MAT 1110:

Chapter 2

2019/2020

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Sets of numbers

1.1. Integers

We are already familiar with the set of whole numbers $\{0, 1, 2, 3, 4, 5, 6, 7, \ldots\}$ and the set of natural numbers $\{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$ which are written down using the numerals. We denote the set of natural numbers and whole numbers respectively, as $\mathbb{N}$ and $\mathbb{W}$,. Thus

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$$
$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$$

The whole numbers can be represented by equally spaced points on a horizontal line where 0 is taken to be the first number.

Whole numbers are ordered. That is they progress from small to large. On a line, numbers to the left of a given number are less than ($<$) the given number, and numbers to the right are greater ($>$) than the given number. For example, $8 > 5$ while $3 < 5$.

If the straight line displaying the natural numbers is extended to the left, we get equally spaced points to the left of zero.

These points represent negative numbers which are written as the natural numbers preceded by a minus sign, for example -4. The positive and negative natural numbers including 0 make up the set of *integers*, denoted by $\mathbb{Z}$. Here, the concept of order still plays a significant role. For example, $-5 < -4$ and $-2 > -4$, this is because -5 appears to the left of -4 while -2 appears to the right of -4. Thus

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$$

It is important so far to note that $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z}$.

The set of positive integers is denoted by $\mathbb{Z}^+$ and note that

$$\mathbb{Z}^+ = \mathbb{N} = \{1, 2, 3, 4, 5, 6, \cdots\}.$$

That is, the set of positive integers is just the set of natural numbers.

Note that the number zero is neither negative nor positive.

Definition 1.2.1 *Rational number are numbers that can be expressed in the form of* $\frac{a}{b}$, *where* a *and* b *are integers and* $b \neq 0$.

We denote by $\mathbb{Q}$ the set of rational numbers.

Examples of rational numbers are $\frac{1}{3}, \frac{4}{9}, \frac{-4}{9}, \frac{200}{13}, 57 = \frac{57}{1}, -\frac{78}{10096}$. Rational numbers are numbers with decimal expansions that are either:

(a) Terminating (ending in an infinite string of zeroes), for example,

$$\frac{3}{4} = 0.75000\cdots = 0.75$$

or

(b) Eventually repeating (ending with a block of digits that repeats over and over), for example

$$\frac{23}{11} = 2.090909\cdots = 2.\overline{09} \quad \text{(the bar indicates the block of repeating digit}$$

Thus,

$$\frac{1}{7} = 0 \cdot 142857142857142857 1\cdots$$

can be written as

$$\frac{1}{7} = 0 \cdot \overline{142857}$$

and

$$\frac{35}{6} = 5 \cdot 83333 \cdots = 5 \cdot 8\overline{3}$$

These properties of rational numbers makes it easier for us to reconstruct a rational number given in a decimal form to its standard form of $\frac{a}{b}$ where a and b are integers with $b \neq 0$.

Example 1.2.0.1 *Show that* $0 \cdot 8\overline{3}$ *is a rational number.*

Solution: We need to show that $0 \cdot 8\overline{3}$ can be expressed in the form $\frac{a}{b}$ where a and b are integers. Recall that

$$0 \cdot 8\overline{3} = 0 \cdot 83333333333 \cdots$$

Now, let $x = 0.8\overline{3} = 0.83333333333 \cdots$. Then

$$10x = 8.33333333 \cdots \tag{1.1}$$

and

$$100x = 83.3333333 \cdots \tag{1.2}$$

Note now that the digits after the decimal points in equation (1.1) and in equation (1.2) are same. If we subtract the two equations we get a string of zeros after the decimal point. So, subtracting equation (1.1) from equation (1.2) gives

$$100x - 10x = 75.000000\cdots$$

That is,

$$90x = 75$$

Dividing both sides by 90 we get

$$x = \frac{75}{90} = \frac{5}{6}$$

But $x = 0.8\overline{3}$. Thus

$$0.8\overline{3} = \frac{5}{6}$$

Since we have expressed the $0.8\overline{3}$ in standard form of rational numbers, we conclude that it is a rational number.

Example 1.2.0.2 *Express* $3.\overline{45}$ *in the form* $\frac{m}{n}$ *where* m *and* n *are integers and* $n \neq 0$.

Solution: Let $x = 3.\overline{45}$, then

$$x = 3.\overline{45} = 3.4545454545\cdots . \tag{1.3}$$

Since the repeating block comes immediately after the decimal point, our next multiplication should move the whole one block of repeating decimals to the left of the decimal point. Thus, multiplying equation (1.3) by 100 we get

$$100x = 345.\overline{45} = 345.4545454545\cdots \tag{1.4}$$

Subtract equation (1.3) from equation (1.4) to get

$$100x - x = 342.0000000\cdots$$

Thus

$$99x = 342$$

Divide by 99 both sides to get

$$x = \frac{342}{99} = \frac{38}{11}$$

Thus

$$3.\overline{45} = \frac{38}{11}$$

The reconstruction is even easier if the decimal expansion is terminating.

Example 1.2.0.3 *Show that* 2.75 *is a rational number.*

Solution: Let $x = 2.75$. Note here that since the decimal expansion is terminating, there is no need for subtraction. An appropriate multiplication by a power of 10 will leave only the zeros after the decimal point. Since there are 2 decimal places, we multiply x by 100. This gives

$$100x = 275.$$

Divide both sides by 100

$$x = \frac{275}{100} = \frac{11}{4}.$$

Note the following inclusions

$$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q}$$

Exercise

Express each of the following in the form $\dfrac{a}{b}$ where a and b are integers and $b \neq 0$.

(i) 1.25

(ii) $0.\overline{3}$

(iii) $0.3\overline{59}$.

(iv) $-2.\overline{13}$

1.3. Irrational numbers

An irrational number is a number which cannot be expressed in the form $\frac{a}{b}$ for some integers a and b. The decimal representation of an irrational number is non terminating and non repeating. The set of irrational numbers will be denoted by **Irr**.

Some examples of irrational numbers are

$$\pi, \sqrt{2}, \sqrt[3]{5}, e \text{ and } \log_{10} 3.$$

where

$$\begin{aligned} \pi &\approx 3.141592654\cdots \\ e &\approx 2.718281828\cdots \end{aligned}$$

We shall now prove that $\sqrt{2}$ is not a rational number by showing that it cannot be expressed in the form $\frac{a}{b}$ for some integers a and $b \neq 0$. But before we do that, we shall first show that if $p \in \mathbb{Z}$ is an integer such that p^2 is divisible by 2, then p itself is divisible by 2.

Proof. Suppose $p \in \mathbb{Z}$ is an integer such that p^2 is divisible by 2. Now, every integer is either divisible by 2 or leaves a remainder of 1 when divided by 2. Therefore, every integer can be written in one of the following:

$$p = \begin{cases} 2n \\ 2n+1 \end{cases}$$

where n is an integer. Then

$$p^2 = \begin{cases} 4n^2 \\ 4n^2 + 4n + 1 \end{cases}$$

where $n \in \mathbb{Z}$.

But since p^2 is divisible by 2, it must be of the form

$$p^2 = 4n^2.$$

This then gives p to be of the form

$$p = 2n, \ n \in \mathbb{Z}.$$

so that p is divisible by 2.

We shall now prove that $\sqrt{2}$ is not a rational number. We shall prove this by contradiction.

To prove a mathematical statement by contradiction, you first assume something about the statement (usually what it is not). Then work through the statement according to the assumption. If you come up with a conclusion which is not consistent with the assumption, that is, the conclusion which contradicts your earlier assumption, then your earlier assumption about the statement must be false.

Example 1.3.0.4 *Prove that* $\sqrt{2}$ *is an irrational number.*

Proof. Assume that $\sqrt{2}$ is a rational number. Then there are integers a and $b \neq 0$ with no common factor and such that $a > 0$ and $b > 0$ such that

$$\sqrt{2} = \frac{a}{b}.$$

Squaring both sides yields

$$2 = \frac{a^2}{b^2},$$

so that $a^2 = 2b^2$. But now a^2 has a factor 2 which means that a^2 is even. Then a itself is also even by the above proof.Thus a has a factor 2. Thus, we can write a in the form

$$a = 2p, \ p \in \mathbb{Z}.$$

From $a^2 = 2b^2$ and

$$a = 2p,$$

we get

$$2b^2 = 4p^2,$$

so that

$$b^2 = 2p^2.$$

Thus b^2 is even. Again we get that b is also even. This means that both a and b have a common factor 2. But now this contradicts our earlier assumption that a and b have no common factor. This contradiction implies that our earlier statement about $\sqrt{2}$ is false. That is, we cannot find integers a and $b \neq 0$ so that $\sqrt{2} = \frac{a}{b}$ in its lowest terms. Hence,

$$\sqrt{2} \neq \frac{a}{b}.$$

Therefore it is irrational.

We remark that the set of rational numbers and the set of irrational numbers are two disjoint sets. That is, there is no number which is both rational and irrational at the same time. Thus

$$\mathbb{Q} \cap \mathbf{Irr} = \emptyset.$$

Example 1.3.0.5 *Given that x and y are integers, and that $\sqrt{3}$ is an irrational number, prove that $x+y\sqrt{3}$ is an irrational number.*

Proof. It is easier to prove such statement by contradiction. So, we assume that $x + y\sqrt{3}$ is rational and aim for the contradiction. Then there are integers a and $b \neq 0$ such that

$$x + y\sqrt{3} = \frac{a}{b}.$$

Solving for $\sqrt{3}$ gives the following

$$y\sqrt{3} = \frac{a}{b} - x$$

so that

$$y\sqrt{3} = \frac{a - bx}{b}$$

and

$$\sqrt{3} = \frac{a - bx}{by}. \tag{1.5}$$

Now the left hand side of equation (1.5) is irrational while the right hand side is a rational number since both the numerator and the denominator are integers. But there is no number which is both rational and irrational. Thus equation (1.5) is a contradiction. Therefore, there are no integers a and b such that $x + y\sqrt{3} = \frac{a}{b}$. This contradiction implies that $x + y\sqrt{3}$ is not rational and hence it is an irrational number.

Note by taking $x = 5$ and $y = -2$ in the above example we can show that $5 - 2\sqrt{3}$ is an irrational number. Just follow the above steps for the proof.

1.4. Real numbers

The union of the set of rational numbers with the set of irrational numbers is the set of real numbers which we denote by the symbol $\mathbb{R}$. That is,

$$\mathbb{Q} \cup \mathbf{Irr} = \mathbb{R}.$$

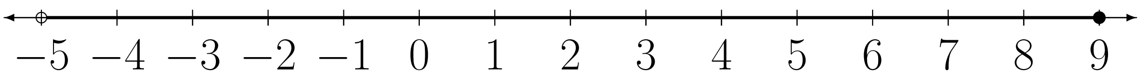
The set of real numbers is uncountable. As a result, we can represent the set of real numbers in one of the following ways:

(i) Interval notation. We use brackets $(,)$, $[,)$ or $[,]$ to give the subset of the set of real numbers. For example, $(-5, 9]$. This is the set of all numbers rationals or irrationals between -5 and 9.

Open brackets indicate that the boundary number is not part of the set while the closed bracket indicates that the boundary number is a member of the set. Note that we do not use braces on a subset of the set of real numbers unless it is a subset of integers.

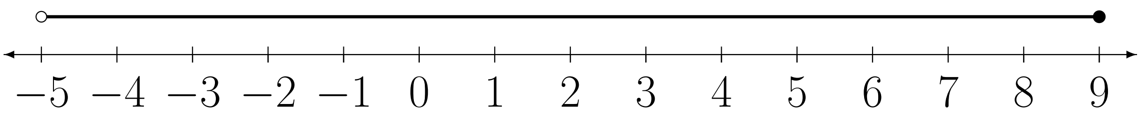
(ii) Set builder notation. We can use set builder notation to describe a subset of the set of real numbers. For example, $\{x \in \mathbb{R} : -5 < x \leq 9\}$. This is the same set as one given above. Sometimes we omit $\mathbb{R}$ and just write $\{x : -5 < x \leq 9\}$ if it is clear that the universal set is the set of real numbers.

(iii) We can also use a number line to display the subset of the set of real numbers. For example,



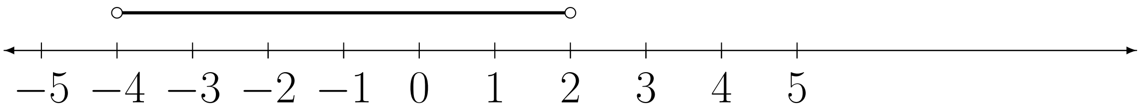
Note that the small circle at -5 means that the boundary number -5 is not included in the set, and the small black circle at 9 means that the boundary number 9 is included in the set.

In most cases, instead of drawing the required subset of $\mathbb{R}$ on the number line itself we draw it slightly above the number line for clarity. For example, the set $(-5, 9]$ would be shown as follows:

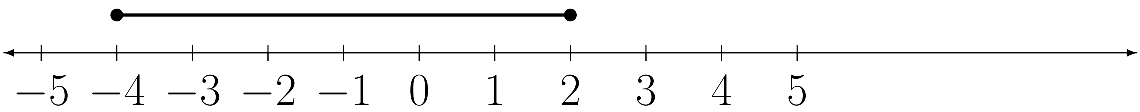


In general, the interval notations are:

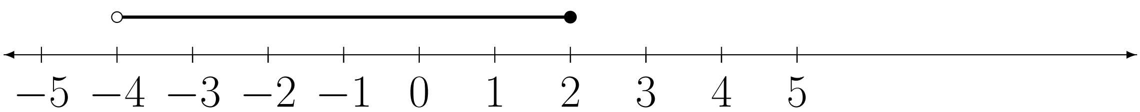
(i) (a, b) represents all the real numbers between a and b, not including a and b. This is an open interval. In set builder notation, we write $\{x : a < x < b\}$. For example, the graph of $(-4, 2)$ is



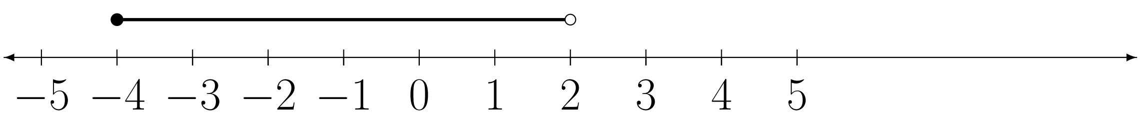
(ii) $[a, b]$ represents all the real numbers between a and b, including a and b. This is a closed interval. In set builder notation, we write $\{x : a \leq x \leq b\}$. For example, the graph of $[-4, 2]$ is



(iii) $(a, b]$ represents all the real numbers between a and b, not including a but including b. This is a half open interval. In set builder notation, we write $\{x : a < x \leq b\}$. For example, the graph of $(-4, 2]$ is



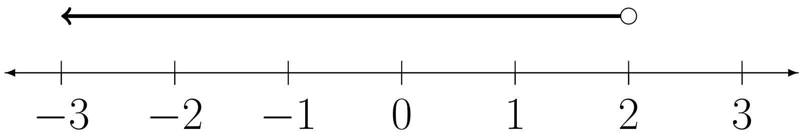
(iv) $[a, b)$ represents all the real numbers between a and b, including a but not b. This is a half open interval. In set builder notation, we write $\{x : a \leq x < b\}$. For example, the graph of $[-4, 2)$ is



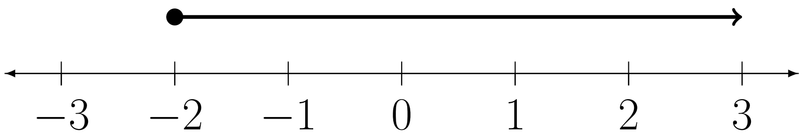
Subsets of the real numbers whose graphs extend forever in the one or both directions can be represented by interval notation using the infinity symbol ∞ or the negative infinity symbol $-\infty$.

Note that infinity (∞) is not a real number. Therefore, wherever this symbol appears you must put an open bracket because it does not belong to the set. The following are the examples of sets which involve infinity (∞) symbol.

(a) $(-\infty, a)$. In set builder notation is given by $\{x : x < a\}$. For example, the set $(-\infty, 2)$ can be displayed on a number line as follows:



(b) $[b, \infty)$. In set builder notation it is given by $\{x : b \leq x\}$. For example, the set $[-2, \infty)$ can be shown on a number line as follows:



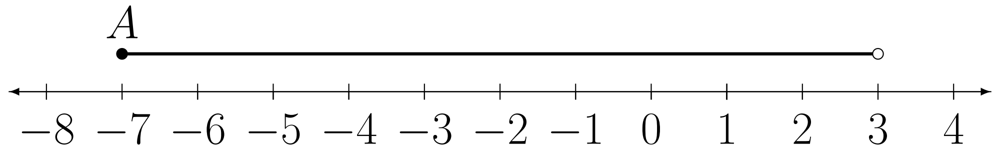
Example 1.4.0.6 *Let* $A = \{x : -7 \leq x < 3\}$ *and* $B = [-1, \infty)$. *Find the following sets and display them on a number line.*

(i) $A \cap B$

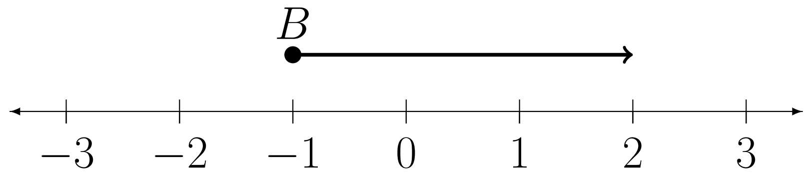
(ii) A'

Solution:

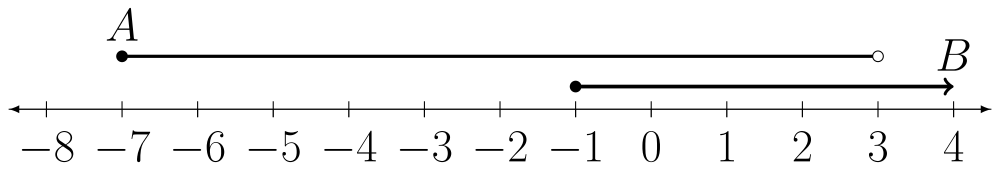
(i) We first display the two sets on a number line so that we can determine the solutions. We have set A given by the number line



While that of B is given by



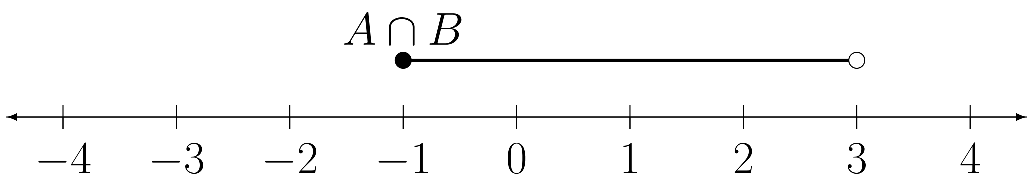
Combining the two diagrams we get



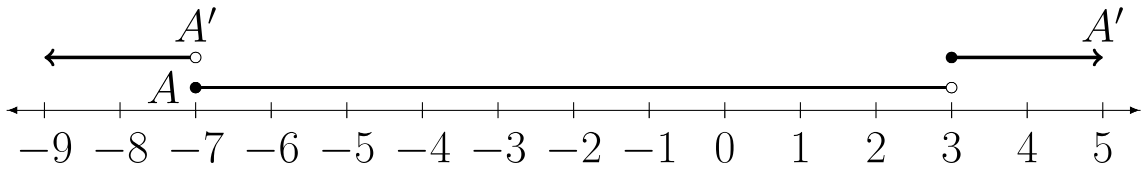
Clearly the two sets have their intersection from -1 to +3 because that is where both graphs appear. While -1 is in the intersection since it is in both sets, +3 is not in the intersection because it is not an element of A. Therefore,

$$A \cap B = [-1, 3).$$

The display on the number line is



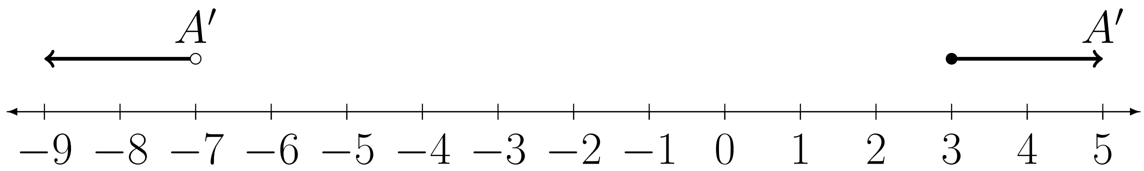
(ii) It is again necessary to draw the number line because we shall see clearly the required solution set.



From the diagram, the set A' is represented by the union of the sets represented by the two arrows. Thus

$$A' = (-\infty, -7) \cup [3, \infty).$$

The number line is given by



Definition 1.4.1 *Absolute value of a number*

The absolute value of the real number a, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \textit{if } a \geq 0 \\ -a & \textit{if } a < 0 \end{cases}.$$

For example $|7| = 7$ and $|-7| = -(-7) = 7$

Thus, the absolute value of a real number is always positive.

Example 1.4.0.7 *Calculate the value of* $|a - b|$ *in each of the following cases.*

(i) $a = 7, b = 4$

(ii) $a = -6, b = 8$

(iii) $a = 2, b = 11$

Solution:

(i) $|7 - 4| = |3| = 3$

(ii) $|-6 - 8| = |-14| = -(-14) = 14$

(iii) $|2 - 11| = |-9| = -(-9) = 9$.

We shall now use the properties of absolute value of a real number to solve equations. As you will see, using the definition of absolute value can sometimes simplify calculations of equations that may otherwise result in quadratic equations. This is because such calculations are reduced to dealing with linear equations instead.

Example 1.4.0.8 *Solve the equation* $|2x - 3| = 5$.

Solution: Using the definition of absolute value we see that there are two possibilities we need to consider. Either $2x - 3$ is positive or it is negative.

Case 1. $2x - 3$ is positive.

If $2x - 3 > 0$ then $|2x - 3| = 2x - 3$. In this case we then have $2x - 3 = 5$. Solving this equation gives $x = 4$.

Case 2. $2x - 3$ is negative.

If $2x-3 < 0$ then $|2x-3| = -(2x-3)$. Then we have $-(2x-3) = 5$ or $2x - 3 = -5$. Solving this equation yields $x = -1$.

Combining the two solutions we get $x = 4$ or $x = -1$.

Let k be a positive real number. Consider the inequality

$$|x| \leq k$$

If x is positive, then $|x| \leq k$ implies that $x \leq k$. On the other hand, if x is negative, then $|x| \leq k$ implies that $-x \leq k$, or $x \geq -k$. Combining the two inequalities we have

$$|x| \leq k \text{ implies that } -k \leq x \leq k.$$

Example 1.4.0.9 *Solve the inequality* $|2x - 3| \leq 5$.

Solution: From the above discussion we have

$$|2x-3| \leq 5 \text{ implies that } \quad -5 \leq 2x-3 \leq 5.$$

Solving each of the two inequalities separately we get

$$-1 \leq x \leq 4.$$

Remark 1.4.1 *Note that if x and y are real numbers, then*

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

Example 1.4.0.10 *Given that $a < b$ and that $\dfrac{|p-a|}{|b-a|} = \dfrac{3}{4}$, express p in terms of a and b if $p < a$.*

Solution: Since $a < b$, we have $b-a > 0$ so that $|b-a| = b-a$. Thus,

$$\frac{|p-a|}{|b-a|} = \frac{|p-a|}{b-a} = \frac{3}{4}.$$

Since $b-a \neq 0$, we cross multiply to get $|p-a| = \frac{3}{4}(b-a)$. But $p-a < 0$, so we have $|p-a| = -(p-a) = \dfrac{3b}{4} - \dfrac{3a}{4}$, or $-p+a = \dfrac{3b}{4} - \dfrac{3a}{4}$. Solving for p we get $p = \dfrac{7a}{4} - \dfrac{3b}{4}$.

A complex number is an expression of the form $a+ib$, where a and b are real numbers and $i^2 = -1$. For convenience, we shall mainly use the letter z to represent a complex number. For example $z = 4+5i$.

The set of all such numbers is called the set of complex numbers and is denoted by $\mathbb{C}$.

Remark 1.5.1 *Note that when a letter is used for the imaginary part we put i before the letter, e.g ib. However, when a number is written for the imaginary part we write i after the number, e.g 7i. This is just a matter of preference as the position of i be placed before or after in any of the situations without causing confusion.*

If $z = a + ib \in \mathbb{C}$, then we call a the real part of z and b the imaginary part of z. We write this as $Re(z) = a$ and $Im(z) = b$. Note that the imaginary part of the complex number $z = a + ib$ is just the real number b without the i. The representation of a complex number in the form $a + ib$ is called the Cartesian form of the complex number. If $b = 0$, $z = a$, and in this case z is said to be purely real. Similarly, if $a = 0$ then $z = ib$ and z is said to be purely imaginary. From these comments we see that the set of real numbers is a subset of the set of complex numbers. That is, $\mathbb{R} \subset \mathbb{C}$

Example 1.5.0.11 *Let $z = -3 - 4i$. State $Re(z)$ and $Im(z)$.*

Solution: We have $Re(z) = -3$ and $Im(z) = -4$.

1.5.1 Equal complex numbers

Definition 1.5.1 *Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. Then $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$. That is, two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.*

Example 1.5.1.1 *(i) Find the value of x and the value of y given that $x + iy = 5 + 4i$.*

(ii) Find the value of a and b if $(a + b) + i(a - b) = 7 - 3i$.

Solution:

(i) Equating the real parts and the imaginary parts we have $x = 5$ and $y = 4$.

(ii) Again equating the real parts and the imaginary parts gives the simultaneous equations

$$\begin{aligned} a + b &= 7 \\ a - b &= -3 \end{aligned}$$

Solving the system of equations yields $a = 2$ and $b = 5$.

1.5.2 Addition and subtraction of complex numbers

Let $z = a + ib$ and $w = u + iv$ be two complex numbers. Then, the sum $z + w$ is given by

$$\begin{aligned} z + w &= (a + ib) + (u + iv) \\ &= (a + u) + ib + iv \\ &= (a + u) + (b + v)i. \end{aligned}$$

Example 1.5.2.1 *Find the sum* $z + w$ *if* $z = 4 + 5i$ *and* $w = 3 - 2i$.

Solution: Adding the real parts and the imaginary parts we have

$$\begin{aligned} z + w &= (4 + 3) + (5 - 2)i \\ &= 7 + 3i. \end{aligned}$$

Example 1.5.2.2 *Find* x *and* y *such that* $(3x - iy) + (2 + 13i) = -7 + 3i$.

Solution: We first add the real parts and the imaginary parts on the left and we get

$$(3x + 2) + i(13 - y) = -7 + 3i.$$

Equating the real parts and the imaginary parts we get

$$
\begin{aligned}
3x + 2 &= -7 \\
13 - y &= 3
\end{aligned}
$$

Solving the two equations we get

$$x = -3 \text{ and } y = 10.$$

Similarly, if $z = a + ib$ and $w = u + iv$, then we define subtraction $z - w$ by

$$
\begin{aligned}
z - w &= (a + ib) - (u + iv) \\
&= (a - u) + i(b - v).
\end{aligned}
$$

Example 1.5.2.3 *Find $z - w$ if $z = 4 + 5i$ and $w = 3 - 2i$.*

Solution: We have

$$
\begin{aligned}
z - w &= (4 - 3) + (5 - (-2))i \\
&= 7 + 7i.
\end{aligned}
$$

1.5.3 Multiplication of complex numbers

Let $z = a + ib$ and $w = u + iv$ be two complex numbers. We define the multiplication of z and w by

$$
\begin{aligned}
zw &= (a + ib)(u + iv) \\
&= au + iav + ibu + i^2bv \\
&= au + i(av + bu) - bv \\
&= au - bv + i(av + bu).
\end{aligned}
$$

We see that $Re(zw) = au - bv$ and $Im(zw) = av + bu$.

Example 1.5.3.1 *Given that* $z = 3 - 4i$ *and* $w = 2 + 5i$. *Compute* zw.

Solution: We have

$$\begin{aligned} zw &= (3-4i)(2+5i) \\ &= 6 + 15i - 8i - 20(i^2) \\ &= 6 + 20 + (15-8)i \\ &= 26 + 7i. \end{aligned}$$

If the expression contains more that two factors, we multiply the factors together in stages.

Example 1.5.3.2 *Multiply* $(3+4i)(5-8i)(1-2i)$.

Solution: Starting with the first two factors and then multiplying the result with the third factor we have the following

$$\begin{aligned} (3+4i)(5-8i)(1-2i) &= (15 + 32 + (-24+20)i)(1-2i) \\ &= (47-4i)(1-2i) \\ &= 47 - 8 + (-94-4)i \\ &= 39 - 98i. \end{aligned}$$

From the previous example, we observed that $i^3 = -i$, because $i^2 = -1$. Continuing, we see that $i^4 = i^2 \times i^2 = -1 \times -1 = 1$, $i^5 = i$ and $i^6 = -1$.

In summary, to calculate any high power of i, you can convert it to a lower power by taking the closest multiple of 4 that is no longer bigger than the exponent and subtract this from the exponent. For example, $i^{99} = i^{(96+3)} = i^3 = -i$.

Example 1.5.3.3 *Simplify each of the following:*

1. i^{17}
2. i^{120}
3. i^{64002}

1.5.4 Conjugate complex numbers

Definition 1.5.2 *Let $z = x + iy$ be a complex number. Then the conjugate of z is denoted by $\overline{z}$ and it is defined by*

$$\overline{z} = x - iy.$$

Note that we only change the sign of the imaginary part.

For example, the conjugate of $4 - 7i$ is $4 + 7i$, and the conjugate of $-9 + 13i$ is $-9 - 13i$.

Remark 1.5.2 *If $z = x + iy$ is a complex number, then the product $z\overline{z}$ is a real number given by*

$$z\overline{z} = (x+iy)(x-iy) = x^2 + y^2.$$

For example, compute $(5+8i)(5-8i)$.

Solution:

$$\begin{aligned}(5+8i)(5-8i) &= 25 - 40i + 40i + 64 \\ &= 25 + 64 \\ &= 89\end{aligned}$$

Remark 1.5.3 *If $z = x+iy$ is a complex number, then $z+\overline{z} = 2Re(z)$ and $z - \overline{z} = 2Im(z)$.*

Example 1.5.4.1 *Given that $z = 5+4i$ compute*

(i) $z - \bar{z}$

(ii) $z + \bar{z}$

(iii) $z\bar{z}$.

1.5.5 Modulus of a complex number

Definition 1.5.3 *Let $z = x+iy$ be a complex number. Then, the modulus of z is denoted by $|z|$ and it is defined by*

$$|z| = \sqrt{x^2+y^2}.$$

Note that $|z|^2 = x^2 + y^2 = z\overline{z}$.

Example 1.5.5.1 *Calculate the modulus of each of the following complex numbers.*

(i) $5 - 12i$

(ii) $-7 - 5i$

(iii) $15i$

Solution:

(i) $|5 - 12i| = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$

(ii) $|-7 - 5i| = \sqrt{(-7)^2 + (-5)^2} = \sqrt{49 + 25} = \sqrt{74}.$

(iii) $|15i| = \sqrt{0^2 + 15^2} = \sqrt{225} = 15.$

1.5.6 Division of complex numbers

Dividing a complex number by a real number is easily done by the basic processes of algebra. For example, if $z = 9 - 12i$, then $\frac{z}{3} = \frac{9-12i}{3} = \frac{3(3-4i)}{3} = 3 - 4i$. Similarly, $\frac{7+2i}{5} = \frac{7}{5} + \frac{2}{5}i$. However, to deal with a situation where the denominator is also a complex number such as $\frac{7-4i}{4+3i}$, we must find other means to do it.

The rule with regard to dividing two complex numbers is that you multiply both the numerator and the denominator by the conjugate of the denominator. This transforms the denominator into a real number. The simplification can then be done as above. Thus, if z and w are two complex numbers, then the division $\frac{z}{w}$ is given by

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}}.$$

Example 1.5.6.1 *Let $z = 7 - 5i$ and $w = -3 + 2i$ be two complex numbers. Find $\frac{z}{w}$.*

Solution:

$$\begin{aligned}
\frac{z}{w} &= \frac{7-5i}{-3+2i} \\
&= \frac{(7-5i)(-3-2i)}{(-3+2i)(-3-2i)} \\
&= \frac{-21-14i+15i-10}{(-3)^2+(2)^2} \\
&= \frac{-31+i}{13} \\
&= \frac{-31}{13} + \frac{1}{13}i.
\end{aligned}$$

Example 1.5.6.2 *Simplify each of the following:*

(a)
$$\frac{3+2i}{1-3i}$$

(b)
$$\frac{1+i}{3-2i} + 2 + 4i$$

Solution:

(a) $\frac{3+2i}{1-3i} = \frac{(3+2i)(1+3i)}{(1^2+(-3)^2)} = \frac{9-7i}{10} = \frac{9}{10} - \frac{7}{10}i$

(b)

$$\begin{aligned}
\tfrac{1+i}{3-2i} + 2 + 4i &= \tfrac{(1+i)(3+2i)}{9+4} + (2+4i) \\
&= \tfrac{1+5i}{13} + (2+4i) \\
&= \tfrac{1}{13} + \tfrac{5}{13}i + 2 + 4i \\
&= (\tfrac{1}{13} + 2) + (\tfrac{5}{13} + 4)i \\
&= \tfrac{27}{13} + \tfrac{57}{13}i.
\end{aligned}$$