent.

Thus, the sine of the angle is the opposite side of the angle divided by the hypotunese, the cosine of the angle is the adjacent side of the angle divided by the hypotunese and the tangent of the angle is the opposite side divided by the adjacent side.

Note also that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Example 1.1.1.2 *Given that* $\sin\theta = \frac{3}{5}$*, find the value of* $\cos\theta$ *and of* $\tan\theta$*.*

We use a right angled triangle as follows; since the sine of the angle is given we take the opposite side of the angle to be 3cm and the hypotunese to be 5cm.



$BC = 3cm$ *and* $AC = 5cm$*.*

We need to find AB, the adjacent side first. By equation (1.1) we find that $AB = 4cm$*. Therefore,*

$\cos\theta = \frac{4}{5}$ *and* $\tan\theta = \frac{3}{4}$*.*

Other ratios which are derived from the three ratios above are their reciprocals cosecant, secant and cotangent respectively. Thus, if θ is an angle, then

$$\begin{aligned} cosec\theta &= \frac{1}{\sin\theta} \\ sec\theta &= \frac{1}{\cos\theta} \\ cot\theta &= \frac{1}{\tan\theta}. \end{aligned}$$

If for example $\cos\theta = \frac{12}{17}$ then

$$\begin{aligned} sec\theta &= \tfrac{1}{\cos\theta} \\ &= 1 \div \tfrac{12}{17} \\ &= \tfrac{17}{12}. \end{aligned}$$

A radian measure is the angle subtended at the centre of a circle by an arc with length equal to that of the radius. Thus, one radian is the measure of the central angle of a circle in which the sides of the angle intercept an arc equal in length to the radius of the circle.

There are 2π radians in one complete revolution. Thus, 2π radians is equivalent to $360°$, or π rad $= 180°$.

The following identities are useful in converting degrees to radians and radians to degrees.

$$1\text{rad} = \frac{180}{\pi}\text{degrees.} \qquad (1.2)$$

$$1\text{degree} = \frac{\pi}{180}\text{rad.} \qquad (1.3)$$

Example 1.1.1.3 *Express each of the following as radians*

a $90°$

(b) $-30°$

(c) $150°$

(d) $270°$

(e) $-240°$

Solution: Since we want radians from degrees we use equation (1.3)

(a) $90° = 90 \times \frac{\pi}{180}\text{rad} = \frac{\pi}{2}\text{rad}.$

(b) $-30° = -30 \times \frac{\pi}{180}\text{rad} = -\frac{\pi}{6}\text{rad}.$

(c) $150° = 150 \times \frac{\pi}{180}\text{rad} = \frac{5\pi}{6}\text{rad}.$

(d) $270° = 270 \times \frac{\pi}{180}\text{rad} = \frac{3\pi}{2}\text{rad}.$

(e) $-240° = -240 \times \frac{\pi}{180}\text{rad} = -\frac{4\pi}{3}\text{rad}$.

Example 1.1.1.4 *Express the following radian measures in degrees.*

(a) $\frac{5\pi}{6}$

(b) $\frac{3\pi}{4}$

(c) $\frac{-7\pi}{5}$

(d) 4π

(e) $\frac{-2\pi}{3}$

Solution: Since we want degrees from radians we equation (1.2).

(a) $\frac{5\pi}{6} = (\frac{5\pi}{6} \times \frac{180}{\pi})° = 150°$.

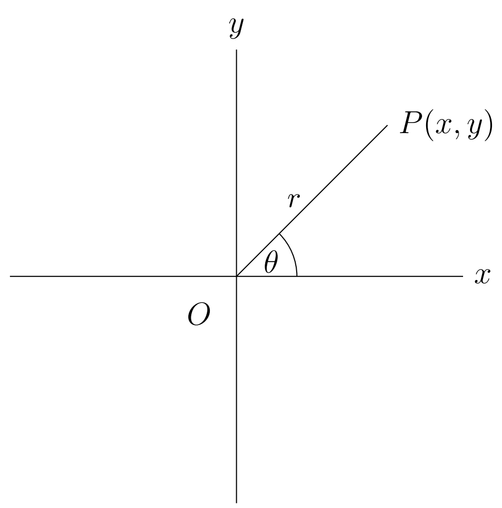
(b) $\frac{3\pi}{4} = \frac{3\pi}{4} \times \frac{180}{\pi} = 135°$.

(c) $-\frac{7\pi}{5} = -\frac{7\pi}{5} \times \frac{180}{\pi} = 252°$.

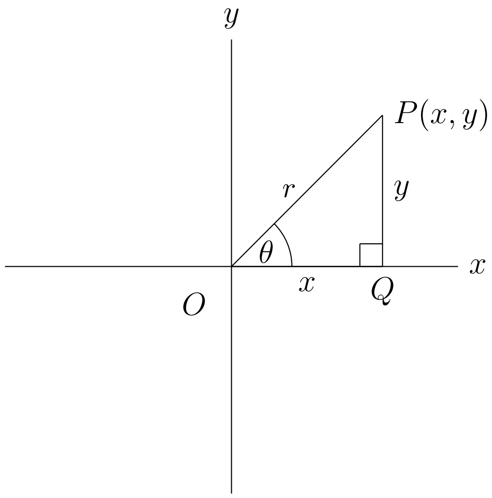
(d) $4\pi = 4\pi \times \frac{180}{pi} = 720°$.

(e) $-\frac{2\pi}{3} = -\frac{2\pi}{3} \times \frac{180}{\pi} = -120°$.

Denote by θ the angle which the ray OP makes with the positive x-axis as below:



To describe the ratios of the angle θ we complete the right angled triangle by dropping a perpendicular from P.

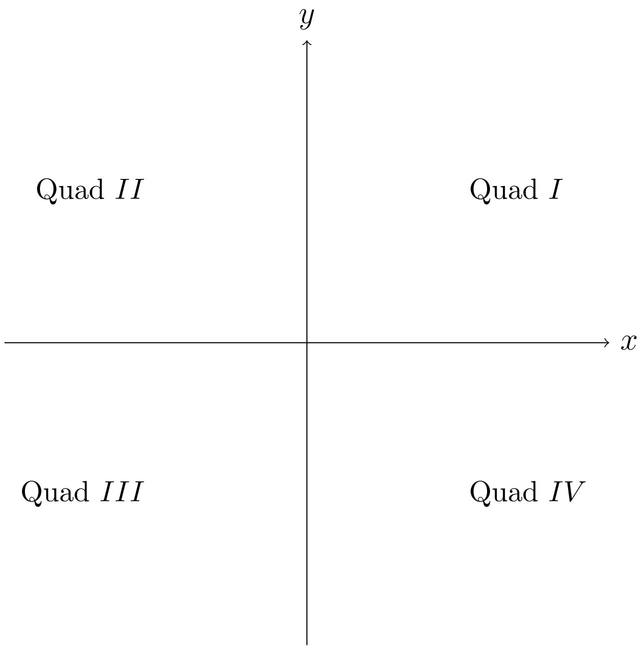


We then have

$$\begin{aligned} \sin\theta &= \tfrac{y}{r} \\ \cos\theta &= \tfrac{x}{r} \\ \tan\theta &= \tfrac{y}{x} \end{aligned}$$

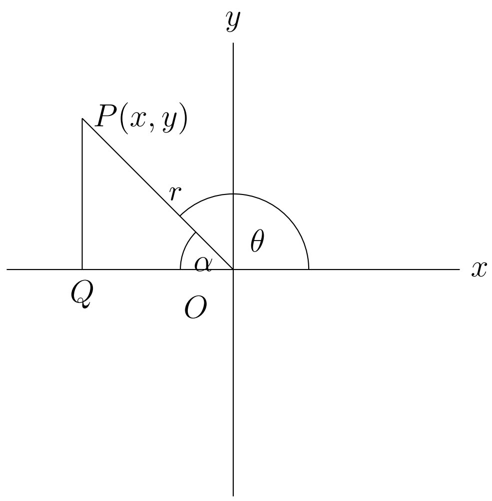
1.1.2 trigonometric ratios of angles greater than $90°$

We shall on many occasions be required to determine trigonometric ratios of angles which are greater than $90°$, we shall use what we shall term, an associated acute angle. We first divide the plane into four quadrants.



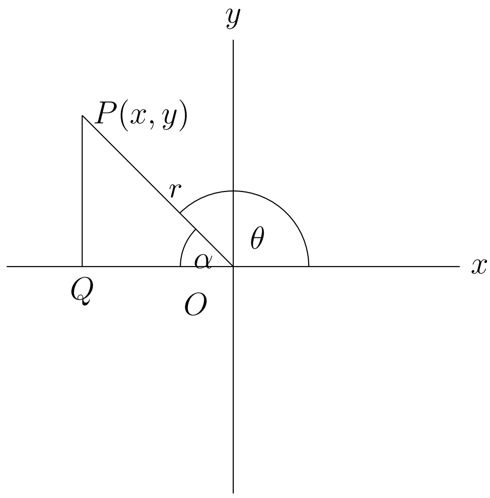
Note: We shall always take θ to be the required angle made by the positive x-axis and the ray OP of the angle.

Consider now the angle POQ formed when a vertical line is drawn from P to meet the x-axis at Q.



The angle POQ formed by the ray OP and the x-axis is always an acute angle regardless of the quadrant and the value of θ. We denote this acute angle by α, and call it the associated acute angle to θ. However, we must understand fully how to calculate this acute angle.

When the angle θ is in the second quadrant, i.e $90^\circ < \theta < 180^\circ$, to get the associated acute angle we subtract θ from $180°$ ($\alpha = 180° - \theta$).

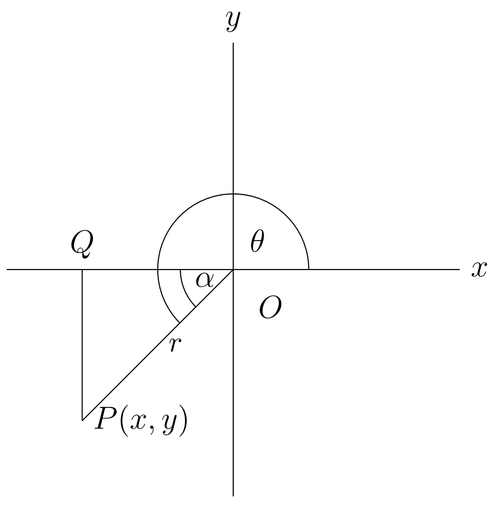


$$\alpha = 180 - \theta \text{ or}$$
$$\alpha = \pi - \theta$$

For example, if $\theta = 135°$, then $\alpha = 180° - 135° = 45°$. Or

if $\theta = \frac{2\pi}{3}$, then $\alpha = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

When the angle θ is in the third quadrant, i.e $180° < \theta < 270°$, to get the associated acute angle we subtract $180°$ from θ, ($\alpha = \theta - 180°$).

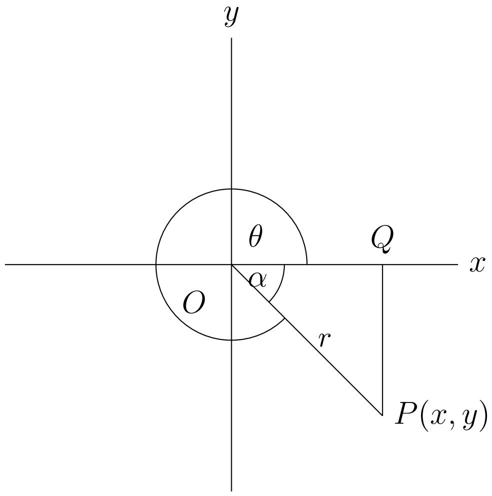


$$\alpha = \theta - 180° \text{ or}$$
$$\alpha = \theta - \pi$$

For example, if $\theta = 240°$ then $\alpha = 240° - 180° = 60°$, or

if $\theta = \frac{5\pi}{4}$ then $\alpha = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$.

When the angle θ is in the fourth quadrant, i.e $270° < \theta < 360°$, to get the associated acute angle we subtract θ from $360°$ ($\alpha = 360° - \theta$).



$$\begin{aligned} \alpha &= 360 - \theta, \text{ or} \\ \alpha &= 2\pi - \theta. \end{aligned}$$

Note that in the first quadrant, the angle θ is an acute angle, therefore theta coincides with alpha, that is, $\theta = \alpha$.

The signs of the trigonometric ratios depend on the signs of x and y, the coordinates of P. This definitely means that the sign of a ratio depends on the quadrant in which the angle falls.

In the first quadrant both x and y are positive, and since r is always positive, we have

$$\sin\theta = \frac{y}{r} > 0, \ \cos\theta = \frac{x}{r} > 0, \ \tan\theta = \frac{y}{x} > 0. \tag{1.4}$$

In the second quadrant x is negative and y is positive, so that we have

$$\sin\theta = \frac{y}{r} > 0, \ \cos\theta = \frac{x}{r} < 0, \ \tan\theta = \frac{y}{x} < 0. \tag{1.5}$$

In the third quadrant both x and y are negative so that

$$\sin\theta = \frac{y}{r} < 0, \ \cos\theta = \frac{x}{r} < 0, \ \tan\theta = \frac{y}{x} > 0. \tag{1.6}$$

Similarly in the fourth quadrant x is positive and y is negative so that

$$\sin\theta = \frac{y}{r} < 0, \ \cos\theta = \frac{x}{r} > 0, \ \tan\theta = \frac{y}{x} < 0. \tag{1.7}$$

Example 1.1.2.1 *(i) Determine the trigonometric ratios of the angle* $120°$*.*

Solution: The angle $120°$ is greater than $90°$ but less that $180°$. Thus, the angle is in the second quadrant. So the associated acute angle is $\alpha = 180° - 120° = 60°$. Now recall that sine is positive in the second quadrant, cosine is negative and tangent is negative. Therefore, we have

$$\begin{aligned} \sin 120° &= \sin 60° \\ \cos 120° &= -\cos 60° \\ \tan 120° &= -\tan 60° \end{aligned}$$

(ii) Determine the trigonometric ratios of the angle $\frac{4\pi}{3}$.

Solution: We see that the angle $\frac{4\pi}{3}$ is greater than π but less than $\frac{3\pi}{2} = 270°$. Therefore, the angle is in the third quadrant. The associated acute angle is $\alpha = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$. Now we recall that sine is negative in the third quadrant, cosine is also negative in the third quadrant but tangent is positive there. So, we have

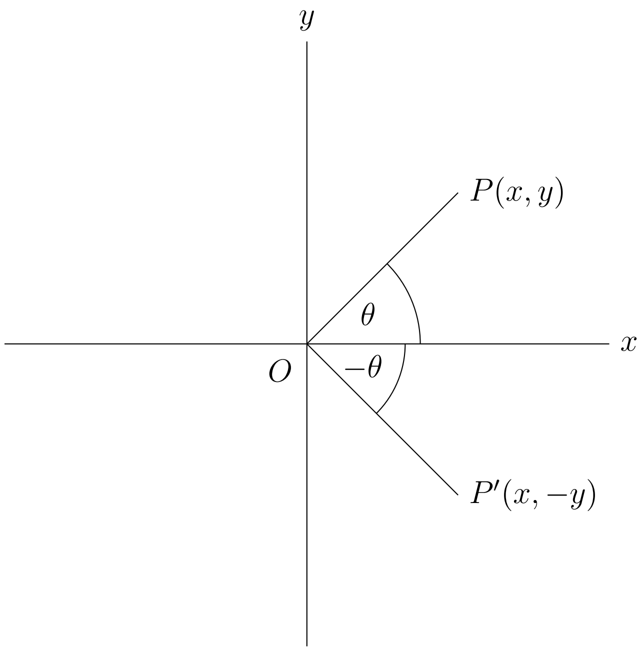
$$\begin{aligned} \sin\tfrac{4\pi}{3} &= -\sin\tfrac{\pi}{3} \\ \cos\tfrac{4\pi}{3} &= -\cos\tfrac{\pi}{3} \\ \tan\tfrac{4\pi}{3} &= \tan\tfrac{\pi}{3}. \end{aligned}$$

(iii) Determine the ratios of the angle $330°$.

Solution: The angle is in the fourth quadrant. The associated acute angle is $360° - 330° = 30°$. Only cosine is positive in the fourth quadrant. Thus

$$\begin{aligned} \sin 330° &= -\sin 30° \\ \cos 330° &= \cos 30° \\ \tan 330° &= -\tan 30°. \end{aligned}$$

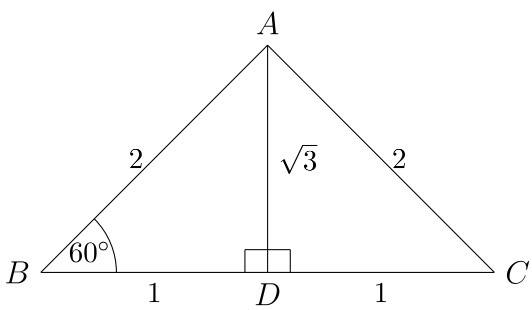
Angles measured in the anticlockwise direction are positive while angles measured in clockwise direction are negative.

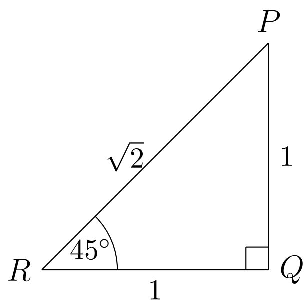


We have already seen that sine is negative in the fourth quadrant while cosine is positive there. We may conclude that for any angle θ;

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

You are required to memorize the trigonometric ratios for the commonly occuring (standard) angles $30°, 45°, 60°,$. We shall now construct their values from right angled triangles and using the two triangles below.





From the equilateral triangle ABC, first note that angle $BAD = 30°$. We then have the following:

$$\begin{aligned}
\sin 60° &= \tfrac{\sqrt{3}}{2} \\
\cos 60° &= \tfrac{1}{2} \\
\tan 60° &= \sqrt{3} \\
\sin 30° &= \tfrac{1}{2} \\
\cos 30° &= \tfrac{\sqrt{3}}{2} \\
\tan 30° &= \tfrac{1}{\sqrt{3}} = \tfrac{\sqrt{3}}{3}.
\end{aligned}$$

From the isosceles trianglePQR, we have

$$\begin{aligned}
\sin 45° &= \tfrac{1}{\sqrt{2}} = \tfrac{\sqrt{2}}{2} \\
\cos 45° &= \tfrac{1}{\sqrt{2}} = \tfrac{\sqrt{2}}{2} \\
\tan 45° &= 1.
\end{aligned}$$

Other angled whose values of their ratios are known are

$$
\begin{aligned}
\sin 90^\circ &= 1 \\
\cos 90^\circ &= 0 \\
\tan 90^\circ &= \infty(\text{undefined}) \\
\sin 180^\circ &= 0 \\
\cos 180^\circ &= -1 \\
\tan 180^\circ &= 0 \\
\sin 270^\circ &= -1 \\
\cos 270^\circ &= 0 \\
\tan 270^\circ &= -\infty(\text{undefined}) \\
\sin 0^\circ &= \sin 360^\circ = 0 \\
\cos 0^\circ &= \cos 360^\circ = 1 \\
\tan 0^\circ &= \tan 360^\circ = 0.
\end{aligned}
$$

IMPORTANT: Memorise these values since you will quite often be asked to find them without using calculators or tables!

The term 'cosine' is not accidental. Cosine of θ is the sine of $90^\circ - \theta$. If α and β are two angles, which sum up to 90°, then they are said to be complementary. β is the complement of α and vice versa. If α and β are complementary angles in a right-angled triangle, then we see that $\sin\alpha = \cos\beta = \cos(90^\circ - \alpha)$, $\cos\alpha = \sin\beta = \sin(90^\circ - \alpha)$, $\tan\alpha = \cot\beta = \cot(90^\circ - \alpha)$ and $\cot\alpha = \tan\beta = \tan(90^\circ - \alpha)$, Thus, the 'co-trig ratio' is the ratio of the complementary angle.

Example 1.1.2.2 *Write down the exact value of each of the following:*

(a) $\sin 120^\circ$,

(b) $\cos 120^\circ$,

(c) $\sin(-150^\circ)$,

(d) $\cos(-45^\circ)$,

(e) $\cot 60^\circ$,

(f) $\sec 30^\circ$,

(g) $\csc \frac{5\pi}{4}$,

(h) $\tan(-\frac{11\pi}{6})$.

Solution:

(a) $120°$ is an angle in the second quadrant. The associated acute angle is $\alpha = 180° - 120° = 60°$. Sine is positive in the second quadrant. Therefore,

$$\sin 120° = \sin 60° = \tfrac{\sqrt{3}}{2}.$$

(b) Cosine is negative in the second quadrant. Therefore,

$$\cos 120° = -\cos 60° = -\tfrac{1}{2}.$$

(c) We use the identity $\sin(-\theta) = -\sin\theta$ we have seen earlier. Thus, $\sin(-150°) = -\sin 150°$. So, we have

$$\sin(-150°) = -\sin 150° = -\sin 30° = -\tfrac{1}{2}.$$

(d) Recall the identity $\cos(-\theta) = \cos\theta$. Thus,

$$\cos(-45°) = \cos 45° = \tfrac{\sqrt{2}}{2}.$$

(e) We have seen that $\cot\theta = \frac{1}{\tan\theta}$. Therefore,

$$\cot 60° = \tfrac{1}{\tan 60°} = \tfrac{1}{\sqrt{3}} = \tfrac{\sqrt{3}}{3}.$$

(f)

$$\sec 30° = \tfrac{1}{\cos 30°} = 1 \div \tfrac{\sqrt{3}}{2} = \tfrac{2}{\sqrt{3}} = \tfrac{2\sqrt{3}}{3}.$$

(g) csc is short for cosecant. Now, $\frac{5\pi}{4}$ is an angle in the third quadrant. The associated acute angle is $\alpha = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$. Sine is negative in the third quadrant so that

$$\sin\tfrac{5\pi}{4} = -\sin\tfrac{\pi}{4} = -\tfrac{1}{\sqrt{2}},$$

since $\frac{\pi}{4} = 45°$. Therefore, we have

$$\csc\tfrac{5\pi}{4} = \tfrac{1}{\sin\frac{5\pi}{4}} = -\tfrac{1}{\sin\frac{\pi}{4}} = -1 \div \tfrac{1}{\sqrt{2}} = -\sqrt{2}.$$

(h) We have $\tan(-\theta) = -\tan\theta$ so that $\tan(-\frac{11\pi}{6}) = -\tan\frac{11\pi}{6}$. Now, $\frac{11\pi}{6}$ is an angle in the fourth quadrant. So, the associated acute angle is $\alpha = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. Tangent is negative in the fourth quadrant. Therefore

$$\tan(-\tfrac{11\pi}{6}) = -\tan\tfrac{11\pi}{6} = -(-\tan\tfrac{\pi}{6}) = \tan\tfrac{\pi}{6} = \tfrac{1}{\sqrt{3}} = \tfrac{\sqrt{3}}{3}.$$

1.1.3 Equations of Trigonometric ratios

Equations of trigonometric ratios are equations that involve finding an angle whose ratio is a given value. For example $\tan x = 1$. This is an equation in which we are required to find the angle x such that its tangent is 1. Care must be taken when solving such equations since most of these equations have more that one solution. You are therefore required to find all the angle whose ratio is the given value.

Consider for example finding x in the equation

$$\sin x = \frac{1}{2} \tag{1.8}$$

First we note from our earlier construction that the angle whose sine is $\frac{1}{2}$ is $30°$. As $\frac{1}{2}$ is a positive number, we know that sine is also positive in the second quadrant. So, there must be an angle in the second quadrant whose associated acute angle is $\alpha = 30°$. To find this angle, we know that in the second quadrant $\alpha = 180° - \theta$. Thus

$$30° = 180° - \theta$$

Solving this equation for θ we get $\theta = 150$. Indeed we now know that $\sin 150° = \sin 30° = \frac{1}{2}$.

Therefore, the solution of the equation

$$\sin x = \tfrac{1}{2}$$

is

$$x = 30°, 150°.$$

Example 1.1.3.1 *Given that θ is an angle in the interval $0 < \theta \leq 360°$, find all values of θ in each of the equations below.*

(a) $2\cos\theta + 1 = 0$

(b) $\cos\theta\tan\theta = \cos\theta$

(c) $2\sin\theta\tan\theta = -\sqrt{3}\tan\theta$

Solution:

(a) We have

$$\begin{aligned} 2\cos\theta + 1 &= 0 \\ 2\cos\theta &= -1 \\ \cos\theta &= -\tfrac{1}{2} \end{aligned}$$

Since cosine of the angle θ is negative, we are looking for the angles in the second and third quadrants where cosine is negative. However, we have to find the associated acute angle first. To find α, the associated acute angle we must first ignore the negative sign. But the acute angle such $\cos\alpha = \frac{1}{2}$ is $60°$. Therefore, the associated acute angle $\alpha = 60°$.

In the second quadrant, $\alpha = 180° - \theta$. Thus, we have $60° = 180° - \theta$. This gives $\theta = 120°$.

In the third quadrant, $\alpha = \theta - 180°$. Thus, we have $60° = \theta - 180°$. This gives $\theta = 240°$.

Therefore, the solution to the equation

$$2\cos\theta + 1 = 0$$

is

$$\theta = 120°, 240°.$$

(b) We solve this equation as follows

$$\begin{aligned} \cos\theta\tan\theta &= \cos\theta \\ \cos\theta\tan\theta - \cos\theta &= 0 \\ \cos\theta(\tan\theta - 1) &= 0 \end{aligned}$$

At this point, never be tempted to divide both sides by $\cos\theta$. This will result in losing some solutions. Instead, proceed as follows:

$$\begin{aligned}\cos\theta(\tan\theta-1) &= 0\\ \cos\theta &= 0 \text{ or}\\ \tan\theta &= 1.\end{aligned}$$

The solution corresponding to $\cos\theta = 0$ are $\theta = 90°, 270°$.

As for $\tan\theta = 1$, we know that tangent is positive in the first quadrant and in the third quadrant. In the first quadrant, $\theta = 45°$ is the solution. To find the angle in the third quadrant, we take $\alpha = 45°$ as our associated acute angle. Then $\theta = 180° + 45° = 225°$ is the required angle. Thus, the solution of the equation

$$\cos\theta\tan\theta = \cos\theta$$

is

$$\theta = 45°, 90°, 225°, 270°.$$

(c) In a similar way we have

$$\begin{aligned}2\sin\theta\tan\theta &= -\sqrt{3}\tan\theta\\ 2\sin\theta\tan\theta+\sqrt{3}\tan\theta &= 0\\ \tan\theta(2\sin\theta+\sqrt{3}) &= 0\\ \tan\theta &= 0 \text{ or}\\ \sin\theta &= -\tfrac{\sqrt{3}}{2}\end{aligned}$$

For $\tan\theta = 0$ we get $\theta = 180°, 360°$. Note we have left out $0°$ because it is not included in the interval.

For $\sin\theta = -\frac{\sqrt{3}}{2}$ we first note that sine is negative in the third quadrant and in the fourth quadrant. We get the associated acute angle by solving the equation

$$\sin\alpha = \tfrac{\sqrt{3}}{2}$$

This gives the associated acute angle to be $\alpha = 60°$. Then we get the required angles as follows:

In the third quadrant $\theta = 180° + 60° = 240°$.

In the fourth quadrant $\theta = 360° - 60° = 300°$.

Therefore, the solution of the equation

$$2\sin\theta\tan\theta = -\sqrt{3}\tan\theta$$

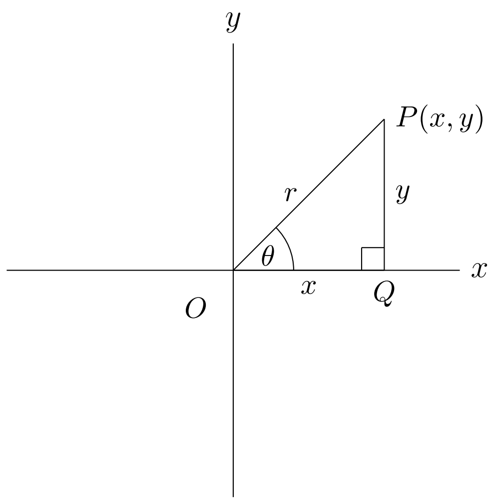
is

$$\theta = 180°, 240°, 300°, 360°.$$

1.2. Trigonometric identities

1.2.1 Basic Identities

Consider again the diagram:



By Pythagoras theorem we have:

$$x^2 + y^2 = r^2 \tag{1.9}$$

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r} \tag{1.10}$$

Now divide both sides of equation (1.9) by r^2. This gives

$$\left(\tfrac{x}{r}\right)^2 + \left(\tfrac{y}{r}\right)^2 = 1$$

Substitute equations in (1.10) into this equation. We get the identity

$$\cos^2\theta + \sin^2\theta = 1 \tag{1.11}$$

The identity (1.11) is true for any value of θ. It is a basic identity and easy to remember. From this equation we also get

$$\begin{aligned}\cos^2\theta &= 1-\sin^2\theta \\ \sin^2\theta &= 1-\cos^2\theta.\end{aligned}$$

Divide equation (1.11) both sides by $\cos^2\theta$ to get

$$1+\left(\tfrac{\sin\theta}{\cos\theta}\right)^2 = \tfrac{1}{\cos^2\theta}$$

or

$$1+\tan^2\theta = \sec^2\theta \tag{1.12}$$

Divide both sides of equation (1.11) to get

$$\left(\tfrac{\cos\theta}{\sin\theta}\right)^2 + 1 = \tfrac{1}{\cos^2\theta}$$

or

$$\cot^2\theta + 1 = \csc^2\theta \tag{1.13}$$

We shall now prove some basic trigonometric identities using (1.11), (1.12) and (1.13). To prove an identity, you start with one side of the equation, work through until you reduce it to the expression on the other side.

Example 1.2.1.1 *Prove that* $(1-\cos A)(1+\sec A) = \sin A\tan A$.

Proof. We shall start with the left hand side (LHS).

$$\begin{aligned}
(1-\cos A)(1+\sec A) &= 1+\sec A-\cos A-\cos A\sec A \\
&= 1+\sec A-\cos A-\cos A(\tfrac{1}{\cos A}) \\
&= 1+\sec A-\cos A-1 \\
&= 1-1+\sec A-\cos A \\
&= \sec A-\cos A \\
&= \tfrac{1}{\cos A}-\cos A \\
&= \tfrac{1-\cos^2 A}{\cos A} \\
&= \tfrac{\sin^2 A}{\cos A} \\
&= \sin A(\tfrac{\sin A}{\cos A}) \\
&= \sin A\tan A, \quad \text{the} \quad \text{right} \quad \text{hand} \quad \text{side.}
\end{aligned}$$

Example 1.2.1.2 *Prove the following identity*

$\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=\frac{2}{\sin x}$

Proof. Starting with the left hand side (LHS) we have

$$\begin{aligned}
\tfrac{\sin x}{1+\cos x}+\tfrac{1+\cos x}{\sin x} &= \tfrac{\sin^2 x+(1+\cos x)(1+\cos x)}{\sin x(1+\cos x)} \\
&= \tfrac{\sin^2 x+1+2\cos x+\cos^2 x}{\sin x(1+\cos x)} \\
&= \tfrac{\sin^2 x+\cos^2 x+1+2\cos x}{\sin x(1+\cos x)} \\
&= \tfrac{2+2\cos x}{\sin x(1+\cos x)} \\
&= \tfrac{2(1+\cos x)}{\sin x(1+\cos x)} \\
&= \tfrac{2}{\sin x}, \quad \text{as required.}
\end{aligned}$$

1.2.2 Compound angle formulae

We now give the compound angle formulas.

Let A and B be angles. We have

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \tag{1.14}$$

For example

$$\begin{aligned}\sin 90^\circ &= \sin(60^\circ + 30^\circ)\\ &= \sin 60^\circ \cos 30^\circ + \sin 30^\circ \cos 60^\circ\\ &= \tfrac{\sqrt{3}}{2} \times \tfrac{\sqrt{3}}{2} + \tfrac{1}{2} \times \tfrac{1}{2}\\ &= \tfrac{3}{4} + \tfrac{1}{4}\\ &= \tfrac{4}{4}\\ &= 1.\end{aligned}$$

If we take $B = A$ then we have

$$\sin 2A = \sin(A + A) = \sin A \cos A + \sin A \cos A = 2 \sin A \cos A \tag{1.15}$$

If we put $-B$ in equation (1.14) and recall that $\sin -B = -\sin(B)$ and $\cos(-B) = \cos B$, then we have

$$\sin(A - B) = \sin A \cos(-B) + \sin(-B) \cos A = \sin A \cos B - \sin B \cos A \tag{1.16}$$

As for cosine we have

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \tag{1.17}$$

If $B = A$ in equation (1.17), then

$$\cos 2A = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A \tag{1.18}$$

We also have on replacing B with $-B$ in equation (1.17)

$$\cos(A - B) = \cos A \cos(-B) - \sin A \sin(-B) = \cos A \cos B + \sin A \sin B \tag{1.19}$$

Using equations (1.14), (1.16), (1.17) and (1.18) it can be easily shown that

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{1.20}$$

giving

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \tag{1.21}$$

We also have

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \tag{1.22}$$

Example 1.2.2.1 *Evaluate each of the following without using a calculator or tables:*

(i) $\cos 75^\circ$

(ii) $\sin 105^\circ$

(iii) $\tan(15^\circ)$

Solution:

(i)

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \tfrac{\sqrt{2}}{2} \times \tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{2}}{2} \times \tfrac{1}{2}\\ &= \tfrac{\sqrt{6}}{4} - \tfrac{\sqrt{2}}{4}\\ &= \tfrac{1}{4}(\sqrt{6} - \sqrt{2})\end{aligned}$$

(ii)

$$\begin{array}{rcl}\sin 105^\circ & = & \sin(60^\circ + 45^\circ)\\ & = & \sin 60^\circ \cos 45^\circ + \sin 45^\circ \cos 60^\circ\\ \& = & \tfrac{\sqrt{3}}{2} \times \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\tfrac{1}{2} & \\ & = & \tfrac{1}{4}(\sqrt{6} + \sqrt{2}).\end{array}$$

(iii)

$$\begin{aligned}\tan 15^\circ &= \tan(60^\circ - 45^\circ)\\ &= \tfrac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ}\\ &= \tfrac{\sqrt{3}+1}{1-\sqrt{3}}.\end{aligned}$$

Example 1.2.2.2 *Prove the following identities.*

(i) $\frac{1-\cos 2A+\sin 2A}{1+\cos 2A+\sin 2A} = \tan A$

(ii) $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

(iii) $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$

Proof.

(i) Starting with the left hand side (LHS) we have

$$\begin{aligned}
\frac{1-\cos 2A+\sin 2A}{1+\cos 2A+\sin 2A} &= \frac{1-(\cos^2 A-\sin^2 A)+2\sin A\cos A}{1+\cos^2 A-\sin^2 A+2\sin A\cos A} \\
&= \frac{1-\cos^2 A+\sin^2 A+2\sin A\cos A}{1-\sin^2 A+\cos^2 A+2\sin A\cos A} \\
&= \frac{2\sin^2 A+2\sin A\cos A}{2\cos^2 A+2\sin A\cos A} \\
&= \frac{\sin A(\sin A+\cos A)}{\cos A(\cos A+\sin A)} \\
&= \frac{\sin A}{\cos A} \\
&= \tan A
\end{aligned}$$

(ii) We start from the left hand side.

$$\begin{aligned}
\sin 2x &= \frac{2\sin x\cos x}{1} \\
&= \frac{2\sin x\cos x}{\cos^2 x+\sin^2 x} \text{ divide both the numerator and denominator by } \cos^2 x \\
&= \frac{\frac{2\sin x\cos x}{\cos^2 x}}{\frac{\cos^2 x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}} \\
&= \frac{2\tan x}{1+\tan^2 x}.
\end{aligned}$$

(iii) Just follow the procedure in (ii).